Entry in Beauty-Contest Games

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Abstract

We study how voluntary participation in Beauty Contest Games (BCGs) affects the actions and payoffs of type-heterogenous players. In a BCG, players have two goals –one personal, the other social– and so BCGs appropriately model relevant economic situations like participating in a social network, partaking in the coding of an open-source software, or the choice of research topics by academics, among others. Key in these and other cases is the concept of "social norm" that will emerge in the associated "community", and so people's entry choices will depend crucially on their expectations regarding not only how many but who (which types) will join in.

We find that in equilibrium there is entry as long as the BCG is "attractive" and that there might be multiple equilibria, each indexed by its associated social norm. We also find that, when finite, there is an odd number of equilibria and that –if ordered based on the value of the associated social norm– odd/even equilibria are stable/unstable.

Further, for low attractiveness, equilibrium social norms are univocally associated with the extrema of the distribution of types in the economy, so that stable/unstable equilibria are linked to maxima/interior minima.

We find that "universal" communities in which everybody joins the BCG (as implicitly assumed by the literature) only occur when the BCG is sufficiently attractive and the economy's average type is not extreme.

In non-universal communities, social norms are affected by the attractiveness of the BCG and the functional form of the distribution of types in the economy (especifically, its skewness around extrema).

Attractiveness affects both the size and the composition of the community. Thus, an increase in attractiveness could lead both to the entry of new members and to the exit of some others.

Keywords: Beauty contest game, endogenous entry, social norms

JEL codes: C7, Z1, L17

1 Introduction

There are more than 30 million registered users of Wikipedia and more than 137,000 of them have performed an action in the last 30 days. Instagram claims a community of more than 500 million users that upload more than 95 million photos every day. Linux has the largest installed base of all general-purpose operating systems because of the dominance of Android on smartphones, it is also the leading operating system on servers and other big iron systems such as mainframe computers, and is used on 99.6% of the TOP500 supercomputers. And more than 150 million guests of Airbnb rented their own and others' abodes in more than 65,000 cities in more than 191 countries.[1]

Behind all these "collaborative economy" examples is the "produser" (Bruns (2006)), a new economic actor that merges the roles of user and producer of contents. On the one hand, she acts as a user when she modifies and runs an open-source software like Linux in her own computer or when she browses the photos posted by others in a social network like Instagram. On the other hand, she acts as a producer when she makes her code modification available to others or when she uploads her own photos for others to see.

But while as an open-source software user she only cares about fixing the bug that hinders her own work or developing a new feature to satisfy an individual need, as a producer she attempts to solve the "big problem" that restrains the growth of the software or focuses on "hot issues" with the hope of attracting her peers' recognition and intra-community status.[2] In other words, "produsers" have two distinct type of goals: a personal goal (her own preferences) and a social goal (the community's "preferences"): the first one requires taking actions that suit the person's attributes (tastes, needs, skills, i.e., her "type"), the latter one demands taking actions that conform to the community's average action (the standard, the "hot topic", the big issues, the popular action, i.e., the "social norm").

The produser, however, has only one choice variable: choosing a piece of coding to work on, or posting a photo/video. Thus, unlike the user-only and producer-only, she will face a trade-off between the personal and social goals: the more she focuses on her own bug or on her own photo style, the less popular she will be in the community, and viceversa.[3]

[1] For wikipedia: https://en.wikipedia.org/wiki/Wikipedia:About.
For Instagram: https://www.instagram.com/about/us/.
For Linux: https://en.wikipedia.org/wiki/Linux#cite_note-22.
For Airbnb: https://www.airbnb.com/about/about-us.
All webpages were accessed on the 20 February 2017.

[2] Likewise, as a user of a social network like Instagram she will only pay attention to her own tastes and will look for photos that fit them, but as a producer she will also consider what photograph styles and topics are "popular" and hence are likely to attract many viewers as well as positive feedback.

[3] In fact, this concept can be extrapolated beyond the realm of technology, the internet, social networks or the "collaborative economy". Indeed, the trade-off between individual and social goals is present in many other applications. For example, in political economy, a member of a political party might face a dilemma when choosing a platform because, on the one hand, she has her own principles to attend to but, on the other hand, in order to become a competitive candidate in a primary election she needs to also cater her campaign promises to match the tastes of the whole party (community). Likewise, an academic has to balance, when selecting a topic for research, her own interests (a particular subfield, methodology) as well as the overall

This crucial trade-off can be appropriately captured by means of a slightly modified version of Morris and Shin (2002)'s Beauty Contest Game (BCG) such that a player's best response function is a weighted average of her own type (personal goal) and the community's average action (social goal). Thus, like in Nagel (1995), a player's best response (or "target function") is an increasing, less-than-proportional function of the average action in the community, and players' actions are thus strategic complements.[4] But, unlike in Nagel (1995)'s and closer to Costa-Gomes and Crawford (2006)'s, in our setup the target functions are different for players with different types.

Going back to the leading examples, a concept as important as –and inseparable from– that of produser is that of "community": one cannot think of Wikipedia without its editors and contributors, or Instagram without its myriad of users. Yet the literature has consistently identified "community" with "economy", implicitly assumming that everyone in the economy was also a member of the community.[5] But clearly, participation in Wikipedia, Instagram and the like is voluntary, and while some people are part of the community others are not. We thus intend to fill this gap in the literature and analyse how voluntary participation affects BCGs.[6]

This is a relevant task as it intends to uncover why the communities in the motivating examples are so successful while others are not. This implies not only finding out the factors that determine the size of those communities, but also, and more importantly, to reveal how the equilibrium social norms that emerge in them are endogenously and simultaneously co-

interest of the academic community at the time (the "hot debates", the trendy topics). And migrants in a new country might be torn between keeping their own ancestral traditions (different for migrants with different origins and/or cultural backgrounds) or adopting the local social norms of the host country (the same for all migrants).

[4] In Angeletos and Lian (2016)'s terminology, the strategic complementarity is "weak" and thus ensures the survival of a unique equilibrium. If the incentive to coordinate was too strong (proportional or more-than-proportional) then multiple equilibria would ensue. This would happen, e.g., if players focused only on being "popular" and ignored their personal goals.

[5] Indeed, the literature on coordination games (which include the BCG), considers that, for example, everyone is forced to choose between attacking a currency or not (Morris and Shin (1998)), or between running against a bank or not (Kiss et al. (2014)), or between evasion and compliance (Sanchez Villalba (2015)). Yet, it is reasonable to assume that many people are not investors and thus ignore the possibility of attacking a currency, that people with no deposits in a bank will not even consider the option of running against it, and that people that do not submit a tax return do not need to choose between evasion and compliance. The specific literature on BCG is not unlike the broader one on coordination games, with both seminal papers (Morris and Shin (2002) and Nagel (1995)) ignoring the possibility of voluntary entry, and the rest of the literature following the same pattern.

[6] There is a long literature on entry in different settings. For example, there are articles on public good contributions in which players have the option to opt out of the mechanism (Norman (2004)), but there the actions are strategic *substitutes*, not complements like in BCGs. There are also examples in which actions are strategic complements (Selten and Guth (1982), Cachon and Camerer (1996)), but unlike our model, they have entry fees, *multiple equilibria*, and their focus is on equilibrium selection. There are other scenarios where the *entry choices* are strategic complements (eg, typical network good games with entry (Katz and Shapiro (1986), Augereau et al. (2004)), but we need the strategic complementarity to arise in the *BCG*, not at the entry stage. We also differ from club goods (Scotchmer (2002)) because we rely on a *spatial* structure that is absent there. Maybe the closest references are in the "citizen-candidate" literature (Osborne and Slivinski (1996), Besley and Coate (1997)) because they consider a spatial structure and entry followed by a location game, but their candidates do not coordinate but *compete* with each other and thus there is no resulting "community". There is also a connection with the new economic geography (Fujita et al. (1999), etc.), although they are mostly concerned with *economies of scale* and *congestion* issues, thus leading to the "agglomeration of the different" unlike our case where we obtain a "community of the alike".

determined by the entry choices of the players. Furthermore, these communities and their associated social norms can be significantly different from the ones obtained when everyone is forced to partake in the BCG (as it is assumed in the literature): the most popular photos among Instagrammers could be quite different from those of the whole population, just like the issues of concern for Linux programmers are likely to be very different from those of the economy as a whole.

We find that non-universal communities do indeed exist in equilibrium as long as the BCG is not too "attractive" and the average type in the economy is rather extreme. Moreover, their associated equilibrium social norms can differ dramatically from the ones corresponding to "compulsory" BCGs, depending on the attractiveness of the BCG and on the distribution of types in the economy. In particular, if the BCG is "barely attractive" then the number of stable equilibria is given by the number of maxima in the distribution of types.[7] Likewise, the associated social norms are linked to the critical points corresponding to those maxima, and thus economies in which the distribution of types has "peaks" quite apart from the average type in the economy (for example, a politically polarized society) can yield voluntary communities (political parties) in which the emergent social norms (party platforms/manifestos) are radically different from the one that would arise in a compulsory BCG (one in which everyone is forced to join a single party, à la PRI in Mexico or the Communisty party in China).

The heterogeneity of types (as reflected in a non-degenerate density function) is another novelty in our analysis, and a crucial one since otherwise we cannot really understand entry choice diversity: It is only normal to assume that different programmers are interested in different bugs/functions of the same open-source software, and that different "Instagrammers" have different preferences regarding photography styles and topics. The heterogeneity in personal preferences will thus yield heterogeneity in choices at the entry stage, which themselves are endogenously co-determined with the social norm that emerges in the BCG. This way, the decision to enter or not depends not only on the size of the community (as in typical network good games (Katz and Shapiro (1986)) or market entry games (Selten and Guth (1982)), but on the (distribution of the) types of the members. Thus, a player's decision to enter can improve or worsen the payoff of members, depending on the type of the entrant, the types of the current members, and on the social norm.[8] This means that the composition of the community can change as the result of some shocks (for example, a change in the attractiveness of the BCG), with some members exiting and, simultaneously, some new people joining the community. For example, an increase in attractiveness (say, better photo-manipulation tools in Instagram) can lead to both a larger community but also to the exit of some people and the entry of others (depending on their preferences regarding

[7]Only stable equilibria/communities are likely to be observed in the data. Unstable equilibria are, by definition, likely to disappear and to lead to a new and stable equilibrium as a result of even minor shocks.

[8]For example, if a "moderate" joins a leftist party, its "radical" wing would be worse off, since the average action in the party (the social norm) will move towards the center. On the other hand, the "moderate" wing of the party will be better off for the same reason.

the manipulation of photos). In other words, the heterogeneity in types allows us not only to explain the size of a community, but also its composition, and to analyze how they would change in response to an external shock.[9] More specifically, we find out that exit can only occur in interior communities, and when the social norm is very responsive to changes in the attractiveness of the *BCG*.

We thus study the entry choices of heterogeneous players that have to decide whether to join a BCG or not. Given the multiple variations possible, in this first attempt we restrict our attention to non-repeated, full-information games, in which there exists a unique, exogenous BCG, the distribution of types is exogenously given, and the outside option (i.e., the utility when opting out) is fixed and exogenous.

The rest of paper is organised as follows. In section 2 we present the theoretical model. In section 3 we undertake a comparative statics analysis regarding how changes in the attractiveness of the *BCG* affects the variables of interest. In section 4 we discuss the results and consider future research paths. In section 5 we conclude. All proofs, as well as some illustrations, are in the Appendix.

2 Model

Players are indexed by their type θ. The population of players is of mass 1 and distributed on the $[0, 1]$ interval according to a publicly known, real-valued, atomless, continuous and differentiable density function $f(\theta)$, with cumulative function $F(\theta)$ and population average (mean) given by $E\theta$. Let $f'(\cdot)$ denote the first derivative of the density function $f(\theta)$. Let $\hat{\Theta}$ denote the set of *interior* local extrema of $f(\theta)$.

The timing of the game is as follows:

Stage 1 Having observed their own types, players simultaneously decide whether to enter the community or not.

Stage 2 Entrants play the (simultaneous) *BCG*; non-entrants get a fixed payoff normalised (without loss of generality) to 0.

If a player of type θ joins the community at stage 1, then at stage 2 she plays the *BCG* by choosing an action $x_\theta \in [0, 1]$ to maximise the normalised[10] utility function:

[9] For example, continuing the example of the previous footnote, if as a result of issues of corruption linked to the Right party, the political support shifts (i.e., the distribution of types shifts mass) from the right to the left, we could find not only that the Left party is likely to increase in size, but that it is the result of some centrist, formerly unaffiliated citizens that decide to join it. However, it could also be the case that some radical members decide to abandon it due to the party's new, "milder" ideological position ("social norm"), and some could even even enrol in a far-left party as a consequence.

[10] This function is not necessarily equal to the utility in case of entering the BCG, but rather the net gain from doing it vis a vis the outside option.

$$u_{IN}(x_\theta;\theta) = \kappa - (1-p)(x_\theta - \theta)^2 - p(x_\theta - \bar{x})^2$$

where $\bar{x}$ is the average action among entrants. This utility function is basically a transformation of the loss function used by Morris and Shin (2002), where parameter $\kappa > 0$ represents the fixed gain from joining the community.[11] The second and third terms are a convex combination of the losses associated with deviating from the "fundamental" and "strategic" goals respectively, where parameter $p \in (0,1)$ reflects the players' preference for the strategic goal.[12] In the global game literature, the first one refers to a parameter of the model that is imperfectly known by the player yet relevant for her decision-making (like the strength of the central bank's reserves in a country with a currency peg), and the second one to the coordination component by which everyone wants to do as the majority does (equivalent to matching the average action). In our model we maintain the interpretation of the strategic component as held by Morris and Shin (2002), but we deviate from their interpretation of the fundamental one: in our case it reflects the player's own type.[13]

The game is one of complete information: every player knows her own type (θ), the distribution of types in the population ($f(\theta)$), and the preferences of every player. The game is solved by backwards induction.

2.1 Stage 2: Beauty-Contest Game

The problem of a member of the community is $\max_{x_\theta} u_{IN}(x_\theta;\theta) = \kappa - (1-p)(x_\theta - \theta)^2 - p(x_\theta - \bar{x})^2$

We follow Morris and Shin (2002) by assuming that the size of the entrant population is sufficiently large as to lead every individual player to ignore her own impact on the average action $\bar{x}$.[14] From the FOC $\frac{\partial u_{IN}(x_\theta;\theta)}{\partial x_\theta} = 0$, we get

$$x_\theta^* = (1-p)\theta + p\bar{x}^* \tag{1}$$

This is the same FOC than Morris and Shin (2002) obtain: the best response function for

[11] If $\kappa < 0$, then nobody would ever enter the *BCG* and the end result is trivial and uninteresting.

[12] In standard BCG experiments, p is usually made equal to 2/3. But players could in principle differ in terms of their p-values (for instance, refer to Costa-Gomes and Crawford (2006)).

[13] Indeed, the global games literature assumes that θ is an unknown parameter. However, in order to solve the model, it resorts to the introduction of private, informative signals that are received by players. These signals are noisy and thus make players heterogeneous regarding their information sets. Here we basically skip the intermediate step and assume that players are heterogeneous from the start, and their heterogeneity is embodied by their type θ (just like it is embodied by their signals in the global games setup).

[14] This is a question to consider in this setting since entry is endogenous and thus small groups of entrants cannot be ruled out. However, since the density function is continuous, this is not an issue in our model because an individual player's mass is negligible. This might not be the case in real world applications with discrete types. Yet experimental evidence (Grosskopf and Nagel (2008)) shows that in *BCG*s with as few as two people, players often ignore their contribution to the average action.

any individual player is a weighted average of the fundamental and strategic goals, with weights given by $1-p$ and p, respectively. That is, players try to balance their own bliss point –given by their types– and the coordination with others –given by the average action $\bar{x}$. Clearly the higher (lower) the importance attached to their own bliss point, i.e, the lower (higher) is p, the closer (farther) will be the optimal action to each player's bliss point θ. Nonetheless, since $p \in (0,1)$ players' optimal actions are strategic complements as $p > 0$.[15]

Let $G \subseteq [0,1]$ denote the subset of entrants, $g(\theta)$ denote the density function of the entrant subpopulation and $\bar{\theta}$ denote the average type among entrants. Thus,

Proposition 1 *In the BCG equilibrium, the average optimal action coincides with the average type among entrants. Formally,* $\bar{x}^* = \bar{\theta}$.

Proof. $\bar{x}^* = \int_{\theta \in G} x_\theta^* g(\theta) d\theta = (1-p)\bar{\theta} + p\bar{x^*}$. Thus, $\bar{x}^* = \bar{\theta}$ ■

Intuitively, every player wants to coordinate with every other, but everyone has a different type. Thus, having the average action be equal to the average type is somewhat like minimising the strategic losses of entrants. This is something to be expected in places like Instagram: the average (or let us say, "typical" or "popular") photo will reflect quite accurately the average tastes of the members of the community.

Corollary 1 *A player θ's optimal action is a weighted average of her own type and of the average type among entrants. Formally,* $x_\theta^* = (1-p)\theta + p\bar{\theta}\ \forall\theta$. *As a result,* $u_{IN}(x_\theta^*;\theta) = \kappa - p(1-p)\left(\theta - \bar{\theta}\right)^2$. *Therefore, κ is the highest payoff that can be attained by members, and only by members of type $\bar{\theta}$.*

Intuitively, members face a trade-off between their own individual goals (embodied by their types θ) and the entrant community's tastes (embodied by the average type among entrants $\bar{\theta}$), which they solve by choosing an action which is a weighted average of both.

As in Guth et al. (2002), different players (i.e. different types) will have different payoffs from entering the BCG:[16] the farther the player's type is from the average type in the entrant community, the worse she is. The players with the highest payoff among the entrants are always those whose type coincides with the average type in the community (those for which $\theta = \bar{\theta}$). Thus, κ is the highest payoff that can be attained by entrants.

[15] Thus, our setup extends the standard beauty-contest game by introducting heterogeneity among players *à la* Costa-Gomes and Crawford (2006). Nagel (1995)'s model is a special case in which everyone is homogeneous (i.e. $\theta = 0$ for every agent), and so the best response for every player is to choose an action as close as possible to a given proportion p of the average action ($p = 2/3$ in Nagel (1995)).

[16] This differs from the standard experimental literature, where the "winner" (the one closest to the target) gets a fixed prize, while the rest get nothing. This setup, though, fits well with our motivating examples (Linux, Instagram, etc.), since everyone in the community gets some benefit from membership, but it differs depending on how close to the "spirit of the community" they are.

A remark is due at this point: all these results are qualified by the phrase "among entrants". Thus, so far, we considered only the average choice or the average type "among entrants". In the literature, this matter is ignored since, by construction, the community included every player in the economy: everyone is forced to choose between attacking a currency peg or not, running against a bank or not, underdeclaring taxes or not. On the contrary, in the present model entry is endogenous: attacking or not is only an option if you first decided to be an investor, running against a bank or not only matters to you if you first deposited your money there, underdeclaring your taxes or not only affects those who do submit their tax returns. This is not a minor issue: the "average type" *among entrants*, $\bar{\theta}$, is likely to be different from the "average type" in the whole population, $E\theta$.

So far, the results in this subsection are very similar to those in the literature, because we also focused on a restricted subpopulation, namely, the community. This will change in the next section.

2.2 Stage 1: Entry Game

Our goals are mainly two: (i) to determine the types of members, $\theta \in G$, and the size of the community, $s \in [0,1]$; (ii) to perform comparative statics analysis.

A player of type θ at stage 1 has to compare her payoff if she decides to play the game, $u_{IN}\left(x_{\theta}^{*};\theta\right) = \kappa - p\left(1-p\right)\left(\theta - \bar{\theta}\right)^{2}$, to her payoff if she stays out of it, $u_{OUT}\left(\theta\right) = 0$. Thus, she optimally enters the BCG if and only if $u_{IN}\left(x_{\theta}^{*};\theta\right) \geq 0$. Define

Definition 1 $h := \sqrt{\kappa p^{-1}\left(1-p\right)^{-1}}$ *is the net gain from entering the BCG.*

Proposition 2 *For any given $\bar{\theta} \in [0,1]$, the entrant community is given by the compact interval around $\bar{\theta}$: $G(\bar{\theta}) := \left[L(\bar{\theta}), U(\bar{\theta})\right] \subseteq [0,1]$ where:*

1. *The lowest type that enters is defined as $L\left(\bar{\theta}\right) := \max\left\{0, \bar{\theta} - h\right\}$.*
2. *The highest type that enters is defined as $U\left(\bar{\theta}\right) := \min\left\{1, \bar{\theta} + h\right\}$.*

Definition 2 *For any given $\bar{\theta} \in [0,1]$,*

- *the size of the community is given by: $s(\bar{\theta}) := \int_{L\left(\bar{\theta}\right)}^{U\left(\bar{\theta}\right)} f\left(\theta\right) d\theta$.*
- *the average type among entrants is given by: $a(\bar{\theta}) := \int_{L\left(\bar{\theta}\right)}^{U\left(\bar{\theta}\right)} \theta \cdot \frac{f(\theta)}{s(\bar{\theta})} d\theta$.*

The size and average type in the community (as well as the lowest- and highest-type among entrants in proposition 2) are not constants, but functions of $\bar{\theta}$ (and h). In other words, for every social norm (and value of the BCG) that we can think of, these functions tell us the size of the associated community and the corresponding average type. However, they are not necessarily the equilibrium values. For an equilibrium to occur, the average type among entrants $a(\bar{\theta})$ must equal the proposed social norm $\bar{\theta}$. Formally,

Definition 3 *The equilibrium average type among entrants is the fixed-point given by the equation:*

$$a\left(\bar{\theta}^{*}\right) := \bar{\theta}^{*}. \tag{2}$$

Equation (2) can be rewritten as: $\int_{L(\bar{\theta}^{*})}^{U(\bar{\theta}^{*})}(\theta-\bar{\theta}^{*})f(\theta)d\theta = 0$

Integrating by parts, and defining $\gamma(\bar{\theta}^{*}) \equiv \left(\bar{\theta}^{*} - L(\bar{\theta}^{*})\right) / \left(U(\bar{\theta}^{*}) - L(\bar{\theta}^{*})\right)$, we have:

$$\left(1-\gamma(\bar{\theta}^{*})\right) F(U(\bar{\theta}^{*})) + \gamma(\bar{\theta}^{*})F(L(\bar{\theta}^{*})) = \int_{L(\bar{\theta}^{*})}^{U(\bar{\theta}^{*})} \frac{F(\theta)}{U(\bar{\theta}^{*}) - L(\bar{\theta}^{*})} d\theta \tag{3}$$

On the LHS of equation (3) we have the expectation of F using a distribution (call it $\Gamma(\theta)$) between the end points of the interval assigning $\gamma(\bar{\theta}^{*})$ to $L(\bar{\theta}^{*})$ and $(1-\gamma(\bar{\theta}^{*}))$ to $U(\bar{\theta}^{*})$. On the RHS we have the expectation of F using a uniform distribution.[17]

From definition 2 it can be shown that:

Proposition 3 *The function $a\left(\bar{\theta}\right)$ is a continuous, weakly increasing function of $\bar{\theta}$. Furthermore, $a\left(\bar{\theta}\right) \in (0,1) \forall \bar{\theta}$.*

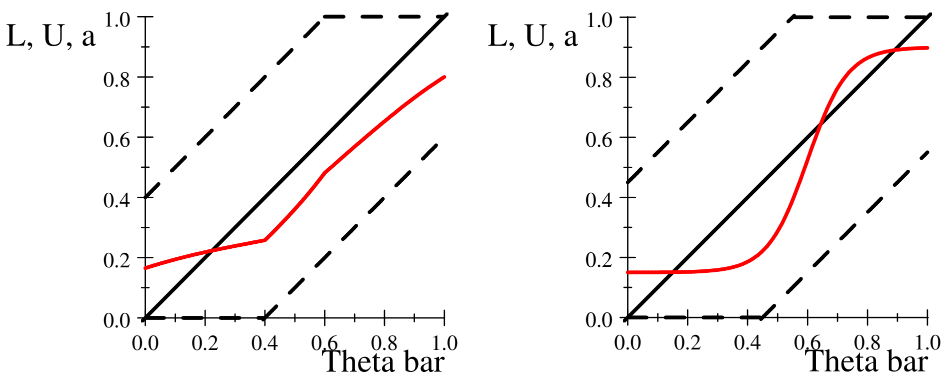
From the definition of equilibrium (definition 3) and the characterisation in proposition 3 we obtain:

Proposition 4 *The game has at least one Nash equilibrium. All equilibria are interior.*

[17] Note that (i) $F(\theta)$ is nondecreasing in θ by definition; (ii) the even distribution, which assigns 1/2 to each of the end points of the interval, is a mean-preserving spread of the uniform distribution and therefore, the uniform distribution second-order stochastically dominates (SOSD) the even distribution; (iii) (a)The even distribution FOSD $\Gamma(\theta)$ if and only if $\bar{\theta}^{*} > \frac{1}{2}\left(U(\bar{\theta}^{*}) + L(\bar{\theta}^{*})\right)$; (b) $\Gamma(\theta)$ and the even distribution coincide if and only if $\bar{\theta}^{*} = \frac{1}{2}\left(U(\bar{\theta}^{*}) + L(\bar{\theta}^{*})\right)$; (c) $\Gamma(\theta)$ first-order stochastically dominates (FOSD) the even distribution if and only if $\bar{\theta}^{*} < \frac{1}{2}\left(U(\bar{\theta}^{*}) + L(\bar{\theta}^{*})\right)$.

Hence, for any $\bar{\theta} \geq \frac{1}{2}\left(U(\bar{\theta}) + L(\bar{\theta})\right)$ $\left(\bar{\theta} \leq \frac{1}{2}\left(U(\bar{\theta}) + L(\bar{\theta})\right)\right)$, if $F(\theta)$ is strictly concave (convex) in θ for $\theta \in (L(\bar{\theta}), U(\bar{\theta}))$, it must be the case that $LHS < (>) RHS$ and therefore $a(\bar{\theta}) < (>) \bar{\theta}$.

Display 4 depicts the derivation of the equilibrium fixed point.[18]



Unique-equilibrium scenario   Multiple-equilibrium scenario

(4)

Equilibria, thus, are indexed by their associated social norms $\bar{\theta}^*$ and can be ranked according to them. Without loss of generality, the equilibrium with the lowest (highest) social norm will be labeled "first" ("last"). In order to characterise the equilibria we use the following definition:

Definition 4 *A Nash equilibrium is stable (unstable) if and only if* $\frac{\partial a(\bar{\theta}^*)}{\partial \bar{\theta}} < (>)1$*, where –in equilibrium–*

$$\frac{\partial a(\bar{\theta}^*)}{\partial \bar{\theta}} = \frac{\left(U(\bar{\theta}^*) - \bar{\theta}^*\right) f(U(\bar{\theta}^*))U' + \left(\bar{\theta}^* - L(\bar{\theta}^*)\right) f(L(\bar{\theta}^*))L'}{s(\bar{\theta}^*)} \tag{5}$$

Note that if $\frac{\partial a(\bar{\theta}^*)}{\partial \bar{\theta}} = 1$, then though there is a fixed point (hence, an equilibrium) there is no "crossing" of the $a(\bar{\theta})$ function and the 45° line, but only "tangency". We will restrict our attention to the first kind of equilibria ("standard equilibria") unless indicated otherwise.

Corollary 2 *Assume a finite number of Nash equilibria. Then, the entry game has an odd number of equilibria: "odd-ranked" equilibria (including the first and last ones) are stable; "even-ranked" equilibria are unstable.*

Definition 5 *Communities can be classified into four mutually exclusive types depending on its social norm* $\bar{\theta}$ *and its value* h*, as depicted in Figure 1:*

[18] Equilibria are obtained when the function $a(\bar{\theta})$ (the red curve) intersects the 45^o line. Functions $L(\bar{\theta})$ and $U(\bar{\theta})$ are depicted by the dashed lines below and above the 45^o line, respectively.

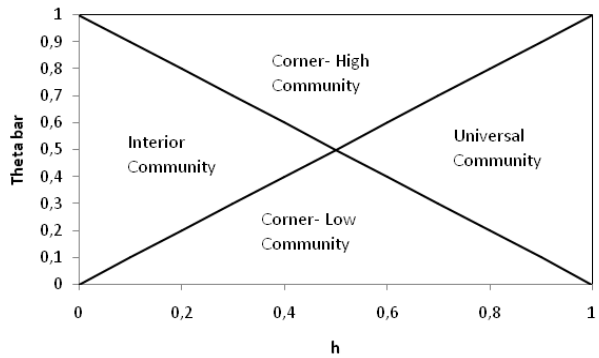


Figure 1: Classification of communities

1. *Corner-low community: in which only low types enter:* $L = 0 \geq \bar{\theta} - h$ *and* $U = \bar{\theta} + h < 1$.
2. *Interior community: in which only intermediate types enter:* $L = \bar{\theta} - h > 0$ *and* $U = \bar{\theta} + h < 1$.
3. *Corner-high community: in which only high types enter:* $L = \bar{\theta} - h > 0$ *and* $U = 1 \leq \bar{\theta} + h$.
4. *Universal community: in which everyone enters:* $L = 0 \geq \bar{\theta} - h$ *and* $U = 1 \leq \bar{\theta} + h$.

In corner and interior communities, the value of h affects the existence and stability of equilibria as well as the equilibrium values of our parameters of interest both directly and indirectly (via $\bar{\theta}^*$) . Instead, it is immediate from equation (2) and definition 5 that $\bar{\theta}^* = E\theta$ and $s(E\theta) = 1$ in a universal community.

Corollary 3 *A universal community can be supported in equilibrium if and only if* $h \geq \max\{E\theta, 1 - E\theta\}$.

Thus, when the BCG is not very valuable (h is low) and/or the population is very radical ($E\theta$ very low or very high), only some types enter (non-universal community). For valuable $BCGs$ and/or moderate societies, everyone joins (universal community).[19]

Proposition 5 *Assume* $h < \max\{E\theta, 1 - E\theta\}$ *and* $\hat{\Theta}$ *is an empty set. Then only corner communities can be supported in equilibrium. A corner-low (high) community is supported in equilibrium if and only if* $F(\theta)$ *is strictly concave (convex) in* θ $\forall\theta$*. Furthermore, (1)* $\bar{\theta}^* < h$ $(\bar{\theta}^* > 1 - h)$ *and (2)* $\bar{\theta}^* \to 0$ $(\bar{\theta}^* \to 1)$ *as* $h \to 0$.

[19] The literature only considers the latter case, so it implicitly assumes that the gain from entering the BCG is quite high and/or that society is rather moderate.

Proposition 6 *Assume $\hat{\Theta}$ is a nonempty, finite set. For arbitrarily low values of h, there are as many Nash equilibria as twice the total number of local maxima minus one. Furthermore, in any equilibrium which supports a corner-low/interior/corner-high community we have that $\bar{\theta}^* \to 0/\hat{\theta} \in \hat{\Theta}/1$ as $h \to 0$.*

Propositions 5 and 6 show that if the number of local extrema is finite, as h is made arbitrarily small, then the equilibria converge to *interior* local extrema that support interior communities, and to corner local maxima that support corner communities.[20] Therefore, the set of local extrema of the density function is crucial in determining the number of equilibria and type of communities supported in equilibrium, at least for sufficiently low values of h.[21] It is also connected to the stability of equilibria, as indicated by the following proposition and corollary:

Proposition 7 *An equilibrium which supports an interior community is stable (unstable) if the entrant community $[\bar{\theta}^* - h, \bar{\theta}^* + h]$ is entirely located in a concave (convex) region of f.*

The following corollary follows from lemma 7 and proposition 6.

Corollary 4 *The equilibria associated with interior local maxima (minima) are stable (unstable) for sufficiently low values of h.*

In the case of corner communities, equilibria are usually stable.[22]

Alternatively, if $F(\theta)$ is linear for some range, then $F(\theta)$ is both convex and concave in that range, and so $\hat{\Theta}$ is not finite. As a result, there is a continuum of interior equilibria for h sufficiently small.[23]

3 Comparative statics

This section illustrates the effects of changes in the net benefit h on the equilibrium values of our variables of interest. This is a relevant matter because the net benefit of the BCG is

[20] Equilibria do not converge to potential corner local minima as h is made arbitrarily small because of SOSD.

[21] If $f(\theta)$ is locally symmetric around a critical point, the equilibrium condition (equation 3) holds at the critical point not just for h arbitrarily small but also for greater values of h.

[22] From proposition 2 and definitions 4 and 5, an equilibrium corner-low community is usually stable since by a first-order Taylor expansion: $F(\bar{\theta}^* + h) - f(\bar{\theta}^* + h)h \approx F(\bar{\theta}^*) > 0$ for $\bar{\theta}^* > 0$. Likewise, an equilibrium corner-high community is usually stable since by a first-order Taylor expansion: $F(\bar{\theta}^* - h) + f(\bar{\theta}^* - h)h \approx F(\bar{\theta}^*) < 1$ for $\bar{\theta}^* < 1$. Only when a first-order Taylor expansion is considered a bad approximation for the cummulative density function we could have unstable equilibria which support corner communities. Example B.1.3 in the appendix illustrates this scenario.

[23] To see this, suppose that $f(\theta) = \eta$ for $\theta \in (\eta_1, \eta_2)$ where η is a positive constant. Then there exists at least a continuum of equilibria which supports a moderate community characterized by $\bar{\theta}^* \in (\eta_1 + h, \eta_2 - h)$ for $h < \frac{1}{2}$.

related to both its "intrinsic" value (e.g., the user-friendliness of the Instagram website, the versatiliy of the Linux programming language) and its cost (most importantly, the price paid to join –i.e., the "membership fee"– but also other costs like the time and effort dedicated to community-related activities).

The impact of h on the average type function $a\left(\bar{\theta}\right)$ is given by:

$$\frac{\partial a\left(\bar{\theta}\right)}{\partial h}=\frac{1}{s\left(\bar{\theta}\right)}\left[\left(U(\bar{\theta})-a(\bar{\theta})\right)f(U(\bar{\theta}))\frac{\partial U(\bar{\theta})}{\partial h}+\left(a(\bar{\theta})-L(\bar{\theta})\right)f(L(\bar{\theta}))\frac{\partial L(\bar{\theta})}{\partial h}\right] \tag{6}$$

As a result, an increase in h shifts[24] up (down) the part of the average function $a(\bar{\theta})$ corresponding to a corner-low (corner-high) community, that is, for $\bar{\theta} < \min\{h, 1-h\}$ ($\bar{\theta} > \max\{h, 1-h\}$) whereas for the part corresponding to an interior community (i.e., for $\bar{\theta} \in (h, 1-h)$) we have that $\frac{\partial a(\bar{\theta})}{\partial h} > 0$ if and only if

$$a(\bar{\theta}) < \bar{\theta} + \left(\frac{f(\bar{\theta}+h)-f(\bar{\theta}-h)}{f(\bar{\theta}+h)+f(\bar{\theta}-h)}\right)h$$

Proposition 8 *Assume $h < \max\{E\theta, 1-E\theta\}$. At any stable equilibrium[25], the equilibrium average type among entrants $\bar{\theta}^*$ is continuous in h and:*

- *increasing in h for a corner-low community.*
- *increasing (decreasing) in h for an interior community if and only if $f(\theta)$ is locally skewed to the right (left) around the underlying critical point.*
- *independent of h for an interior community if and only if $f(\theta)$ is locally symmetric around the underlying critical point.*
- *decreasing in h for a corner-high community.*

Formally,

$$\frac{d\bar{\theta}^*}{dh}=\begin{cases}\frac{f(\bar{\theta}^*+h)h}{F(\bar{\theta}^*+h)-f(\bar{\theta}^*+h)h} & \textit{if corner-low}\\ \frac{[f(\bar{\theta}^*+h)-f(\bar{\theta}^*-h)]h}{[F(\bar{\theta}^*+h)-F(\bar{\theta}^*-h)]-[f(\bar{\theta}^*+h)+f(\bar{\theta}^*-h)]h} & \textit{if interior}\\ \frac{(-1)f(\bar{\theta}^*-h)h}{1-[F(\bar{\theta}^*-h)+f(\bar{\theta}^*-h)h]} & \textit{if corner-high}\end{cases} \tag{7}$$

Thus, when the BCG becomes more valuable (h goes up), corner communities become more moderate (its social norm $\bar{\theta}^*$ moves away from extreme positions, i.e., away from the

[24] Clearly, it does not shift at all the part of the average function corresponding to a universal community, i.e., for $1-h \leq \bar{\theta} \leq h$.

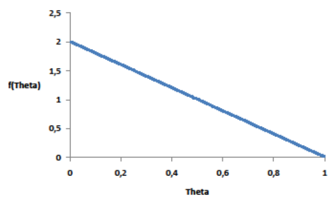
[25] For the rest of the analysis, we will focus on the stable equilibria because they are the ones that will be observed empirically and because unstable equilibria (by definition) are unlikely to survive shocks like the ones considered in this Comparative Statics section.

boundaries of the $[0, 1]$ segment). Intuitively, the community becomes more attractive and is joined by new members whose type is more moderate than those of the old members[26], and so the community becomes less extreme (on average). Interior communities, on the other hand, can become more or less moderate, depending on the skewness of the density function in the neighbourhood of the associated local extreme. Intuitively, as the BCG becomes more valuable, new members join: if the density is locally skewed to the right then most of them will have higher-than-average types and a few will have lower-than-average types, so the social norm will go up as a result. Analogous stories explain the cases when the density is locally skewed to the left and symmetric. Displays 8 and 9 illustrate these results.[27]

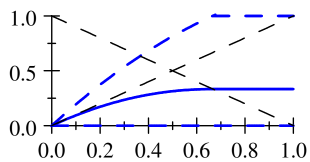
[26] In corner-low communities only higher-than-average types enter ($\theta > \bar{\theta}^*$) since there are no players with $\theta < 0$. Likewise, in corner-high communities only lower-than-average types enter ($\theta < \bar{\theta}^*$) since there are no players with $\theta > 1$.

[27] For each density function $f(\theta)$ (vertical sub-display): Top panel: density function $f(\theta)$; Central panel: equilibrium social norm(s), $\bar{\theta}^*$, as a function of the attractiveness of the BCG, h; Bottom panel: equilibrium size of the community, $s\left(\bar{\theta}^*\right)$, as a function of h. In Central panels: Solid blue line: equilibrium social norm $\bar{\theta}^*$; Dashed lines: lower- and upper-bounds of the community, $L\left(\bar{\theta}^*\right)$ and $U\left(\bar{\theta}^*\right)$. In the bimodal (uniform) distribution, for low values of h, there are multiple but finite (infinite) equilibria (hence multiple $\bar{\theta}^*$). In the bottom panel of the bimodal distribution, the curve first convex then concave (first concave first convex) depicts the size of the community corresponding to the central equilibrium, $\bar{\theta}^* = 0.5$ (upper and lower equilibria).

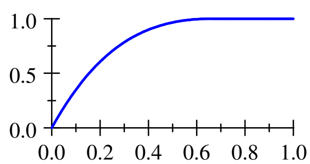
Strictly decreasing $f(\theta)$



Density function

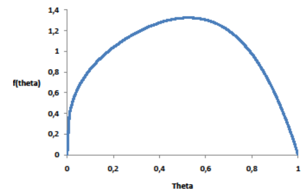


Equilibrium social norm

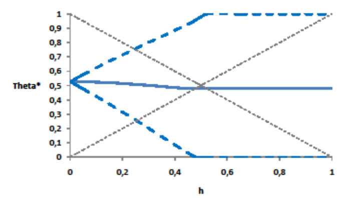


Size of community

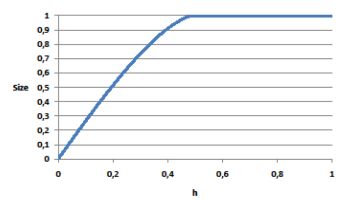
Unimodal $f(\theta)$



Density function

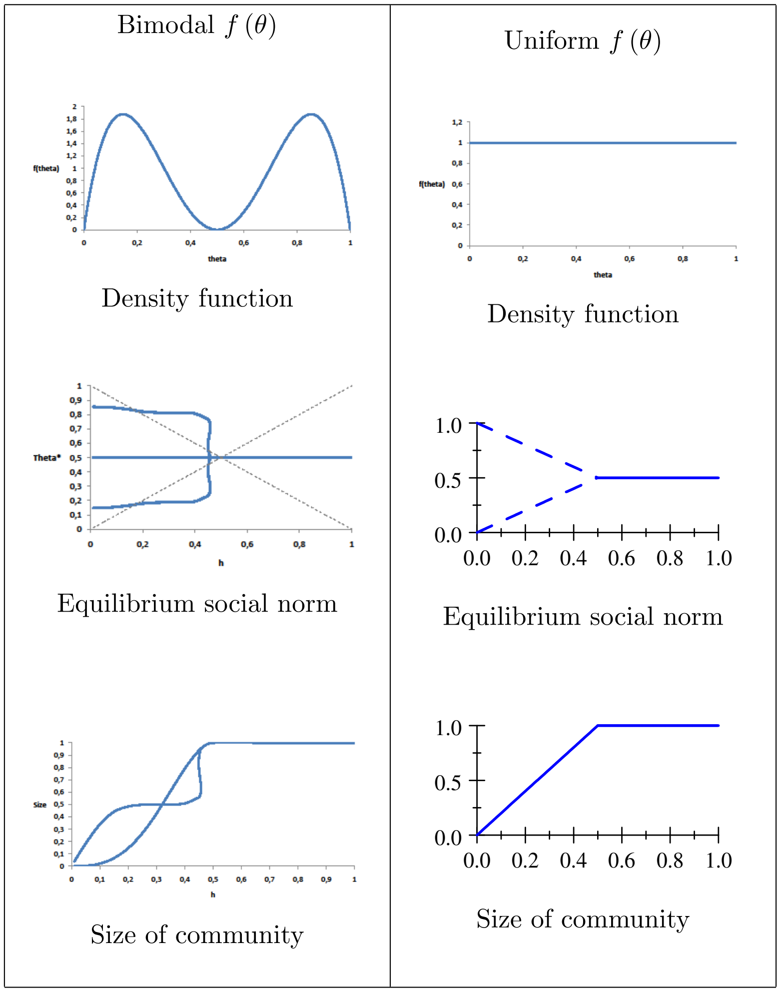


Equilibrium social norm



Size of community

(8)



(9)

Proposition 9 *Assume $h < \max\{E\theta, 1 - E\theta\}$. The equilibrium size of the entrant community is increasing in h at any stable equilibrium.*

Not surprisingly, a more valuable *BCG* attracts new members and thus the community grows larger.[28] Yet, despite a larger size, there may be some exit in equilibrium:

Proposition 10 *Assume $h < \max\{E\theta, 1 - E\theta\}$. An increase in h can lead to the exit of some entrants only in an interior community: low (high) types entrants exit the game in a stable equilibrium if and only if $\frac{d\bar{\theta}^*}{dh} > 1$ ($\frac{d\bar{\theta}^*}{dh} < -1$).*

That is, the exit of community members can only occur in interior communities and when the social norm is sufficiently responsive to h. In said scenario, a small change in the value of

[28] At least in stable equilibria. But as mentioned before, these are the empirically relevant cases. The bottom panels of displays 8 and 9 illustrate this proposition.

the BCG could significantly change the community's equilibrium social norm, and so former members might find themselves so far from the new social norm that they now prefer to exit the community.[29]

Though not enough to kick members out, the increase in the value of the BCG could still have a negative impact on some members of the community:[30]

Proposition 11 *Assume $h < \max\{E\theta, 1 - E\theta\}$. Necessary and sufficient conditions for some members of a corner and/or interior community to be worse off by an increase in the gain from entering the BCG, κ (hence, in the attractiveness h), are:*

(i) F is strictly convex (concave) when evaluated at $U(\bar{\theta}^*)$ $(L(\bar{\theta}^*))$ in a corner-low (-high) community.

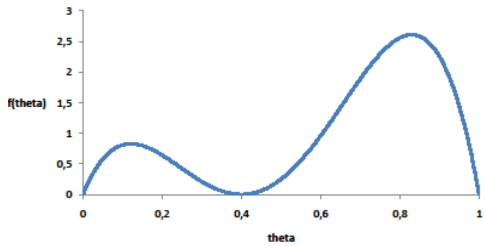
(ii) $|\frac{d\bar{\theta}^*}{dh}| > 1$ for an interior community.

Thus, as the BCG gets more valuable, the entry of new members changes the social norm in the community according to proposition 8, and as a result some former members might be worse off: they are now farther from the new social norm than they were from the old one, so their utility from participating in the BCG goes down. For some, the extra loss is small and so it is still profitable to remain in the community (proposition 11); for others, the extra loss is large and thus prefer to exit (propostion 10). This is shown in display 10:[31] in the central (purple) equilibrium, as the attractiveness h increases from 0.25 to around 0.33, the social norm drops sharply from around 0.35 to just above 0.2 and thus some members (those with types above 0.6) are so worse-off that they decide to leave the community, while those just below 0.6 stay in the community but suffer a large decrease in wellbeing.

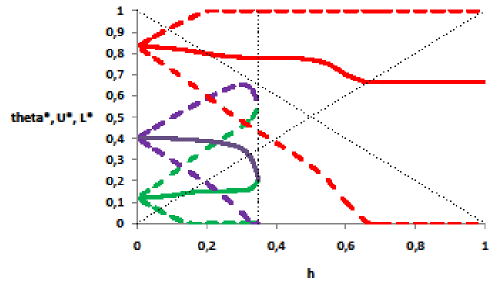
[29] Note, also, that a necessary condition for high(low)-type members to exit the interior community.is the density function's left (right) skewness around $\bar{\theta}^*$ (from proposition 8).

[30] In a universal community there is no additional entry with an increase in κ since the whole population is already playing the BCG. As a result, the equilibrium average type among entrants is fixed at $E\theta$ and it does not respond to changes in h. Therefore, an increase in h is Pareto-improving.

[31] The density distribution is bimodal and three equilibria coexist for low values of h ($\lesssim 0.35$), as indicated by the colours in the right panel (solid lines: social norms $\bar{\theta}^*$; dashed lines: lower and upper bounds L^* and U^*).



Density function



Social norm, bounds of community

(10)

From propositions 9 and 10, it is possible that, as the BCG becomes more valuable, the community gets larger, yet some former members leave. In other words, entry and exit can happen simultaneously, so that we observe a change both in the size and in the *composition* of the community: new members join as the community's value goes up, but the shift in the social norm will worsen the situation of some old members, to the point of ejecting a few of them from the community. This could be the result, for example, of an equilibrium "disappearing" when attractiveness goes up: if, in display 10, we started in the green equilibrium when $h = 0.25$, the community would be a corner-low; but if h goes up to 0.4, the green equilibrium is not available anymore and we move to the red one, and the community is now a corner-high. This happens because the average function $a\left(\bar{\theta}\right)$ gets "flatter" as a result of the increase in h (from the discussion of equation 6) and so it now intersects the 45^o line only once (instead of thrice as before): in display 4, the right panel illustrates the situation when h is low and the left one when it is high.[32] Thus, the increase in attractiveness leads to a significant change in the composition of the community due to the exit of low types and its replacement by high types.

4 Discussion

The result of stable equilibria associated to maxima of the density function (proposition 6 and corollary 4) brings up a rather unexpected outcome: though not present in the preferences of players, the *size* of of the community matters in equilibrium. Indeed, the utility of participants is quadratic in the distance to the average action and own type, but as long as these distances do not change, there is no explicit gain from joining a larger community over a smaller one. However, a stable community will always be around a maximum of the type distribution and so –among all possible communities in the neighborhood of that

[32] Display 4 does not plot the scenario given by the density function of display 10, but it does accurately illustrate the qualitative results. Note that, for the value of h where the equilibrium "vanishes", the $a\left(\theta\right)$ function becomes tangent to the 45^o-line.

maximum– the one that emerges in equilibrium is the largest one[33]. This results from the fact that if the density function is strictly increasing or decreasing in a given range, then the equilibrium community cannot be interior (proposition 5). Further, it has practical consequences regarding the the sustainability of arbitrary BCGs. Consider, for example, that an entrepreneur creates an economic journal and that her target are papers with both theoretical and empirical sections (thus, θ is the "theory/empirics mix", with $\theta = 0$ for purely empirical papers and $\theta = 1$ for purely theoretical ones). In terms of the model, let us assume that she wants to appeal to academics with papers in the $[l, u]$ range, $l > 0$ and $u < 1$, so that the implicit social norm that she intends to generate is $z = (l + u)/2$: this is the "typical paper" or average theory/empirics mix she hopes to attract. If everyone in the $[l, u]$ range submit papers to the journal, however, the *actual* theory/empirics mix in the submissions, $a(z)$, will be determined by the distribution of types of academics in that range. In particular, if the density is increasing (i.e., in this range there are more theory-oriented than empirically-oriented academics), then the "typical" submission mix, $a(z)$, will be higher than the one intended by the entrepreneur, z. [34] Moreover, the founder will have to adapt or risk the loss of potential submitters: when academics see the actual theory/empirics mix published, $a(z)$, their best reply will be submitting new manuscripts in the $[a(z) - h, a(z) + h]$ range, where $a(z) - h > l$ and $a(z) + h > u$.[35] In fact, this process will not stop here, but will continue as long as the density of types is increasing: if it is so in $[a(z) - h, a(z) + h]$, then a new mix $a(a(z)) > a(z)$ will emerge, and so on until an equilibrium is reached and there are no more incentives to change the mix. This means that creating an arbitrary "community" with arbitrary social norms and range of types is not easy, as it is the result of the players' actions and cannot be overruled by an individual, even the founder/owner of the BCG. If the founder wanted to stick to the original range of papers $[l, u]$ she will end up losing potential customers and/or a clever competitor could offer an alternative outlet for publication that better suits the academics' tastes and needs.[36]

Another element worth discussing is the impact of a more valuable BCG on the size and composition of communities obtained in section 3. Indeed, as mentioned there, the net

[33] If the density function was approximated by a histogram with bin size $2h$, in equilibrium the community would coincide with the tallest bar (the local mode of the distribution). In other words, since the size of a community equals the area of the corresponding bar and all bars are equally wide, the equilibrium community is the largest one. We focus here on an interior scenario, but the analysis can be applied to the others as well.

[34] In general, and despite the journal founder's intentions, submissions will lead to an outcome different from the one she was looking for, and this without violating the journal's submission guidelines regarding acceptable topics.

[35] Indeed, academics of type l (supposed to be originally indifferent between submitting or not their papers to the journal), observe that the actual mix of papers published $a(z)$ is higher than the announced one z and so farther than expected from their preferred mix l, and thus end up not submitting their manuscripts. Likewise, those of type u (also supposed to be indifferent), observe that the actual mix $a(z)$ is higher than the announced z and thus closer to their own preferred mix u, and so they strictly prefer to submit their manuscripts.

[36] This story can be extrapolated to many other areas: the types of problems tackled by Linux programmers will follow their own concerns, not those of Linus Torvalds (although they may play a role); the community of Instagrammers will be the result of the distribution of tastes, not necessarily the one intended by its creators; a political party could degenerate into a radical group, even though it was not founded as such; a neighbourhood could become a slum (or an elitist enclave) in spite of the City Hall's intention to populate it with middle-class people; a club created for "cool" people could end up being an ensemble of losers; etc.

benefit of the BCG is negatively related to its price, and so the comparative statics can be re-phrased in terms of the "membership fee" f: we need to replace "if the BCG becomse more valuable" with "if the BCG gets cheaper."[37] Thus, a greedy monopolist that charges "too much" for joining the BCG could thus force some participants to exit the community (when h goes down from 0.8 to 0.4 in the central panel of display 8).[38] On the other hand, another one that lowers the fee could attract more members, although she should be aware that the composition of the community could change as a result (as discussed in section 3), and this might be taken into account too (when in display 10 we are in the green equilibrium/corner-low community for low h but switch to the red equilibrium/corner-high community for high h).[39] Yet, even more interestingly, the relationship is not symmetric over time: that is, an increase in the price followed by a decrease of the same magnitude does not necessarily return the situation to the original one. This is due to the multiplicity of equilibra in some scenarios, like the one in display 10: at the beginning, the community is not very valuable (say $h = .25$) because, e.g., it is related to a new, expensive technology (e.g., blueprint design for 3D printers) and so only a small, niche group joins (the green equilibrium in the figure). As printers become more ubiquitous and less expensive, sharing designs becomes more and more profitable, and more people join the community of designers, until eventually everyone does it ($h > .67$; red equilibrium). If, however, the monopolist now decided to charge for the distribution of designs to the point of pushing h back to 0.25, the equilibrium will now be the upper, red one. In other words, the composition of the equilibrium will be entirely different from the original one, even though h is the same (mostly low types before v mostly high types after). Furthermore, the transition from one equilibrium to another and back could also happen at different values of h (at different fees f): consider display 11 (a zoomed in version of the central panel of display 9), and let us start at the upper equilibrium when $h = 0.44$. Similar to the previous case, we switch to the universal community at $h \approx 0.458$. But now, if the monopolist charged a larger fee (so that h is sligtly below 0.458) there will be no switch back to the corner-high community: it will remain a universal one, and will only revert to a corner-high community if $h \approx 0.456 < 0.458$.[40] In other words, the switch from one community to another and back happens at different values (fees), and thus the monopolist of the example could capture the whole market by decreasing the fee until $h = 0.458$, but then immediately increase the price again so that h is just above 0.456, thus increasing her margin per user without

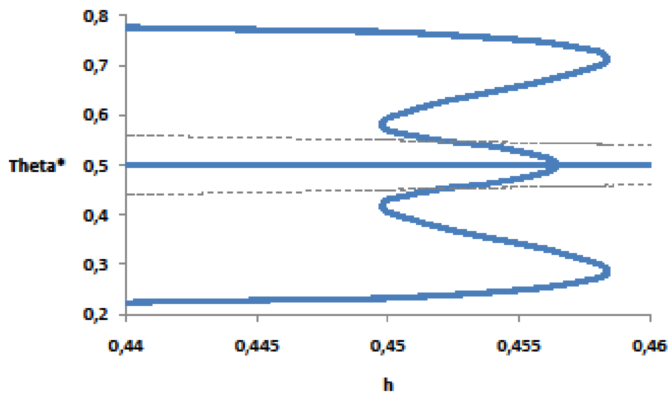
[37] Of course, as said in section 3, other factors influence the net benefit of the BCG, but for the present analysis we will focus on its price and how it impacts on the variables of interest.

[38] In particular, a near-universal community (maybe a giant like Google or Facebook?) that decided to charge for its product could end up giving up its dominant position as users defect and it becomes a smaller, non-universal community. This is not too different from the situation that news companies (like the New York Times) faced when they decided to charge for suscriptions to their (until then free) online newspaper contents.

[39] For example, an exclusive resort or restaurant intending to attract a certain type of customers (the "social norm") may be wary of the impact of lowering the price. Likewise, communities with a strict (i.e., costly) code of conduct (like the Amish) are aware that a loosening in the penalties for misconduct might attract more people to the community, but they could be of the "wrong" type, and the resulting social norm will suffer as a result.

[40] Or switch to a corner-low community, due to symmetry.

any negative impact on the number of users.[41] Likewise, if the monopolist started with a universal community ($0.456 < h < 0.458$) and then increased the price so $h = 0.25$ and the community turns into a corner-high one, an immediate return to the original price will not restore the universal community: it will require a further decrease in the fee, enough to push h beyond 0.458.



"Tomahawk"

(11)

5 Conclusions

From the ever-expanding "collaborative economy" to the formation of political parties, from the choice of research topic by academics to the creation of "urban tribes", economic actors have to balance out the pursuit of two different goals: one personal, the other social. This creates a tradeoff between the player's "type" (her needs, skills, preferences) and the community's "social norm" (the trend, the rule, the tradition) that is appropriately modelled as a Beauty Contest Game (BCG).

Unlike the literature on BCGs, we allow players to voluntarily decide whether to participate or not in one of these BCGs and –consistent with the observation of social networks, political parties, academic fields or urban tribes– we find that in equilibrium non-universal communities do indeed arise as long as the "attractiveness" of the BCG is not too high and/or the average type in the economy is rather extreme.

Furthermore, we highlight the fact that players' entry choices are inextricably and simultaneously co-determined by the social norm that they expect to emerge in equilibrium in the

[41] This asymmetry resembles the "tomahawk" scenario in the literature on new economic geography (Fujita et al. (1999), etc.).

community. Thus, we emphasize the fundamental role of a non-degenerate distribution of types in the economy, as different ones lead to different equilibrium communities, both in terms of size and –more importantly– of its composition.

Indeed, if we focus on the cases in which our "voluntary" setup differs from the "compulsory" one present in the liteature, we find that stable equilibria –the ones that we are likely to observe in the data– are closely connected to the maxima of the type distribution, as are the associated social norms to the corresponding critical points in the type domain. Thus, a simple inspection of the type distribution is quite informative about the number and stability properties of the equilibria we can expect, as well as about the associated social norm that is likely to emerge in each case. Furthermore, said social norm can differ dramatically from the one in a "compulsory" setup, as is the case in a polarized society in which the maxima of the type distribution are quite apart from the average type in the economy.

Since entry choices and social norms are jointly determined, changes in the attractiveness of the BCG will affect both the size and the composition of the community. Like in network good games or market entry games, an increase in the attractiveness of the BCG usually leads to a larger community. But unlike them, it can lead to both the entry of new players as well as the exit of some former members, because the resulting change in the social norm makes some players better off and others worse off. Also unlike them, entry choices are not inherently strategic complements or substitutes, but their relationships depend on the types of the members, on the type of the potential entrant, and on the social norm.

Thus, the BCG+entry model is a useful tool to understand and analyse many situations of economic relevance, from social networks to academia, from political economy to cultural economics. The present analysis is however a simple one, and just a first step towards a better understanding of these and other scenarios. In the future, thus, we plan to relax some of the assumptions made here to better reflect the features of those situations. They include, among others, allowing for more and endogenously set up BCGs, and a dynamic version that will permit us to analyse the impact of education and immigration policies on a community's size and composition.

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Appendix

A Proofs

- **Proof of proposition 2**:

  Proof. Note that $u_{IN}\left(x^*_\theta;\theta\right) = p(1-p)[\bar{\theta}+h-\theta][\theta-(\bar{\theta}-h)]$. A player of type θ optimally enters the BCG if and only if $u_{IN}\left(x^*_\theta;\theta\right) \geq 0$. Since $\kappa > 0$, (at least) some types of players will enter the BCG. To determine who enters the game, we need to first find the types of the players who are indifferent between entering the BCG and not. Technically, this means finding the roots of $u_{IN}\left(x^*_\theta;\theta\right) = 0$, which yield $\theta_L = \bar{\theta} - h$ and $\theta_U = \bar{\theta} + h$. Since types are only defined in the [0,1] interval, the entrant community must be a subset of that real-valued interval, and hence the highest type that enters must be 1 if and only if $\theta_U \geq 1$ and the lowest type that enters must be 0 if and only if $\theta_L \leq 0$. As a result, the entrant community is given by the interval around $\bar{\theta}$: $\left[L(\bar{\theta}), U(\bar{\theta})\right] \subset [0,1]$. Since it is closed and bounded, it is compact. ■

- **Proof of proposition 3**:

  Proof.

  1. Continuous: From the definition of the average function we know that $a\left(\bar{\theta}\right) := \frac{\int_{L(\bar{\theta})}^{U(\bar{\theta})} \theta\cdot f(\theta)d\theta}{\int_{L(\bar{\theta})}^{U(\bar{\theta})} f(\theta)d\theta}$. Continuity follows from the fact that: (i) $L\left(\bar{\theta}\right)$ and $U\left(\bar{\theta}\right)$ are continuous in $\bar{\theta}$ $\forall\bar{\theta}$; (ii) the integrand is also continuous; and (iii) $h > 0$ so that the size of the community (the denominator of the quotient) is strictly positive.

  2. Bounded: By definition, $0 \leq L(\bar{\theta}) < U(\bar{\theta}) \leq 1$:

  $$0 \leq L\left(\bar{\theta}\right) = \frac{\int_{L(\bar{\theta})}^{U(\bar{\theta})} L\left(\bar{\theta}\right)\cdot f\left(\theta\right) d\theta}{\int_{L(\bar{\theta})}^{U(\bar{\theta})} f\left(\theta\right) d\theta} < \frac{\int_{L(\bar{\theta})}^{U(\bar{\theta})} \theta\cdot f\left(\theta\right) d\theta}{\int_{L(\bar{\theta})}^{U(\bar{\theta})} f\left(\theta\right) d\theta} < \frac{\int_{L(\bar{\theta})}^{U(\bar{\theta})} U\left(\bar{\theta}\right)\cdot f\left(\theta\right) d\theta}{\int_{L(\bar{\theta})}^{U(\bar{\theta})} f\left(\theta\right) d\theta} = U(\bar{\theta})$$

  This implies $a\left(\bar{\theta}\right) \in (L(\bar{\theta}), U(\bar{\theta})) \subset (0,1)$.

  3. Weakly increasing: Let $U' := \frac{\partial U(\bar{\theta})}{\partial\bar{\theta}} \in \{0,1\}$ and $L' := \frac{\partial L(\bar{\theta})}{\partial\bar{\theta}} \in \{0,1\}$. Taking a partial derivative with respect to $\bar{\theta}$ for $\bar{\theta} \neq \{h, 1-h\}$[42]:

  $$\frac{\partial a\left(\bar{\theta}\right)}{\partial\bar{\theta}} = \frac{1}{s\left(\bar{\theta}\right)}\left[\left(U(\bar{\theta}) - a(\bar{\theta})\right) f(U(\bar{\theta}))U' + \left(a(\bar{\theta}) - L(\bar{\theta})\right) f(L(\bar{\theta}))L'\right]$$

  By the above boundary property, all factors in each term are nonnegative, and so is the derivative. Thus, $\frac{\partial a(\bar{\theta})}{\partial\bar{\theta}} \geqslant 0$.

  ■

[42] At these particular values, there is a switch in the type of the community and the average function $a(\bar{\theta})$ may not be differentiable.

- **Proof of proposition 4**:

  Proof. By proposition 3, at least a fixed point of equation 2 exists by Brouwer's Fixed Point Theorem. Since $a(\bar{\theta}) \in (0,1) \forall \bar{\theta}$, the equilibria must be interior. ■

- **Proof of proposition 5**:

  Proof. Given that $\hat{\Theta}$ is empty, an equilibrium which supports an interior community cannot exist by the following lemma:

Lemma 1 *An interior community can be supported in equilibrium only if $\hat{\Theta}$ is non-empty.*

Proof. We prove this lemma by contradiction. So suppose that $\hat{\Theta}$ is the empty set and a Nash equilibrium which supports an interior community exists. Equation (3) applied to an interior community can be written as:

$$\frac{1}{2}[F(\bar{\theta}^* + h) + F(\bar{\theta}^* - h)] = \int_{\bar{\theta}^* - h}^{\bar{\theta}^* + h} F(\theta) \frac{1}{2h} d\theta \tag{12}$$

If $f(\theta)$ has no local extrema in the interior of its domain (i.e., $\hat{\Theta}$ is empty) then $f(\theta)$ is either strictly decreasing in θ, implying that F is strictly concave $\forall \theta$, or $f(\theta)$ is strictly increasing in θ, implying that $F(\theta)$ is strictly convex $\forall \theta$. If $F(\theta)$ is strictly concave (convex) $\forall \theta$, then $LHS < (>) RHS$ of equation 12 by SOSD (refer to footnote 17) so that $a(\bar{\theta}) < (>) \bar{\theta}$ $\forall \bar{\theta} \in (h, 1-h)$. But this implies that equation 12 is not satisfied with equality for any $\bar{\theta} \in (h, 1-h)$, a contradiction. ■

Moreover, an equilibrium which supports a universal community cannot exist by corollary 3. However, an equilibrium must exist by proposition 4 and therefore, the equilibria must support either a corner-low community or a corner-high community or both.

We prove this proposition by contradiction. Suppose not so that there is an equilibrium characterized by F being strictly convex $\forall \theta$ and a corner-low community being supported in equilibrium. By definition 5, $\bar{\theta}^* \leq h$. Furthermore, equation 3 is satisfied in equilibrium:

$$\frac{h}{\bar{\theta}^* + h} F(\bar{\theta}^* + h) + \frac{\bar{\theta}^*}{\bar{\theta}^* + h} F(0) = \int_0^{\bar{\theta}^* + h} F(\theta) \frac{1}{\bar{\theta}^* + h} d\theta \tag{13}$$

Given that $F(\theta)$ is nondecreasing and strictly convex $\forall \theta$, it must be the case that $LHS > RHS$ of equation (13) by FOSD and SOSD respectively with respect to the even distribution (refer to footnote 17). Therefore $a(\bar{\theta}^*) > \bar{\theta}^*$. A contradiction.

Suppose now that there exists an equilibrium which supports a corner-high community and it is characterized by F being strictly concave $\forall \theta$. By definition 5, $\bar{\theta}^* \geq 1 - h$.

Then equation 3 is satisfied in equilibrium:

$$\frac{1-\bar{\theta}^*}{1-\bar{\theta}^*+h}F(1)+\frac{h}{1-\bar{\theta}^*+h}F(\bar{\theta}^*-h)=\int_{\bar{\theta}^*-h}^{1}F(\theta)\frac{1}{1-\bar{\theta}^*+h}d\theta \tag{14}$$

Given that F is nondecreasing and strictly concave $\forall\theta$, it must be the case that $LHS < RHS$ of equation 14 by FOSD and SOSD respectively with respect to the even distribution (refer to footnote 17) and therefore $a(\bar{\theta}^*) < \bar{\theta}^*$. A contradiction.

Taking limits using equation 13 (14), we have that $\lim_{h\to 0} LHS = 0$ ($\lim_{h\to 0} LHS = 1$). Therefore $\bar{\theta}^* \to 0$ ($\bar{\theta}^* \to 1$) as $h \to 0$. Finally, the statement $\bar{\theta}^* < h$ ($\bar{\theta}^* > 1-h$) follows from definition 5. ■

- **Proof of proposition 6**:

  Proof. Given that $\hat{\Theta}$ is nonempty, f has at least one interior critical point which is an inflexion point of F. Given that the equilibrium must be interior, note that for any $\theta \in (0,1)$, we can always find sufficiently low values of h (i.e., $0 < h < \min\{\theta, 1-\theta\}$) such that $\theta - h > 0$ and $\theta + h < 1$.

  We first prove that equation (12) is satisfied for an arbitrarily small h only at a critical point. To see this, note that for all real-values $\tilde{\theta} \notin \hat{\Theta}$, there exists some $h \in (0, \min\{\tilde{\theta}, 1-\tilde{\theta}\})$ such that $F(\theta)$ is either "locally" strictly concave or "locally" strictly convex in θ in the entire arbitrarily small vicinity of $\tilde{\theta}$ given by $(\tilde{\theta}-h, \tilde{\theta}+h)$. In the first (second) case, $LHS < (>) RHS$ of equation (12) so that $a(\tilde{\theta}) < (>) \tilde{\theta}$ for h arbitrarily small. Thus, as $h \to 0$, equation (12) is satisfied if and only if it is evaluated at a critical point as this critical point is an inflexion point of the corresponding cumulative distribution function $F(\theta)$. Hence, at any equilibrium which supports an interior community, $\bar{\theta}^* \to \hat{\theta} \in \hat{\Theta}$ as $h \to 0$.

  On the other hand, from the proof of proposition 5 we know that if an equilibrium which supports a corner-low (high) community exists for arbitrarily small values of h, then $\bar{\theta}^* \to 0$ ($\bar{\theta}^* \to 1$) as $h \to 0$.

  Now $\hat{\Theta}$ is a chain whose elements can be ordered by the less-than relation. Let $\hat{\theta}_{min}$ and $\hat{\theta}_{max}$ denote the least and greatest elements in the set $\hat{\Theta}$ respectively.

  Firstly, suppose that $\hat{\theta}_{min}$ ($\hat{\theta}_{max}$) is a local maximum. Then $F(\theta)$ is strictly convex (concave) for $\theta < \hat{\theta}_{min}$ ($\theta > \hat{\theta}_{max}$). This implies that for an arbitrarily small h, there is no equilibrium characterized by $\bar{\theta}^* < \hat{\theta}_{min}$ ($\bar{\theta}^* > \hat{\theta}_{max}$) which supports a corner-low (high) community as $LHS > RHS$ ($LHS < RHS$) of equation 13 (equation 14) for all $\bar{\theta}^* < \hat{\theta}_{min}$ ($\bar{\theta}^* > \hat{\theta}_{max}$) and h made arbitrarily small. Instead, suppose now that $\hat{\theta}_{min}$ ($\hat{\theta}_{max}$) is a local minimum. Then $F(\theta)$ is strictly concave (convex) for $\theta < \hat{\theta}_{min}$ ($\theta > \hat{\theta}_{max}$). The equilibrium characterized by $\bar{\theta}^* \to \hat{\theta}_{min}$ ($\bar{\theta}^* \to \hat{\theta}_{max}$) as $h \to 0$ is unstable by proposition 7. This together with corollary 2 implies that for an arbitrarily small h, there must a stable equilibrium characterized by $\bar{\theta}^* < \hat{\theta}_{min}$ ($\bar{\theta}^* > \hat{\theta}_{max}$) which supports a corner-low (high) community.

All these statements imply that for an arbitrarily small h there are as many Nash equilibria as twice the total number of local maxima (including the exterior ones as well) minus one. ■

- **Proof of proposition 7**:

  Proof. By taking second-order Taylor approximations evaluated at $\bar{\theta}^*$ around $\bar{\theta}^* + h$ and $\bar{\theta}^* - h$ respectively, we have:

$$F(\bar{\theta}^*) \approx F(\bar{\theta}^* + h) - f(\bar{\theta}^* + h)h + \frac{1}{2}f'(\bar{\theta}^* + h)h^2$$
$$F(\bar{\theta}^*) \approx F(\bar{\theta}^* - h) + f(\bar{\theta}^* - h)h + \frac{1}{2}f'(\bar{\theta}^* - h)h^2$$

  which implies:

$$[F(\bar{\theta}^* + h) - F(\bar{\theta}^* - h)] - [f(\bar{\theta}^* + h) + f(\bar{\theta}^* - h)]h \approx \frac{h^2}{2}[f'(\bar{\theta}^* - h) - f'(\bar{\theta}^* + h)] \quad (15)$$

  Note that the term on the RHS is positive (negative) if the compact interval $[\bar{\theta}^* - h, \bar{\theta}^* + h]$ is entirely located on a concave (convex) region of f. ■

- **Proof of proposition 8**:

  Proof. By totally differentiating equation (2), we get the following result:

$$\frac{d\bar{\theta}^*}{dh} = \begin{cases} \frac{f(\bar{\theta}^*+h)h}{F(\bar{\theta}^*+h)-f(\bar{\theta}^*+h)h} & \text{if corner-low} \\ \frac{[f(\bar{\theta}^*+h)-f(\bar{\theta}^*-h)]h}{[F(\bar{\theta}^*+h)-F(\bar{\theta}^*-h)]-[f(\bar{\theta}^*+h)+f(\bar{\theta}^*-h)]h} & \text{if interior} \\ \frac{(-1)f(\bar{\theta}^*-h)h}{1-[F(\bar{\theta}^*-h)+f(\bar{\theta}^*-h)h]} & \text{if corner-high} \end{cases} \quad (16)$$

  The denominator of $\frac{d\bar{\theta}^*}{dh}$ is positive (negative) for corner-low (-high) communities if and only if the equilibrium is stable (unstable). At any stable equilibrium, the sign of the derivative is given by the sign of the numerator. In an interior community, if the density is locally skewed to the right (left) around a local maximum $\hat{\theta}$ then $f(\hat{\theta} + \epsilon) > (<) f(\hat{\theta} - \epsilon)$ for a sufficiently small $\epsilon > 0$. Finally, continuity follows from the fact that the function $a(\bar{\theta})$ is continuous in h. ■

- **Proof of proposition 9**:

  Proof. By equation (16) in the proof of proposition 8, $dU^*/dh = d(\bar{\theta}^* + h)/dh > 0$ for a corner-low community whereas $dL^*/dh = d(\bar{\theta}^* - h)/dh < 0$ for a corner-high community. This implies that the equilibrium community size is increasing in h for these communities.

  By total differentiation, we get the following result:

$$\frac{ds(\bar{\theta}^*)}{dh} = \frac{[F(\bar{\theta}^* + h) - F(\bar{\theta}^* - h)][f(\bar{\theta}^* + h) + f(\bar{\theta}^* - h)] - 4f(\bar{\theta}^* + h)f(\bar{\theta}^* - h)h}{[F(\bar{\theta}^* + h) - F(\bar{\theta}^* - h)] - [f(\bar{\theta}^* + h) + f(\bar{\theta}^* - h)]h} \quad (17)$$

If the equilibrium is stable (equation 5) then, for an interior community, $f(\bar{\theta}^* + h) + f(\bar{\theta}^* - h) < \frac{1}{h}[F(\bar{\theta}^* + h) - F(\bar{\theta}^* - h)]$ and therefore $\frac{ds(\bar{\theta}^*)}{dh} > 0$.[43] ■

- **Proof of proposition 10**:

  Proof. By equation (16) in the proof of proposition 8, $U(\bar{\theta}^*)$ is strictly increasing in h ($d(\bar{\theta}^* + h)/dh > 0$) for a stable corner-low community whereas $L(\bar{\theta}^*)$ is strictly decreasing in h ($d(\bar{\theta}^* - h)/dh < 0$) for a stable corner-high community. Therefore, there is no exit in stable corner communities as h increases.

  By equation (16), $d(\bar{\theta}^* + h)/dh < 0$ for an interior community if and only if:

$$\frac{1}{2h}[F(\bar{\theta}^* + h) - F(\bar{\theta}^* - h)] < f(\bar{\theta}^* - h)$$

  in a stable equilibrium (opposite sign if unstable). Hence, the left skewness around $\bar{\theta}^*$ (i.e., $f(\bar{\theta}^* - h) > f(\bar{\theta}^* + h)$) is a necessary condition in a stable equilibrium given equation 4.

  Furthermore, $d(\bar{\theta}^* - h)/dh > 0$ for an interior community if and only if:

$$\frac{1}{2h}[F(\bar{\theta}^* + h) - F(\bar{\theta}^* - h)] < f(\bar{\theta}^* + h)$$

  in a stable equilibrium (opposite sign if unstable). Thus, the right skewness around $\bar{\theta}^*$ (i.e., $f(\bar{\theta}^* - h) < f(\bar{\theta}^* + h)$) is a necessary condition in a stable equilibrium given equation 4. ■

- **Proof of proposition 11**:

  Proof. Given that $u_{IN}(x_\theta^*; \theta) = \kappa - p(1-p)\left(\theta - \bar{\theta}\right)^2$, then

$$\frac{du_{IN}(x_\theta^*; \theta)}{d\kappa} = 1 + \frac{1}{h}\frac{d\bar{\theta}^*}{dh}\left(\theta - \bar{\theta}\right)$$

  Hence, $\frac{du_{IN}(x_\theta^*;\theta)}{d\kappa} < 0$ if and only if:

$$\frac{d\bar{\theta}^*}{dh}\left(\theta - \bar{\theta}\right) < -h \tag{18}$$

  Subsets (i) and (ii) follow from this condition and equation (16).

  (i) Since $\frac{d\bar{\theta}^*}{dh} > 0$ for corner-low community, $\bar{\theta}^* - \left(\frac{d\bar{\theta}^*}{dh}\right)^{-1} h > 0$ if and only if $\frac{d\bar{\theta}^*}{dh} > \frac{h}{\bar{\theta}^*}$. Alternatively, $\bar{\theta}^* + h - \frac{F(\bar{\theta}^*+h)}{f(\bar{\theta}^*+h)} > 0$ if and only if $\frac{F(\bar{\theta}^*+h)}{\bar{\theta}^*+h} < f(\bar{\theta}^* + h)$ which is equivalent to strict convexity of F at $\bar{\theta}^* + h$.

  (ii) It follows from condition (18) and the inequality $|\theta - \bar{\theta}^*| \leq h$ for the entrants.

[43] As a result, for an interior community, $\frac{ds(\bar{\theta}^*)}{dh} < 0$ if and only if $\frac{1}{h}[F(\bar{\theta}^* + h) - F(\bar{\theta}^* - h)] \in \left(\frac{4f(\bar{\theta}^*+h)f(\bar{\theta}^*-h)}{f(\bar{\theta}^*+h)+f(\bar{\theta}^*-h)}, f(\bar{\theta}^* + h) + f(\bar{\theta}^* - h)\right)$ and $\frac{ds(\bar{\theta}^*)}{dh} > 0$ if and only if $\frac{1}{h}[F(\bar{\theta}^* + h) - F(\bar{\theta}^* - h)] \notin \left[\frac{4f(\bar{\theta}^*+h)f(\bar{\theta}^*-h)}{f(\bar{\theta}^*+h)+f(\bar{\theta}^*-h)}, f(\bar{\theta}^* + h) + f(\bar{\theta}^* - h)\right]$

(iii) Since $\frac{d\bar{\theta}^*}{dh} < 0$ for corner-high community, $\bar{\theta}^* + \left(-\frac{d\bar{\theta}^*}{dh}\right)^{-1} h < 1$ if and only if $-\frac{d\bar{\theta}^*}{dh} > \frac{h}{1-\bar{\theta}^*}$. Alternatively, $\bar{\theta}^* - h + \frac{1-F(\bar{\theta}^*-h)}{f(\bar{\theta}^*-h)} < 1$ if and only if $\frac{1-F(\bar{\theta}^*-h)}{1-\bar{\theta}^*+h} < f(\bar{\theta}^* - h)$ which is equivalent to strict concavity of F at $\bar{\theta}^* - h$.

■

B Illustrations with different distribution functions

The computation of the size and average type of the community depends on the density function of types. We proceed to impose more structure on it to illustrate some of the results of sections 2 and 3. In particular, we explore the outcomes from different kinds of density functions: specifically, the cases of monotonically decreasing/increasing, single-peaked, multi-peaked and uniform density functions. The specific parameters used are indicated in each case.

B.1 Cumulative distribution F is strictly concave/convex/linear

B.1.1 Strictly concave F

If the density function is monotonically decreasing in θ, then the cumulative distribution F is strictly concave everywhere. It depicts an economy in which there are many low types and just a few high types. It would be good to represent a typical income distribution.

There are many functions that are monotonically decreasing, so we use here an example with a triangular distribution. Specifically, we assume $f(\theta) = 2 - 2\theta$ for every $\theta \in [0, 1]$ and 0 everywhere else. As a result, $F(\theta) = \theta(2 - \theta)$ and $E(\theta) = 1/3$.

The equilibrium value of $\bar{\theta}$ is found by looking for a fixed point such that $a(\bar{\theta}^*) = \bar{\theta}^*$. This value depends on the value of h. In this case an equilibrium always exists and it is unique and stable.

For sufficiently low values of h ($h < 2/3 = \max\{1/3, 2/3\}$), only low types enter and so a corner-low community emerges.[44]

The equilibrium is given by the lowest root of equation:

$$2h^2 + (3 - \bar{\theta}^*)(\bar{\theta}^* - h) = 0$$

[44] Note that an interior community cannot be supported in equilibrium because $a(\bar{\theta}) = \frac{\bar{\theta}-\bar{\theta}^2-(h^2/3)}{1-\theta} < \bar{\theta}$ since $h > 0$. Additionally, corner-high communites cannot be supported in equilibrium either because $a(\bar{\theta}) = \frac{1}{3}(1 + 2(\bar{\theta} - h)) \leq \frac{1}{3}(1 + 2\bar{\theta} - 2(1 - \bar{\theta})) < \frac{1}{3}(1 + 2\bar{\theta} - (1 - \bar{\theta})) = \bar{\theta}$ where the first inequality follows from the fact that $h \geq 1 - \bar{\theta}$ for a corner-high community.

Therefore, $\bar{\theta}^*$ is increasing and concave in h, as are the highest type that enter the BCG. and the size of the community, $s(\bar{\theta}^*) = (\bar{\theta}^* + h)(2 - \bar{\theta}^* - h)$.

For sufficiently high values of h ($h \geq 2/3 = \max\{1/3, 2/3\}$), everyone joins so that we have a universal community. The average type in the community is then flat and equal to the average type of the whole population, i.e, $\bar{\theta}^* = E(\theta) = 1/3$.

B.1.2 Strictly convex F

This is basically a mirror image of the previous scenario. For example, it could reflect the distribution of "poverty gaps". Now, the density f is monotonically increasing in θ.

B.1.3 Linear F

A uniform distribution depicts an economy in which all types are equally represented. It would reflect well the case in which the number of high, medium and low income people in the economy are roughly the same. Or the proportion of people of each type drafted by the army.

Formally, the uniform distribution corresponds to $f(\theta) = 1 \ \forall \theta \in [0,1]$ (and 0 everywhere else) with $E\theta = 1/2$. Thus, the cumulative distribution F is linear in θ so that the size function becomes $s(\bar{\theta}) = U(\bar{\theta}) - L(\bar{\theta})$, while the average function is $a(\bar{\theta}) = \frac{1}{2}[U(\bar{\theta}) + L(\bar{\theta})]$.

The equilibrium value of $\bar{\theta}$ is found by looking for a fixed point such that $a(\bar{\theta}^*) = \bar{\theta}^*$. There exists a continuum of Nash equilibria for $h < 1/2$:

- Community corner-low: $\bar{\theta}^* = h$ implying $U(\bar{\theta}^*) = 2h \ \forall h < 1/2$.
- Community interior: $\bar{\theta}^* \in (h, 1-h) \ \forall h < 1/2$. These equilibria are *peculiar* as they satisfy $\frac{\partial a(\bar{\theta}^*)}{\partial \bar{\theta}} = 1$ (i.e., $a(\bar{\theta}^*)$ is tangent to the 45^o-line, as opposed to the "crossing" equilibria we focused on).
- Community corner-high: $\bar{\theta}^* = 1 - h$ implying $L(\bar{\theta}^*) = 1 - 2h \ \forall h < 1/2$.

If $h \geq 1/2$ there is only one equilibrium in which everyone enters (universal community: $s^* = 1$ ($L^* = 0$ and $U^* = 1$) and so the average type of the entrants equals the population average type: $\bar{\theta}^* = E\theta = 1/2$.

Size is weakly increasing in h: $s(\bar{\theta}^*) = 2h$ if $h < 1/2$ (in all equilibria) and $s(\bar{\theta}^*) = 1$ if $h \geq 1/2$. The infinite number of equilibria can give rise to a "first-mover advantage": the first BCG to arise will disencourage the creation of others nearby, and so a random element could have lasting effects.

B.2 Single-peak distribution

This distribution has an interior mode: the most frequent type is somewhere in the middle (unlike the previous two cases, in which it was in one of the ends of the domain). This can be a good depiction of interior society's political leanings, where most people are in the interior, central regions of the political spectrum, while radicals at the end are relatively few. It could also, if the distribution is rather concentrated in a small subdomain, be used to model homogeneous societies where almost everyone has the same type or very similar ones. This could be useful to model scenarios as disparate as the (post-tax) income distribution in Scandinavian countries, or culturally homogeneous societies like North Korea.

Specifically, let's assume $f(\theta) = \beta\theta^{\alpha}(1-\theta^{\delta})$ for every $\theta \in [0,1]$ and 0 everywhere else with $\beta \equiv \frac{(1+\alpha)(1+\alpha+\delta)}{\delta}$. The mode is given by $\hat{\theta} = \left(\frac{\alpha}{\alpha+\delta}\right)^{1/\delta}$ whereas $E\theta = \frac{(1+\alpha)}{(2+\alpha)}\frac{(1+\alpha+\delta)}{(2+\alpha+\delta)}$. For instance, we set $\alpha = 1/3$ and $\delta = 4$ so that $E\theta = 0,481203008 < \frac{1}{2} < 0,526640388 = \hat{\theta}$. Hence, the density is "locally" skewed to the left around the mode ($Modeskewness \equiv (E\theta - \hat{\theta})/\sigma = -0,186506013$, where σ denotes the standard deviation) but as the domain gets bigger, it becomes more and more skewed to the right. Overall, Pearson's moment coefficient of skewness is: $E[\left(\frac{\theta-E\theta}{\sigma}\right)^3] = -0,012773536$.

The equilibrium value $\bar{\theta}^*$ is found by looking for a fixed point such that $a(\bar{\theta}^*) = \bar{\theta}^*$. There is a unique and stable equilibrium. The only type of community that is ruled out (in this example though!) is the one in which only high types enter (i.e., corner-high community).

The novelty, compared to the monotonically increasing/decreasing density functions, is that for h sufficiently low ($h <\simeq 0,48$) only the medium types enter the BCG (interior community: note that as $h \to 0$, $\bar{\theta}^* \to \hat{\theta}$). Then, for $0.48 \approx< h < 1 - E\theta$, as h increases, high types enter (since $L(\bar{\theta}^*) = 0$, it is a corner-low community) until everyone enters (universal community) at $h \geq 1 - E\theta$).

As it is illustrated in the central panel of display 8, the equilibrium is nonmonotonic in h for the interior community. It is first decreasing in h for $h <\approx 0.47$ due to the "local" left-skewness around the mode but it is increasing in h for $h \in [0.47, 0.48]$ due to the "local" right-skewness property of the distribution outside the mode's neighborhood.

Finally, all players benefit from an increase in κ.

B.3 Double-peak distribution

These distributions could reflect societies that are politically polarised. In extreme scenarios, they could depict economies populated by men ($\theta = 0$) and women ($\theta = 1$).

B.3.1 Monotonic Equilibria

Consider the family of density functions indexed by $\theta \in \Theta = [0,1]$, $f(\theta) = \beta\theta(1-\theta)(c-\theta)^2$ for all $c \geq 0$ and $\beta \equiv \frac{60}{10c(c-1)+3}$.

This distribution is double-peaked for $c \in (0,1)$ whereas it is single-peaked and skewed to the right (left) for $c \geq 1$ ($c = 0$). For the purpose of illustration, we fix $c = 2/5$. As a result, $\beta = 100$ and $E\theta = 2/3$; the local maxima are given by the roots to equation $4\theta^2 - \left(3 + \frac{4}{5}\right)\theta + \frac{2}{5} = 0$ (approximately $\hat{\theta}_1 = 0.12$ and $\hat{\theta}_2 = 0.83$); and the local minimum is given by $c = 2/5$. Note that the distribution is "locally" skewed to the left (right) around the second (first) local maximum.

Three equilibria coexist for sufficiently low values of h ($h \leq 0.349135$). When h is arbitrarily small, all equilibria support interior communities. The outer (odd) equilibria are stable while the inner (even) one is unstable. The one at the top (bottom) is decreasing (increasing) in h due to the local left (right) skewness of the distribution in the neighborhood of the local maximum[45] whereas the unstable middle one is decreasing in h because of the asymmetry of the density around the local minimum: it is a bit skewed to the left. Note that as $h \to 0$, the equilibria converge to the critical points of the density function. At $h = 0.25$ ($h = 0.15$), $U(\bar{\theta}^*) = 1$ ($L(\bar{\theta}^*) = 0$) on the top (bottom) equilibrium and the supported community becomes one of corner-high (corner-low). At approximately $h \approx 0.32$, the middle equilibria starts supporting a corner-low community as well. Both the middle and bottom equilibria converge to the same equilbrium value, $\bar{\theta}^* = 0.2075$, at $h \approx 0.3491356$. At this equilibrium value the $a(\bar{\theta}^*)$ function is "tangent" to the 45^o degree line: $\frac{\partial a(\bar{\theta}^*)}{\partial \bar{\theta}} = 1$. Given that $\frac{\partial^2 a(\bar{\theta}^*)}{\partial \bar{\theta} \partial h} > 0$, and $E\theta > 1/2$ only the top equilbria survives for $h > 0.35$ (refer to display 10). Furthermore, the display shows the exit of several high types due to an increase in h at the unstable equilibrium for $h > 0,3$.

B.3.2 NonMonotonic Equilibria

Consider the following density function defined for $\theta \in [0,1]$:

$$f(\theta) = \beta\left(\frac{1}{2} - \theta\right)^2 [\min\{(\theta + 0.1)^4, (1.1 - \theta)^4\} - 0,0001]$$

In this case, the distribution is symmetric so that $E\theta = Median = 1/2$. The local minimum is located on the axis of symmetry ($\check{\theta} = 1/2$) whereas the local maxima are approximately $\hat{\theta}_1 \approx 0.3$ and $\hat{\theta}_2 \approx 0.7$). Unlike the previous case, now the distribution is "locally" skewed to the left (right) around the lower (higher) local maximum.

[45] Thus, these outer communities converge as h increases. This implies that in this society, as attractiveness increases (the membership fee goes down), sub-cultures will assimilate into a unique, integrating one.

Three equilibria coexist for sufficiently low values of h ($h \leq 0.3$). Unlike the previous case, all equilibria refer to interior communities for $h < 1/2$. Hence, corner communities cannot be supported in equilibria. The outer equilibria are stable while the inner one is unstable for $h \leq 0.3$ but becomes stable for $h > 0.3$. Note that as $h \to 0$, the equilibria converge to the critical points of the density function. The ones at the top and bottom are nonmonotonic in h: the one at the top is increasing (decreasing) in h for $h < 0.24$ ($h < 0.24$) due to the local right (left) skewness of the distribution in the neighborhood of the local maximum and decreasing in h for $h \in (0.24, 0.3)$.[46] The size functions of these two equilibria are the same. The middle equilibrium is flat in h because of the symmetry of the density around the local minimum. Given that $E\theta = 1/2$, the middle equilibria exist for all values of h (the community is of interior for $h < 1/2$ and of universal for $h \geq 1/2$) whereas the exterior equilibria do not exist for $h > 0.3$ (Refer to the bimodal example in display 9).

[46] Thus, the outer communities diverge (their differences increase) as h increases, and so in this society differences are exacerbated and radicalisation and seggregation result. However, if h goes sufficiently up, a universal community would emerge.