Convexity, concavity, super-additivity, and sub-additivity of cost function without fixed cost

Tanaka, Yasuhito and Hattori, Masahiko

31 July 2017

Convexity, concavity, super-additivity, and sub-additivity of cost function without fixed cost

Masahiko Hattori*
Faculty of Economics, Doshisha University,
Kamigyo-ku, Kyoto, 602-8580, Japan.
and
Yasuhito Tanaka†
Faculty of Economics, Doshisha University,
Kamigyo-ku, Kyoto, 602-8580, Japan.

Abstract

With zero fixed cost, convexity of a cost function implies super-additivity, and concavity of a cost function implies sub-additivity. But converse relations do not hold. However, in addition to the zero fixed cost condition we put the following assumption.

(1) If a cost function is convex in some interval, it is convex throughout the domain.

(2) If a cost function is concave in some interval, it is concave throughout the domain.

Then, super-additivity implies convexity and sub-additivity implies concavity. Subsequently, super-additivity and convexity are equivalent, and sub-additivity and concavity are equivalent.

Keywords: cost function, convexity, concavity, super-additivity, sub-additivity

JEL Classification: D43, L13

*mhattori@mail.doshisha.ac.jp
†yasuhito@mail.doshisha.ac.jp

1 Introduction

Convexity and concavity are important properties for cost functions of firms. Also super-additivity and sub-additivity are other important properties for them. A cost function $c(x)$ is convex when it satisfies

$$c(\lambda x + (1-\lambda)y) \leq \lambda c(x) + (1-\lambda)c(y) \text{ for } 0 \leq \lambda \leq 1,\ x \geq 0,\ y \geq 0.$$

It is concave when it satisfies

$$c(\lambda x + (1-\lambda)y) \geq \lambda c(x) + (1-\lambda)c(y) \text{ for } 0 \leq \lambda \leq 1,\ x \geq 0,\ y \geq 0.$$

It is super-additive if it satisfies

$$c(x+y) \geq c(x) + c(y), \text{ for } x \geq 0,\ y \geq 0.$$

It is sub-additive if it satisfies

$$c(x+y) \leq c(x) + c(y), \text{ for } x \geq 0,\ y \geq 0.$$

It is well known that with zero fixed cost, that is, $c(0) = 0$, convexity implies super-additivity, and concavity implies sub-additivity. But converse relations do not hold. See Bruckner and Ostrow (1962) and Sen and Stamatopoulos (2016). Referring to Bourin and Hiai (2015), Sen and Stamatopoulos (2016) pointed out that the following function is super-additive but it is not convex.

$$xe^{-\frac{1}{x^2}},\ x \geq 0.$$

However, if, in addition to the zero fixed cost condition, we put the following assumption, we can show that super-additivity implies convexity, and sub-additivity implies concavity.

Assumption 1. (1) *If a cost function is convex in some interval, it is convex throughout the domain.*

(2) *If a cost function is concave in some interval, it is concave throughout the domain.*

Then, super-additivity and convexity are equivalent, and sub-additivity and concavity are equivalent.

Assumption 1 excludes a case where a cost function is strictly convex in some interval and strictly concave in another interval. Above mentioned $xe^{-\frac{1}{x^2}}$ is such a function.

We assume that a cost function, $c(x)$, is positive for positive x, increasing and continuous.

2 Main results

We show the following theorems.

Theorem 1. *(1) If there is no fixed cost, convexity of a cost function implies its super-additivity.*

(2) If there is no fixed cost, concavity of a cost function implies its sub-additivity.

Proof. (1) Convexity ⇒ super-additivity:

For $0 \le \lambda \le 1$ and $x \ge 0$ convexity implies

$$c(\lambda x + (1-\lambda)\cdot 0) \le \lambda c(x) + (1-\lambda)c(0) = \lambda c(x).$$

Thus,

$$c(\lambda x) \le \lambda c(x).$$

Then, for $x > 0$ and $y \ge 0$

$$\begin{aligned} c(x) + c(y) =& c\left(\frac{x}{x+y}(x+y)\right) + c\left(\frac{y}{x+y}(x+y)\right) \\ \le& \frac{x}{x+y}c(x+y) + \frac{y}{x+y}c(x+y) = c(x+y). \end{aligned}$$

(2) Concavity ⇒ sub-additivity:

For $0 \le \lambda \le 1$ and $x \ge 0$ concavity implies

$$c(\lambda x + (1-\lambda)\cdot 0) \ge \lambda c(x) + (1-\lambda)c(0) = \lambda c(x).$$

Thus,

$$c(\lambda x) \ge \lambda c(x).$$

Then, for $x > 0$ and $y \ge 0$

$$\begin{aligned} c(x) + c(y) =& c\left(\frac{x}{x+y}(x+y)\right) + c\left(\frac{y}{x+y}(x+y)\right) \\ \ge& \frac{x}{x+y}c(x+y) + \frac{y}{x+y}c(x+y) = c(x+y). \end{aligned}$$

□

Theorem 2. *Suppose that a cost function satisfies Assumption 1, and there is no fixed cost.*

(1) Super-additivity of a cost function implies convexity.

(2) Sub-additivity of a cost function implies concavity.

Proof. (1) Super-additivity ⇒ convexity:

Let $x > 0$. By super-additivity, for any $0 \le \lambda \le 1$,

$$c(x) \ge c(\lambda x) + c((1-\lambda)x).$$

This implies, for any $0 \le \lambda \le 1$,

$$c(x) \ge c(\lambda x + (1-\lambda)\cdot 0) + c(\lambda \cdot 0 + (1-\lambda)x). \tag{1}$$

By the zero fixed cost condition, for any $0 \le \lambda \le 1$,

$$c(x) = \lambda c(x) + (1-\lambda)c(0) + \lambda c(0) + (1-\lambda)c(x). \tag{2}$$

(1) and (2) imply that, for any $0 \le \lambda \le 1$,

$$\lambda c(x) + (1-\lambda)c(0) \ge c(\lambda x + (1-\lambda)\cdot 0), \tag{3a}$$

or

$$\lambda c(0) + (1-\lambda)c(x) \ge c(\lambda \cdot 0 + (1-\lambda)x) \tag{3b}$$

holds. Suppose that for some x and some $0 < \lambda < 1$, (3a) does not hold, that is,

$$\lambda c(x) + (1-\lambda)c(0) < c(\lambda x + (1-\lambda)\cdot 0). \tag{4a}$$

Then, (3b) must hold with strict inequality, that is,

$$\lambda c(0) + (1-\lambda)c(x) > c(\lambda \cdot 0 + (1-\lambda)x). \tag{4b}$$

(4a) means that $c(x)$ can not be convex (nor linear) throughout $[0, x]$. Similarly, (4b) means that $c(x)$ can not be concave (nor linear) throughout $[0, x]$. It contradicts to Assumption 1.

> If $c(x)$ is not differentiable at, for example, y, it may be not concave nor convex (nor linear) in an interval including y. However, if we exclude a pathological function which is continuous everywhere but differentiable nowhere, any function is concave or convex in some interval. Then, by Assumption 1 it is concave or convex throughout the domain.

Therefore, both (3a) and (3b) must hold for any $x > 0$ and any $0 \le \lambda \le 1$. Alternatively, we can prove the same result assuming that (3b) instead of (3a) does not hold.

Let $y = \alpha x$ for $x > 0$ and $0 < \alpha < 1$. Assume that for some $0 < \lambda < 1$,

$$\lambda c(x) + (1-\lambda)c(y) < c(\lambda x + (1-\lambda)y). \tag{5}$$

Let

$$z = \lambda x + (1-\lambda)y = [\lambda + (1-\lambda)\alpha]x.$$

By (3a)

$$\alpha c(x) \geq c(\alpha x) = c(y).$$

Then, from (5)

$$\left(\frac{\lambda}{\alpha} + 1 - \lambda\right) c(y) < c(z).$$

This means

$$c(y) < \beta c(z) + (1 - \beta) c(0), \tag{6}$$

where

$$\beta = \frac{\alpha}{\lambda + (1-\lambda)\alpha} > 0 \text{ and } 1 - \beta = \frac{\lambda(1-\alpha)}{\lambda + (1-\lambda)\alpha} > 0.$$

(5) means that $c(x)$ can not be convex (nor linear) throughout $[y, x]$. (6) means that $c(x)$ can not be concave (nor linear) throughout $[0, z]$. It contradicts to Assumption 1. Therefore, for any $x > y \geq 0$ and any $0 \leq \lambda \leq 1$

$$\lambda c(x) + (1-\lambda) c(y) \geq c(\lambda x + (1-\lambda) y).$$

Thus, $c(x)$ is convex throughout the domain.

(2) Sub-additivity ⇒ concavity:

Let $x > 0$. By sub-additivity, for any $0 \leq \lambda \leq 1$,

$$c(x) \leq c(\lambda x) + c((1-\lambda)x).$$

This implies, for any $0 \leq \lambda \leq 1$,

$$c(x) \leq c(\lambda x + (1-\lambda) \cdot 0) + c(\lambda \cdot 0 + (1-\lambda) x). \tag{7}$$

By the zero fixed cost condition, for any $0 \leq \lambda \leq 1$,

$$c(x) = \lambda c(x) + (1-\lambda) c(0) + \lambda c(0) + (1-\lambda) c(x). \tag{8}$$

(7) and (8) imply that, for any $0 \leq \lambda \leq 1$,

$$\lambda c(x) + (1-\lambda) c(0) \leq c(\lambda x + (1-\lambda) \cdot 0), \tag{9a}$$

or

$$\lambda c(0) + (1-\lambda) c(x) \leq c(\lambda \cdot 0 + (1-\lambda) x) \tag{9b}$$

holds. Suppose that for some x and some $0 < \lambda < 1$, (9a) does not hold, that is,

$$\lambda c(x) + (1-\lambda) c(0) > c(\lambda x + (1-\lambda) \cdot 0). \tag{10a}$$

Then, (9b) must hold with strict inequality, that is,

$$\lambda c(0) + (1-\lambda) c(x) < c(\lambda \cdot 0 + (1-\lambda) x). \tag{10b}$$

(10a) means that $c(x)$ can not be concave throughout $[0, x]$. Similarly, (10b) means that $c(x)$ can not be convex throughout $[0, x]$. It contradicts to Assumption 1. Therefore, both (9a) and (9b) must hold for any $x > 0$ and any $0 \leq \lambda \leq 1$. Alternatively, we can prove the same result assuming that (9b) instead of (9a) does not hold.

Let $y' = \alpha' x$ for $x > 0$ and $0 < \alpha' < 1$. Assume that for some $0 < \lambda < 1$,

$$\lambda c(x) + (1 - \lambda)c(y') > c(\lambda x + (1 - \lambda)y'). \tag{11}$$

Let

$$z' = \lambda x + (1 - \lambda)y' = [\lambda + (1 - \lambda)\alpha']x.$$

By (9a)

$$\alpha' c(x) \leq c(\alpha' x) = c(y').$$

Then, from (11)

$$\left(\frac{\lambda}{\alpha'} + 1 - \lambda\right) c(y') > c(z').$$

This means

$$c(y') > \beta' c(z') + (1 - \beta')c(0), \tag{12}$$

where

$$\beta' = \frac{\alpha'}{\lambda + (1 - \lambda)\alpha'} > 0, \text{ and } 1 - \beta' = \frac{\lambda(1 - \alpha')}{\lambda + (1 - \lambda)\alpha'} > 0.$$

(11) means that $c(x)$ can not be concave throughout $[y', x]$. (12) means that $c(x)$ can not be convex throughout $[0, z']$. It contradicts to Assumption 1. Therefore, for any $x > y' \geq 0$ and any $0 \leq \lambda \leq 1$

$$\lambda c(x) + (1 - \lambda)c(y') \leq c(\lambda x + (1 - \lambda)y').$$

Thus, $c(x)$ is concave throughout the domain.

□

If $c(x)$ is twice continuously differentiable, convexity of $c(x)$ is equivalent to non-negativity of its second order derivative, that is, $c''(x) \geq 0$, and its concavity is equivalent to non-positivity of its second order derivative, that is, $c''(x) \leq 0$. We present a proof of Theorem 2 using differentiability.

Proof of Theorem 2 with differentiability. (1) Super-additivity ⇒ convexity:

Let $x > 0$ and $y > 0$. By super-additivity

$$c(x + y) \geq c(x) + c(y).$$

With the zero fixed cost condition, we have

$$\frac{c(x + y) - c(x)}{y} \geq \frac{c(y) - c(0)}{y}.$$

Let $y \to 0$,

$$\lim_{y\to 0} \frac{c(x+y) - c(x)}{y} \geq \lim_{y\to 0} \frac{c(y) - c(0)}{y}.$$

Thus,

$$c'(x) \geq c'(0).$$

This means

$$\frac{c'(x) - c'(0)}{x} \geq 0.$$

Let $x \to 0$, we obtain

$$c''(0) = \lim_{x\to 0} \frac{c'(x) - c'(0)}{x} \geq 0.$$

Suppose that there exist $x_2 > x_1 > 0$ such that $c'(x_2) < c'(x_1)$. Since $c'(x_2) \geq c'(0)$, we have $c'(x_1) > c'(0)$. Then, $c(x)$ is strictly concave in an interval within $[x_1, x_2]$ and at the same time strictly convex in an interval within $[0, x_1]$. It contradicts to Assumption 1. Thus, $c''(x) \geq 0$ throughout the domain and $c(x)$ is convex.

(2) Sub-additivity ⇒ concavity:

Let $x > 0$ and $y > 0$. By sub-additivity

$$c(x+y) \leq c(x) + c(y).$$

With the zero fixed cost condition, we have

$$\frac{c(x+y) - c(x)}{y} \leq \frac{c(y) - c(0)}{y}.$$

Let $y \to 0$,

$$\lim_{y\to 0} \frac{c(x+y) - c(x)}{y} \leq \lim_{y\to 0} \frac{c(y) - c(0)}{y}.$$

Thus,

$$c'(x) \leq c'(0).$$

This means

$$\frac{c'(x) - c'(0)}{x} \leq 0.$$

Let $x \to 0$, we obtain

$$c''(0) = \lim_{x\to 0} \frac{c'(x) - c'(0)}{x} \leq 0.$$

Suppose that there exist $x_2' > x_1' > 0$ such that $c'(x_2') > c'(x_1')$. Since $c'(x_2') \leq c'(0)$, we have $c'(x_1') < c'(0)$. Then, $c(x)$ is strictly convex in an interval within $[x_1', x_2']$ and at the same time strictly concave in an interval within $[0, x_1']$. It contradicts to Assumption 1. Thus, $c''(x) \leq 0$ throughout the domain and $c(x)$ is concave.

□

3 Concluding Remark

We usually assume that a cost function is convex throughout the domain or concave throughout the domain. Therefore, if there is no fixed cost, we can use interchangeably convexity and super-additivity of a cost function, and can use interchangeably concavity and subadditivity of a cost function.

References

Sen, D. and Stamatopoulos, G. (2016) "Licensing under general demand and cost functions," *European Journal of Operations Research*, **253**, pp. 673-680.

Bruin, J.-C. and Hiai, F. (2015) "Anti-norms on finite von Neumann algebras," *Publications of the Research Institute for Mathematical Sciences*, **51**, pp. 207-235.

Bruckner, A. M. and Ostrow, E. (1962) "Some function classes related to the class of convex functions," *Pacific Journal of Mathematics*, **14**, pp. 1203-1215.