# Duality in land rental problems

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28 July 2017

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### Abstract

A single tenant demands a particular amount of land from several
lessors. The way lessors rent their land can be seen as equivalent to a
bankruptcy problem. We extend self-duality in bankruptcy problems
to these land rental situations. We provide a complete characterization
of the family of rules that satisfy self-duality. Moreover, self-duality
is enough to assure that the amount of land is proportionally shared
among the lessors. Adding other reasonable properties, we pick up a
single rule.

**Keywords:** Self-duality; land rental problems; bankruptcy problems;
proportional rule

## 1   Introduction

Duality is an important concept in mathematics which usually implies the
involution operation, i.e. given $x$ and its dual $y$, then the dual of $y$ is $x$.

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Even though duality has different interpretations depending on the area, we focus on the interpretation in bankruptcy problems (O'Neill, 1982), which is consequently extended to a land rental problem context.

A land rental problem involves several lessors and a single tenant, who needs to rent land, but less than all land that is owned by the lessors. The main question is knowing how much land each lessor should rent and the price per unit of land (Sarkar, 2017; Valencia-Toledo and Vidal-Puga, 2017).

Bankruptcy problems are situations where a group of agents $(N)$ have some claims $(c \in \mathbb{R}_+^N)$ that sum up more than the particular amount of land demanded by the tenant $(E)$, i.e. $\sum_{i \in N} c_i > E$. A bankruptcy rule is a way to share the total estate among the claimants. For each claimant, her share can be seen either as an award of part of her claim or as a loss of the other part.

The self-duality principle in bankruptcy problems (Aumann and Maschler, 1985) states that the criterion used to grant awards should coincide with the one used to grant losses. Many important rules in the context of bankruptcy problems satisfy self-duality, for example, the proportional rule (Young, 1988; Herrero and Villar, 2001, 2002; Herrero, 2003), the Talmut rule (Chun, 1988), the Contested Garment principle (Dagan, 1996), and the reverse Talmud rule (van den Brink et al., 2013). For a survey, we refer to Thomson (2003, 2015); Moulin (2002).

We extend the idea of self-duality to land rental problems. The lessors can be seen as claimants in a bankruptcy problem. Lessors' claims are the land they own, and the estate is the amount of land that the tenant needs. Moreover, land renting problems have other elements to take into account, as for example the fact that the tenant is also an agent[1]. There is a revenue, or reservation price, that every lessor obtains by each unit of non-rented land, which is denoted by $r$. As opposed, the revenue that the tenant can get by each unit of rented land is denoted by $R$. We always assume that there exists

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[1]See Csóka and Herings (2017) for another extension of bankruptcy problems in which the estate holder is an explicit agent.

benefit of cooperation, i.e. $R > r$.

We apply the concept of self-duality to the land share. We can think in terms of how to share the amount of land that is not rented instead of the amount of rented land, since they are equivalent problems.

In order to study self-duality, we define the dual problem of a land rental problem as follows. The agents and the amount of available land remain unchanged. Meanwhile, the dual of the amount of land that the tenant needs is given by $E^d = c(N) - E$. The dual of the reservation price per unit of land (denoted by $r^d$) satisfies $r^d E^d = r E$. This is done so that the revenue for the lessors are equal in both the original problem and its dual problem. Also, the dual of the revenue per unit of land for the tenant (denoted by $R^d$) satisfies $R^d E^d = RE$. This means that the revenue for the tenant are equal in both the original problem and its dual problem.

We provide the complete characterization of the family of rules that satisfy self-duality. Under these rules, the amount of land is shared by the proportional rule. As opposed, in bankruptcy problems, it is necessary at least to consider some additional properties apart from self-duality.

The remainder of this paper is organized as follows. In Section 2, we present the model. In Section 3, we define self-duality and we provide a complete characterization of the family of rules that satisfy it. Also, we present charaterizations of a subfamily of rules, adding price independence (The price does not depend on the distribution among the lessors of the available land.), and a unique rule, adding a property that states how the share should be done in the 2-person case[2]. In Section 4, we give some concluding remarks.

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[2]Properties that provide conditions on how the share should be in the 2-person case have played an important role in the axiomatization of many solutions in different contexts, e.g., in cooperative games (Hart and Mas-Colell, 1989; van den Brink and Funaki, 2009; Huettner, 2015; van den Brink et al., 2016), bankruptcy problems (Thomson, 2008), and land rental problems (Valencia-Toledo and Vidal-Puga, 2017).

# 2 Notation and model

We denote the set of nonnegative real numbers as $\mathbb{R}_+$. We denote the set of positive real numbers as $\mathbb{R}_{++}$. Let $\mathbb{N}_+ = \{1, 2, \ldots\}$ be the set of potential lessors and let $\mathcal{N}$ be the set of non-empty finite subsets of $\mathbb{N}_+$. Let $N = \{1, 2, \ldots, n\} \in \mathcal{N}$ be a generic set of lessors, and let $S$ be a generic subset of $N$. Let $\mathbb{R}^S$ be the $|S|$-dimensional Euclidean space, whose coordinates are indexed by the elements of $S$. Given $y \in \mathbb{R}^S$, we write $y(S) = \sum_{i \in S} y_i$. Given $x, y \in \mathbb{R}^S$, we write $x \leq y$ when $x_i \leq y_i$ for all $i \in S$. Given $x \in \mathbb{R}^N$ and $S \subset N$, we denote the restriction of $x$ on $S$ as $x_S$ i.e. $x_S = (x_i)_{i \in S}$. Moreover, the vectors $(0, \ldots, 0) \in \mathbb{R}^S$ and $(1, \ldots, 1) \in \mathbb{R}^S$ are denoted by $0_S$ and $1_S$, respectively.

A *land rental problem* is a tuple $L = (N_0, c, r, K, E)$ where $N_0 = \{0\} \cup N$ is the set of agents with $0$ the unique tenant and $N$ the set of lessors, $c \in \mathbb{R}_{++}^N$ is the vector whose coordinates represent the amount of available land for each lessor, $r \in \mathbb{R}_{++}$ is the reservation price per unit of land for the lessors, $E \in \left]0, c(N)\right[$ is the amount of land that the tenant needs to rent, and $K \in \mathbb{R}_{++}$ is the revenue that the tenant can obtain from that amount of rented land. In order to assure that there exists benefit of cooperation, we assume $K > rE$. We define $R = \frac{K}{E}$ as the revenue per unit of land that the tenant gets with her activity. For convenience, we denote $(N, c, r, R, E)$ instead of $(N_0, c, r, K, E)$. Let $\mathcal{L}$ be the set of all land rental problems.

An efficient feasible agreement is a pair $(p, x) \in \mathbb{R}_+ \times \mathbb{R}_+^N$ where $p$ denotes the price per unit of land, $x \leq c$ and $x(N) = E$, so that $x_i$ denotes the land rented by lessor $i \in N$. We denote $A^L$ as the set of feasible agreements on a land rental problem $L$. Let $\mathscr{A} = \bigcup_{L \in \mathcal{L}} A^L$ be the set of all potential feasible agreements.

Given $(p, x) \in A^L$, the utility for tenant and each lessor $i \in N$ are $u_0^L(p, x) = (R - p)E$ and $u_i^L(p, x) = (p - r)x_i$, respectively. Notice that with a slight notation abuse and for simplicity, we write $u_i^L(p, x)$ instead of $u_i^L((p, x))$ for all $i \in N_0$.

A *rule* as a function $\psi : \mathcal{L} \longrightarrow \mathscr{A}$ that assigns to each problem $L =$

$(N, c, r, R, E) \in \mathcal{L}$ a pair $\left(p^{\psi}(L), x^{\psi}(L)\right) = \psi(L) \in A^{L}$ satisfying the following conditions:

(i) *Scale invariance* says that the final price and the amount of rented land are independent of changes of scale i.e. $p^{\psi}\left(N, \beta c, \frac{\alpha}{\beta}r, \frac{\alpha}{\beta}R, \beta E\right) = \frac{\alpha}{\beta}p^{\psi}(N, c, r, R, E)$ and $x^{\psi}\left(N, \beta c, \frac{\alpha}{\beta}r, \frac{\alpha}{\beta}R, \beta E\right) = \beta x^{\psi}(N, c, r, R, E)$ for all $\alpha, \beta > 0$.

(ii) *Strict individual rationality* says that the price is greater than $r$ and less than $R$, i.e. $r < p^{\psi}(N, c, r, R, E) < R$, in order to assure the benefit of the lessors and the tenant respectively.

(iii) *Continuity* says that $p^{\psi}$ and $x^{\psi}$ are continuous functions.

Let $\Psi$ be the set of all rules $\psi$.

# 3 Duality

Given a land rental problem $L = (N, c, r, R, E)$, we define its dual as $L^{d} = \left(N, c, r^{d}, R^{d}, E^{d}\right) \in \mathcal{L}$ where $E^{d} = c(N) - E$ and $r^{d}$ and $R^{d}$ are such that $r^{d}E^{d} = rE$ and $R^{d}E^{d} = RE$, respectively. Hence, $r^{d} = \frac{rE}{c(N)-E}$ and $R^{d} = \frac{RE}{c(N)-E}$.

**Definition 1.** *A rule $\psi$ satisfies self-duality if for all $L \in \mathcal{L}$, $u^{L}\left(p^{\psi}(L), x^{\psi}(L)\right) = u^{L^{d}}\left(p^{\psi}\left(L^{d}\right), x^{\psi}\left(L^{d}\right)\right)$ and $x^{\psi}(L) = c - x^{\psi}\left(L^{d}\right)$.*

Let $\Theta = \left\{(e, t) \in \left]0, \frac{1}{2}\right] \times \left]0, 1\right[\right\}$, which is illustrated in Figure 1, and $\Delta^{\mathcal{N}} = \left\{\hat{c} \in \mathbb{R}_{++}^{N} : N \in \mathcal{N}, \hat{c}(N) = 1\right\}$. Let $\mathcal{F}$ be the set of continuous functions $f : \Theta \times \Delta^{\mathcal{N}} \to \mathbb{R}$ with $f(e, t, \hat{c}) \in \left]t, 1\right[$ for all $(e, t) \in \Theta$ and $\hat{c} \in \Delta^{\mathcal{N}}$.
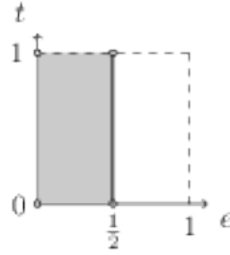
Figure 1: Dual area ($\Theta$).

In Proposition 1, we state the characterization result.

**Proposition 1.** *A rule $\psi$ satisfies self-duality iff there exists $f \in \mathcal{F}$ such that for all $L = (N, c, r, R, E) \in \mathcal{L}$, the price is given by*

$$p^\psi(L) = \begin{cases} Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right), & \text{if } c(N) > 2E \\ Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right), & \text{if } c(N) \leq 2E, \end{cases}$$

*and the amount of land is given by $x^\psi(L) = \frac{E}{c(N)}c$.*

*Proof.* ($\Rightarrow$) Let $\psi$ be a rule that satisfies self-duality. Let $L = (N, c, r, R, E) \in \mathcal{L}$. Preliminarily, we prove that the dual price satisfies

$$p^\psi(L^d) = \frac{E}{c(N) - E}p^\psi(L). \tag{1}$$

By self-duality, the tenant's utility is $\left(R - p^\psi(L)\right)E = \left(R^d - p^\psi(L^d)\right)E^d$. This is equivalent to write $\left(R - p^\psi(L)\right)E = \left(\frac{RE}{c(N)-E} - p^\psi(L^d)\right)(c(N) - E)$, from which we deduce that (1) holds.

Now, we prove that there exists a function $f \in \mathcal{F}$ such that the price is given by $p^\psi(L) = Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$ if $c(N) > 2E$ and $p^\psi(L) = Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$ if $c(N) \leq 2E$. By scale invariance,

$$p^\psi(N, c, r, R, E) = Rp^\psi\left(N, \frac{c}{c(N)}, \frac{r}{R}, 1, \frac{E}{c(N)}\right). \tag{2}$$

We define $f(e, t, \hat{c}) = p^\psi(N, \hat{c}, t, 1, e)$ for all $(e, t, \hat{c}) \in \Theta \times \Delta^\mathcal{N}$. By strict individual rationality, $t < f(e, t, \hat{c}) < 1$. Since $p^\psi$ is a continuous function, $f$ is continuous. Hence, $f \in \mathcal{F}$.

6

On the one hand, when $c(N) > 2E$,

$$p^{\psi}(L) \stackrel{(2)}{=} Rp^{\psi}\left(N, \frac{c}{c(N)}, \frac{r}{R}, 1, \frac{E}{c(N)}\right) = Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right).$$

On the other hand, when $c(N) \leq 2E$,

$$\begin{aligned}
p^{\psi}(L) &\stackrel{(1)}{=} \frac{c(N) - E}{E} p^{\psi}\left(L^d\right) \\
&= \frac{c(N) - E}{E} p^{\psi}\left(N, c, \frac{rE}{c(N) - E}, \frac{RE}{c(N) - E}, c(N) - E\right) \\
&\stackrel{(2)}{=} \frac{c(N) - E}{E} \frac{RE}{c(N) - E} p^{\psi}\left(N, \frac{c}{c(N)}, \frac{r}{R}, 1, \frac{c(N) - E}{c(N)}\right) \\
&= Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right).
\end{aligned}$$

Finally, we prove that the amount of land is given by $x^{\psi}(L) = \frac{E}{c(N)} c$. By self-duality, for all $i \in N$, $\left(p^{\psi}(L) - r\right) x_i^{\psi}(L) = \left(p^{\psi}\left(L^d\right) - r^d\right) x_i^{\psi}\left(L^d\right)$. By (1) and the last part of self-duality $(x^{\psi}(L) = c - x^{\psi}(L^d))$, it is equivalent to write

$$\left(p^{\psi}(L) - r\right) x_i^{\psi}(L) = \left(\frac{Ep^{\psi}(L)}{c(N) - E} - \frac{rE}{c(N) - E}\right)\left(c_i - x_i^{\psi}(L)\right). \quad (3)$$

Since $p^{\psi}(L) > r$, (3) is simplified to $(c(N) - E) x_i^{\psi}(L) = E\left(c_i - x_i^{\psi}(L)\right)$, from where we deduce $x_i^{\psi}(L) = \frac{Ec_i}{c(N)}$ for all $i \in N$.

$(\Leftarrow)$ Let $\psi$ be a rule given by $p^{\psi}(L) = Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$ if $c(N) > 2E$ and $p^{\psi}(L) = Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$ if $c(N) \leq 2E$, for some $f \in \mathcal{F}$, and $x^{\psi}(L) = \frac{E}{c(N)} c$.

Since the land is shared proportionally, i.e. $x^{\psi}(L) = \frac{E}{c(N)} c$, it is straightforward to check $x^{\psi}(L) = c - x^{\psi}(L^d)$.

We now check that $u^L(\psi(L)) = u^{L^d}\left(\psi(L^d)\right)$. By efficiency it is enough to prove $u_i^L(\psi(L)) = u_i^{L^d}\left(\psi(L^d)\right)$ for all $i \in N$. We have three cases.

First, $c(N) > 2E$. For every $i \in N$, $u_i^L(\psi(L)) = \left(p^{\psi}(L) - r\right) x_i^{\psi}(L)$. Since $c(N) > 2E$, for every $i \in N$, we have

$$u_i^L(\psi(L)) = \left[Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - r\right] x_i^{\psi}(L).$$

Furthermore, for every $i \in N$, we have $u_i^{L^d}\left(\psi(L^d)\right) = \left(p^\psi(L^d) - r^d\right) x_i^\psi(L^d)$. Since $c(N) > 2E$ or $c(N) - 2E > 0$, $2E^d = 2(c(N) - E) = c(N) + c(N) - 2E > c(N)$, i.e. $c(N) < 2E^d$, so that

$$
\begin{aligned}
u_i^{L^d}\left(\psi(L^d)\right) &= \left(p^\psi(L^d) - r^d\right) x_i^\psi(L^d) \\
&= \left[R^d f\left(1 - \frac{E^d}{c(N)}, \frac{r^d}{R^d}, \frac{c}{c(N)}\right) - r^d\right]\left[c_i - x_i^\psi(L)\right] \\
&= \left[\frac{RE}{c(N) - E} f\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - \frac{rE}{c(N) - E}\right]\left[c_i - \frac{Ec_i}{c(N)}\right] \\
&= \frac{E}{c(N) - E}\left[Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - r\right]\frac{(c(N) - E)c_i}{c(N)} \\
&= \left[Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - r\right]\frac{Ec_i}{c(N)} \\
&= \left[Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - r\right] x_i^\psi(L) = u_i^L(\psi(L)).
\end{aligned}
$$

Second, $c(N) < 2E$. For every $i \in N$, $u_i^L(\psi(L)) = \left(p^\psi(L) - r\right) x_i^\psi(L)$. Since $c(N) < 2E$, for every $i \in N$, we have

$$
u_i^L(\psi(L)) = \left[Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - r\right] x_i^\psi(L).
$$

Furthermore, for every $i \in N$, we have $u_i^{L^d}\left(\psi(L^d)\right) = \left(p^\psi(L^d) - r^d\right) x_i^\psi(L^d)$. Since $c(N) < 2E$ or $c(N) - 2E < 0$, $2E^d = 2(c(N) - E) = c(N) + c(N) - 2E < c(N)$, i.e. $c(N) > 2E^d$, so that

$$
\begin{aligned}
u_i^{L^d}\left(\psi(L^d)\right) &= \left(p^\psi(L^d) - r^d\right) x_i^\psi(L^d) \\
&= \left[R^d f\left(\frac{E^d}{c(N)}, \frac{r^d}{R^d}, \frac{c}{c(N)}\right) - \frac{E}{c(N) - E}r\right]\left[c_i - x_i^\psi(L)\right] \\
&= \left[\frac{RE}{c(N) - E} f\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - \frac{rE}{c(N) - E}\right]\left[c_i - \frac{c_i E}{c(N)}\right] \\
&= \frac{E}{c(N) - E}\left[Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - r\right]\frac{(c(N) - E)c_i}{c(N)} \\
&= \left[Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - r\right]\frac{Ec_i}{c(N)} \\
&= \left[Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - r\right] x_i^\psi(L) = u_i^L(\psi(L)).
\end{aligned}
$$

Third, $c(N) = 2E$, for every $i \in N$, we have

$$u_i^L(\psi(L)) = \left(p^\psi(L) - r\right) x_i^\psi(L)$$

$$= \left[Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right) - r\right] \frac{Ec_i}{c(N)}$$

$$= \left[Rf\left(1 - \frac{E}{2E}, \frac{r}{R}, \frac{c}{2E}\right) - r\right] \frac{Ec_i}{2E}$$

$$= \left[\frac{RE}{2E - E}f\left(1 - \frac{2E - E}{2E}, \frac{r}{R}, \frac{c}{2E}\right) - \frac{rE}{2E - E}\right] \left[c_i - \frac{Ec_i}{2E}\right]$$

$$= \left[R^d f\left(1 - \frac{E^d}{c(N)}, \frac{r^d}{R^d}, \frac{c}{c(N)}\right) - r^d\right] \left[c_i - x_i^\psi(L)\right] \tag{4}$$

Since $c(N) = 2E$, $2E^d = 2(c(N) - E) = 2(2E - E) = 2E = c(N)$, i.e. $c(N) = 2E^d$. Then, (4) equals $\left(p^\psi(L^d) - r^d\right) x_i^\psi(L^d)$. Hence, $u_i^L(\psi(L)) = u_i^{L^d}\left(\psi(L^d)\right)$. $\qquad\square$

The rule associated to $f \in \mathcal{F}$ is denoted as $\psi^f$.

The next property states that the price does no depend on the distribution of the land.

**Definition 2.** *A rule $\psi$ satisfies price independence if for all $L \in \mathcal{L}$ and $c' \in \mathbb{R}^N_{++}$ such that $c'(N) = c(N)$, we have $p^\psi(L') = p^\psi(L)$ where $L' = (N, c', r, R, E)$.*

Let $\mathcal{G}$ be the set of continuous functions $g : \Theta \to \mathbb{R}$ with $g(e, t) \in ]t, 1[$ for all $(e, t) \in \Theta$.

**Proposition 2.** *A rule $\psi$ satisfies self-duality and price independence iff there exists $g \in \mathcal{G}$ such that for all $L = (N, c, r, R, E) \in \mathcal{L}$, the price is given by*

$$p^\psi(L) = \begin{cases} Rg\left(\frac{E}{c(N)}, \frac{r}{R}\right), & \text{if } c(N) > 2E \\ Rg\left(1 - \frac{E}{c(N)}, \frac{r}{R}\right), & \text{if } c(N) \le 2E, \end{cases}$$

*and the amount of land is given by $x^\psi(L) = \frac{E}{c(N)}c$.*

*Proof.* ($\Rightarrow$) Let $\psi$ be a rule that satisfies self-duality and price independence. Let $L = (N, c, r, R, E) \in \mathcal{L}$. Under self-duality, by Proposition 1 there exists $f \in \mathcal{F}$ such that $p^\psi(L) = Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$ if $c(N) > 2E$ and

$p^\psi(L) = Rf\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$ if $c(N) \le 2E$, and $x^\psi(L) = \frac{E}{c(N)}c$. Under price independence, for every $c' \in \mathbb{R}^N_{++}$ with $c'(N) = c(N)$, we have $p^\psi(L') = p^\psi(L)$. We prove the existence of $g \in \mathcal{G}$ so that $p^\psi(L) = Rg\left(\frac{E}{c(N)}, \frac{r}{R}\right)$ if $c(N) > 2E$ and $p^\psi(L) = Rg\left(1 - \frac{E}{c(N)}, \frac{r}{R}\right)$ if $c(N) \le 2E$. First, assume that $c(N) > 2E$. Since $c(N) = c'(N)$,

$$p^\psi(L') = Rf\left(\frac{E}{c'(N)}, \frac{r}{R}, \frac{c'}{c'(N)}\right) = Rf\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c'}{c(N)}\right).$$

Then, $f\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c'}{c(N)}\right) = f\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$. Consequently, $f$ does not depend on $\frac{c}{c(N)}$, and hence $g\left(\frac{E}{c(N)}, \frac{r}{R}\right) = f\left(\frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$ is well-defined. By continuity of $f$, we can define $g(e, t) = f\left(e, t, \frac{c}{c(N)}\right)$ for all $(e, t) \in \Theta$. Second, assume now that $c(N) \le 2E$. By definition, $g\left(1 - \frac{E}{c(N)}, \frac{r}{R}\right) = f\left(1 - \frac{E}{c(N)}, \frac{r}{R}, \frac{c}{c(N)}\right)$.

($\Leftarrow$) Let $\psi$ be a rule given by, for all $L \in \mathcal{L}$, $p^\psi(L) = Rg\left(\frac{E}{c(N)}, \frac{r}{R}\right)$ if $c(N) > 2E$ and $p^\psi(L) = Rg\left(1 - \frac{E}{c(N)}, \frac{r}{R}\right)$ if $c(N) \le 2E$ for some $g \in \mathcal{G}$, and $x^\psi(L) = \frac{E}{c(N)}c$. Let $f \in \mathcal{F}$ defined as $f(e, t, \hat{c}) = g(e, t)$ for all $(e, t) \in \Theta$ and $\hat{c} \in \Delta^N$. By Proposition 1, $\psi$ satisfies self-duality. Let $L' \in \mathcal{L}$ be given as in the definition of price independence. We prove $p^\psi(L') = p^\psi(L)$. First, assume that $c(N) > 2E$. Since $c'(N) = c(N)$, we deduce $p^\psi(L') = Rg\left(\frac{E}{c'(N)}, \frac{r}{R}\right) = Rg\left(\frac{E}{c(N)}, \frac{r}{R}\right) = p^\psi(L)$. Second, assume that $c(N) \le 2E$. Since $c'(N) = c(N)$, we deduce $p^\psi(L') = Rg\left(1 - \frac{E}{c'(N)}, \frac{r}{R}\right) = Rg\left(1 - \frac{E}{c(N)}, \frac{r}{R}\right) = p^\psi(L)$. $\qquad\square$

The rule corresponding to $g \in \mathcal{G}$ is denoted (with a slight abuse of notation) as $\psi^g$.

Let $\mathcal{L}^2 \subset \mathcal{L}$ be the set of all rental problems with a unique lessor. A rule is standard for two if the tenant and the unique lessor share equally the benefit of cooperation.

**Definition 3.** *A rule $\psi$ satisfies standard for two if for all $L \in \mathcal{L}^2$, $u_0^L(\psi(L)) = u_i^L(\psi(L))$ where $N = \{i\}$.*

**Theorem 1.** *A rule $\psi$ satisfies self-duality, price independence and standard for two iff for all $L \in \mathcal{L}$, $p^\psi(L) = \frac{R+r}{2}$ and $x^\psi(L) = \frac{E}{c(N)}c$.*

*Proof.* ($\Rightarrow$) Let $\psi$ be a rule that satisfies self-duality, price independence and standard for two. Let $L = (N, c, r, K, E) \in \mathcal{L}$. By Proposition 2, there exists $g \in \mathcal{G}$ such that $p^\psi(L) = Rg\left(\frac{E}{c(N)}, \frac{r}{R}\right)$ if $c(N) > 2E$ and $p^\psi(L) = Rg\left(1 - \frac{E}{c(N)}, \frac{r}{R}\right)$ if $c(N) \leq 2E$, and $x^\psi(L) = \frac{E}{c(N)}c$. We need to prove that $Rg\left(\frac{E}{c(N)}, \frac{r}{R}\right) = \frac{R+r}{2}$ if $c(N) > 2E$ and $Rg\left(1 - \frac{E}{c(N)}, \frac{r}{R}\right) = \frac{R+r}{2}$ if $c(N) \leq 2E$, or equivalently $g(e, t) = \frac{1+t}{2}$ if $\frac{1}{2} > e$ and $g(1 - e, t) = \frac{1+t}{2}$ if $\frac{1}{2} \leq e$ for all $(e, t) \in \Theta$. Let $e = \frac{E}{c(N)} \in \left]0, \frac{1}{2}\right]$ and $t = \frac{r}{R} \in [0, 1[$. By standard for two, for every $i \in N$, $u_0^{L'}(\psi(L')) = u_i^{L'}(\psi(L'))$. Assume first $\frac{1}{2} > e$. By efficiency, we have $(R - Rg(e, t))E = (Rg(e, t) - r)E$. Then, we deduce that $g(e, t) = \frac{1+t}{2}$. Assume now $\frac{1}{2} \leq e$. By efficiency, we have $(R - Rg(1 - e, t))E = (Rg(1 - e, t) - r)E$. Then, we deduce that $g(1 - e, t) = \frac{1+t}{2}$.

($\Leftarrow$) Let $\psi$ be a rule given by $p^\psi(L) = \frac{R+r}{2}$ and $x^\psi(L) = \frac{E}{c(N)}c$. It is straightforward to check that $\psi = \psi^g$ for $g(e, t) = \frac{1+t}{2}$ for all $(e, t) \in \Theta$. By Proposition 2, $\psi$ satisfies self-duality and price independence. Now, we prove that $\psi$ satisfies standard for two. It is enough to prove that, when $N = \{1\}$, $u_0^L(\psi(L)) = u_1^L(\psi(L))$. We have $u_0^L(\psi(L)) = \left(R - \frac{R+r}{2}\right)E = \frac{R-r}{2}E$. Analogously, $u_1^L(\psi(L)) = \left(\frac{R+r}{2} - r\right)E = \frac{R-r}{2}E$. $\qquad\square$

We denote by $\psi^2$ the unique rule which is characterized in Theorem 1.

The properties are independent. Whether we remove one the properties, we obtain more rules. The rule given by $p^\psi(L) = \frac{R + \alpha r}{1 + \alpha}$ for $\alpha \in ]0, 1[$ and $x^\psi(L) = \frac{E}{c(N)}c$ satisfies self-duality and price independence, but fails standard for two. The rule given by $p^\psi(L) = r + \frac{R-r}{2c(N)}\max_{i \in N} c_i$ and $x^\psi(L) = \frac{E}{c(N)}c$ satisfies self-duality and standard for two, but fails price independence. The rule given by $p^\psi(L) = \frac{R+r}{2}$ and $x_i^\psi(L) = \min\{c_i, \lambda\}$ for every $i \in N$, where $\lambda$ solves $\sum_{i \in N} \min\{c_i, \lambda\} = E$ satisfies standard for two and price independence, but fails self-duality.

11

# 4    Concluding remarks

We study the duality in land rental problems. We present the self-duality based on the idea of self-duality in bankruptcy problems. In bankruptcy problems, for every agent, the final outcome coincides with the amount obtained from the share of the estate. But, in land rental problems, we have considered that for every agent, the utility (outcome) depends on the share of the rented land (estate) and the price by each unit of land.

The proportional land share is obtained by only self-duality in land rental problems. As opposed, for example, in bankruptcy problems, it is necessary at least to consider some other properties. Furthermore, we have characterized the family of rules that satisfy self-duality. A rule in this family is denoted as $\psi^f$. We have characterized the subfamily of rules that satisfies self-duality and price independence. A rule in this subfamily is denoted as $\psi^g$ . Finally, we have characterized the unique rule that satisfies self-duality, price independence and standard for two, which is denoted as $\psi^2$. We also provided the discussion about independence of properties.

Table 1 shows a summary of these results.

| Property | $\psi^f$ | $\psi^g$ | $\psi^2$ |
|---|---|---|---|
| Self-duality | YES | YES | YES |
| Price independence | - | YES | YES |
| Standard for two | - | - | YES |

Table 1: Properties of rules in land rental problems.

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