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## **How Borda voting rule can respect Arrow IIA and avoid Cloning manipulation**

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## How Borda voting rule can respect Arrow IIA and avoid Cloning manipulation

### ABSTRACT

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This paper proposes a new formulation of the Borda rule in order to deal with the problem of cloning manipulation. This new Borda voting specification will be named: Dynamic Borda Voting (DBV) and it satisfies Arrow's IIA condition. The calculations, propositions with proof and explanations are made to show the effectiveness of this method. From DBV, the paper presents a method to measure and quantify the magnitude of the shock due to change in irrelevant alternatives over a scale moving from 0 to 100.

Keywords: Voting rules; Arrow IIA; Cloning manipulation;

JEL Classification: D8; D7; C6.

### INTRODUCTION

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Is the individual rational making his choices? If so, is that the whole society is rational? In an election by ballot, the plurality of votes always indicates the wishes of the voters, that is to say that the candidate who obtains this plurality is necessarily the one that the voters prefer to his opponents (Borda, 1781). Borda (1781) shows that this opinion, is true in the situation where the vote is made between only two subjects, but it may be misleading in all the other situations. Since the eighteenth century, two voting methods claim to be able to provide solutions to this problem of aggregation of individual choices: the main works are from Borda (1781) and Condorcet (1785).

Unfortunately, these works are not faultless and face many criticisms. This paper focuses on voting Borda. One of the main criticisms of the Borda rule is that it is highly vulnerable to strategic voting. Voting strategically for/against a candidate means giving a higher or lower Borda score than the voter's preference ordering would imply (Lehtinen, 2007). According to Black (1958), defending the susceptibility of his rule to strategic manipulation, Borda claimed, "My scheme is intended only for honest men". About that, In Borda count a defeated candidate can manipulate the election result in his favour in sincere way by introducing a candidate which is a clone of him and voters ranked this clone candidate immediately below him. In this situation Borda rule is strictly follows but manipulation is possible (Islam, Mohajan, & Moolio, 2012).

Borda rule faces a major constraint. This is Arrow's impossibility theorem. Arrow (1963) states that when voters have three or more distinct alternatives, no ranked voting electoral system can convert the ranked preferences of individuals into a community-wide ranking while also satisfying a specified set of criteria: Universal Domain (UD), Non-

Dictatorship (ND), Strict Pareto (SP) and Independence of Irrelevant Alternatives (IIA) (Arrow, 1963). In fact, although Borda's rule satisfies the first 3 conditions, it does not respect IIA one.

This paper proposes a new formulation of the Borda rule in order to deal with the problem of cloning manipulation. This new Borda voting specification will be named: Dynamic Borda Voting (DBV) and it satisfies Arrow's IIA condition. From DBV, the paper presents a method to measure and quantify the magnitude of the shock due to change in irrelevant alternatives over a scale moving from 0 to 100.

The DBV opens the way to practical applications. The paper will present as an example, its application on the risk aversion behaviour of investors on the stock market due to exogenous shocks. Modelling this will focus on changes in individual stock portfolio selections and their influence on the overall market situation.

## 1 RELATED LITERATURE

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One of the best known and most important results of social choice theory is the theorem of Gibbard (1973) and Satterthwaite (1975) which states that the only rules of ‘non-manipulable’ collective choice by agents are dictatorial rules (Béhue, Favardin, & Lepelley, 2009). In other words, any rule of collective choice that is somewhat democratic comes up against the following difficulty: there are situations in which an agent (or a coalition of agents) is induced to express a non-sincere preference in order to obtain a result Which he prefers to the one he would get with a sincere strategy (Béhue, Favardin, & Lepelley, 2009).

According to Satterthwaite (1975) almost every participant in the formal deliberations of a commission realizes that situations may occur where he can manipulate the outcome of the commission’s vote by misrepresenting his preferences. For example, a voter in choosing among a Democrat, a Republican, and a minor party candidate may decide to follow the “sophisticated strategy” of voting for his second choice, the Democrat, instead of his “sincere strategy” of voting for his first choice, the minor party candidate, because he thinks that a vote for the minor party candidate would be a wasted vote on a hopeless cause. Satterthwaite (1975) investigates if a committee can eliminate use of sophisticated strategies among its members by constructing a voting procedure that is “strategy-proof” in the sense that under it, no committee member will ever have an incentive to use a sophisticated strategy. He found that every strategy-proof voting procedure is dictatorial (Satterthwaite, 1975).

Voting paradoxes are numerous. Lepelley, Moyouwou & Smaoui (2017) study scoring elimination rules (SER). SER gives points to candidates according to their rank in voters’ preference orders and eliminates those with the lowest number of points, constitute an important class of voting rules. This class of rules, that includes some famous voting methods such as Plurality Runoff or Coombs Rule, suffers from a severe pathology known as monotonicity paradox or monotonicity failure, that is, getting more points from voters can make a candidate a loser and getting fewer points can make a candidate a winner (Lepelley, Moyouwou, & Smaoui, 2017). Focusing on the profiles that create the strict and the strong Borda paradoxes Diss & Tlidi (2016) provide an organized knowledge of the conditions for a profile to show or to never show Borda’s paradox. The framework they use allows them to determine the

minimum number of voters needed for a profile to show either the strong or the strict Borda paradoxes when the differences between candidates and the weighted scoring rule are already determined. It also allows them to give the differences between candidates in the pairwise election outcomes required for a profile to never exhibit one of the two paradoxes for a given weighted scoring rule and a fixed number of voters. Finally, they are able to describe what range of weighted scoring rules could possibly accompany a given number of voters and specified differences between candidates in the pairwise election outcomes in order for a profile to never exhibit one of the two paradoxes.

Saari & McIntee (2013) work on pairwise and positional election outcomes. According to them, while it has been known since the eighteenth century that the Borda and Condorcet winners need not agree, it had not been known, for instance, in which settings the Condorcet and plurality winners can disagree, or must agree. These relationships are based on an easily used method that connects pairwise tallies with admissible positional outcomes (Saari & McIntee, 2013).

Another important issue of voting is Arrow's Impossibility paradox. By the way, Barbie, Puppe & Tasnádi (2006) characterize the preference domains on which the Borda count satisfies Arrow's "Independence of Irrelevant Alternatives (IIA)" condition. According to them, these domains are obtained by fixing one preference ordering and including all its cyclic permutations (Condorcet cycles). Therefore, the Borda count is "non-manipulable" on a broader class of domains when combined with appropriately chosen tie-breaking rules (Barbie, Puppe, & Tasnádi, 2006). On the other hand, they also prove that the rich domains on which the Borda count is "non-manipulable" for all possible tie-breaking rules are again the cyclic permutation domains. Ever since their work, the two most important results of social choice theory, the impossibility theorems of Arrow and Gibbard-Satterthwaite (see Barbie, Puppe, & Tasnádi, 2006), have led to a steady search for possibility results on restricted domains.

The common method is to fix an appropriate set of admissible preferences, and to determine which social welfare functions satisfy Arrow's conditions. This is done considering that social choice functions are "non-manipulable", on that preference domain. Another view on the question is presented by Dasgupta and Maskin (2003) based on Maskin (1995) work. They consider specific preference aggregation rules such as Borda count, and ask on what domains these rules satisfy desirable conditions in the spirit of Arrow's conditions.

## **2 DYNAMIC BORDA VOTING (DBV) AND IIA ARROW'S CONDITION**

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### **2.1 SIMPLE BORDA VOTING AND ARROW'S IIA PRESENTATION**

#### **2.1.1 Theoretical presentation**

Let's adopt Borda Voting presentation in (Islam, Mohajan, & Moolio, 2012). Let  $M = \{1, 2, \dots, m\}$  be the set of individual voters, and let  $N = \{x, y, \dots, z\}$  be the finite set of alternatives where  $\text{Card}(M) = m$  and  $\text{Card}(N) = n$ .

Each voter has to rank the candidates in his preference order and then, we proceed to the count of the number of times each candidate is ranked first. At the end, the candidate who receives a relative majority is elected.

If  $x$  is strictly preferred to  $y$  we write  $xPy$  and so on. If  $x$  is related to  $y$ , the binary relations according to Arrow (1963) is as follows:

- Reflexivity:  $\forall x \in N; xRx$ .
- Completeness:  $\forall x, y \in N \ \& \ x \neq y \Rightarrow xRy \text{ or } yRx$ .
- Transitivity:  $\forall x, y, z \in N \text{ if } , xRy \ \& \ yRz \Rightarrow xRz$ .
- Anti-symmetry:  $\forall x, y \in N \text{ if } , xRy \ \& \ yRx \Rightarrow x = y$ .
- Asymmetry:  $\forall x, y \in N, \text{ such that } xRy \Rightarrow \sim (yRx)$ .

According to Arrow (1963), the social welfare function should satisfy independence of irrelevant alternatives. It says that if we're trying to figure out whether society prefers  $x$  to  $y$ , what people think of  $z$  shouldn't matter.

### 2.1.2 Arithmetical Calculations

Let us assume that there are 4 voters and 4 alternatives  $x, y, z$  and  $t$  and the preference profile be as follows:

Voter 1:  $xPyPzPt$  / Voter 2:  $xPyPzPt$  /

Voter 3:  $yPxPtPz$

/

Voter 4:  $yPtPxPz$

In the social preferences matrix, we have for the 1<sup>st</sup> profile ( $R_1^N$ ):

	$i_1$	$i_2$	$i_3$	$i_4$
3	$x$	$x$	$y$	$y$
2	$y$	$y$	$x$	$t$
1	$z$	$z$	$t$	$x$
0	$t$	$t$	$z$	$z$

With  $i_i$  representing voter  $i_i \in [1,4]$ .  $\text{Card}(N) = 4$  and  $n-1 = 3$ .

Let's note Borda score for  $x$ :  $B_x$  Therefore, Borda count for the different alternatives will be as follows:

- ❖ For  $x$ :  $B_x = 3 \times 2 + 2 \times 1 + 1 \times 1 + 0 \times 0 = 9$
- ❖ For  $y$ :  $B_y = 3 \times 2 + 2 \times 2 + 1 \times 0 + 0 \times 0 = 10$
- ❖ For  $z$ :  $B_z = 3 \times 0 + 2 \times 0 + 1 \times 2 + 0 \times 2 = 2$
- ❖ For  $t$ :  $B_t = 3 \times 0 + 2 \times 1 + 1 \times 1 + 0 \times 2 = 3$

Here  $y$  gets the highest marks that is  $B_y = 10$ , so  $y$  wins.

The social preference order is  $(S^1): \overline{y, x, t, z}$ .

### 2.1.3 The problem

According to Arrow (1963), there is no social welfare rule that satisfies the all 4 impossibility theorem conditions. Borda rule respect the 3 first Arrow's conditions, but doesn't satisfies the last one that is IIA.

Proof:

Let's consider the same example, but let's assume in respect to IIA, that some voters  $(i_1, i_2, i_4)$  change their preferences (Satterthwaite, 1975), but keep the same first choice.  
 Voter 1:  $xPzPtPy$  / Voter 2:  $xPtPzPy$  / Voter 3:  $yPxPtPz$  / Voter 4:  $yPxPzPt$

The 2<sup>nd</sup> profile matrix of social preference  $(R_2^N)$  will be as follow:

	$i_1$	$i_2$	$i_3$	$i_4$
3	x	x	y	y
2	z	t	x	x
1	t	z	t	z
0	y	y	z	t

Borda count for the different alternatives will be as follows:

- ❖ For  $x$ :  $\dot{B}_x = 3 \times 2 + 2 \times 2 + 1 \times 0 + 0 \times 0 = 10$
- ❖ For  $y$ :  $\dot{B}_y = 3 \times 2 + 2 \times 0 + 1 \times 0 + 0 \times 2 = 6$
- ❖ For  $z$ :  $\dot{B}_z = 3 \times 0 + 2 \times 1 + 1 \times 2 + 0 \times 1 = 4$
- ❖ For  $t$ :  $\dot{B}_t = 3 \times 0 + 2 \times 1 + 1 \times 2 + 0 \times 1 = 4$

Here x gets the highest marks that is  $\dot{B}_x = 10$ , so x wins instead of y.

The social preference order  $(S^2)$  in  $(R_2^N)$  is:  $\overline{x, y, (t, z)} \neq \overline{y, x, t, z}$  in  $(R_1^N)$

Conclusion: Borda voting does not respect IIA.

## 2.2 DYNAMIC BORDA VOTING (DBV): A SOLUTION?

Now let's bring some adjustments to the Borda rule. The DBV consist on computing (in the 1<sup>st</sup> profile  $R_k^N$ ) weight linked to Borda winner through dynamic steps, and used them to determine (in the 2<sup>nd</sup> profile  $R_l^N$ ) the social preferences after changes in individual preferences.

**Statement:**

$$\forall x, \dots, y, z \in N, \text{ and } \forall (S^k) \neq (S^l),$$

$$\exists \dot{B}_{N_\omega} = f(B_{x_\omega}, \dots, B_{y_\omega}, B_{z_\omega}) \text{ such as } (S^k) = (S^l)$$

The objective is to find from the  $B_{x_\omega}$  the  $\dot{B}_{N_\omega}$  that, respect:  $(S^k) = (S^l)$ ;

$B_{x_\omega}$  follows a geometrical progression:  $B_{x_j} = B_{x_0} \times (n - 1)^{j_x - 1}$ ;

$j$  is the level of dynamic floor.

So that:  $\lim_{(n,j) \rightarrow \infty} B_{x_j} = B_{x_0} \times (n - 1)^{j_x - 1} = \infty$ ;

If  $\text{Card}(M) = m \rightarrow j \in [1, m]$

$B_{N_i}$  are computed according to an order of entry in a floor chain. The order considered is the  $(S^k)$  one.

All the  $B_{x_\omega}$  obtained from  $R_k^N$  are used to compute the  $\dot{B}_{N_\omega}$  in  $R_l^N$ . With  $k < l$ .

$$\dot{B}_{N_\omega} = \dot{B}_N \times B_{N_\omega}$$

### 2.2.1 Arithmetical Calculations

Let's consider the same above example. There are 4 voters and 4 alternatives  $x, y, z$  and  $t$  and the preference profile be as follows:

In  $R_{k=1}^N$ ,

	$i_1$	$i_2$	$i_3$	$i_4$
3	$x$	$x$	$y$	$y$
2	$y$	$y$	$x$	$t$
1	$z$	$z$	$t$	$x$
0	$t$	$t$	$z$	$z$

In  $R_{l=2}^N$ ,

	$i_1$	$i_2$	$i_3$	$i_4$
3	$x$	$x$	$y$	$y$
2	$z$	$t$	$x$	$x$
1	$t$	$z$	$t$	$z$
0	$y$	$y$	$z$	$t$

Using the simple Borda Voting, we demonstrated that: The social preference order  $(S^2)$  in  $(R_2^N)$  is:  $\overline{x, y, (t, z)} \neq \overline{y, x, t, z}$  in  $(R_1^N) \rightarrow (S^1) \neq (S^2)$ .

❖ Now let's find the  $B_{N_\omega} : B_{x_\omega}, B_{y_\omega}, B_{z_\omega}$  &  $B_{t_\omega}$

Number of floor is  $\text{Card}(N) = 4$ .

Order of entrance in floor<sup>1</sup>:  $(S^1): \overline{y, x, t, z}$

$B_y = B_{y_0} = 10; B_x = B_{x_0} = 9; B_t = B_{t_0} = 3$  &  $B_z = B_{z_0} = 2$ .

i. First Floor:  $j_y = 1$ .

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<sup>1</sup> In the event of a tie, choose the Condorcet winner.

According to  $(S^1)$ , the individual entering in this floor is  $y$ .

Therefor,

- $B_{y_1} = B_{y_0} \times (n - 1)^{1-1} = 10 \times (4 - 1)^0 = 10$

In all cases,  $B_N = B_{N_0} = B_{N_1}$ .

ii. Second Floor:  $j_y = 2; j_x = 1$ .

According to  $(S^1)$ , the individual entering in this floor is  $x$ .

- $B_{y_2} = B_{y_0} \times (n - 1)^{2-1} = 10 \times (4 - 1)^1 = 30$
- $B_{x_1} = B_{x_0} \times (n - 1)^{1-1} = 9 \times (4 - 1)^0 = 9$

This is what happen in the matrix for  $j_y = 2; j_x = 1$

	$i_1$	$i_2$	$i_3$	$i_4$
3	$x$	$x$	$3y$	$3y$
2	$3y$	$3y$	$x$	$t$
1	$z$	$z$	$t$	$x$
0	$t$	$t$	$z$	$z$

iii. Third Floor:  $j_y = 3; j_x = 2; j_t = 1$ .

According to  $(S^1)$ , the individual entering in this floor is  $t$ .

- $B_{y_3} = B_{y_0} \times (n - 1)^{3-1} = 10 \times (4 - 1)^2 = 90$
- $B_{x_2} = B_{x_0} \times (n - 1)^{2-1} = 9 \times (4 - 1)^1 = 27$
- $B_{t_1} = B_{t_0} \times (n - 1)^{1-1} = 3 \times (4 - 1)^0 = 3$

This is what happen in the matrix for  $j_y = 3; j_x = 2; j_t = 1$

	$i_1$	$i_2$	$i_3$	$i_4$
3	$3x$	$3x$	$9y$	$9y$
2	$9y$	$9y$	$3x$	$t$
1	$z$	$z$	$t$	$3x$
0	$t$	$t$	$z$	$z$

iv. Last Floor:  $j_y = 4; j_x = 3; j_t = 2; j_z = 1$ .

According to  $(S^1)$ , the individual entering in this floor is  $z$ .

- $B_{y_4} = B_{y_0} \times (n - 1)^{4-1} = 10 \times (4 - 1)^3 = 270$
- $B_{x_3} = B_{x_0} \times (n - 1)^{3-1} = 9 \times (4 - 1)^2 = 81$
- $B_{t_2} = B_{t_0} \times (n - 1)^{2-1} = 3 \times (4 - 1)^1 = 9$
- $B_{z_1} = B_{z_0} \times (n - 1)^{1-1} = 2 \times (4 - 1)^0 = 2$

This is what happen in the matrix for  $j_y = 4; j_x = 3; j_t = 2; j_z = 1$ .

	$i_1$	$i_2$	$i_3$	$i_4$
3	$9x$	$9x$	$27y$	$27y$
2	$27y$	$27y$	$9x$	$3t$
1	$z$	$z$	$3t$	$3x$
0	$3t$	$3t$	$z$	$z$

As we are in the last floor, the  $B_{N_\omega}$  are:  $B_{x_\omega} = 81$ ,  $B_{y_\omega} = 270$ ,  $B_{z_\omega} = 2$  &  $B_{t_\omega} = 9$ .

❖ Now let's derived the  $\dot{B}_{N_\omega}$ :  $\dot{B}_{x_\omega}$ ,  $\dot{B}_{y_\omega}$ ,  $\dot{B}_{z_\omega}$  &  $\dot{B}_{t_\omega}$

They are as:  $\dot{B}_{N_\omega} = \dot{B}_N \times B_{N_\omega}$

We already have the  $\dot{B}_N$ . They are Borda points of different alternatives in  $R_2^N$ .

Previously, we have demonstrated that in  $R_2^N$ , the Borda winner is not the same as in  $R_1^N$  due to change in irrelevant alternatives. That led to  $(S^1) \neq (S^2)$ .

Now, let's consider the same changes in the irrelevant alternatives.

$$\dot{B}_x = 10, \quad \dot{B}_y = 6, \quad \dot{B}_z = 4, \quad \& \quad \dot{B}_t = 4.$$

Let's consider our weights:

	$i_1$	$i_2$	$i_3$	$i_4$
3	$B_{x_\omega}x$	$B_{x_\omega}x$	$B_{y_\omega}y$	$B_{y_\omega}y$
2	$B_{z_\omega}z$	$B_{t_\omega}t$	$B_{x_\omega}x$	$B_{x_\omega}x$
1	$B_{t_\omega}t$	$B_{z_\omega}z$	$B_{t_\omega}t$	$B_{z_\omega}z$
0	$B_{y_\omega}y$	$B_{y_\omega}y$	$B_{z_\omega}z$	$B_{t_\omega}t$

$$\begin{aligned} \rightarrow \dot{B}_{x_\omega} &= \dot{B}_x \times B_{x_\omega} = 10 \times 81 = 810 \\ \rightarrow \dot{B}_{y_\omega} &= \dot{B}_y \times B_{y_\omega} = 6 \times 270 = 1620 \\ \rightarrow \dot{B}_{z_\omega} &= \dot{B}_z \times B_{z_\omega} = 4 \times 2 = 6 \\ \rightarrow \dot{B}_{t_\omega} &= \dot{B}_t \times B_{t_\omega} = 4 \times 9 = 36 \end{aligned}$$

$$\dot{B}_{y_\omega} > \dot{B}_{x_\omega} > \dot{B}_{t_\omega} > \dot{B}_{z_\omega} \rightarrow \overrightarrow{y, x, t, z} \text{ in } R_2^N \rightarrow (S^1) = (S^2). \text{ Q.E.D.}$$

### 3 DBV AND CLONING MANIPULATION

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According to Islam, Mohajan & Moolio (2012), in Borda count a defeated candidate can manipulate the election result in his favour in sincere way by introducing a candidate which is a clone of him and voters ranked this clone candidate immediately below him. In this situation Borda rule is strictly follows but manipulation is possible. Giving to them, the possibility of manipulation of the result of an election through the misrepresentation of preferences was considered neither by Borda nor by Condorcet. What about Dynamic Borda Voting?

#### 3.1.1 The problem

Let us assume that there are 17 voters of three types and three alternatives  $x, y, z$  and the preference profile be as follows (Islam, Mohajan, & Moolio, 2012):

- Type 1:  $xPyPz$  by 8 voters,
- Type 2:  $yPzPx$  by 5 voters,
- Type 3:  $zPxPy$  by 4 voters.

Borda count in this profile be as follows:

$$\begin{aligned} \text{For } x: B_x &= 8 \times 2 + 5 \times 0 + 4 \times 1 = 20 \text{ marks,} \\ \text{For } y: B_y &= 8 \times 1 + 5 \times 2 + 4 \times 0 = 18 \text{ marks,} \\ \text{For } z: B_z &= 8 \times 0 + 5 \times 1 + 4 \times 2 = 13 \text{ marks.} \end{aligned}$$

Islam Mohajan & Moolio (2012) modify the above example by adding two alternatives  $u$  and  $v$ . Making preference profile being as follows:

- Type 1:  $xPyPzPuPv$  by 8 voters,

Type 2:  $yPzPxPuPv$  by 5 voters,

Type 3:  $zPxPyPuPv$  by 4 voters.

Now Borda counts in  $R_1^N$  would be as follows:

For $x$ : $B_x = 8 \times 4 + 5 \times 2 + 4 \times 3 = 54$ marks,
For $y$ : $B_y = 8 \times 3 + 5 \times 4 + 4 \times 2 = 52$ marks,
For $z$ : $B_z = 8 \times 2 + 5 \times 3 + 4 \times 4 = 47$ marks,
For $u$ : $B_u = 8 \times 1 + 5 \times 1 + 4 \times 1 = 17$ marks,
For $v$ : $B_v = 8 \times 0 + 5 \times 0 + 4 \times 0 = 0$ mark.

Here,  $x$  wins again. Type-3 voters have realized that  $x$  would win in the election then they would have change their preference profile as: Type 3:  $zPyPuPvPx$  by 4 voters, so that the Borda counts in  $R_2^N$  would be:

For $x$ : $\dot{B}_x = 8 \times 4 + 5 \times 2 + 4 \times 0 = 42$ marks,
For $y$ : $\dot{B}_y = 8 \times 3 + 5 \times 4 + 4 \times 3 = 56$ marks,
For $z$ : $\dot{B}_z = 8 \times 2 + 5 \times 3 + 4 \times 4 = 47$ marks,
For $u$ : $\dot{B}_u = 8 \times 1 + 5 \times 1 + 4 \times 2 = 21$ marks,
For $v$ : $\dot{B}_v = 8 \times 0 + 5 \times 0 + 4 \times 1 = 4$ marks.

In this case  $y$  would have won:  $\overline{y, z, x, u, v} \neq \overline{x, y, z, u, v} \rightarrow (S^2) \neq (S^1)$ .

### 3.1.2 Solution

Let's use DBV.

❖ First, let's find the  $B_{N_\omega}$ :  $B_{x_\omega}, B_{y_\omega}, B_{z_\omega}, B_{u_\omega}$  &  $B_{v_\omega}$

We use  $(S^1)$ :  $\overline{x, y, z, u, v}$  to determine the entering order in the floors. Here,  $n=5$ .

- $B_{x_5} = B_{x_\omega} = B_{x_0} \times (n-1)^{5-1} = 54 \times (5-1)^4 = 13824$
- $B_{y_4} = B_{y_\omega} = B_{y_0} \times (n-1)^{4-1} = 52 \times (5-1)^3 = 3328$
- $B_{z_3} = B_{z_\omega} = B_{z_0} \times (n-1)^{3-1} = 47 \times (5-1)^2 = 752$
- $B_{u_2} = B_{u_\omega} = B_{u_0} \times (n-1)^{2-1} = 17 \times (5-1)^1 = 68$
- $B_{v_1} = B_{v_\omega} = B_{v_0} \times (n-1)^{1-1} = 0 \times (5-1)^0 = 0$

❖ Now let's derived the  $\dot{B}_{N_\omega}$ :  $\dot{B}_{x_\omega}, \dot{B}_{y_\omega}, \dot{B}_{z_\omega}, \dot{B}_{u_\omega}$  &  $\dot{B}_{v_\omega}$

There are as:  $\dot{B}_{N_\omega} = \dot{B}_N \times B_{N_\omega}$

We already have the  $\dot{B}_N$ . They are Borda points of different alternatives in  $R_2^N$

- $\dot{B}_{x_\omega} = \dot{B}_x \times B_{x_\omega} = 42 \times 13824 = 580608$
- $\dot{B}_{y_\omega} = \dot{B}_y \times B_{y_\omega} = 56 \times 3328 = 186368$
- $\dot{B}_{z_\omega} = \dot{B}_z \times B_{z_\omega} = 47 \times 752 = 35344$

$$\rightarrow \dot{B}_{u\omega} = \dot{B}_u \times B_{u\omega} = 21 \times 68 = 1428$$

$$\rightarrow \dot{B}_{v\omega} = \dot{B}_v \times B_{v\omega} = 4 \times 0 = 0$$

$$\dot{B}_{x\omega} > \dot{B}_{y\omega} > \dot{B}_{z\omega} > \dot{B}_{u\omega} > \dot{B}_{v\omega} \rightarrow \overline{x, y, z, u, v} \text{ in } R_2^N \rightarrow (S^1) = (S^2). \text{ Q.E.D.}$$

#### 4 DBV AND EXOGENOUS SHOCK

The gap between  $B_{N\omega}$  and  $\dot{B}_{N\omega}$  can pave the way to a new analysis of behavioural adjustment due to exogenous shocks. The aim here is to analyse the importance of the gap between these two vectors. The higher the distance, the greater the impact.

Let consider the first example. The two vectors are:

$$B_{N\omega} = (B_{x\omega} = 81, B_{y\omega} = 270, B_{z\omega} = 2, B_{t\omega} = 9) \rightarrow B_{-1} = \begin{pmatrix} 270 \\ 81 \\ 9 \\ 2 \end{pmatrix}$$

$$\dot{B}_{N\omega} = (\dot{B}_{x\omega} = 810, \dot{B}_{y\omega} = 1620, \dot{B}_{z\omega} = 6, \dot{B}_{t\omega} = 36) \rightarrow B_{-2} = \begin{pmatrix} 1620 \\ 810 \\ 36 \\ 6 \end{pmatrix}$$

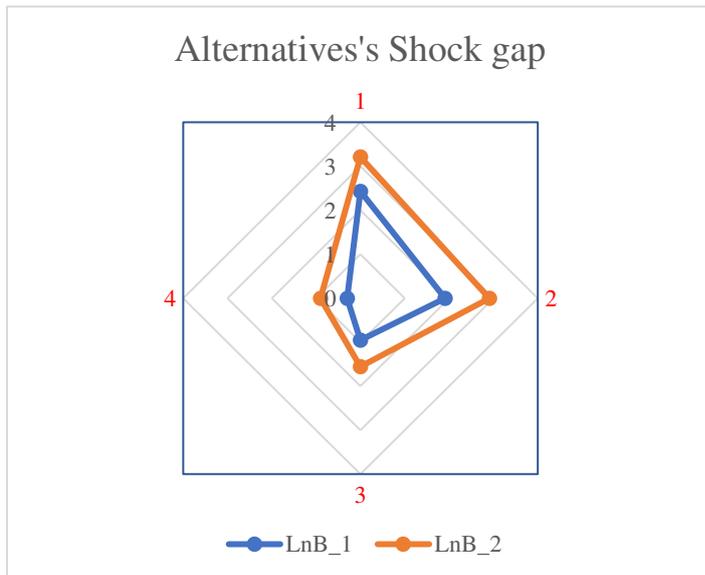


Figure 1: Alternatives's Shock gap. Source: Author

This radar graph shows the gap between  $R_1^N$  (in blue) and  $R_2^N$  (in brown). The graph shows alternatives y (which is represented by the red number 1) and x (which is represented by the red number 2) are most influenced by the shock.

To capture the behaviours of individuals, let's study elasticity.

The following regression<sup>2</sup> shows that when there is no change in the profile preference of individuals, the slope tends to 1. The effect here is therefore  $1-1 = 0\%$ .

```
. regress LnB_1 LnB_1, noconstant
```

Source	SS	df	MS	Number of obs =	4
Model	10.5550426	1	10.5550426	F( 1, 3) =	.
Residual	0	3	0	Prob > F =	.
Total	10.5550426	4	2.63876066	R-squared =	1.0000
				Adj R-squared =	1.0000
				Root MSE =	0

LnB_1	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]
LnB_1	1	.	.	.	.

Figure 2: Level of slope when there is no change in alternative preferences. Source: Author from Stata.

From the graph bellow, the effect from introducing change in alternatives, leads to a slope of 0.6869. Meaning that the impact is  $1 - 0.6869 = 0.3131$ . The impact of the change in ( $S^1$ ), due to the change in  $R_2^N$  has a magnitude of 31 on a scale of 100.

```
. regress LnB_1 LnB_2, noconstant
```

Source	SS	df	MS	Number of obs =	4
Model	10.3805151	1	10.3805151	F( 1, 3) =	178.43
Residual	.174527526	3	.058175842	Prob > F =	0.0009
Total	10.5550426	4	2.63876066	R-squared =	0.9835
				Adj R-squared =	0.9780
				Root MSE =	.2412

LnB_1	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]
LnB_2	.6869397	.0514257	13.36	0.001	.5232801 .8505994

Figure 3: Level of slope when there is a change in alternative preferences. Source: Author from Stata.

The shock had therefore a 31% impact on the aggregated behaviour of individuals.

Let us assume that we are in a stock market, the individual preferences are ordered according to their risk aversion ( $R_1^N$ ). They therefore choose the composition of their portfolio according to their risk aversion. After an exogenous short-term shock (drastic decline in oil prices, bad economic conditions, etc.) Individuals decide to change ( $R_2^N$ ) the composition of their portfolios (increase their investments in some assets and reduce in others).

The shock leading to ( $R_2^N$ ) has therefore an impact on individuals' risk aversion. The magnitude of that shock is 31 over the scale we've defined. This can be the market volatility or

<sup>2</sup> These statistical regressions are only illustrative.

the market oversight. It has in this case, increased their aversion if  $y$  and  $x$  are risky assets (according to figure 1).

## CONCLUSION

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Finally, collective rationality in the aggregation of choices is sometimes difficult to establish. Arrow (1963) said that there was no voting rule that is not subject to bias. Thus, the impossibility theorem sets 4 conditions to be satisfied for any rational voting rule. The Borda rule respected 3 of them, but stumbled on the last one: Indifference of Irrelevant Alternatives.

By redesigning the Borda rule's weight determination technique and making it dynamic, this paper allows the rule to respect the Arrow IIA. The new rule is called Dynamic Borda Voting (DBV). The DBV also makes it possible to deal with the problem of cloning manipulation.

Calculations, propositions with proof and explanations are made to show the effectiveness of this method. From DBV, the paper presents a method to measure and quantify the magnitude of the shock due to change in irrelevant alternatives over a scale moving from 0 to 100.

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