Equilibrium and Optimal Fertility with Increasing Returns to Population and Endogenous Mortality

Cuberes, David and Tamura, Robert

Clark University, Clemson University

23 June 2017

Equilibrium and Optimal Fertility with Increasing Returns to Population and Endogenous Mortality

David Cuberes and Robert Tamura*

Clark University and Clemson University

June 23, 2017

Abstract

We present a general equilibrium dynamic model that characterizes the gap between optimal and equilibrium fertility and investment in human capital. In the model, the aggregate production function exhibits increasing returns to population arising from specialization but households face the standard quantity-quality trade-off when deciding how many children they have and how much education these children receive. In the benchmark model, we solve for the equilibrium and optimal levels of fertility and investment per child and show that competitive fertility is too low and investment per child too high. We next introduce mortality of young adults in the model and assume that households have a precautionary demand for children. Human capital investment raises the likelihood that a child survives to the next generation. In this setup, the model endogenously generates a demographic transition but, since households do not internalize the positive effects of a larger population on productivity and the negative effects of human capital on mortality, the demographic transition takes place much later in the equilibrium solution compared with the efficient solution. The efficient solution produces a demographic transition 10000 years earlier than the equilibrium solution. Our model can be interpreted as a bridge between the literature on endogenous demographic transitions and the scarce papers that study welfare issues associated with fertility and human capital decisions. Moreover, our results can be used to shed light on understanding demographic transitions in currently developing countries and to formulate policy recommendations to enhance welfare during these transitions.

1 Introduction

In this paper we present a theoretical model that characterizes the equilibrium and efficient fertility rate and investment per children. Due to the presence of an agglomeration economy, arising from specialization, the competitive equilibrium results in a too low number of children and too much investment per child. With high constant mortality, there can be no difference between the efficient and equilibrium solutions for fertility and human capital investment. However if mortality declines as a function of human capital, then there can be dramatic differences between the efficient and equilibrium solutions. The mechanism through which this gap is generated is a strong decrease in the precautionary demand for children as a response

*We thank Michal Jerzmanowski, Kevin Tsui, and seminar participants at University of Barcelona, University of Vigo, GREQAM (Marseille), and the 2013 Warwick Summer School for useful comments. All remaining errors are ours.

to the fall in mortality. These results still hold if one includes coordination costs of specialization. The only difference is that in this case, human capital accumulates faster along the efficient transition to the balanced growth path. Finally, the model produces a closed form solution for the quantity-quality children trade-off that may be easily estimated.

An important contribution of our model is that we use it to have quantitative predictions on the time series of several variables like fertility, mortality, human capital, and income per capita, as well as the timing of the demographic transition.

The paper is organized as follows. The next section presents a summary of the related literature. The theoretical model is developed in Section 3. Section 4 adds mortality to the model and presents results on the demographic transition. Section 5 shows the numerical solution of the model with mortality. Section 6 concludes the paper.

2 Related Literature

Our paper relates to two different strands of the literature. The first one is the study of welfare properties associated with fertility and human capital decisions. The concept of an optimal population growth rate and level has long been discussed by Dasgupta (1969), Samuelson (1975), Razin and Ben-Zion (1975), Nerlove et al. (1982, 1987), Gigliotti (1983), Willis (1987), Zimmermann (1989), and, more recently, Golosov et al. (2007). These papers often discuss how to apply the concept of Pareto optimality to questions related to population, acknowledging the fact that an increase in population by one member increases the welfare of this individual, even if it decreases the welfare of all the existing population (Eckstein and Wolpin, 1985). In our model, parents maximize a utility function that depends on their children's income. We show that this equilibrium is inefficient because parents do not internalize the positive effect that a larger population and stock of human capital have on next generation's income.

Second, our paper contributes to the theoretical literature on the demographic transition. The demographic transition is typically composed of two stages: a mortality transition, characterized by sharp declines in mortality rates, and a fertility transition where after a brief lag, fertility rates decline much faster than mortality rates. Altogether, the typical demographic transition then displays a hump-shaped evolution of a country's population growth rate. Most of the existing theoretical literature focuses on analyzing the triggers of the fertility transition, which include an increase in technological progress and human capital (Galor and Weil, 1999, 2000; Galor and Moav, 2002), an increase in income per capita (Becker, 1960; Becker and Lewis, 1973, Jones, 2001), a reduction in gender gaps (Goldin, 1990; Galor and Weil, 1996; Lagerlöf, 2003; Tertilt, 2005, 2006; Doepke and Tertilt; 2009), and a fall in infant mortality rates (Sah, 1991; Kalemli-Ozcan, 2002). The causes of the mortality transition are better understood. For instance, Weil (2005) highlights the importance of improvements in the standards of living - mainly the quantity and quality of food consumed-, improvements in housing and more often washing of clothes, and investments in public health - clean water and food-, as well as new medical treatments. However, few formal models endogenize mortality and even fewer analyze the potential effect of declines in mortality rates on the fertility transition.[1]

Our paper contributes by explaining both the mortality and the fertility transition in a unified frame-

[1] Tamura 2006, does endogenize mortality by assuming that the human capital of the adult child raises the survival probability of the adult child.

work. The key economic mechanism in our model is the secular decline in mortality rates, combined with the household's response in terms of human capital accumulation. While there is strong empirical evidence that the number of children produced by a couple declines as infant and/or child mortality declines, the effect that reductions in this mortality have on fertility sharply differs across theoretical models. In the framework of the Barro-Becker model (Becker and Barro, 1988; Barro and Becker, 1989), Doepke (2004) and Fernandez-Villaverde (2001) show that a drop in child mortality rates reduces the cost of raising survivor children, hence increasing net fertility. In contrast, Boldrin and Jones (2002) show that if one modifies the standard Barro-Becker model so that parents consumption when old directly enters the utility function of the children (the old-security hypothesis), it is possible to generate a positive correlation between infant and/or child mortality rates and fertility. One important difference between our paper and Boldrin and Jones (2002) is that child mortality rates are endogenous in our case, decreasing as the stock of human capital in the society rises. Moreover, while our model generates a positive correlation between young adult mortality rates and fertility as in Boldrin and Jones (2002), we emphasize a different channel, namely a fall in the precautionary demand for children. The implicit assumption we make is that a decline in the likelihood that children die before adulthood induces a reduction in fertility if parents seek to have an optimal number of surviving offspring (see Kalemli-Ozcan, 2002, 2003, Tamura 2006, Tamura et al. 2016, Tamura and Simon 2017).

Another important paper that endogenously links the fall in mortality rates and demographic transitions is Jones (2001). In his model, the occasional generation of new ideas in a Malthusian economy translates into a larger population size. This larger population in turn produces more ideas and this raises consumption per capita, reducing mortality rates and hence triggering a demographic transition. In his model, as in ours, it is the case that, had agents taken into account the positive effects of a larger population, the industrial revolution (and the demographic transition) would have taken place much earlier. However, Jones does not solve for the optimal problem explicitly. Soares (2005) develops a model where reductions in mortality (but not infant, child, or young mortality) are the main force behind economic development. His model also generates a demographic transition, where gains in life expectancy at birth are followed by reductions in fertility and increases in the rate of human capital accumulation. The onset of the transition is characterized by a critical level of life expectancy at birth, which marks the movement of the economy from a Malthusian equilibrium to an equilibrium with investments in human capital and the possibility of long-run growth. Finally, Kalemli-Ozcan (2002) develops a model in which parents have a precautionary demand for children and so reductions in infant mortality rates induce a fall in fertility and an increase in investment per child. One important difference with our paper is that she does not use the model to have predictions on the timing of the demographic transition, nor does she compare the optimal and equilibrium problem.

3 Theoretical Benchmark

3.1 The Baseline Model

3.1.1 Setup

Consider an economy populated by P_t workers. There is a single consumption good which is produced using the human capital of N_t workers, where $1 \leq N_t < P_t$. These N_t workers use the following reduced form for aggregate output:[2]

$$Y_t = \left\{ \sum_{i=1}^{N_t} h_{it}^{\frac{1}{\omega}} \right\}^{\omega} \tag{1}$$

where Y_t is output at period t, and h_{it} is human capital provided by agent i at period t. The parameter $\omega > 1$ represents the degree of increasing returns to population at the aggregate level, arising from specialization returns as in Rosen (1982) and Tamura (1992, 2002, 2006). Note that there are diminishing returns to individual human capital since ω is larger than one, however there is constant returns to the entire distribution of human capital. Within the production coalition, workers are paid the marginal product of their human capital. Let y_{jt} be a typical worker j's earnings. Then:

$$\begin{aligned} y_{jt} &= \left\{ \sum_{i=1}^{N_t} h_{it}^{\frac{1}{\omega}} \right\}^{\omega-1} \overline{h}_{jt}^{\frac{1-\omega}{\omega}} h_{jt} \\ &= \left\{ \sum_{i=1}^{N_t} h_{it}^{\frac{1}{\omega}} \right\}^{\omega-1} h_{jt}^{\frac{1}{\omega}} \end{aligned} \tag{2}$$

where the last equality arises because in the equilibrium that we consider $\overline{h}_{jt} = h_{jt}$, for any j and t. It is evident that earnings of all N_t members of the production coalition exactly exhausts output:

$$Y_t = \sum_{j=1}^{N_t} y_{jt} \tag{3}$$

This formulation is convenient because it allows us to have increasing returns but also firms that behave competitively, in a similar spirit as in Romer (1986). There are two restrictions on the number of workers in a production coalition. First, it cannot fall below 1, which represents autarky. Second, we impose a restriction that the number of workers cannot exceed either a size determined by coordination costs, or a fixed proportion, ξ, of the total population.[3] Thus we assume:

$$N_t = \min\{\max\{1, [\sigma \overline{h}_t]^{\lambda}\}, \max\{1, \xi P_t\}\}. \tag{4}$$

Thus while market specialization can increase, that is the number of distinct workers cooperating to produce the single consumption good can increase, it can never exceed the population of the economy.[4] What determines the scope of specialization? Since the number of workers in the production coalition is determined by the average human capital $\overline{h}_t$, or the population of the economy, ξP_t, there is an external effect of human capital and population. During autarkic production, human capital accumulation does not affect the number of workers within a production coalition. Therefore, absent any other external effects of human capital, there would be no difference in the human capital accumulation of workers in equilibrium versus workers in an efficient allocation. Once production moves out of autarky, $N_t > 1$, there are two long run possibilities. First the coordination costs may not bind, and market coalition is given by $N_t = \xi P_t$, and grows at the rate of population growth. The second possibility is that the coordination costs always binds,

[2]See Tamura (1992) for microfoundations of this functional form.

[3]All we assume is $0 < \xi \leq 1$.

[4]We have in mind a population P_t that can exceed the population of a small country like, for instance, Denmark. However it seems reasonable that much specialization can be restricted to a large metropolitan area. Hence any increasing returns to specialization are inherently smaller than a country like, for example, the United States.

and $N_t = [\sigma\overline{h}_t]^\lambda$. In this case the market grows at the constant rate equal to the human capital growth rate raised to the λ power. We present more on the long run balanced growth path in the numerical solutions section.

Returning to production, under symmetry output is given by:

$$Y_t = N_t^\omega h_t$$

or, in per capita terms

$$y_t = N_t^{\omega-1} h_t = Z_t h_t$$

In an equilibrium solution, individuals do not take into account the positive external benefit their human capital accumulation or their fertility provides for future specialization gains. Furthermore they do not take into account that their human capital is complementary to other workers, and by extension the human capital investments they make on their children's human capital has effects on wages of all other workers in that generation. Hence they treat the time path of total factor productivity, Z_t as exogenous technological progress.[5]

The economy is populated by individuals who live for two periods and a family consists of a parent and his children. In the first period individuals receive education provided by their parent. In the second they form their own household, work, choose their fertility, rear and educate their children.

We first focus on the stationary balanced growth world without mortality.[6] Parents care about their own consumption, c, the number of children, x, they have, and the income, y, of their typical child. A parent receives no utility from leisure, so they only work or spend time rearing and educating their children. Thus the typical generation t parent i wishes to maximize:

$$\alpha lnc_{it} + (1-\alpha)lnx_{it} + \alpha\beta lny_{it+1} \tag{5}$$

The typical t period parent's budget constraint is

$$c_{it} = w_{it}h_{it}\left[1 - x_{it}(\theta + \tau_{it})\right] \tag{6}$$

where θ is the unavoidable time cost of child rearing, and τ_{it} is time spent teaching each child. Human capital next period depends on the investment time per child τ_{it} i.e.

$$h_{it+1} = Ah_{it}\tau_{it}^\mu \tag{7}$$

where $\mu > 0$ and $A > 1$ is an efficiency parameter.[7]

3.1.2 Competitive Equilibrium

In the competitive equilibrium we have price taking behavior. Individuals are paid the marginal product of their human capital. The typical parent takes the wage per unit of human capital as given, and does

[5]Of course in equilibrium we have $\frac{Z_{t+1}}{Z_t} = \left(\frac{N_{t+1}}{N_t}\right)^{\omega-1} = g_n^{\omega-1}$

[6]Later on we will introduce mortality to show that the efficient solution involves a different emphasis altogether.

[7]In previous work, Tamura (1991, 1996, 2006) an externality of human capital is posited in the accumulation technology. In those models the externality allowed for convergence in human capital along the accumulation path. Since in all simulations we assume identical agents, we abstract from this additional externality.

not assume that his/her investment decisions have any effect on this wage, nor on the wage of any other worker. Given that the production technology is constant returns to scale in the distribution of human capital in the economy, paying each worker their marginal product exactly exhausts output. To see this, we note that for the typical worker j the wage per unit of human capital for worker j is given by:[8]

$$w_{it} = \left\{ \sum_{j=1}^{N_t} h_{jt}^{\frac{1}{\omega}} \right\}^{\omega-1} h_{it}^{\frac{1}{\omega}-1} \tag{8}$$

$$y_{it} = \left\{ \sum_{j=1}^{N_t} h_{jt}^{\frac{1}{\omega}} \right\}^{\omega-1} h_{it}^{\frac{1}{\omega}} \tag{9}$$

Observe that the wage per unit of human capital for worker i is *decreasing* in human capital of worker i. However (9) shows that *earnings*, y_{it} for worker i is increasing in worker i's human capital. Utility for a parent i in generation t can be written as:

$$\begin{aligned} U(h_{it}) &= \max_{\{h_{it+1}, x_{it}\}} \{\alpha \ln c_{it} + (1-\alpha) \ln x_{it} + \alpha\beta ln y_{it+1}\} \\ &= \alpha \max_{\{h_{it+1}, x_{it}\}} \left\{ ln w_{it} + ln h_{it} + ln(1 - x_{it}[\theta + \tau_{it}]) + \frac{1-\alpha}{\alpha} ln x_{it} + \beta ln h_{it+1} + \beta ln w_{it+1} \right\} \end{aligned}$$

The first order condition with respect to fertility produces the following.

$$\begin{aligned} \frac{\partial}{\partial x_{it}} = 0 \Leftrightarrow \frac{\alpha(\theta + \tau_{it})}{1 - x_{it}(\theta + \tau_{it})} &= \frac{1-\alpha}{x_{it}} \\ x_{it}(\theta + \tau_{it}) &= 1 - \alpha \end{aligned} \tag{10}$$

The first-order condition shows that the fraction of resources spent on the next generation is constant and equal to $1-\alpha$. Now consider the first order condition with respect to a child's human capital:

$$\begin{aligned} \frac{\partial}{\partial h_{it+1}} = 0 \Leftrightarrow \frac{\alpha x_{it} \tau_{it}}{(1 - x_{it}[\theta + \tau_{it}])\mu h_{it+1}} &= \frac{\alpha\beta}{h_{it+1}} \\ x_{it}\tau_{it} &= \alpha\beta\mu \end{aligned} \tag{11}$$

Combining the results in (10) and (11), we find the equilibrium stationary fertility and investment time:

$$x^{eq} = \frac{1 - \alpha - \alpha\beta\mu}{\theta} \tag{12}$$

$$\tau^{eq} = \frac{\alpha\beta\mu\theta}{1 - \alpha - \alpha\beta\mu} \tag{13}$$

One immediate parameter restriction is evident:

$$1 - \alpha - \alpha\beta\mu > 0$$

[8] One can assume that there are N_t different types of workers, each type with an initial measure of 1, and each worker is a set of measure zero of the number of workers of their type. In the equilibrium solution, a parent completely ignores the effect of human capital investment on their children's wage or the wage of any one of that generation.

Intuitively, an increase in β - i.e. if parents care more about their children's income - reduces fertility and increases the investment in children's human capital. Similarly, a higher θ obviously reduces fertility and increases investment per child since the fixed cost of rearing a child is now higher. Finally, an increase in μ also increases investment in children's human capital, since the return to this investment is now higher.[9]

3.1.3 Efficient Solution

In the efficient solution, we consider the case in which a social planner internalizes the positive externality of population growth on TFP growth, as well as the positive externality of human capital investment on the next generation wages. For convenience and without loss of generality we focus on the efficient solution in which all parents are treated equally.[10] We consider a social planner that maximizes the utility of adults within a production coalition, that is for the members of the current production coalition N_t.[11] The social planner's problem can be written as:

$$\max_{\{c_{jt},x_{jt},h_{jt+1}\}_{j=1}^{N_t}} \left\{ \frac{1}{N_t}\sum_{j=1}^{N_t}[\alpha ln c_{jt} + (1-\alpha) ln x_{jt} + \alpha\beta ln y_{jt+1}] \right\} \tag{14}$$

subject to the resource constraint:

$$\sum_{j=1}^{N_t} c_{jt} \leq \left\{ \sum_{j=1}^{N_t} h_{jt}^{\frac{1}{\omega}}(1 - x_{jt}[\theta + \tau_{jt}])^{\frac{1}{\omega}} \right\}^{\omega} \tag{15}$$

Let Λ be the multiplier on the resource constraint. The first order condition with respect to consumption for the typical parent i is given by:

$$\frac{\partial}{\partial c_{it}} = 0 \Leftrightarrow \frac{\alpha}{N_t c_{it}} = \Lambda \tag{16}$$

Thus all parents receive the same consumption.[12] Assume for the time being the possibility that production coalition size is bigger than 1, and is already determined by ξP_t, that is the production coalition is a constant proportion of the population. We will focus on the homogeneous population case, but for now we allow the social planner to pick individual parental fertility by agent type, $1, ..., N_t$. The first order condition for optimal fertility, x_{it}, can be written as:

$$\frac{1}{N_t}\left\{ \frac{1-\alpha}{x_{it}} + \alpha\beta \sum_{j=1}^{N_t} \frac{(\omega-1)\left\{\sum_{s=1}^{N_t} x_{st} h_{st+1}^{\frac{1}{\omega}}\right\}^{\omega-2} h_{jt+1}^{\frac{1}{\omega}} h_{it+1}^{\frac{1}{\omega}}}{y_{jt+1}} \right\} =$$

$$\Lambda \left\{ \sum_{j=1}^{N_t} h_{jt}^{\frac{1}{\omega}}(1 - x_{jt}[\theta + \tau_{jt}])^{\frac{1}{\omega}} \right\}^{\omega-1} h_{it}^{\frac{1}{\omega}}(1 - x_{it}[\theta + \tau_{it}])^{\frac{1}{\omega}-1}[\theta + \tau_{it}] \tag{17}$$

[9] These results are similar to those in Tamura (2002).

[10] It is simple to show that this has no effect on the human capital investment decision in the cases with unequal Pareto weights.

[11] We ignore cases where the population is not integer divisible by N_t.

[12] Again, if the Pareto weights were different, then two parents could receive different consumption values, but it would not effect the accumulation path of human capital.

The top line represents the marginal benefits of fertility. The second term in the curly brackets is the full effect of additional fertility on the earnings of the generation t+1 adults. Notice that we can keep the number of types of workers constant, and adjust the population of each type in generation t+1 by x_{jt}. The bottom line represents the marginal cost of fertility in terms of foregone current output.

The first order condition with respect to human capital investment, h_{it+1}, can be written as:

$$\frac{\alpha\beta}{\omega N_t}\left\{\sum_{j=1}^{N_t}\frac{(\omega-1)\left\{\sum_{s=1}^{N_t} x_{st}h_{st+1}^{\frac{1}{\omega}}\right\}^{\omega-2} h_{jt+1}^{\frac{1}{\omega}}x_{it}h_{it+1}^{\frac{1}{\omega}-1}}{y_{jt+1}}+\frac{\left\{\sum_{s=1}^{N_t} x_{st}h_{st+1}^{\frac{1}{\omega}}\right\}^{\omega-1} h_{it+1}^{\frac{1}{\omega}-1}}{y_{it+1}}\right\} =$$
$$\Lambda\left\{\sum_{j=1}^{N_t} h_{jt}^{\frac{1}{\omega}}(1-x_{jt}[\theta+\tau_{jt}])^{\frac{1}{\omega}}\right\}^{\omega-1}\frac{h_{it}^{\frac{1}{\omega}}(1-x_{it}[\theta+\tau_{it}])^{\frac{1}{\omega}-1}x_{it}\tau_{it}}{\mu h_{it+1}} \tag{18}$$

Looking at the marginal benefits term, the top line of equation (18), the first term in the curly brackets is the effect of higher human capital for the i^{th} type generation t+1 adult on all wages of t+1 workers. The second term in the curly brackets is the direct effect higher human capital on earnings of the i^{th} type generation t+1 worker. In both cases the social planner internalizes the effect on both the wage per unit of human capital as well as the direct effect, the second term. The bottom line is the marginal cost of additional human capital for the i^{th} type generation t+1 adult.

We now impose symmetry, that is we assume that all individuals are of the same type, $h_{it} = h_{jt}, \forall\ i,\ j,\ t$. Under symmetry, utilizing the definition of y_{jt+1}, our Euler equation with respect to fertility can now be written as:

$$\frac{1-\alpha+\alpha\beta(\omega-1)}{x_t} = \Lambda N_t^{\omega} h_t(\theta+\tau_t) \tag{19}$$

Similarly we can write our Euler equation with respect to human capital investment as:

$$\alpha\beta\mu = \Lambda y_t N_t x_t \tau_t \tag{20}$$

Using the Euler equation for consumption and the resource constraint we can solve for Λ:

$$\Lambda = \frac{\alpha}{N_t^{\omega} h_t(1-x_t[\theta+\tau_t])} \tag{21}$$

The fraction of resources spent on the next generation is given by:

$$x_t(\theta+\tau_t) = \frac{1-\alpha+\alpha\beta(\omega-1)}{1+\alpha\beta(\omega-1)} \tag{22}$$

Differentiating with respect to $\omega - 1$, it is easy to show that the share of resources spent on the next generation is increasing in the gains from specialization. Obviously when there is no aglommeration return to market size, $\omega = 1$, then the efficient and equilibrium solutions coincide. Thus for economies in which $\omega > 1$, the efficient solution involves a greater share of current output spent on the next generation. This occurs via a rise in fertility as we show below.

Using this we can solve for the efficient fertility and efficient investment rate:

$$x^{eff} = \frac{1-\alpha-\alpha\beta\mu+\alpha\beta(\omega-1)}{\theta[1+\alpha\beta(\omega-1)]} \quad (23)$$

$$\tau^{eff} = \frac{\alpha\beta\mu\theta}{1-\alpha-\alpha\beta\mu+\alpha\beta(\omega-1)} \quad (24)$$

It is clear that an increase in $\omega - 1$ induces a higher fertility rate since the positive external effect of population on output per capita and wages is now higher. Of course, the corresponding level of investment per child decreases as $\omega - 1$ increases.

3.1.4 Comparing Both Setups

In this section we compare the different solutions of the equilibrium problem and the efficient problem. It is easy to show that fertility is higher in the efficient solution than the equilibrium solution. It is also easy to show that human capital investment is slower in the efficient solution compared with the equilibrium solution. Finally, we can show that utility is always higher in the efficient solution compared with the equilibrium solution. Non trivially however, economic growth can be higher in the equilibrium solution than in the efficient solution. However for large enough gains from specialization, higher ω, the growth rate in the efficient solution exceeds that of the equilibrium solution.

Comparing fertility between the two cases:

$$x^{eff} = \frac{1-\alpha-\alpha\beta\mu+\alpha\beta(\omega-1)}{\theta[1+\alpha\beta(\omega-1)]} > \frac{1-\alpha-\alpha\beta\mu}{\theta} = x^{eq}$$
$$\iff 0 > -\alpha(1+\beta\mu)\alpha\beta(\omega-1),$$

which holds $\forall\ \omega > 1$. Next we compare human capital investment rates from the two cases:

$$\tau^{eff} = \frac{\alpha\beta\mu\theta}{1-\alpha-\alpha\beta\mu+\alpha\beta(\omega-1)} < \frac{\alpha\beta\mu\theta}{1-\alpha-\alpha\beta\mu} = \tau^{eq}$$
$$\iff \alpha\beta(\omega-1) > 0,$$

which holds $\forall\ \omega > 1$. Next we show that utility is always higher in the efficient solution compared with the equilibrium solution: Assume that the population is identical between the efficient and equilibrium problems to start, all individuals have the same human capital, h_t, and that the production coalition is the same size as well, $N_t = \xi P_t$. Writing out utility of the typical parent with h_t human capital for both equilibrium and efficient problems, and canceling out identical terms we end up with:

$$V^{eff}(h_t) - V^{eq}(h_t) = (1-\alpha-\alpha\beta\mu+m)ln(\frac{1-\alpha-\alpha\beta\mu+m}{1-\alpha-\alpha\beta\mu}) - (1+m)ln(1+m)$$
$$m = \alpha\beta(\omega-1)$$

Observe that when there is no agglomeration return to specialization, $\omega = 1$, then $m = 0$, and there is no difference between the efficient utility and equilibrium utility. Taking the derivative of utility difference with respect to m we get:

$$\begin{aligned}\frac{\partial(V^{eff}(h_t) - V^{eq}(h_t)}{\partial m} &= ln(\frac{1-\alpha-\alpha\beta\mu+m}{1-\alpha-\alpha\beta\mu}) + 1 - ln(1+m) - 1 \\ \frac{\partial(V^{eff}(h_t) - V^{eq}(h_t)}{\partial m} &= ln(\frac{1-\alpha-\alpha\beta\mu+m}{(1-\alpha-\alpha\beta\mu)(1+m)}) > 0 \\ \iff m > 0 &\end{aligned}$$

So for all values with positive agglomeration economy gains from specialization, $\omega > 1, m > 0$, we have the gap between the efficient utility and the equilibrium utility is positive.

Now we show that growth in output per worker cannot be so easily ordered. For small enough gains in specialization, equilibrium growth can exceed efficient growth. For larger values of specialization gains, efficient growth can exceed equilibrium growth.

Case 1: Assume the following parameter configuration, $\lambda = 5$:

$$\begin{aligned}\alpha = \frac{5}{8}, \beta = \frac{3}{5}, \mu = \frac{1}{2}, \theta = \frac{1}{8}, A = 5, \omega = 1.35 \\ x^{eq} = 0.98, \tau^{eq} = .2066, x^{eff} = 1.85, \tau^{eff} = .096 \\ \Gamma^{eq} = (x^{eq})^{\omega-1}A(\tau^{eq})^{\mu} = 2.2568 \\ \Gamma^{eff} = (x^{eff})^{\omega-1}A(\tau^{eff})^{\mu} = 1.9198\end{aligned}$$

Case 2: Assume the following parameter configuration, $\lambda = 5$:

$$\begin{aligned}\alpha = \frac{5}{8}, \beta = \frac{3}{5}, \mu = \frac{1}{2}, \theta = \frac{1}{8}, A = 5, \omega = 1.75 \\ x^{eq} = 0.98, \tau^{eq} = .2066, x^{eff} = 2.62, \tau^{eff} = .059 \\ \Gamma^{eq} = (x^{eq})^{\omega-1}A(\tau^{eq})^{\mu} = 2.2387 \\ \Gamma^{eff} = (x^{eff})^{\omega-1}A(\tau^{eff})^{\mu} = 2.5060\end{aligned}$$

Thus we have the interesting possibility that while contemporaneous utility is higher under an efficient solution compared with the contemporaneous equilibrium solution, if economic growth is more rapid under the equilibrium solution eventually those born in the future would be happier due to their higher human capital compared with those equivalent generation arriving from the efficient solution.[13] Notice that we present the solutions for $\lambda = 5$. We specified the value of λ because we had to ensure that the production coalition grows at the rate of population growth. For values of λ smaller than 5, the efficient fertility rate could produce a human capital growth rate that is too slow to support the growth rate of the production coalition at the rate of population growth. The critical value of λ is given by:

$$\lambda = \frac{ln[1-\alpha-\alpha\beta\mu+\alpha\beta(\omega-1)] - ln\theta - ln[1+\alpha\beta(\omega-1)]}{lnA + \mu ln[1-\alpha-\alpha\beta\mu+\alpha\beta(\omega-1)]} \tag{25}$$

In the numerical solutions section we will present results where the coordination costs fall too slowly in some cases so that the balanced growth path in the efficient solution is constrained. This produces a feature

[13] Recall that the efficient solution maximizes the utility of the representative parent alive today. The typical parent only cares about the future through fertility and the income of the typical child. Had a parent cared about the infinitely lived dynasty, then this result could be overturned.

quite similar to Galor and Weil (2000) in which reducing the constraint leads to a rise in fertility. In our case the relaxation of the coordination costs arises as λ increases.

4 Mortality

In this section we add mortality of young adults. In particular we assume that human capital investment raises the likelihood that a child survives to the next generation. Second we modify the preferences to introduce a precautionary demand for fertility, as in Tamura (2006), Tamura and Simon (2017) and Tamura, Simon and Murphy (2016). These are inspired by the seminal work of Kalemli-Ozcan (2002, 2003). We show that under some simple assumptions human capital accumulation is more rapid under the efficient solution than the equilibrium solution.

4.1 Equilibrium Solution

Assume that parents care about their own consumption, their number of expected surviving children, and the earnings of their surviving adult children. Assume preferences can be written as:

$$\alpha lnc_{it} + (1-\alpha)ln[x_{it}(1-\delta_{it})] - \frac{\delta_{it}}{2x_{it}(1-\delta_{it})} + \alpha\beta lny_{it+1} \tag{26}$$

We assume as in Tamura (2006) that parents must educate all their children, and after the education investment is made, only $1-\delta_{it}$, of the children survive to adulthood. The term $\frac{\delta_{it}}{2x_{it}(1-\delta_{it})}$ represents a precautionary demand for children (see Kalemli-Ozcan, 2003). Intuitively, for a given mortality of young adults, increases in fertility reduce the disutility generated by the death of a family's child. Therefore the budget constraint for the typical parent is given, as before, by (6):

$$c_{it} = w_{it}h_{it}\left[1 - x_{it}(\theta+\tau_{it})\right],$$

where w_{it} is given above by (8). We assume that human capital accumulation remains as in (7):

$$h_{t+1} = Ah_t\tau_t^{\mu}$$

where $\mu > 0$ and $A > 1$ is an efficiency parameter. Importantly, we assume that the cumulative mortality of young adults is a declining function of the average human capital of their generation:[14]

$$\delta_{it} = \min\{\hat{\delta}, \Delta exp(-\psi_1\bar{h}_{it+1}^{\psi_2})\}, \tag{27}$$

where $\bar{h}_{it+1}$ is the average human capital of adult generation $t+1$ and ψ_1 and ψ_2 are positive parameters. Hence whether a child born of a t generation parent, and hence a t+1 generation adult, survives is an increasing function of the average human capital of their generation.[15] Labeling terms that are not affected

[14] This is similar to that assumed in Tamura (2006).

[15] Tamura (2006) argues that this is the case if modern sanitation and modern personal hygiene are best at reducing early mortality. He allows for international spillovers like development of antibiotics and vaccines, which are abstracted from here. Also note, that unless human capital exceeds a critical threshold, it does not affect the mortality rate. The critical value solves: $\ln\hat{\delta} - \ln\Delta - \psi_1\bar{h}_{it+1}^{\psi_2} = 0$.

by parental choices as $\hat{U}$, the optimization problem for the parent can be written as:

$$\begin{aligned} V(h_{it}) &= \max_{\{h_{it+1},x_{it}\}} \left\{ \alpha \ln c_{it} + (1-\alpha)\ln x_{it} - \frac{\delta_{it}}{2x_{it}(1-\delta_{it})} + \alpha\beta ln(y_{it+1}) \right\} \\ &= \hat{U} + \max_{\{h_{it+1},x_{it}\}} \left\{ \alpha ln(1 - x_{it}[\theta+\tau_{it}]) + (1-\alpha) ln x_{it} - \frac{\delta_{it}}{2x_{it}(1-\delta_{it})} + \alpha\beta ln h_{it+1} \right\} \end{aligned}$$

The first order condition with respect to fertility and human capital investment are:

$$\frac{\alpha[\theta+\tau_{it}]}{1-x_{it}[\theta+\tau_{it}]} = \frac{1-\alpha}{x_{it}} + \frac{\delta_{it}}{2x_{it}^2(1-\delta_{it})} \tag{28}$$

$$\frac{\alpha x_{it}\tau_{it}}{1-x_{it}[\theta+\tau_{it}]} = \alpha\beta\mu \tag{29}$$

Observe that the first order condition on optimal human capital investment in the equilibrium model is identical whether mortality is non-zero or not. The only way in which mortality risk affects human capital investment is through the fertility choice. This is precisely the case since we assumed that the entire effect of human capital investment on young adult mortality is external to the parent. To solve this problem, we create a grid on human capital investment rates given by:

$$\tau_i = \{\frac{i(1-\theta)}{k}\}_{i=1}^{k} \tag{30}$$

Then for each value of τ_i we compute the mortality rate for these young adults:

$$\delta_i = \min\{\hat{\delta}, \Delta exp(-\psi_1[Ah_t\tau_i^{\mu}]^{\psi_2})\}.$$

We use (28) to solve for the fertility choice for each investment rate on the grid. The fertility choice, for a given investment rate solves the following quadratic equation:

$$\begin{aligned} ax_{it}^2 &+ bx_{it} + c = 0 \qquad (31) \\ a &= -(\theta+\tau_{it}) \\ b &= 1-\alpha - \frac{\delta(h_{it+1})(\theta+\tau_{it})}{2[1-\delta(h_{it+1})]} \\ c &= \frac{\delta(h_{it+1})}{2[1-\delta(h_{it+1})]} \\ x_{it} &= \frac{-b-\sqrt{b^2-4ac}}{2a} \end{aligned}$$

If we define $m = \frac{\delta}{2(1-\delta)}$, then it is easy to show that $\frac{\partial x}{\partial m} > 0$. So in a world with mortality risk, for any choice of human capital investment, fertility is higher than a world with no mortality risk. After solving for the fertility choice as a function of the investment rate, we verify that (29) holds. For those values that do not solve the equilibrium we discard these potential equilibrium solutions. From this reduced set of investment and fertility candidate pairs, the utility maximizing choice is selected.

4.2 Efficient Solution

Here we solve the planner's problem, and we only examine the case with identical individuals. Thus we focus on the equal weight solution, and hence the representative parent. There are several regions that must be solved in order to characterize the efficient path. In the earliest days, human capital will be sufficiently low that no specialization occurs, and hence autarky is the method of production. Human capital accumulation may affect the mortality of the young adult, but not determine the scale of the production coalition. In the final stage, either the coordination technology will determine the size of the production coalition, or it will be determined by the fixed proportion of the total population, c.f. (4). Since all individuals are identical, the earnings of a typical worker are given by:

$$\begin{aligned} y_{it+1} &= N_{t+1}^{\omega-1} h_{it+1} \\ N_{t+1} &= \min\{\max\{1, [\sigma \overline{h}_{t+1}]^{\lambda}\}, \max\{1, \xi P_{t+1}\}\} \end{aligned}$$

The planner's problem can be written as:

$$\max_{x_{it}, h_{it+1}} \left\{ \alpha ln(1 - x_{it}[\theta + \tau_{it}]) + (1-\alpha)[ln(1 - \delta(h_{it+1})) + lnx_{it}] - \frac{\delta(h_{it+1})}{2x_{it}[1 - \delta(h_{it+1})]} + \alpha\beta[(\omega - 1)lnN_{t+1} + lnh_{it+1}] \right\}$$

The first order conditions for fertility and human capital investment can be written as:

$$\frac{\alpha(\theta + \tau_t)}{1 - x_t[\theta + \tau_t]} = \frac{1-\alpha}{x_t} + \frac{\delta(h_{t+1})}{2x_t^2[1 - \delta(h_{t+1})]} + \frac{\alpha\beta(\omega - 1)}{N_{t+1}} \frac{\partial N_{t+1}}{\partial x_t} \tag{32}$$

$$\begin{aligned} \frac{\alpha x_{it}}{1 - x_t[\theta + \tau_t]} &= \frac{\alpha\beta\mu}{\tau_t} - \frac{1}{1 - \delta(h_{t+1})} \frac{\partial \delta(h_{t+1})}{\partial \tau_{it}} \left\{ 1 - \alpha + \frac{1}{2x_t[1 - \delta(h_{t+1})]} \right\} \\ &\quad + \frac{\alpha\beta(\omega - 1)}{N_{t+1}} \frac{\partial N_{t+1}}{\partial h_{t+1}} x \frac{\partial h_{t+1}}{\partial \tau_t} \end{aligned} \tag{33}$$

$$\frac{\partial \delta(h_{t+1})}{\partial \tau_t} = \begin{cases} 0, & \text{if } \ln\hat{\delta} - \ln\Delta - \psi_1 h_{t+1}^{\psi_2} > 0. \\ -\frac{\delta(h_{t+1})\psi_1\psi_2\mu h_{it+1}^{\psi_2}}{\tau_t} & \text{if } \ln\hat{\delta} - \ln\Delta - \psi_1 h_{t+1}^{\psi_2} \leq 0. \end{cases}$$

$$\frac{\partial N_{t+1}}{\partial x_t} = \begin{cases} 0, & \text{if } N_{t+1} = \max\{1, [\sigma h_{t+1}]^{\lambda}\} < \xi P_{t+1t+1} \\ N_t(1 - \delta(h_{t+1})) & \text{if } 1 < N_{t+1} = \xi P_{t+1} \leq [\sigma h_{t+1}]^{\lambda} \end{cases}$$

$$\frac{\partial N_{t+1}}{\partial h_{t+1}} \frac{\partial h_{t+1}}{\partial \tau_t} = \begin{cases} 0, & \text{if } N_{t+1} = \max\{1, \xi P_{t+1}\} < [\sigma h_{t+1}]^{\lambda} \\ \frac{\lambda\mu\sigma^{\lambda} h_{t+1}^{\lambda}}{\tau_t} & \text{if } 1 < N_{t+1} = [\sigma h_{t+1}]^{\lambda} \leq \xi P_{t+1} \end{cases}$$

If human capital of the child does not affect the mortality rate of the child, that is h_{t+1} is below the critical threshold, and if $\partial N_{t+1}/\partial x_t = 0$ & $\partial N_{t+1}/\partial h_{t+1} = 0$, then the two first order conditions of the social planner are identical with the first order conditions in the equilibrium solution. Thus for the earliest part of history, there is no difference in the path of human capital taken in the equilibrium solution and the efficient solution.

Now consider the range where children's human capital reduces their mortality, $\partial\delta/\partial\tau_t < 0$. Further assume that the market size could be larger than autarky in the children's adulthood, i.e. $N_{t+1} > 1$. We use the same solution algorithm to solve for the efficient choice of fertility and human capital investment as we employed in solving for the equilibrium choices. We use a grid on possible investment rates, τ_t and the implied young adult mortality rate. We let possible investment rates come from the following grid:

$$\begin{aligned}
\tau_i &= \{\frac{i(1-\theta)}{k}\}_{i=1}^{k} \\
\delta_i &= \min\{\hat{\delta}, \Delta exp(-\psi_1\{Ah_t\tau_i^{\mu}\}^{\psi_2})\}.
\end{aligned}$$

The first order condition for efficient fertility is given by (32). If the production coalition is determined by the evolution of population, then the first order condition for fertility can be written as:

$$\frac{\alpha(\theta+\tau_{it})}{1-x_{it}[\theta+\tau_{it}]} = \frac{1-\alpha+\alpha\beta(\omega-1)}{x_{it}} + \frac{\delta(h_{it+1})}{2x_{it}^2[1-\delta(h_{it+1})]}$$

For any given investment rate τ_{it}, this is a quadratic function in x_{it} given by:

$$\begin{aligned}
ax_{it}^2 &+ bx_{it} + c = 0 \\
a &= -(\theta+\tau_{it})(1+\alpha\beta(\omega-1)) \\
b &= 1-\alpha+\alpha\beta(\omega-1) - \frac{\delta(h_{it+1})(\theta+\tau_{it})}{2[1-\delta(h_{it+1})]} \\
c &= \frac{\delta(h_{it+1})}{2[1-\delta(h_{it+1})]} \\
x_{it} &= \frac{-b-\sqrt{b^2-4ac}}{2a}
\end{aligned}$$

For each investment and ferttilty pair, we compute the utility of the representative parent, and the efficient solution is the maximizing pair. Unlike in the equilibrium solution, we do not constrain the choice of investment other than feasibility. Implicitly the investment rate either satisfies (33) if internal, or is a corner solution. In either case it is generally not equal to the equilibrium choice as (29) and (33) are not identical. If the production coalition is determined by the coordination technology, i.e. $1 < N_{t+1} = [\sigma h_{t+1}]^{\lambda} < \xi P_{t+1}$, then the first order condition for fertility can be written as:

$$\frac{\alpha(\theta+\tau_{it})}{1-x_{it}[\theta+\tau_{it}]} = \frac{1-\alpha}{x_{it}} + \frac{\delta(h_{it+1})}{2x_{it}^2[1-\delta(h_{it+1})]}$$

As with the previous case, for each investment rate τ_{it}, the first order condition of fertility is a quadratic form given by:

$$\begin{aligned}
ax_{it}^2 &+ bx_{it} + c = 0 \\
a &= -(\theta+\tau_{it}) \\
b &= 1-\alpha+\alpha\beta(\omega-1) - \frac{\delta(h_{it+1})(\theta+\tau_{it})}{2[1-\delta(h_{it+1})]} \\
c &= \frac{\delta(h_{it+1})}{2[1-\delta(h_{it+1})]} \\
x_{it} &= \frac{-b-\sqrt{b^2-4ac}}{2a}
\end{aligned}$$

The solution is the investment rate and fertility pair that maximizes utility. For both the equilibrium solution and the efficient solution, we produce the time series on population, fertility, mortality, human capital, coalition size, real output per capita. The evolution is given by:

$$
\begin{aligned}
h_{t+1}^r &= A h_t^r (\tau_t^r)^\mu \\
\delta(h_{t+1}^r) &= \min\{\hat{\delta}, \Delta exp(-\psi_1 \{A h_t^r (\tau_t^r)^\mu\}^{\psi_2})\} \\
P_{t+1}^r &= P_t^r [1 - \delta(h_{t+1}^r)] x_t^r \\
w_{t+1} &= (N_{t+1}^r)^{\omega - 1} \\
r &= \text{equilibrium or efficient}
\end{aligned}
$$

5 Numerical Solution

In this section we present numerical solutions to compare and contrast the equilibrium time series with the efficient time series. One interesting feature displayed is the interplay between human capital growth and growth of the production coalition. If the long run coalition is bounded by the coordination cost, that is $1 < N_t = [\sigma h_t]^\lambda < \xi P_t$, then even with high gains from specialization, a large ω, fertility in the efficient solution will not be given by (23), but rather $\hat{x}^{eff} = [A\hat{\tau}^\mu]^\lambda$. If the long run coalition is bounded by the maximum fraction of the population, $1 < N_t = \xi P_t < [\sigma h_t]^\lambda$, then fertility in the efficient solution will be given by (23).

Table **??** contains the base parameters used in the numerical solutions. Some parameters are time invariant, e.g. α, β, θ, σ, and mortality risk is only a function of human capital. Others are step functions in the human capital stock in the economy, e.g. A and μ. For values of human capital less than or equal to 4.0, they have one value, and for human capital in excess of 4.0, they have a different value. In the case of θ and μ, we used the stationary values of equilibrium fertility and investment time, in order to peg their values. We assumed that in the equilibrium stationary solution, fertility is 1.0001, and $\tau = .375$. Using (12) and (13) it is easy to see that $x^{eq}\tau^{eq} = \alpha\beta\mu$. Thus we solve for the stationary value of $\mu = \frac{x^{eq}\tau^{eq}}{\alpha\beta}$. Given μ, our stationary $\theta = \frac{1-\alpha\alpha\beta\mu}{x^{eq}}$. In Table **??**, we only report values for θ and μ to four significant values.

There are five cases that we examine. These affect only two parameters, A and λ. Most important is the different value of λ. As we increase the magnitude of this parameter we increase the growth rate of the coordinated production coalition. Thus as we increase λ we increase the stationary fertility in the efficient solution. Notice that in Case 1, the efficient stationary fertility, 0.9000, and is independent of the aglomeration economy arising from specialization (λ). This maybe surprising at first look. However what is happening is that the production coalition is bound by $[\sigma h_t]^\lambda$, and population is decreasing. Eventually the declining population will become the binding constraint, which will then lead to a rise in fertility. This however has not happened in the solutions by year 5000.[16] Moving from Case 1 to Case 5, we observe that the efficient stationary fertility rises with ω. Fertility rises for every ω value, except for the first, $\omega = 1.25$, as coordination technology improves, λ increases, until it reaches its stationary value.

As there are 4 values of ω and 5 Cases, that is 5 different coordination technologies, we present the results of the numerical solutions in two ways. First, for each Case, we present the time series for each variable of interest for all values of ω for both the equilibrium and the efficient solutions. The variables

[16] Indeed from 9880 onward fertility is 2.0126, and schooling is 6.84 for $\omega = 1.25$. For $\omega = 1.50$, from 9760 onward fertility is 2.1206, and schooling is 7.06. From 9640 onward fertility is 2.2418, and schooling is 7.37 for $\omega = 1.875$. From 9480 onward fertility is 2.3912, and schooling is 7.74 for $\omega = 2.50$.

Table 1: Parameter Values & Stationary Numerical Solutions

$\alpha = 0.55$	$\beta = 0.65$	$\theta = .075$	$\delta(h_t) = min\left\{.74, .9exp(-.015h_t^3)\right\}$		$\sigma = 1.25$
$\mu = .03295$	if $h \leq 4.0$	$A = 1.5148$	if $h \leq 4.0$		
	Case 1	Case 2	Case 3	Case 4	Case 5
A	11.25 if $h > 4.0$	11.25 if $h > 4.0$	11.25 if $h > 4.0$	20 if $h > 4.0$	30 if $h > 4.0$
μ	1.049 if $h > 4.0$	1.049 if $h > 4.0$	1.049 if $h > 4.0$	1.049 if $h > 4.0$	1.049 if $h > 4.0$
λ	1.25	4.50	9.45	5.66	9.77
ω					
			equilibrium stationary fertility		
1.25	1.0001	1.0001	1.0001	1.0001	1.0001
1.50	1.0001	1.0001	1.0001	1.0001	1.0001
1.875	1.0001	1.0001	1.0001	1.0001	1.0001
2.50	1.0001	1.0001	1.0001	1.0001	1.0001
			equilibrium stationary schooling: 40τ		
1.25	15.0	15.0	15.0	15.0	15.0
1.50	15.0	15.0	15.0	15.0	15.0
1.875	15.0	15.0	15.0	15.0	15.0
2.50	15.0	15.0	15.0	15.0	15.0
			efficient stationary fertility		
1.25	0.4500	2.0126	2.0126	2.0126	2.0126
1.50	0.4500	2.6970	2.8716	2.8716	2.8716
1.875	0.4500	2.9088	3.1142	3.9407	3.9407
2.50	0.4500	3.1764	3.4213	4.3216	5.3079
			efficient stationary schooling: 40τ		
1.25	37.0	6.84	6.84	6.84	6.84
1.50	37.0	4.91	4.43	4.43	4.43
1.875	37.0	4.99	4.47	2.90	2.90
2.50	37.0	5.09	4.51	2.94	1.84

we present are growth rates of income, log of human capital, total fertility rates, mortalityand schooling. Afterwards we present the time series for each ω in their own graph as we vary the coordination technology. In this way, the reader can see the comparative dynamics from ω and the comparative dynamics from λ.

Figure **??** contains the time series for per capita income growth. In order to keep things in scale, we constructed annualized growth rates over 1000 years.[17] Thus each observation is given by:

$$g_y = \frac{\ln y_{t+25} - \ln y_t}{1000}, \tag{34}$$

where t represents adults in generation t, and each period is 40 years. We color code the growth rates; the green curves are the equilibrium growth rates, and the yellow-orange-red curves are the efficient growth rates. We did calibrate the equilibrium model to produce a Demographic Transition between 1600 to 2000. In all 5 cases, the efficient solution has an earlier acceleration in growth, and a much earlier decline in mortality, over 19000 years earlier! Since the equilibrium parent does not internalize the human capital externality on survival rates, the timing of the equilibrium Demographic Transition is independent of ω. It need not be constant across the cases. For Case 1 Figure 2 shows that the equilibrium log human capital time series, which is independent of the ω but not by Case. This is because Case 4 and Case 5 have a much larger value of A.

Figure **??** contains the efficient growth rates of income per capita, arranged by ω. One interesting result is that for every value of ω, Case 1 with the most restrictive coordination technology, lowest λ, has the lowest growth rate until about 1000, but the highest growth rate in the balanced growth path. This is because the fertility rate over the remaining period is actually below the equilibrium balanced growth path fertility. The growth rate for Case 5 is always highest before about 1000, and then has the second highest growth rate of the cases. Case 3 has the same growth rate as Case 2 until about 1000 and then has the lowest growth rate. Finally Case 4 has the second highest growth rate before 1000, and the third highest growth rate after 1000.

Figure **??** contains the time series for log human capital. The demographic transition has accelerated human capital investment in the efficient solution before falling to a lower balanced growth path value. For each case, the higher the returns to specialization, greater ω, the lower the rate of human capital investment, particularly noticeable after 2000. While the rate of human capital investment does not vary by ω, nor λ in the equilibrium solution, since A changes between Cases 1-3 compared with Case 4 and 5, the rate of growth of human capital is higher for Cases 4 and 5 compared with the identical solutions for Cases 1-3. The difference between cases for the efficient solutions is more clearly evident in figure **??**. Although in Case 1 the fertility and investment solutions are nearly identical for all values of ω. This is shown in the first panel of figure **??**, where there is almost no difference between the efficient paths.

Figure **??** contains the time series of total fertility rates. We produced total fertility rates by multiplying fertility in the model solutions by 2. In three of the five cases the efficient solution fertility falls from its initial values before the equilibrium solution enters into its Demographic Transition. Cases 1, 2 and 4 have lower fertility than the equilibrium solution for roughly 19000 years.[18] In Cases 2-5, the balanced growth path efficient total fertility rates begin to differ by ω, with higher fertility rates for higher values

[17] We made no effort to calibrate the model with observed long run growth rates. The incredible growth rates that exist arise from the transition dynamics of expansion of the production coalition.

[18] The change after a very long stationary solution in all 5 cases of the efficient solution arises because the size of the production accelerates, even though it is determined by the coordination technology.

of ω. In Case 5, the long run fertility rates in the efficient solution differ across all four values of ω. The equilibrium time series path of total fertility reproduces the long classical-Malthusian world, interrupted by a brief rapid Demographic Transition, followed by a stationary total fertility rate of near zero population growth. In contrast, each of the five Cases produces efficient fertility time series that typically have an intermediate phase of fertility, lower than the classical-Malthusian fertility, but different from the long run balanced growth path fertility. This intermediate phase, unlike the short Demographic Transition in the equilibrium solution, is much longer lasting. Roughly lasting 19000 years, whereas the equilibrium Demographic Transition lasts on the order of 700 years. Figure **??** contains the mortality rate for these solutions. Here the standard equilibrium demographic transition via the mortality revolution is evident. In this parameterization, the mortality decline is identical across all cases. This occurs because when the value of A changes in Cases 4 and 5, coincides with a decline of mortality to 0. If the mortality function required a higher value of human capital before it declines to 0, then there would be a divergence between Cases 1-3 and Cases 4 and 5. For all Cases, the equilibrium mortality decline begins 1320, but does not drop below .70 until 1680. From 1680 until 1880 mortality declines only from .58 to .57, before dropping to .38 in 1920, .38 in 1960 and .18 in 2000 and then 0 in 2040. The mortality decline in the efficient solution does differ by ω for each case. However the mortality decline occurs between -17640 and -17320 for Case 1. For Case 2, the efficient solution has mortality decline between -17640 and -17360. The efficient mortality decline occurs between -17640 and -17280 for Case 3. For Case 4 again the decline begins in -17640 and ends by -17320. Finally in Case 5 decline begins and ends in -17640 and -17280. If we date the end of the mortality decline in the year when mortality is less than 1%, the ending date for $\omega = 1.25$ varies between -17480 for Cases 1 and 2, -17440 for Cases 3, 4 and 5, At the other end, for $\omega = 2.50$, the ending date of the mortality decline, defined as mortality below 1%, occurs in year -17520 for Case 1, -17480 for Cases 2 and 4, -17440 for Case 3 and 5.

We present the years of schooling per worker, 40τ, in Figure **??**. As with fertility, schooling does not depend on ω for the equilibrium solutions. Fertility is affected by ω in the efficient solution, and hence so is schooling. In all 5 Cases, the efficient solution has an earlier rise in schooling compared with the equilibrium solution, and a higher peak schooling. In cases 2-5, along the balanced growth path, schooling in the equilibrium solution exceeds the schooling in the efficient solution, because fertility is lower in the equilibrium solution due to a quantity-quality trade-off. In Case 1, fertility is actually lower in the balanced growth path in the efficient solution compared with the equilibrium solution. This however is an intermediate run case. Fertility in the efficient solution is significantly lower than 1 at .45. Thus population is declining over a long period of time, because the coordination technology is the limiting factor determining production coalition size. When the population becomes the limiting factor, the efficient solutions will be given by (23) and (24). In results not reported here, this occurs around year 10000.

6 Conclusions

In this paper we characterize the gap between optimal and equilibrium fertility an investment in human capital with and without mortality of young adults. We develop a model in which the aggregate production function exhibits increasing returns to population. Moreover, individuals have a precautionary demand for children when mortality risk is non-zero. These two assumptions result in an equilibrium fertility rate that is sub-optimal and too much investment in human capital per child in equilibrium in the long run.

When we add mortality to the model we find that the optimal solution of the model involves accumulating human capital rapidly to eliminate mortality as soon as possible. Our numerical solutions indicate that the efficient solution produces 0 mortality 19000 years before it occurs in the equilibrium solution. This leads to an early intermediate fertility transition although optimal fertility in the long run increases again above the equilibrium level to take advantage of the increasing returns in population. The model presented here provides a unified framework to understand fertility and mortality in a dynamic context and allows us to establish a clear comparison between the equilibrium and optimal paths of the two variables.

References

Barro, R. J., and Becker, G.S., “Fertility Choice in a Model of Economic Growth.” *Econometrica*, 57, 1989: 481-501.

Becker, G. S., “An Economic Analysis of Fertility,” in *Demographic and Economic Change in Developed Countries*, Conference of the Universities, NBER, Princeton University Press, Princeton, 1960: 209-240.

Becker, G. S., and Barro, R.J., “A Reformulation of the Economic Theory of Fertility.” *Quarterly Journal of Economics*, 103, 1988: 1-25.

Becker, G. S., and Lewis, H.G., “On the Interaction between the Quantity and Quality of Children.” *Journal of Political Economy*, 81, 1973: S279-S288.

Boldrin, M., and Jones, L.E., “Mortality, Fertility and Saving Decisions,” *Review of Economic Dynamics*, 5, 2002: 775-814.

Chakraborty, S., “Endogenous Lifetime and Economic Growth,” *Journal of Economic Theory*, 116, 2004: 119-137.

Cigno, A., “Fertility Decisions when Infant Survival is Endogenous,” *Journal of Population Economics*, 11, 1998: 21-28.

Conde-Ruiz, J. I., Gimenez, E.L.and Perez-Nievaz, M., ”Millian Efficiency with Endogenous Fertility,” *Review of Economic Studies*, 77, 2010: 154-187.

Chomitz, K. M., and Birdsall, N., “Incentives for Small Families: Concepts and Issues.” in *Proceedings of the World Bank Annual Conference on Development Economics* 1990. Washington D.C., World Bank, 1991: 309-339.

Dasgupta, P., “On the Concept of Optimal Population,” *Review of Economic Studies*, 36, 1969: 295-318.

Doepke, M., “Child Mortality and Fertility Decline: Does the Barro-Bekcer Model Fit the Facts?” *Journal of Population Economics*, 18, 2005: 337-366.

Doepke, M., and Tertilt, M., “Women’s Liberation: What’s in it for Men?” *Quarterly Journal of Economics*, 124, 2009: 1541-1591.

Easterlin, R. A., “How Beneficient is the Market? A Look at the Modern History of Mortality,” *European Review of Economic History*, 3, 1999: 257-294.

Eckstein, Z. and Wolpin, K.I., “Endogenous Fertility and Optimal Population Size,” *Journal of Public Economics*, 27, 1985: 93-106.

Eckstein, Z., Mira, P., and Wolpin, K.I., “A Quantitative Analysis of Swedish Fertility Dynamics: 1751-1990,” *Review of Economic Dynamics*, 2, 1999: 137-165.

Ehrlich, I., Lui, F.T., “Intergenerational Trade, Longevity, and Economic Growth,” *Journal of Political Economy*, 99, 1991: 1029-1059.

Fernandez-Villaverde, J., “Was Malthus Right? Economic Growth and Population Dynamics,” University of Pennsylvania working paper, 2001.

Galor, O., "The Demographic Transition: Causes and Consequences," *Cliometrica*, 6, 2012: 1-28.

Galor, O., and Moav, O., "Natural Selection and the Origin of Economic Growth," *Quarterly Journal of Economics*, 117, 2002: 1133-1191.

Galor O, and Weil, D. N., "From Malthusian Stagnation to Modern Growth," *American Economic Review*, 89, 1999: 150-154.

Galor, O., and Weil, D.N., "Population, Technology, and Growth: From Malthusian Stagnation to the Demographic Transition and Beyond," *American Economic Review*, 90, 2000: 806-28.

Gigliotti, A. G., "Total Utility, Overlapping Generations and Optimal Population." *Review of Economic Studies*, 50, 1983: 71-86

Goldin, C., *Understanding the Gender Gap: An Economic History of American Women*. Oxford University Press, New York, 1990.

Golosov, M., Jones, L.E., and Tertilt, M., "Efficiency with Endogenous Population Growth," *Econometrica*, 75, 2007: 1039-1071.

Jones, C.I., "Was an Industrial Revolution Inevitable? Economic Growth Over the Very Long Run," *The B.E. Journal of Macroeconomics, Advances in Macroeconomics*, Volume 1, 2001: Number 2 Article 1.

Kalemli-Ozcan, S., "Does the Mortality Decline Promote Economic Growth?" *Journal of Economic Growth* 7, 2002: 411-39.

Kalemli-Ozcan, S., "A Stochastic Model of Mortality, Fertility and Human Capital Investment," *Journal of Development Economics*, February 2003, Vol. 70(1), 103-118.

Lagerlöf, N-P., "From Malthus to Modern Growth: Can Epidemics Explain the Three Regimes?" *International Economic Review*, 44, 2003: 755-777.

Nerlove, M., Razin, A., and Sadka, E., "Population Size and the Social Welfare Functions of Bentham and Mills," *Economic Letters* 10, 1982: 61-64.

Nerlove, M., Razin, A., and Sadka, E., "Household and Economy: Welfare Economics of Endogenous Fertility," *Economic Theory, Econometrics and Mathematical Economics Series*, Boston: Harcourt, Brace Jovanovich, 1987.

Razin, A. and Ben-Zion, U., "An Intergenerational Model of Population Growth," *American Economic Review*, 65, 1975: 923-933.

Romer, P. M., "Increasing Returns and Long-Run Growth," *Journal of Political Economy*, 94(5), October, 1986: 1002-1037.

Sah, R. K., "The Effects of Child Mortality Changes on Fertility Choice and Parental Welfare," *Journal of Political Economy*, 99, 1991: 582-606.

Samuelson, P. A., "The Optimum Growth Rate for Population," *International Economic Review* 16, 1975: 531-538.

Soares, R., "Mortality Reductions, Educational Attainment, and Fertility Choice," *American Economic Review*, 95, 2005: 580-601.

Tamura, R., “Income Convergence in an Endogenous Growth Model,” *Journal of Political Economy* 99, 1991: 522-540.

Tamura, R., ”Efficient Equilibrium Convergence: Heterogeneity and Growth,” *Journal of Economic Theory* 58, 1992: 355-376.

Tamura, R., “From Decay to Growth: A Demographic Transition to Economic Growth,” *Journal of Economic Dynamics and Control* 20, 1996: 1237-1262.

Tamura, R., “Human Capital and the Switch from Agriculture to Industry,” *Journal of Economic Dynamics and Control* 27, 2002: 207-242.

Tamura, R., “Human Capital and Economic Development,” *Journal of Development Economics* 79, 2006: 26-72.

Tamura, R., Simon, C.J. “Secular Fertility Declines, Baby Booms and Economic Growth: International Evidence,” forthcoming *Macroeconomic Dynamics*, 2017.

Tamura, R., Simon, C.J., Murphy, K.M. “Black and White Fertility, Differential Baby Booms: The Value of Equal Education Opportunity,” *Journal of Demographic Economics* 82, 2016: 27-109.

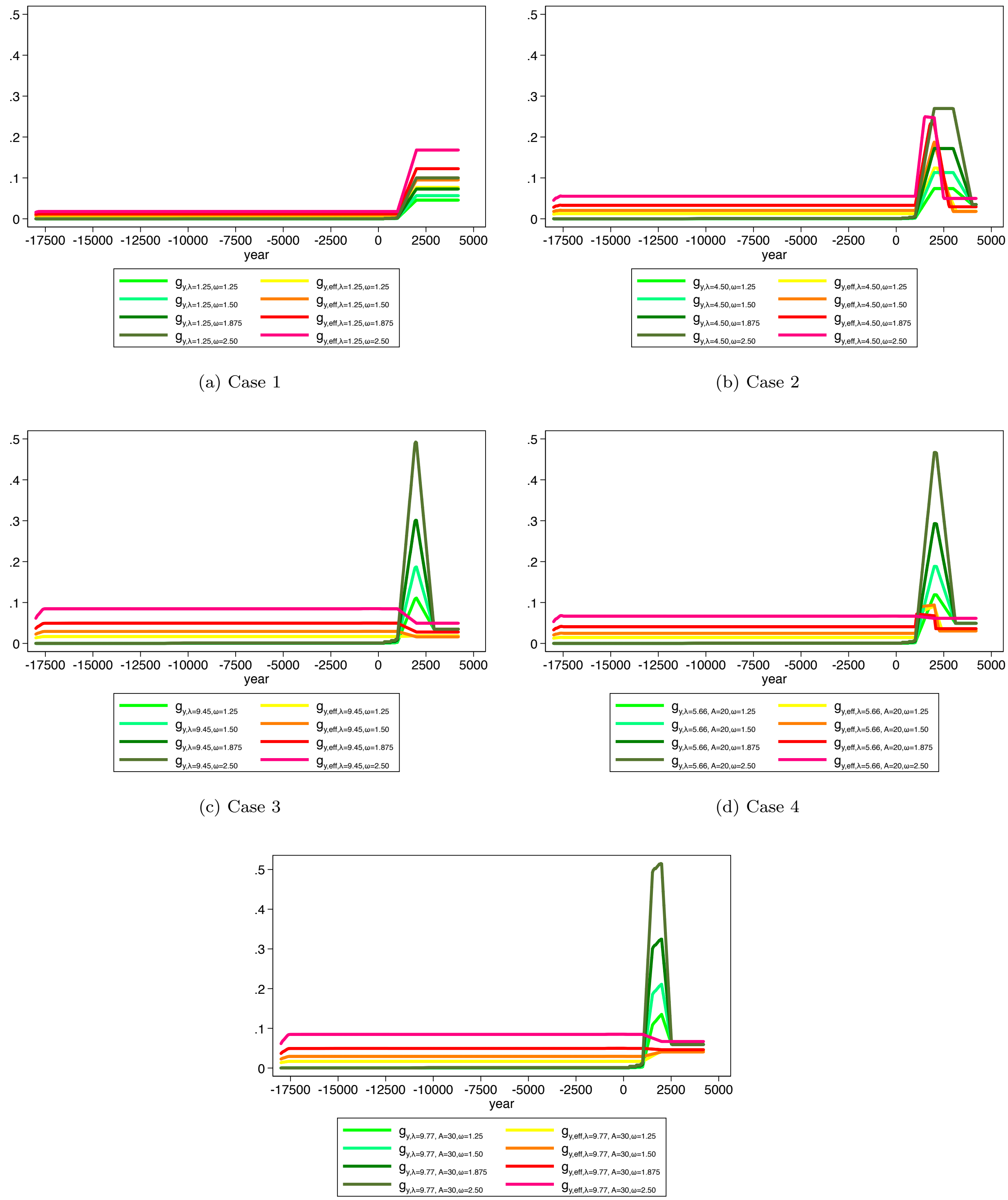
Tertilt, M., “Polygyny, Fertility and Savings,” *Journal of Political Economy*, 113, 2005: 1341-1371.

Tertilt, M.,“Polygyny, Women’s Rights, and Development,” *Journal of the European Economic Association*, 4, 2006: 523-530.

Weil, D. N., *Economic Growth*, Pearson, Addison, 2005.

Willis, R. J., “Externalities and Population,” in *Population Growth and Economic Development: Issues and Evidence,* edited by G. Johnson and R. D. Lee, Madison: University of Wisconsin Press, 1987: 661-702.

Zimmermann, K. F., *Economic Theory of Optimal Population.* (Editor) Springer-Verlag. Berlin. Heidelberg. New York, 1989.



(a) Case 1

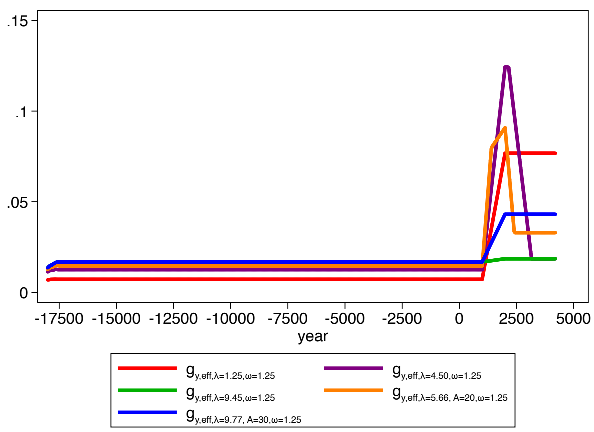
(b) Case 2

(c) Case 3

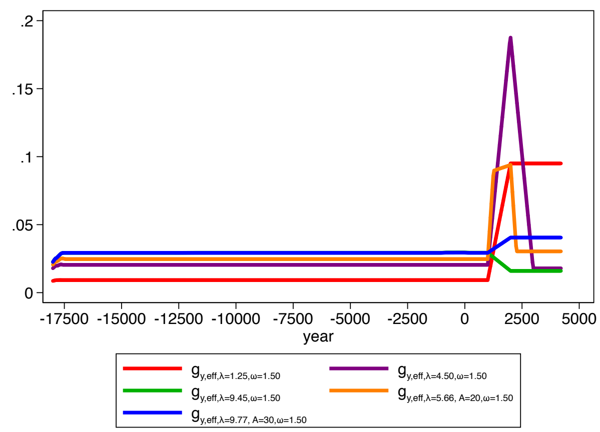
(d) Case 4

(e) Case 5

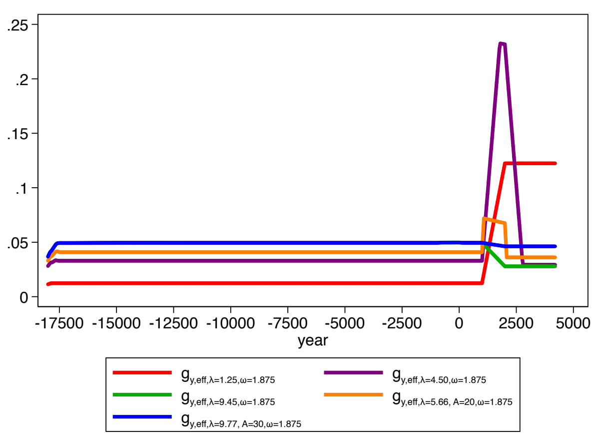
Figure 1: Equilibrium and Efficient Growth Rates of Per Capita Income



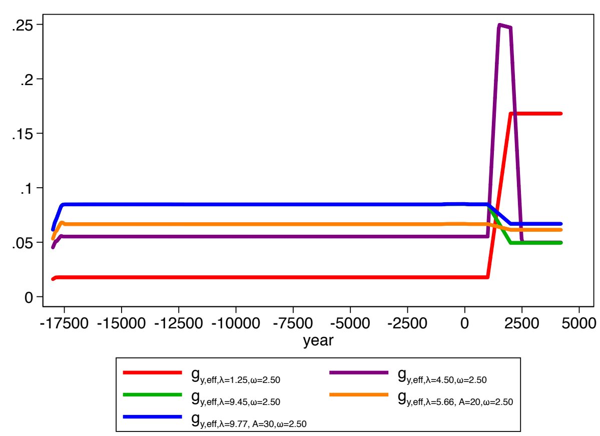
(a) $\omega = 1.25$



(b) $\omega = 1.50$

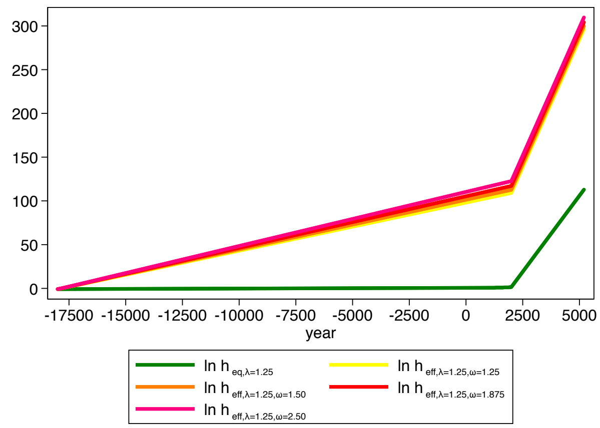


(c) $\omega = 1.875$

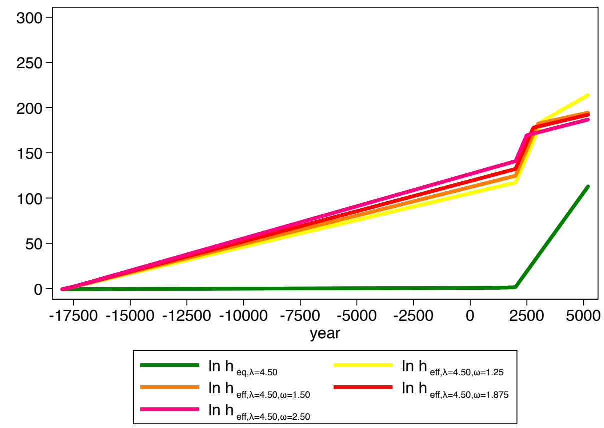


(d) $\omega = 2.50$

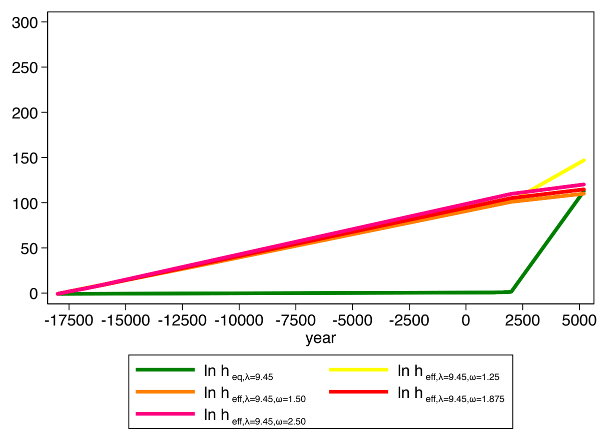
Figure 2: Efficient Growth Rates of Per Capita Income



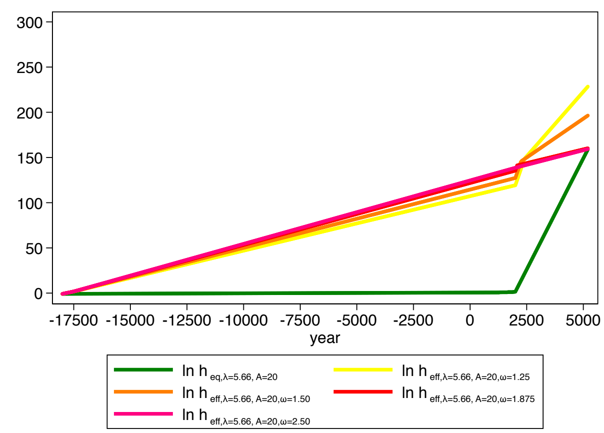
(a) Case 1



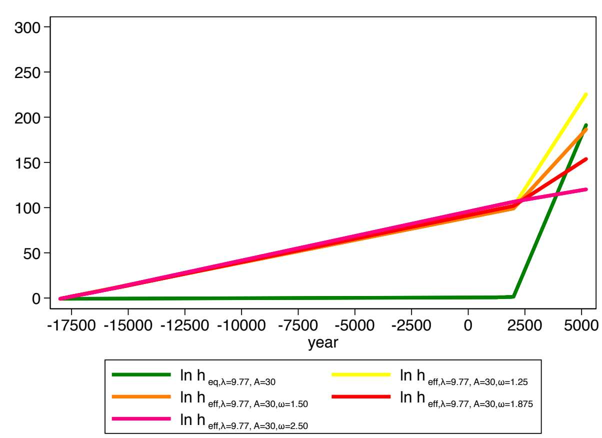
(b) Case 2



(c) Case 3

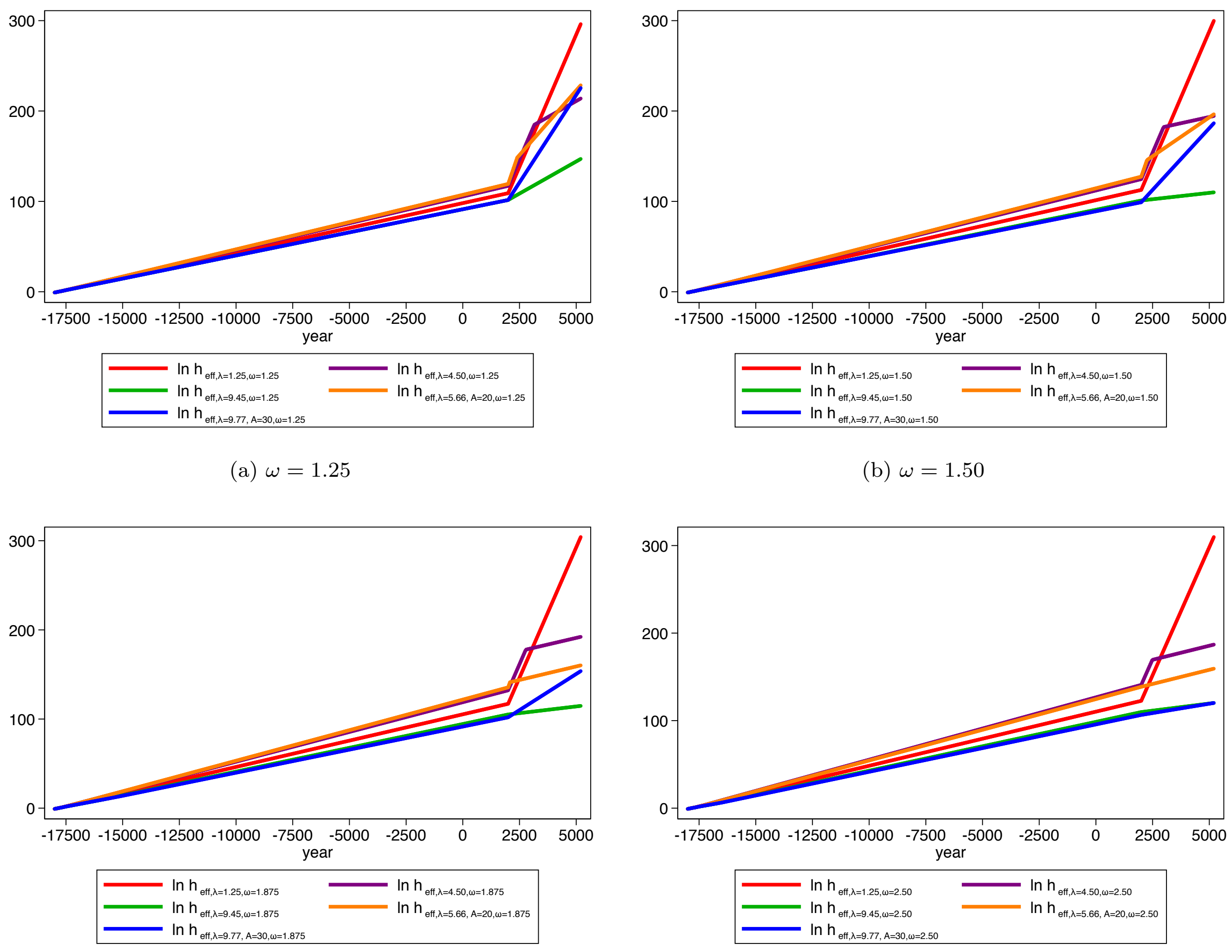


(d) Case 4



(e) Case 5

Figure 3: Equilibrium and Efficient Log Human Capital



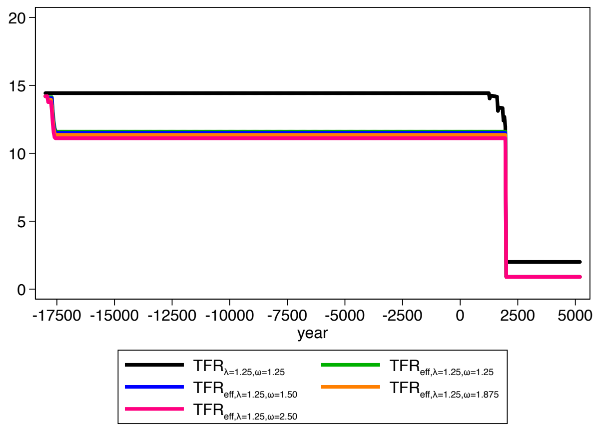
(a) $\omega = 1.25$

(b) $\omega = 1.50$

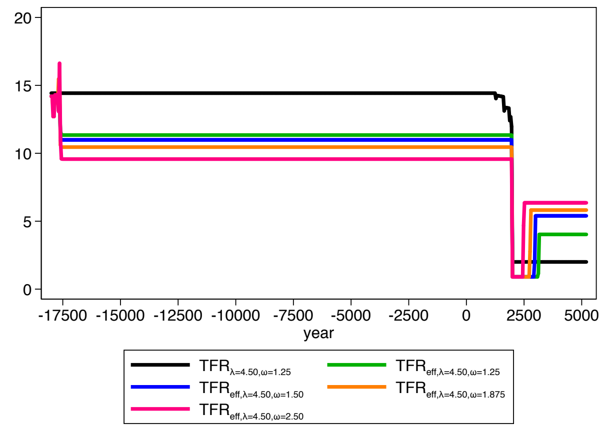
(c) $\omega = 1.875$

(d) $\omega = 2.50$

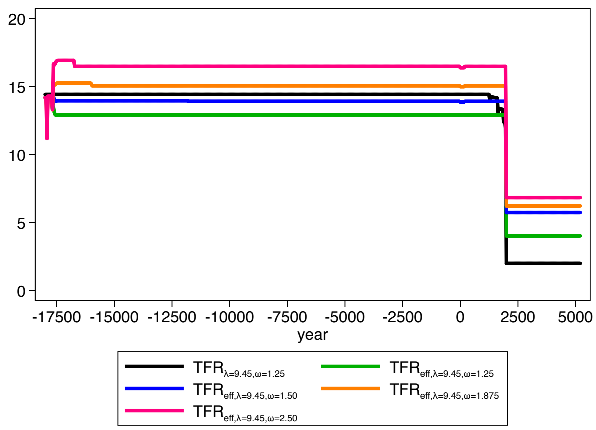
Figure 4: Efficient Log Human Capital



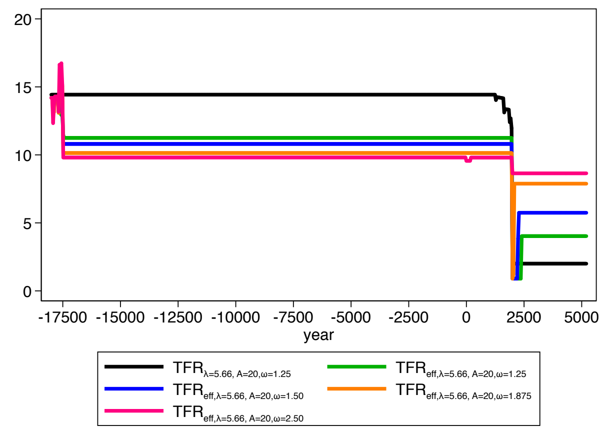
(a) Case 1



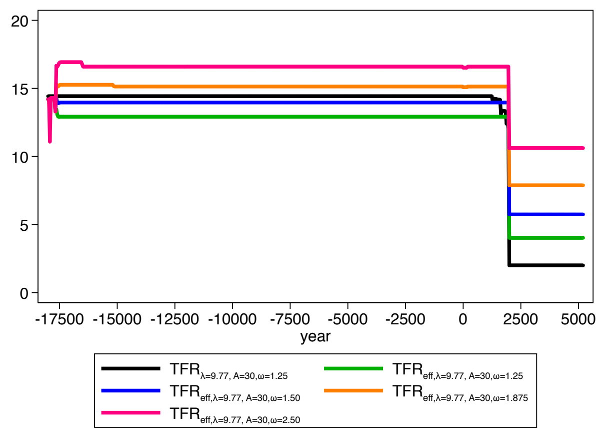
(b) Case 2



(c) Case 3

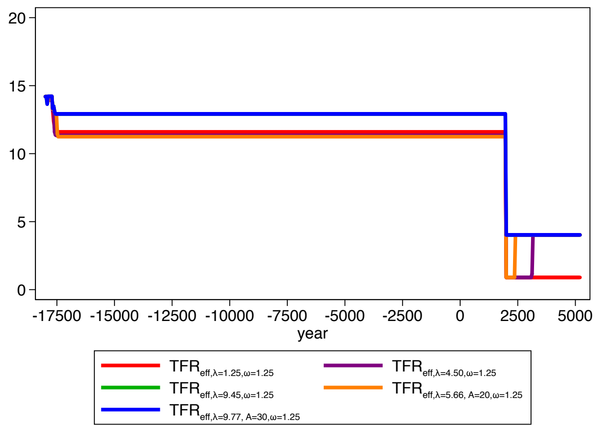


(d) Case 4

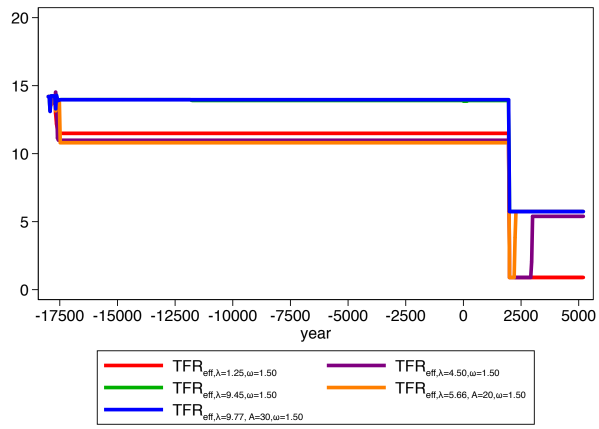


(e) Case 5

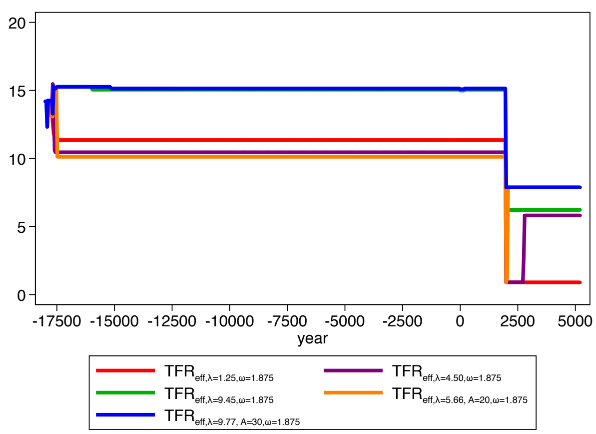
Figure 5: Equilibrium and Efficient Total Fertility Rates



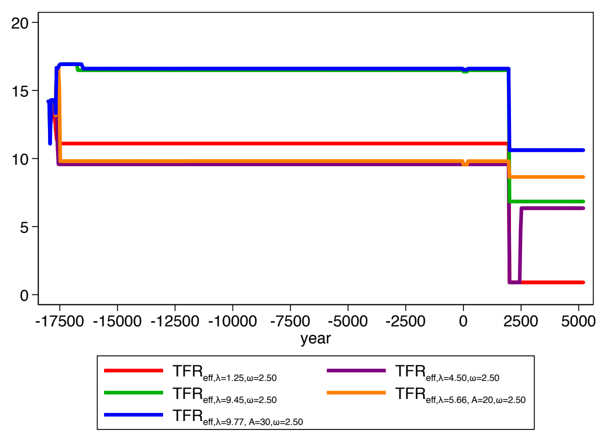
(a) $\omega = 1.25$



(b) $\omega = 1.50$

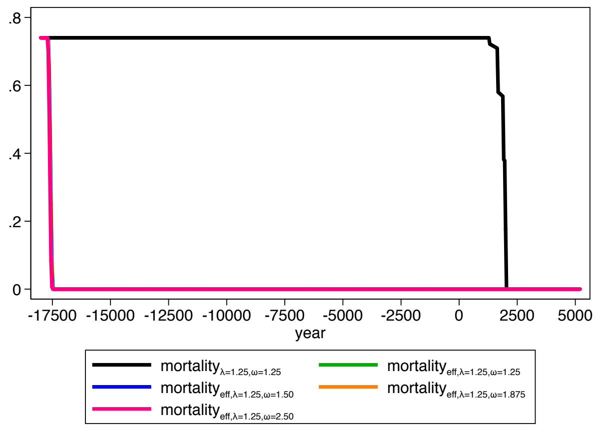


(c) $\omega = 1.875$

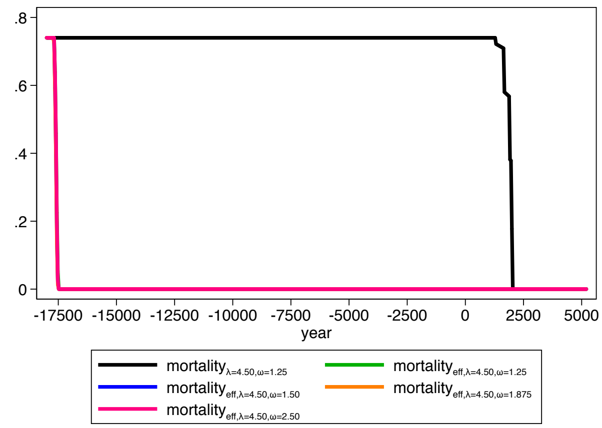


(d) $\omega = 2.50$

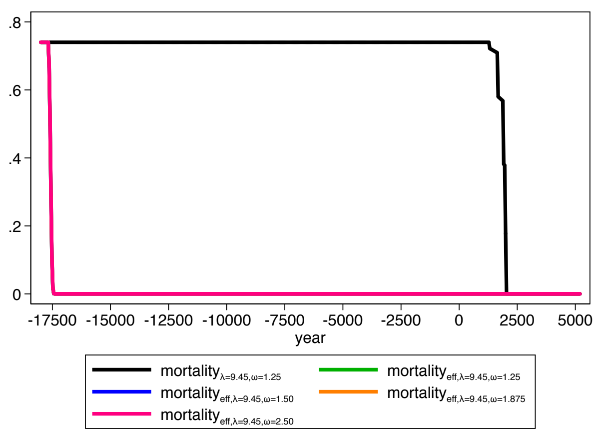
Figure 6: Efficient Total Fertility Rates



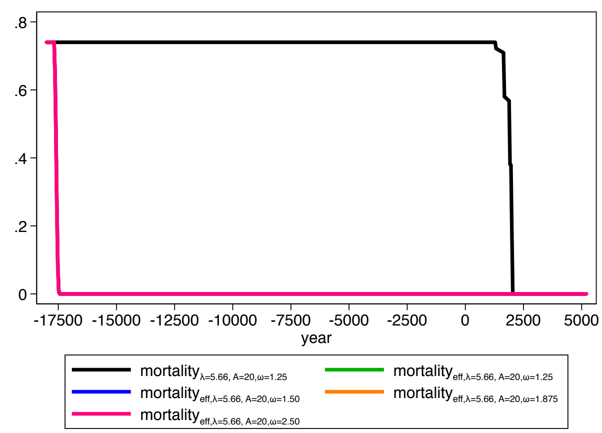
(a) Case 1



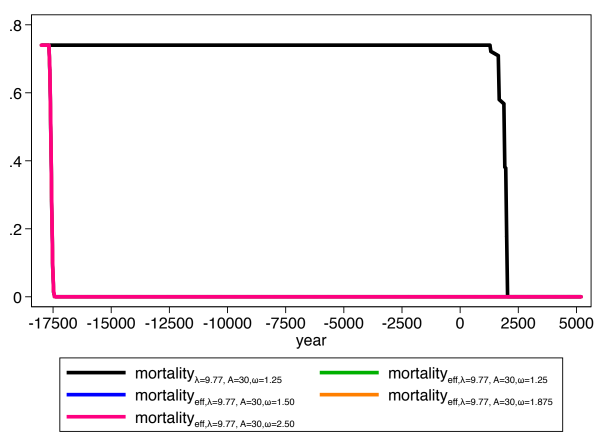
(b) Case 2



(c) Case 3

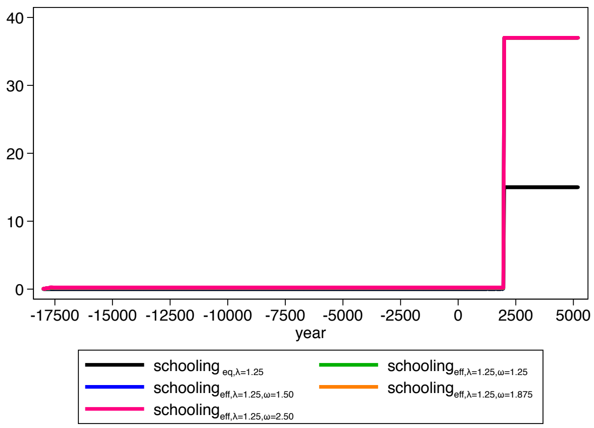


(d) Case 4

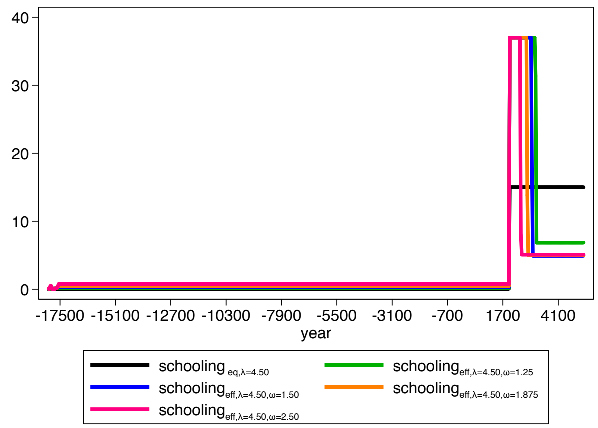


(e) Case 5

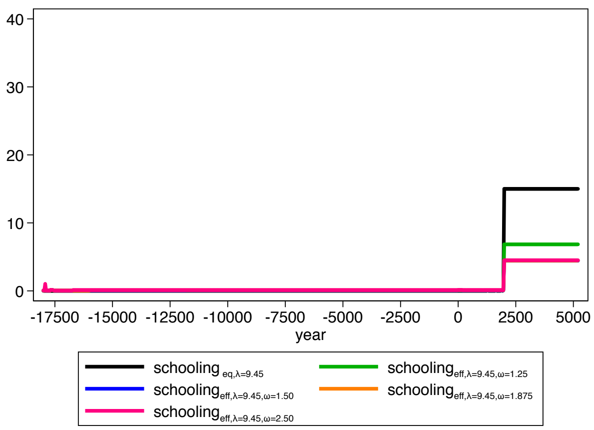
Figure 7: Equilibrium and Efficient Mortality



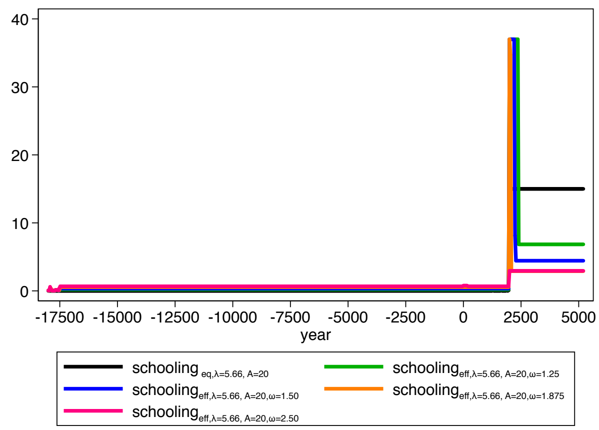
(a) Case 1



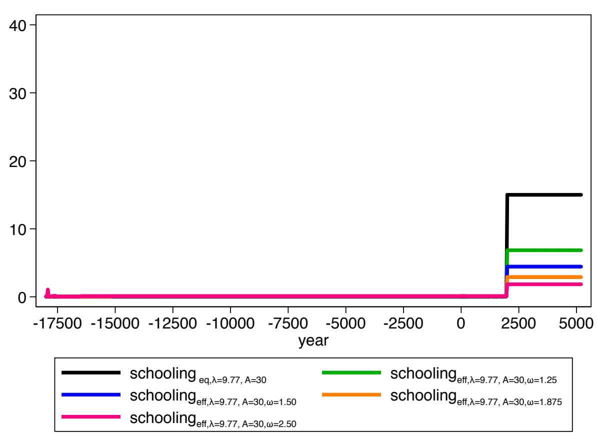
(b) Case 2



(c) Case 3

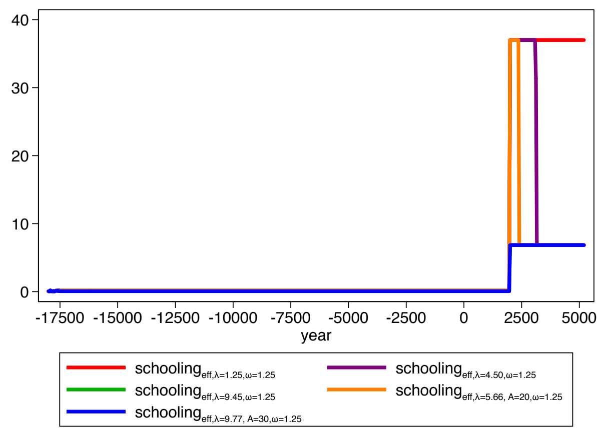


(d) Case 4

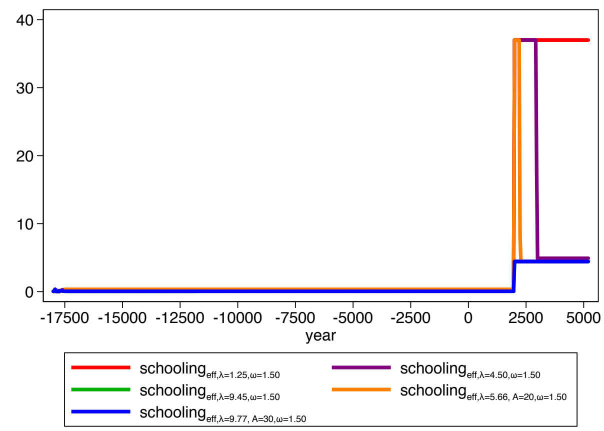


(e) Case 5

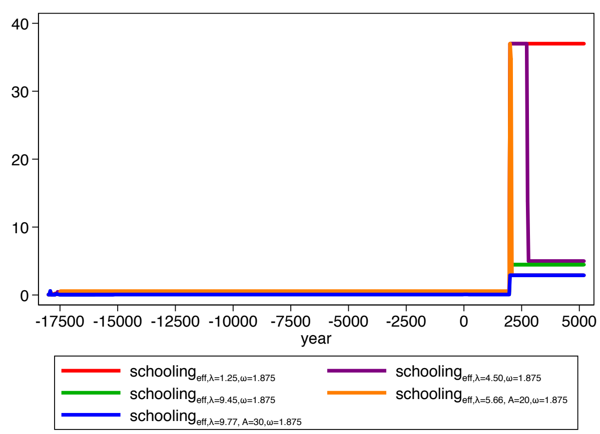
Figure 8: Equilibrium and Efficient Schooling



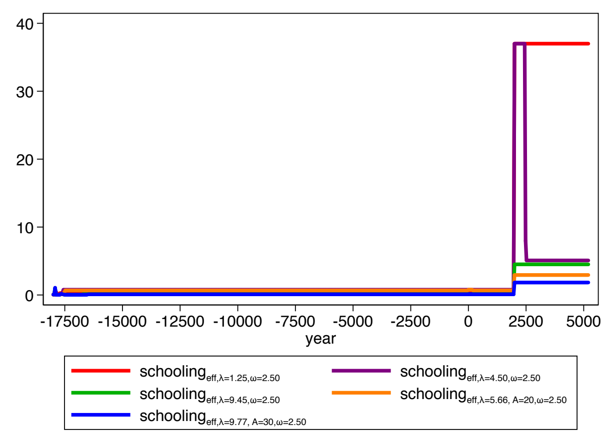
(a) $\omega = 1.25$



(b) $\omega = 1.50$



(c) $\omega = 1.875$



(d) $\omega = 2.50$

Figure 9: Efficient Schooling