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Tanaka, Yasuhito and Hattori, Masahiko

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Masahiko Hattori*
Faculty of Economics, Doshisha University,
Kamigyo-ku, Kyoto, 602-8580, Japan.
and
Yasuhito Tanaka†
Faculty of Economics, Doshisha University,
Kamigyo-ku, Kyoto, 602-8580, Japan.

Abstract

We investigate the existence of a strictly convex and strictly sub-additive cost function with positive fixed cost. If there is a positive fixed cost, any cost function can not be super-additive, and concavity (including linearity) of cost function implies strict sub-additivity. Then, does there exist a strictly convex and strictly sub-additive cost function? We will present such a cost function. It is close to a linear function although it is strictly convex.

Keywords: cost function, strict convexity, strict sub-additivity

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*mhattori@mail.doshisha.ac.jp
†yasuhito@mail.doshisha.ac.jp

1 Introduction

Convexity and concavity are important properties for cost functions of firms. Also super-additivity and sub-additivity are other important properties for them. A cost function $c(x)$ is convex when it satisfies

$$c(\lambda x + (1-\lambda)y) \leq \lambda c(x) + (1-\lambda)c(y) \text{ for } 0 \leq \lambda \leq 1,\ x \geq 0,\ y \geq 0.$$

It is concave when it satisfies

$$c(\lambda x + (1-\lambda)y) \geq \lambda c(x) + (1-\lambda)c(y) \text{ for } 0 \leq \lambda \leq 1,\ x \geq 0,\ y \geq 0.$$

It is super-additive if it satisfies

$$c(x+y) \geq c(x) + c(y),\ \text{ for } x \geq 0,\ y \geq 0.$$

It is sub-additive if it satisfies

$$c(x+y) \leq c(x) + c(y),\ \text{ for } x \geq 0,\ y \geq 0.$$

If these inequalities strictly hold, we say that a cost function is strictly convex, strictly concave and so on. Hattori and Tanaka (2017) have shown the following results about the case of zero fixed cost.

Zero fixed cost case It is well known that with zero fixed cost, that is, $c(0) = 0$, convexity implies super-additivity, and concavity implies sub-additivity. But converse relations do not hold (please see Bruckner and Ostrow (1962) and Sen and Stamatopoulos (2016)). Referring to Bourin and Hiai (2015), Sen and Stamatopoulos (2016) pointed out that the following function is super-additive but it is not convex.

$$xe^{-\frac{1}{x^2}},\ x \geq 0.$$

However, if, in addition to the zero fixed cost condition, we put the following assumption, we can show that super-additivity implies convexity, and sub-additivity implies concavity.

Assumption 1. (1) *If a cost function is convex in some interval, it is convex throughout the domain.*

(2) *If a cost function is concave in some interval, it is concave throughout the domain.*

Then, super-additivity and convexity are equivalent, and sub-additivity and concavity are equivalent.

Assumption 1 excludes a case where a cost function is convex in some interval and concave in another interval. Above mentioned $xe^{-\frac{1}{x^2}}$ is such a function.

In this paper we consider the case where there is a positive fixed cost, and will show the following results.

Positive fixed cost case It is well known that a cost function with a positive fixed cost can not be super-additive throughout the domain, and even with positive fixed cost concavity of a cost function means sub-additivity. Then, does there exist a strictly convex and strictly sub-additive cost function? We will show that the answer to this question is *Yes*. The following function is a strictly convex and strictly sub-additive cost function.

$$c(x) = \frac{x^n}{(x+1)^{n-1}} + n - 1,\ x \geq 0,\ n > 1.$$

In Section 2 we assume that n is a natural number which is larger than 1. In Section 3 we extend the result in that section to a case where n is a real number larger than 1.

2 A strictly convex and strictly sub-additive cost function with positive fixed cost when n is a natural number

2.1 Preliminary results

We write a cost function $c(x)$ with a positive fixed cost f as follows.

$$c(x) = v(x) + f.$$

$v(x)$ is a variable cost.

First we show the following results.

Lemma 1. (1) *A cost function with a positive fixed cost cannot be super-additive throughout the domain.*

(2) *Concavity of a cost function with or without fixed cost means its sub-additivity.*

Proof. (1) Impossibility of super-additivity:

Suppose that $c(x)$ is defined for $x \geq 0$ and super-additive throughout the domain with $c(0) > 0$. Then, for $x > 0$ we have

$$c(x) = c(x+0) \geq c(x) + c(0) > c(x).$$

It is a contradiction.

A cost function may be super-additive in some interval. A cost function may be super-additive in some interval although it can not be super-additive throughout the domain. Assume $c(x) = x^2 + 4,\ x \geq 0$. For $y > 0$ we have

$$(x+y)^2 + 4 - (x^2 + 4 + y^2 + 4) = 2xy - 4.$$

When $xy > 2$, $c(x)$ is super-additive.

(2) Concavity $\Rightarrow$ sub-additivity:

For $x \geq 0$ and $y \geq 0$ concavity implies

$$c(x) = c\left(\frac{x}{x+y}(x+y) + \frac{y}{x+y} \cdot 0\right) \geq \frac{x}{x+y}c(x+y) + \frac{y}{x+y}c(0),$$

and

$$c(y) = c\left(\frac{y}{x+y}(x+y) + \frac{x}{x+y} \cdot 0\right) \geq \frac{y}{x+y}c(x+y) + \frac{x}{x+y}c(0).$$

Then,

$$c(x) + c(y) \geq c(x+y) + c(0).$$

Thus,

$$c(x) + c(y) \geq c(x+y).$$

Linear cost function is strictly sub-additive. Suppose $c(x) = ax + f,\ a \geq 0, f \geq 0$. Then, we have

$$c(x+y) - c(x) - c(y) = a(x+y) + f - ax - f - ay - f = -f < 0.$$

Therefore, $c(x)$ is strictly sub-additive.

□

2.2 General property

Strict convexity of a cost function $c(x)$ is equivalent to strict convexity of a variable cost $v(x)$. Since $v(0) = 0$, strict convexity of $v(x)$ implies strict super-additivity.

Lemma 2. *If a variable cost $v(x)$ is strictly convex, it is strictly super-additive.*

Proof. Strict convexity of $v(x)$ means

$$v(x) = v\left(\frac{x}{x+y}(x+y) + \frac{y}{x+y} \cdot 0\right) < \frac{x}{x+y}v(x+y) + \frac{y}{x+y}v(0),$$

and

$$v(y) = v\left(\frac{y}{x+y}(x+y) + \frac{x}{x+y} \cdot 0\right) < \frac{y}{x+y}v(x+y) + \frac{x}{x+y}v(0).$$

Then,

$$v(x) + v(y) < v(x+y) + v(0).$$

Since $v(0) = 0$,

$$v(x) + v(y) < v(x+y).$$

□

Suppose that a cost function $c(x)$ is sub-additive. Then,

$$c(x+y) \leq c(x) + c(y).$$

This means

$$v(x+y) \leq v(x) + v(y) + f.$$

Thus, we have

$$0 < v(x+y) - v(x) - v(y) \leq f. \tag{1}$$

Therefore, we must search a function which is positive, increasing and satisfies (1) for all $x \geq 0$ and $y \geq 0$.

2.3 An example of strictly convex and strictly sub-additive cost function with positive fixed cost

Now consider the following cost function;

$$c(x) = \frac{x^n}{(x+1)^{n-1}} + n - 1, \; x \geq 0, \tag{2}$$

where n is a natural number which is larger than 1. The first order and the second order derivatives of $c(x)$ are

$$c'(x) = \frac{x^{n-1}(x+n)}{(x+1)^n} > 0,$$

and

$$c''(x) = \frac{(n-1)nx^{n-2}}{(x+1)^{n+1}} > 0.$$

Thus, $c(x)$ is an increasing and strictly convex function. The fixed cost is $n-1 > 0$. Let us check its sub-additivity in the following theorem.

Theorem 1. *The cost function in (2) is strictly sub-additive.*

Proof. We prove this theorem by mathematical induction. Let

$$w^k = [c(x+y) - c(x) - c(y)]|_{n=k},$$

and

$$w^{k+1} = [c(x+y) - c(x) - c(y)]|_{n=k+1}.$$

We have

$$w^k = -\frac{A}{(x+1)^k(y+1)^k(x+y+1)^k},$$

and

$$w^{k+1} = -\frac{B}{(x+1)^k(y+1)^k(x+y+1)^k},$$

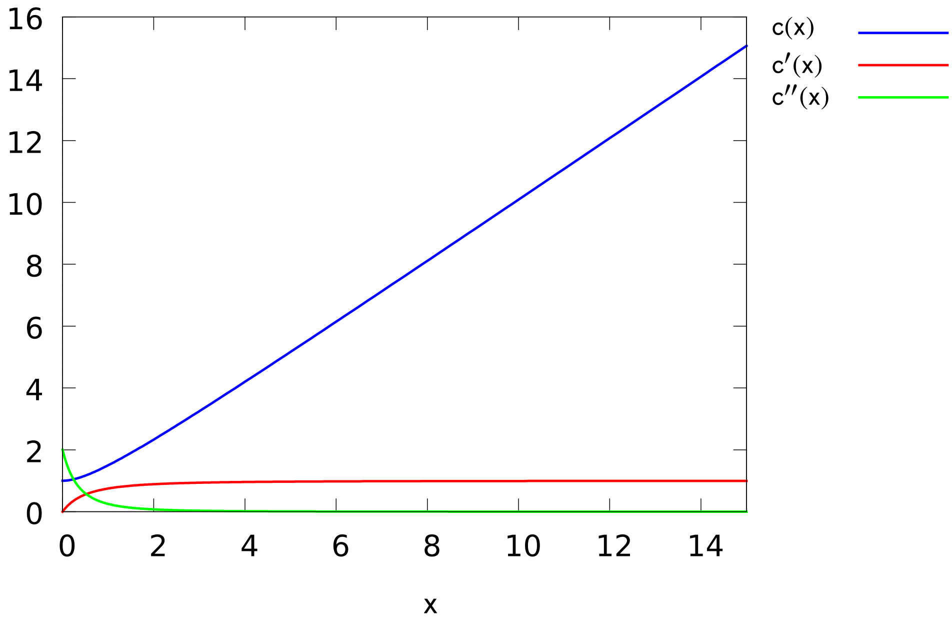

Figure 1: An example of $c(x)$ when $n = 2$

where

$$\begin{aligned}A =& k(x+1)^k(y+1)^k(x+y+1)^k - (x+1)^k(y+1)^k(x+y+1)^k \\ &+ x^{k+1}(y+1)^k(x+y+1)^k + x^k(y+1)^k(x+y+1)^k \\ &+ (x+1)^k y^{k+1}(x+y+1)^k + (x+1)^k y^k(x+y+1)^k \\ &- (x+1)^k y(y+1)^k(x+y)^k - x(x+1)^k(y+1)^k(x+y)^k - (x+1)^k(y+1)^k(x+y)^k,\end{aligned}$$

and

$$\begin{aligned}B =& k(x+1)^k(y+1)^k(x+y+1)^k + x^{k+1}(y+1)^k(x+y+1)^k + (x+1)^k y^{k+1}(x+y+1)^k \\ &- (x+1)^k y(y+1)^k(x+y)^k - x(x+1)^k(y+1)^k(x+y)^k\end{aligned}$$

First consider a case where $n = 2$. Then, we obtain

$$w^2 = -\frac{x^2 + xy + 2x + y^2 + 2y + 1}{(x+1)(y+1)(x+y+1)} < 0.$$

Thus, $c(x)$ is strictly sub-additive when $n = 2$. Now suppose that it is strictly sub-additive when $n = k$, and consider a case where $n = k + 1$. Assume $w^k < 0$. Comparing w^{k+1} and w^k, we get

$$w^{k+1} - w^k = -\frac{C}{(x+1)^k(y+1)^k(x+y+1)^k}.$$

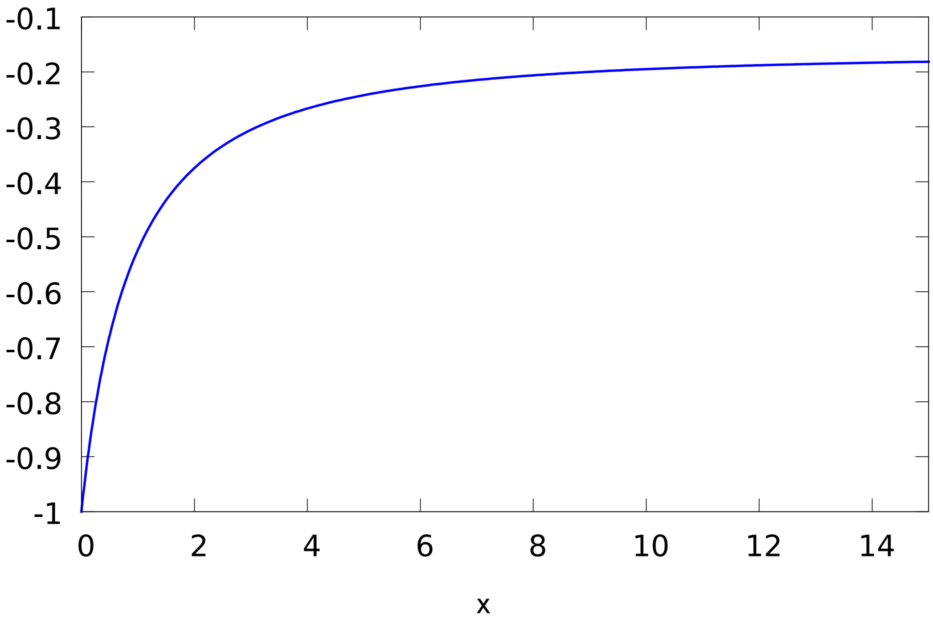

Figure 2: An illustration of w^n when $n = 2$ and $y = 5$

where

$$\begin{aligned} C =&(x+1)^k(y+1)^k(x+y+1)^k - x^k(y+1)^k(x+y+1)^k \\ &-(x+1)^k y^k(x+y+1)^k + (x+1)^k(y+1)^k(x+y)^k. \end{aligned}$$

Since C is reduced to

$$\begin{aligned} C =&[(x+1)^k - x^k](y+1)^k(x+y+1)^k + (x+1)^k[(y+1)^k(x+y)^k - y^k(x+y+1)^k] \\ =&[(x+1)^k - x^k](y+1)^k(x+y+1)^k + (x+1)^k[(y^2+xy+y+1)^k - (y^2+xy+y)^k] > 0, \end{aligned}$$

we find

$$w^{k+1} - w^k < 0.$$

Because we assume $w^k < 0$, this implies $w^{k+1} < 0$. Therefore, $c(x)$ is sub-additive for any value of n. □

In Figure 1 we illustrate $c(x)$, $c'(x)$ and $c''(x)$ assuming $n = 2$. Also in Figure 2 we illustrate the relation between the value of $c(x+y) - c(x) - c(y)$ and x when $n = 2$ and $y = 5$. $c(x+y) - c(x) - c(y) < 0$ means that $v(x+y) - v(x) - v(y)$ is smaller than $n-1$. In Figure 3 we illustrate the value of $v(x+y) - v(x) - v(y)$ assuming $n = 2$ and $y = 5$. $v(x+y) - v(x) - v(y)$ is as follows.

$$v(x+y) - v(x) - v(y) = \frac{D}{(x+1)^n(y+1)^n(x+y+1)^n},$$

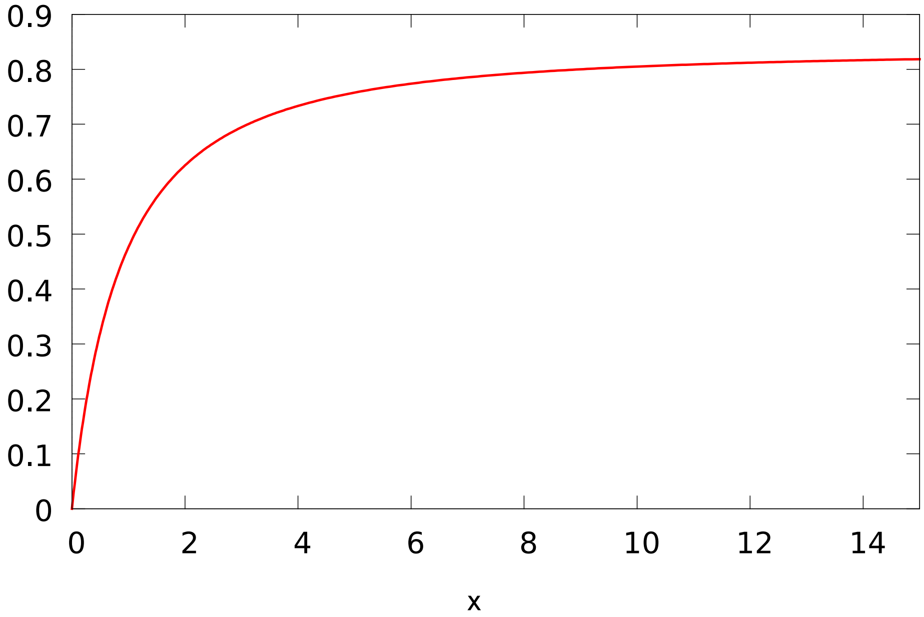

Figure 3: An illustration of $v(x+y)-v(x)-v(y)$ when $n=2$ and $y=5$

where

$$\begin{aligned}D =&(x+1)^n y(y+1)^n(x+y)^n + x(x+1)^n(y+1)^n(x+y)^n\\&+(x+1)^n(y+1)^n(x+y)^n - x^{n+1}(y+1)^n(x+y+1)^n\\&-x^n(y+1)^n(x+y+1)^n-(x+1)^n y^{n+1}(x+y+1)^n-(x+1)^n y^n(x+y+1)^n.\end{aligned}$$

3 Extension to a case where *n* is a real number

We extend the result in the previous section to a case where n is a real number. In this section we assume $x>0$ and $y>0$. Again consider the following cost function.

$$c(x)=\frac{x^n}{(x+1)^{n-1}}+n-1,\ x>0.$$

Now n is a real number larger than 1. Let

$$\varphi(x)=\left(\frac{x+1}{x}\right)^x.$$

As a preliminary result we show

Lemma 3. *$\varphi(x)$ is increasing with respect to x.*

Proof. We have

$$\varphi(x) = e^{\ln\left(\frac{x+1}{x}\right)^x} = e^{x\ln\left(\frac{x+1}{x}\right)}.$$

Then,

$$\varphi'(x) = \varphi(x)\left(\ln\frac{x+1}{x} - \frac{1}{x+1}\right).$$

Let

$$\psi(x) = \frac{\varphi'(x)}{\varphi(x)} = \left(\ln\frac{x+1}{x} - \frac{1}{x+1}\right).$$

Differentiating this with respect to x, we get

$$\psi'(x) = -\frac{1}{x(x+1)} + \frac{1}{(x+1)^2} < 0.$$

Thus, $\psi(x)$ is decreasing with respect to x, and we have $\lim_{x\to\infty}\psi(x) = 0$ because $\lim_{x\to\infty}\frac{x+1}{x} = 1$. Then, $\psi(x) > 0$. Since $\varphi(x) > 0$, we obtain $\varphi'(x) > 0$. □

Let

$$w^n = c(x+y) - c(x) - c(y),\ x > 0,\ y > 0.$$

This lemma means

$$\left(\frac{x+1}{x}\right)^x < e = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x,$$

or

$$\ln\left(\frac{x+1}{x}\right)^x < 1.$$

Also it implies

$$\left(\frac{x+y+1}{x+y}\right)^{x+y} > \left(\frac{y+1}{y}\right)^y \text{ for } x > 0,\ y > 0. \tag{3}$$

We show the following theorem.

Theorem 2. *$c(x)$ is strictly sub-additive when n is a real number larger than 1.*

Proof. First when $n = 1$, $c(x) = x$. Then, $w^1 = 0$. We show that w^n is decreasing with respect to n. We have

$$w^n = \frac{(x+y)^n}{(x+y+1)^n} - \frac{x^n}{(x+1)^n} - \frac{y^n}{(y+1)^n} - (n-1).$$

Differencing w^n with respect to n yields

$$\frac{dw^n}{dn} = -\frac{E}{(x+1)^n(y+1)^n(x+y+1)^n},$$

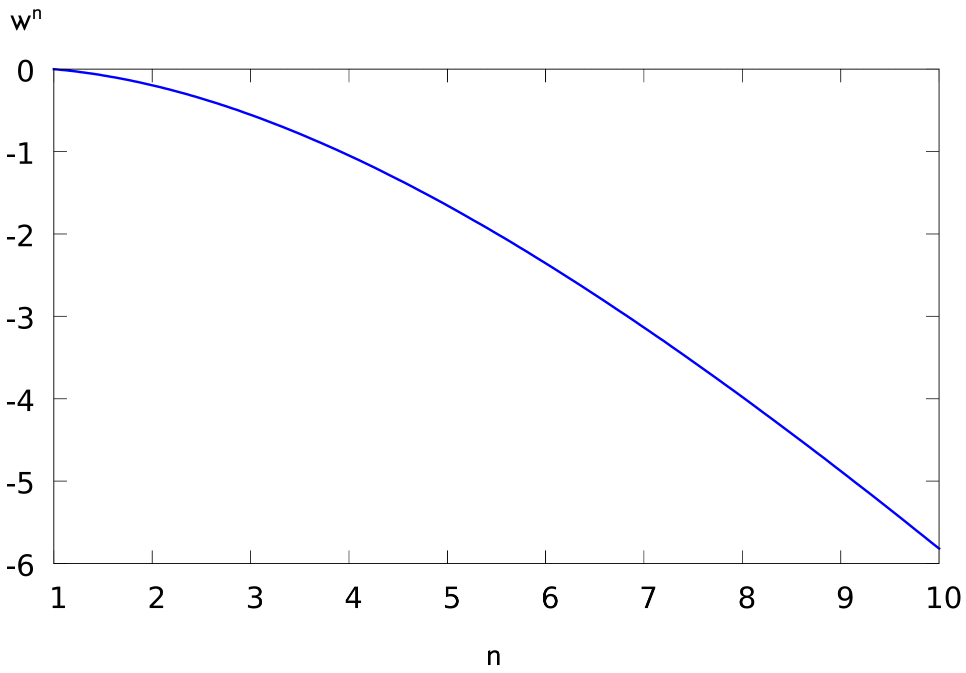

Figure 4: An example of w^n when $x = 10$ and $y = 5$

where

$$\begin{aligned}
E =&(x+1)^n y(y+1)^n(x+y)^n \ln(x+y+1) + x(x+1)^n(y+1)^n(x+y)^n \ln(x+y+1) \\
&+ (x+1)^n(y+1)^n(x+y)^n \ln(x+y+1) - (x+1)^n y(y+1)^n(x+y)^n \ln(x+y) \\
&- x(x+1)^n(y+1)^n(x+y)^n \ln(x+y) - (x+1)^n(y+1)^n(x+y)^n \ln(x+y) \\
&- (x+1)^n y^{n+1}(x+y+1)^n \ln(y+1) - (x+1)^n y^n(x+y+1)^n \ln(y+1) \\
&+ (x+1)^n y^{n+1}(x+y+1)^n \ln(y) + (x+1)^n y^n(x+y+1)^n \ln(y) \\
&- x^{n+1}(y+1)^n(x+y+1)^n \ln(x+1) - x^n(y+1)^n(x+y+1)^n \ln(x+1) \\
&+ x^{n+1}(y+1)^n(x+y+1)^n \ln(x) + x^n(y+1)^n(x+y+1)^n \ln(x) \\
&+ (x+1)^n(y+1)^n(x+y+1)^n.
\end{aligned}$$

D is reduced to

$$\begin{aligned}
D =&(x+y+1)(x+1)^n(y+1)^n(x+y)^n \ln\frac{x+y+1}{x+y} - (x+1)^n y^n(y+1)(x+y+1)^n \ln\frac{y+1}{y} \\
&- x^n(x+1)(y+1)^n(x+y+1)^n \ln\frac{x+1}{x} + (x+1)^n(y+1)^n(x+y+1)^n.
\end{aligned}$$

Since $\ln\left(\frac{x+1}{x}\right)^x < 1$, we obtain

$$D > (x+y+1)(x+1)^n(y+1)^n(x+y)^n \ln\frac{x+y+1}{x+y} - (x+1)^n y^n (y+1)(x+y+1)^n \ln\frac{y+1}{y}$$
$$- x^n(x+1)(y+1)^n(x+y+1)^n \ln\frac{x+1}{x} + (x+1)^n(y+1)^n(x+y+1)^n \ln\left(\frac{x+1}{x}\right)^x.$$

This is rewritten as

$$\begin{aligned} D >&(x+y+1)(x+1)^n(y+1)^n(x+y)^{n-1}\ln\left(\frac{x+y+1}{x+y}\right)^{x+y} \\ &-(x+1)^n y^{n-1}(y+1)(x+y+1)^n\ln\left(\frac{y+1}{y}\right)^y \\ &-x^{n-1}(x+1)(y+1)^n(x+y+1)^n\ln\left(\frac{x+1}{x}\right)^x \\ &+(x+1)^n(y+1)^n(x+y+1)^n\ln\left(\frac{x+1}{x}\right)^x. \end{aligned}$$

From (3) $\left(\frac{x+y+1}{x+y}\right)^{x+y} > \left(\frac{y+1}{y}\right)^y$. Therefore, we obtain

$$\begin{aligned} D >&(x+y+1)(x+1)^n(y+1)^n(x+y)^{n-1}\ln\left(\frac{y+1}{y}\right)^y \\ &-(x+1)^n y^{n-1}(y+1)(x+y+1)^n\ln\left(\frac{y+1}{y}\right)^y \\ &-x^{n-1}(x+1)(y+1)^n(x+y+1)^n\ln\left(\frac{x+1}{x}\right)^x \\ &+(x+1)^n(y+1)^n(x+y+1)^n\ln\left(\frac{x+1}{x}\right)^x \\ =&(x+y+1)(x+1)^n(y+1)[(y+1)^{n-1}(x+y)^{n-1}-y^{n-1}(x+y+1)^{n-1}]\ln\left(\frac{y+1}{y}\right)^y \\ &+(y+1)^n(x+y+1)^n(x+1)[(x+1)^{n-1}-x^{n-1}]\ln\left(\frac{x+1}{x}\right)^x \\ =&(x+y+1)(x+1)^n(y+1)[(xy+y^2+x+y)^{n-1}-(xy+y^2+y)^{n-1}]\ln\left(\frac{y+1}{y}\right)^y \\ &+(y+1)^n(x+y+1)^n(x+1)[(x+1)^{n-1}-x^{n-1}]\ln\left(\frac{x+1}{x}\right)^x > 0. \end{aligned}$$

Thus, we get $\frac{dw^n}{dn} < 0$.

Since $w^1 = 0$, $w^n < 0$ for $n > 1$. Therefore, we conclude that for $x > 0$ $c(x)$ is strictly sub-additive when n is a real number larger than 1. □

In Figure 4 we illustrate w^n assuming $x = 10,\ y = 5$. In Figure 5 we present an example of $c(x)$ assuming $n = \sqrt{2}$.

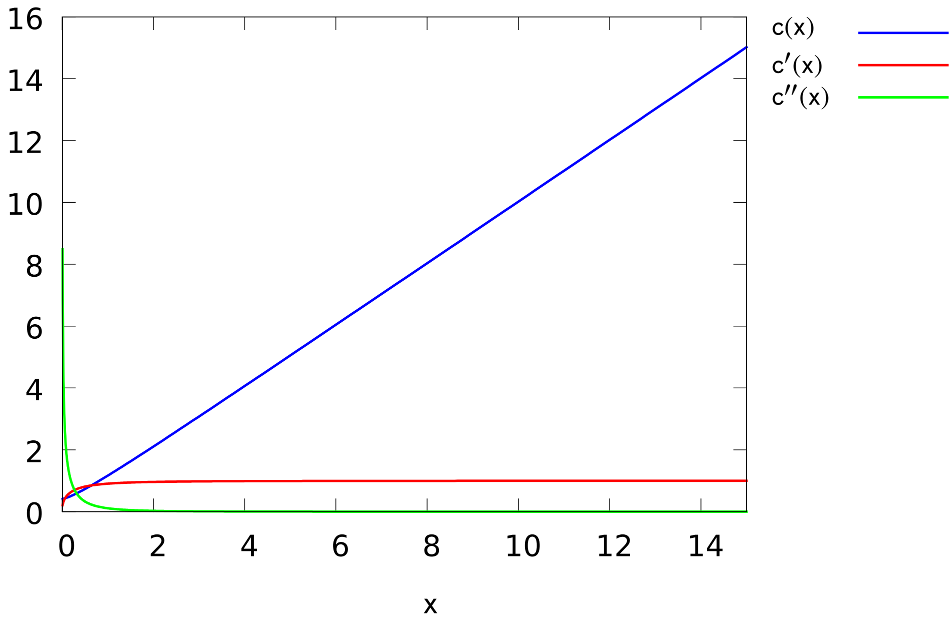

Figure 5: An example of $c(x)$ when $n = \sqrt{2}$

4 Concluding Remark

We have found a cost function which is strictly convex and strictly sub-additive with positive fixed cost. As we see in Figure 1 and 5, $c(x)$ is very close to a linear function although it is strictly convex.

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