Point-in-Time PD Term Structure Models with Loan Credit Quality as a Component

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POINT-IN-TIME PD TERM STRUCTURE MODELS WITH LOAN CREDIT QUALITY AS A COMPONENT *

- Methodologies for IFRS9 ECL estimation and CCAR stress testing

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Abstract

Most point-in-time PD term structure models used in industry for stress testing and IFRS9 expected loss estimation apply only to macroeconomic scenarios. Loan level credit quality is not a factor in these models. In practice, credit profile at assessment time plays an important role in the performance of the loan during its lifetime. A forward-looking point-in-time PD term structure model with loan credit quality as a component is widely expected. In this paper, we propose a forward-looking point-in-time PD term structure model based on forward survival probability, extending the model proposed in [8] by including a loan specific credit quality score as a component. The model can be derived under the Merton model framework. Under this model, the forward survival probability for a forward term is driven by a loan credit quality score in addition to macroeconomic factors. Empirical results show, the inclusion of the loan specific credit score can significantly improve the performance of the model. The proposed approaches provide a tool for modeling point-in-time PD term structure in cases where loan credit profile is essential. The model can be implemented easily by using, for example, the SAS procedure PROC NLMIXED.

Keywords: PD term structure, loan credit quality score, macroeconomic scenario, forward survival probability, maximum likelihood

1. Introduction

For a loan with a non-default risk rating R_i at initial time t_0, the forward probability of default (PD) in the j^{th} forward term is the PD given that the loan has survived for the first $(j-1)$ terms. Given a scenario $x=(x_1, x_2, ..., x_m)$ for the j^{th} forward term, let $\tilde{p}_{ij}(x)$ denote the forward PD for the j^{th} forward term for a loan with a non-default initial rating R_i. The forward survival probability $\tilde{s}_{ij}(x)$ for the j^{th} forward term is $1-\tilde{p}_{ij}(x)$.

A forward-looking point-in-time PD term structure model based on forward probability of default $\tilde{p}_{ij}(x)$ is proposed in [8]) under the Merton model framework ([3], [4], [6]). As reviewed in section 2, the model applies only to macroeconomic scenarios. Loan credit profile or credit quality is not a factor.

In practice, loan credit profile plays an important role in the performance of a loan during its lifetime, and is essential to loan loss assessments. A forward-looking point-in-time PD term structure model with loan credit quality as a component is needed for stress testing and IFRS9 loss projections.

We assume that the loan credit profile known at initial time t_0 has been summarized as a credit quality score x_0. Let $x=(x_1, x_2, ..., x_m)$ denote a macroeconomic scenario for a forward term, and $z=(x_0, x_1, x_2, ..., x_m)$ the mixed scenario adding the loan credit quality score. Let Φ denote the standard normal CDF function.

In this paper, we introduce a general form of point-in-time PD term structure models (see model (2.4A)) based on forward survival probability with $\tilde{s}_{ij}(z)$ being given:

$$\tilde{s}_{ij}(z)=\Phi[b_{ij}+r_j x_0+r_{ij}(a_1x_1+a_2x_2+\ldots+a_mx_m)] \quad (1.1)$$

$$a_1^{\;2}+a_2^{\;2}+\ldots+a_m^{\;2}=1 \quad (1.2)$$

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where $\{b_{i\,j}\}$ are intercepts, $\{a_1, a_2, ..., a_m\}$ are coefficients for the macroeconomic variables (common to all ratings and all forward terms), and $\{r_{i\,j}\}$ are the sensitivities for the loan in responding to the changes of the credit index $ci(x)$ given by

$$ci(x) = a_1 x_1 + a_2 x_2 + ... + a_m x_m$$

Note that the credit quality score x_0 for a loan is a measure of the credit risk of the loan relative to other loans in the portfolio. Thus, the sensitivity parameters $\{r_j\}$ for a loan in responding to the changes of x_0 are required to be differentiated only between forward terms, not between the risk ratings.

Let $\tilde{s}_{i\,j}$ denote the long-run average forward survival probability for the j^{th} forward term for a loan with a non-default initial rating R_i, and let $c_{i\,j}$ be the threshold value given by $c_{i\,j} = \Phi^{-1}(\tilde{s}_{i\,j})$. Note that $\tilde{s}_{i\,j}$ can be estimated directly from the sample. Under the assumption that $x = (x_1, x_2, ..., x_m)$ is independent of x_0, and that both $ci(x)$ and x_0 (at rating level) are normally distributed, model (1.1) can be shown to be equivalent to the equation below (see model (2.6)):

$$\tilde{s}_{i\,j}(z) = \Phi[c_{i\,j}\sqrt{1 + (r_j v_i)^2 + (r_{i\,j} v)^2} + r_j(x_0 - u_i) + r_{i\,j}(ci(x) - u)] \qquad (1.3)$$

where v and u are the standard deviation and mean of $ci(x)$, while v_i and u_i are the standard deviation and mean of x_0 for loans with non-default initial rating R_i.

It can be shown that under model (1.3) (see (2.8)) the expected value of $\tilde{s}_{i\,j}(z)$ (with respect to the changes of $ci(x)$ and x_0) is the long-run average forward survival probability $\tilde{s}_{i\,j}$ $(= \Phi(c_{i\,j}))$. This implies the forward survival probability given by (1.3) is driven upside-down along its long-run average by the credit index $ci(x)$ and the loan credit score x_0.

When credit score x_0 is irrelevant, models (1.1) and (1.3) reduce respectively to (1.4) and (1.5) below:

$$\tilde{s}_{i\,j}(x) = \Phi[b_{i\,j} + r_{i\,j}(a_1 x_1 + a_2 x_2 + ... + a_m x_m)] \qquad (1.4)$$

$$\tilde{s}_{i\,j}(x) = \Phi[c_{i\,j}\sqrt{1 + (r_{i\,j} v)^2} + r_{i\,j}(ci(x) - u)] \qquad (1.5)$$

Model (1.5) is essentially the same point-in-time PD term structure model as proposed in [8]. The only difference is that model (1.5) targets the forward survival probability $\tilde{s}_{i\,j}(x)$, while in [8] the model targets the forward probability of default $\tilde{p}_{i\,j}(x)$.

We propose the point-in-time PD term structure model (1.3). The advantages of model (1.3) include:

(a) Loan level credit quality, essential for loan loss assessments, is a model component. The forward survival probability is given by the loan specific credit score in addition to the credit index $ci(x)$ composed of macroeconomic variables.
(b) Only the sensitivity parameters $\{r_{i\,j}\}$ and $\{r_j\}$ are required to be estimated, given the credit index $ci(x)$ and the long-run average forward survival probabilities.
(c) The model in general outperforms its counterpart that includes macroeconomic factors only.
(d) It can be derived under the Merton model framework (see section 2.2).

The paper is organized as follows: In section 2, we derive the forward survival probability model. In section 3, we determine the log-likelihood used for parameter estimation. A parameter estimation algorithm by maximum likelihood is proposed in section 4. In section 5, we provide an empirical example and use the proposed model to estimate the point-in-time PD term structure for a commercial portfolio.

2. The Mathematics of Forward Survival Probability Models

2.1. Forward probability of default and forward survival probability

For a loan with a non-default risk rating R_i at initial time t_0, the j^{th} forward PD is the PD for the loan in the j^{th} period $(t_{j-1}, t_j]$ given that the loan has survived the period $[t_0, t_{j-1}]$. Given a term structure sample, let $n_{ij}(t_j)$ denote the number of loans that have survived the period $[t_0, t_{j-1}]$ with a non-default initial rating R_i, and $d_{ij}(t_j)$ the number of loans that survived the period $[t_0, t_{j-1}]$ but default in $(t_{j-1}, t_j]$. Then the sample forward probability of default $\tilde{p}_{ij}(t_j)$ and the sample forward survival probability $\tilde{s}_{ij}(t_j)$ for the period $(t_{j-1}, t_j]$ are given respectively by

$$\tilde{p}_{ij}(t_j) = d_{ij}(t_j)/n_{ij}(t_j)$$
$$\tilde{s}_{ij}(t_j) = 1 - \tilde{p}_{ij}(t_j) = 1 - d_{ij}(t_j)/n_{ij}(t_j)$$

A forward-looking point-in-time PD term structure model is proposed in [8] under the Merton model framework ([3], [4], [6]). Let $x = (x_1, x_2, ..., x_m)$ denote a macroeconomic scenario with values given by a list of key macroeconomic variables. It is shown in [8] under some appropriate conditions that the forward probability of default $\tilde{p}_{ij}(x)$ is given by

$$\tilde{p}_{ij}(x) = \Phi[b_{ij}\sqrt{1 + r_{ij}^2} + r_{ij}\, ci(x)] \tag{2.1}$$

where $ci(x)$ is a credit index with zero mean and one standard deviation, derived by a normalization from a linear combination $a_1x_1 + a_2x_2 + ... + a_mx_m$, and $b_{ij} = \Phi^{-1}(\tilde{p}_{ij})$, where $\tilde{p}_{ij}$ denotes the long-run average forward PD for the j^{th} forward term for a loan with initial rating R_i. The coefficients $\{a_1, a_2, ..., a_m\}$ do not depend on the rating index i and forward term number j.

Under model (2.1), the forward PD for a loan with an initial rating R_i and a forward term j is driven by the credit index along the long-run average forward probability of default $\tilde{p}_{ij} = \Phi(b_{ij})$, while r_{ij} measures the sensitivity the forward PD in responding to the changes of credit index.

Model (2.1) proposed in [8] applies only to macroeconomic scenarios. Loan specific credit profile and quality known at initial time are not a factor. In practice, loan credit quality score plays an important role in the performance of a loan during its lifetime, and is essential to loan loss assessments.

For simplicity, we assume that the loan credit profile known at initial time t_0 has been summarized as a credit quality score x_0. For example, for a risk-rated loan portfolio, x_0 can be a credit quality score derived from factors including

1. The ratio of loan to value
2. The debt service ratio
3. The number of notches downgraded in the last two quarters

Let $z=(x_0, x_1, x_2, ..., x_m)$ be the mixed scenario for a forward term. When a specific period $(t_{j-1}, t_j]$ is concerned, we label by $z(t_j)$ the values of z for the period (value of x_0 kept the same as at time t_0).

For a loan with a non-default initial rating R_i at time t_0, let $\tilde{p}_{ij}(z)$ and $\tilde{s}_{ij}(z)=1-\tilde{p}_{ij}(z)$ denote respectively the forward PD and the forward survival probability for the period $(t_{j-1}, t_j]$ given the mixed scenario z. Let $c_i(t_j)$ and $p_{ij}(t_j)$ denote respectively the corresponding cumulative PD and marginal PD for the period $[t_0, t_j]$ given the history of $z(t)$ for $t_0 \le t \le t_j$.

We assume that the following Markov property is satisfied: the forward PD conditional on $z(t_j)$ is equal to the forward PD conditional on the entire history: $z(t), t_0 \le t \le t_j$. This requirement is not unreasonable, as lagged macroeconomic variables are included and used for the forward model based on their contributions to the model.

Proposition 2.1. The following equations hold (assuming the Markov property for (2.2C) and (2.2D)):

$$c_i(t_j) = p_{i1}(t_1) + p_{i2}(t_2) + \ldots + p_{ij}(t_j) \tag{2.2A}$$

$$\tilde{p}_{ij}(z(t_j)) = p_{ij}(t_j)/(1-c_i(t_{j-1})) \tag{2.2B}$$

$$p_{ij}(t_j) = c_i(t_j) - c_i(t_{j-1}) = [(1-c_i(t_{j-1})]\tilde{p}_{ij}(z(t_j)) \tag{2.2C}$$

$$[1-c_i(t_j)] = [1-\tilde{p}_{i1}(z(t_1))][1-\tilde{p}_{i2}(z(t_2))]\ldots[1-\tilde{p}_{ij}(z(t_j))] \tag{2.2D}$$

Proof. Equation (2.2A) is immediate. Equation (2.2B) follows from the Bayesian theorem, while equation (2.2C) follows from (2.2B). For (2.2D), we have by (2.2A) and (2.2C)

$$\begin{aligned} 1-c_i(t_j) &= 1-c_i(t_{j-1}) - p_{ij}(t_j) \\ &= 1-c_i(t_{j-1}) - (1-c_i(t_{j-1}))\tilde{p}_{ij}(z(t_j)) \\ &= [(1-c_i(t_{j-1})][1-\tilde{p}_{ij}(z(t_j))] \end{aligned}$$

Then (2.2D) follows by induction. □

2.2. Forward survival probability model

For a portfolio with k non-default risk ratings, and a loan with a non-default initial rating R_i, we focus on the default risk for the loan in the j^{th} forward term $(t_{j-1}, t_j]$ given that the loan has survived the period $[t_0, t_{j-1}]$. Assume that there exists a latent variable y_{ij} given by

$$y_{ij} = b_{ij} + r_j x_0 + r_{ij}(a_1 x_1 + a_2 x_2 + \ldots + a_m x_m) + \varepsilon_{ij} \tag{2.3}$$

such that the loan will default in the period $(t_{j-1}, t_j]$ when $y_j > 0$, where ε_{ij} is a normal random variable with zero mean and is independent of the mixed scenario $z=(x_0, x_1, x_2, ..., x_m)$. The coefficients $\{a_1, a_2, ..., a_m\}$ do not depend on rating index i and the forward term number j.

By an appropriate rescaling to both sides of (2.3), we can assume that the standard deviation of ε_{ij} is 1. Then the forward survival probability $\tilde{s}_{ij}(z)$ and forward probability of default $\tilde{p}_{ij}(z)$ for the period $(t_{j-1}, t_j]$ can be derived from (2.3) as:

$$\tilde{s}_{ij}(z) = \Phi[b_{ij} + r_j x_0 + r_{ij}(a_1 x_1 + a_2 x_2 + \ldots + a_m x_m)] \tag{2.4A}$$

$$\tilde{p}_{ij}(z) = 1 - \Phi[b_{ij} + r_j x_0 + r_{ij}(a_1 x_1 + a_2 x_2 + ... + a_m x_m)] \quad (2.4B)$$

where $\{b_{ij}\}$ are intercepts, $\{a_1, a_2, ..., a_m\}$ are coefficients for the macroeconomic variables (common to all ratings and all forward terms), and $\{r_{ij}\}$ are the sensitivity parameters for the loan in responding to the changes of the credit index $ci(x)$ defined by

$$ci(x) = a_1 x_1 + a_2 x_2 + ... + a_m x_m$$

The credit quality score x_0 for a loan is a measure of the credit risk of the loan relative to other loans in the portfolio. Therefore, the sensitivity parameters $\{r_j\}$ for a loan in responding to the changes of x_0 are required to be differentiated only between forward terms, not between the risk ratings. Normalization to the credit index $ci(x)$ is not required in (2.4A) and (2.4B).

Disturbance in parameter estimation occurs due to the multiplicative structure between the sensitivity parameters $\{r_{ij}\}$ and the macroeconomic coefficients $(a_1, a_2, ..., a_m)$ in model (2.4A): an arbitrary increase for the norm of $(a_1, a_2, ..., a_m)$ by a rescale as $(\rho a_1, \rho a_2, ..., \rho a_m)$ can be off-set in the model by a scale-down for $\{r_{ij}\}$ as $\{r_{ij}/\rho\}$. We thus impose a constraint for the macroeconomic coefficients as below:

$$a_1^2 + a_2^2 + ... + a_m^2 = 1 \quad (2.5A)$$

In practice, the sign of a coefficient a_i is usually pre-determined. For example, default risk is expected to increase as unemployment rate increases. We thus require the coefficient for unemployment rate in the model be positive. In this way, we can assume that all $\{a_i\}$ are nonnegative by an appropriate sign rescaling. Then a linear constraint as below can be imposed

$$a_1 + a_2 + ... + a_m = 1 \quad (2.5B)$$

Let $\tilde{s}_{ij}$ denote the long-run average forward survival probability for the j^{th} forward term for a loan with an non-default initial rating R_i and c_{ij} the threshold value given by $c_{ij} = \Phi^{-1}(\tilde{s}_{ij})$. Note that $\tilde{s}_{ij}$ can be estimated directly from the sample. Under the assumption that $x = (x_1, x_2, ..., x_m)$ is independent of x_0, and that both $ci(x)$ and x_0 (at rating level) are normally distributed, model (2.4A) becomes (either constraint (2.5A) or (2.5B) is on):

$$\tilde{s}_{ij}(x) = \Phi[c_{ij}\sqrt{1 + (r_j v_i)^2 + (r_{ij} v)^2} + r_j(x_0 - u_i) + r_{ij}(a_1 x_1 + a_2 x_2 + ... + a_m x_m - u)] \quad (2.6)$$

where v and u are the standard deviation and mean of $ci(x)$, while v_i and u_i are the standard deviation and mean of x_0 for loans with non-default initial rating R_i.

We propose the point-in-time PD term structure model (2.6) (i.e., model (1.3)). Model (2.6) is derived from (2.4A) based on a well-known lemma ([5]) for the following expectation with respect to s:

$$E_s[\Phi(a + bs)] = \Phi(a/\sqrt{1+b^2}),\ s \sim N(0,1) \quad (2.7)$$

Applying (2.7) to (2.6), we have

$$\begin{aligned} E[\tilde{s}_{ij}(z)] &= E\{\Phi[c_{ij}\sqrt{1 + (r_j v_i)^2 + (r_{ij} v)^2} + r_j(x_0 - u_i) + r_{ij}(a_1 x_1 + a_2 x_2 + ... + a_m x_m - u)]\} \\ &= \Phi(c_{ij}) = \tilde{s}_{ij} \end{aligned} \quad (2.8)$$

This means, the forward survival probability is as driven upside-down along its long-run average by the credit index $ci(x)$ and the loan credit score x_0.

In the rest of this section, we show that model (2.6) can also be derived under the Merton model framework. For a loan with a non-default rating R_i at initial time t_0, we are interested in the default risk for the loan in the period $(t_{j-1}, t_j]$, assuming that the loan has survived the period $[t_0, t_{j-1}]$. Under the Merton model framework ([3], [4], [6]), the default risk in $(t_{j-1}, t_j]$ is governed by a latent random variable $z_{i\,j}$, called the firm's normalized asset value, which splits into two parts as:

$$z_{i\,j} = s\sqrt{\rho_{i\,j}} + \varepsilon_{i\,j}\sqrt{1-\rho_{i\,j}}\,,\ \ 0 < \rho_{i\,j} < 1,\ \ s \sim N(0,1),\ \ \varepsilon_{i\,j} \sim N(0,1) \tag{2.9}$$

where s denotes the systematic risk (common to all non-default ratings and all terms) and $\varepsilon_{i\,j}$ is the idiosyncratic risk independent of s. The quantity $\rho_{i\,j}$ is called the asset correlation. It is assumed that there exist threshold values $\{b_{i\,j}\}$ such that the borrower will default in period $(t_{j-1}, t_j]$ if the normalized asset value $z_{i\,j}$ falls below the threshold value $b_{i\,j}$.

Assume that s and $\varepsilon_{i\,j}$ decompose further as:

$$s = \lambda_1\,(ci(x)-u)/v + e_1\sqrt{1-\lambda_1^{\,2}}\,,\quad e_1 \sim N(0,1),\ \ 0<\lambda_1<1$$
$$\varepsilon_{i\,j} = \lambda_2\,(x_0-u_i)/v_i + e_2\sqrt{1-\lambda_2^{\,2}}\,,\quad e_2 \sim N(0,1),\ \ 0<\lambda_2<1$$

Then by (2.9) we have

$$\begin{aligned} z_{i\,j} &= \lambda_1\sqrt{\rho_{i\,j}}\,(ci(x)-u)/v + \lambda_2\sqrt{(1-\rho_{i\,j})}\,(x_0-u_i)/v_i + e_1\sqrt{\rho_{i\,j}(1-\lambda_1^{\,2})} + e_2\sqrt{(1-\rho_{i\,j})(1-\lambda_2^{\,2})} \\ &= \lambda_1\sqrt{\rho_{i\,j}}\,(ci(x)-u)/v + \lambda_2\sqrt{(1-\rho_{i\,j})}\,(x_0-u_i)/v_i + e,\quad e \sim N(0,\sigma^2) \end{aligned}$$

where

$$e = e_1\sqrt{\rho_{i\,j}(1-\lambda_1^{\,2})} + e_2\sqrt{(1-\rho_{i\,j})(1-\lambda_2^{\,2})}\,\cdot$$
$$\sigma^2 = \rho_{i\,j}(1-\lambda_1^{\,2}) + (1-\rho_{i\,j})(1-\lambda_2^{\,2}) = 1-\rho_{i\,j}\lambda_1^{\,2} + \rho_{i\,j}\lambda_2^{\,2} - \lambda_2^{\,2} \tag{2.10}$$

Assume that e is independent of $(x_0, x_1, x_2, \ldots, x_m)$. Then by Merton model and using (2.7), we have

$$\begin{aligned} \tilde{p}_{i\,j}(x_0, x_1, x_2, \ldots, x_m) &= E[P(z_{i\,j} < b_{i\,j} \mid x_0, x_1, x_2, \ldots, x_m)] \\ &= E\{P[e < b_{ij} - \lambda_1\sqrt{\rho_{i\,j}}\ (ci(x)-u)/v - \lambda_2\sqrt{1-\rho_{i\,j}}\ (x_0-u_i)/v_i] \mid x_0, x_1, x_2, \ldots, x_m]\} \\ &= \Phi[b_{i\,j}/\sigma - (\lambda_1\sqrt{\rho_{i\,j}}/\sigma)(ci(x)-u)/v - (\lambda_2\sqrt{(1-\rho_{i\,j})}/\sigma)\ (x_0-u_i)/v_i] \\ &= \Phi[b_{i\,j}\sqrt{1+(r_{i\,j}v)^2+(\tilde{r}_{i\,j}v_j)^2} - r_{i\,j}(ci(x)-u) - \tilde{r}_{i\,j}\ (x_0-u_i)\,] \end{aligned} \tag{2.11}$$

where

$$r_{i\,j}v = \lambda_1\sqrt{\rho_{i\,j}}/\sigma,\ \ \tilde{r}_{i\,j}v_i = \lambda_2\sqrt{(1-\rho_{i\,j})}/\sigma$$

Here we use the relationship: $1+(r_{i\,j}v)^2+(\tilde{r}_{i\,j}v_i)^2 = 1/\sigma^2$ shown as below by using (2.10):

$$\begin{aligned} (1+(r_{i\,j}v)^2+(\tilde{r}_{i\,j}v_i)^2 &= 1+[(\lambda_1^{\,2}\rho_{i\,j} + \lambda_2^{\,2}(1-\rho_{i\,j})]/\sigma^2 \\ &= [1-\rho_{i\,j}\lambda_1^{\,2} + \rho_{i\,j}\lambda_2^{\,2} - \lambda_2^{\,2} + \lambda_1^{\,2}\rho_{i\,j} + \lambda_2^{\,2}(1-\rho_{i\,j})]/\sigma^2 = 1/\sigma^2 \end{aligned}$$

By (2.11) and using the relationship $\Phi(-b_{i\,j}) = \Phi(c_{i\,j})$, we have

$$\tilde{s}_{i\,j}(x_0, x_1, x_2, \ldots, x_m) = \Phi[c_{i\,j}\sqrt{1+(r_{i\,j}v)^2+(\tilde{r}_{i\,j}v_j)^2} + r_{i\,j}(ci(x)-u) + \tilde{r}_{i\,j}\ (x_0-u_i)]$$

By collapsing the rating index i (i.e., making no differentiation for the sensitivities between ratings) for $\tilde{r}_{ij}$ and replacing $\tilde{r}_{ij}$ by r_j, we have model (2.6).

3. Log-Likelihood Given Term Default Frequency Sample

In this section, we derive the log-likelihood and demonstrate its concavity given the observed term default frequencies by using forward survival probability. We use the following notations:

(a) $n_{ij}(t_j, x_0)$ - The number of loans that survived the period $[t_0, t_{j-1}]$ with a non-default initial rating R_i and credit quality score x_0 known at initial time t_0.

(b) $d_{ij}(t_j, x_0)$ - The number of defaults in the period $(t_{j-1}, t_j]$ for loans that have survived the period $[t_0, t_{j-1}]$ with a non-default initial rating R_i and credit quality score x_0 known at initial time t_0.

Assume that for each forward term the default count for loans with an initial rating R_i follows a binomial distribution. Then the log-likelihood for observing default frequency for the j^{th} forward term is

$$FL_j = \sum_{i,\, x_0,\, t_j} \{[n_{ij}(t_j, x_0) - d_{ij}(t_j, x_0)] \log(\tilde{s}_{ij}[z(t_j)]) + d_{ij}(t_j, x_0) \log(1 - \tilde{s}_{ij}[z(t_j)])\} \quad (3.1)$$

with t_j sliding through the sample time window. Here we have dropped out the summands corresponding to the logarithms of the binomial coefficients, which are independent of the parameters for $\tilde{s}_{ij}[z(t_j)]$ as given by (2.4A) or (2.6). Here we use the notation $z(t)$ as in section 2.1.

There are cases when we need to estimate sensitivity parameters only over a period $[t_h, t_{h+j}]$ for some $j \geq 1$. This is the case when we assume that the parameters are constant over this period due to, for example, the low default count in the sample for a single forward term. Let $L(h, h+j)$ denote the log-likelihood for a forward period $[t_h, t_{h+j}]$. The following proposition holds.

Proposition 3.1 ([8, Proposition 4.1]). The following equation holds up to a summand which is independent of the parameters for $\{\tilde{s}_{ij}[z(t_j)]\}$ given by model (2.4A) or (2.6):

$$L(h, h+N) = FL_{h+1} + FL_{h+2} + \ldots + FL_{h+N} \quad (3.2)$$

□

A function is log concave if its logarithm is concave. If a function is concave, a local maximum is a global maximum, and the function is unimodal. This property is important for the searching of the maximum likelihood estimates. The proposition below shows the concavity of the log-likelihood (3.2) as a function of $\{a_1, a_2, \ldots, a_m\}$, $\{b_{ij}\}$, and $\{r_{ij}\}$.

Proposition 3.2 ([8, Proposition 4.2]). The following statements hold:

(a) When $\tilde{s}_{ij}(z(t_j))$ is given by (2.4A), (3.2) is concave as a function of the r-parameters $\{r_j, r_{ij}\}$, or a function of the b-parameters $\{b_{ij}\}$ and the a-parameters $\{a_1, a_2, \ldots, a_m\}$

(b) When $\tilde{s}_{ij}(z(t_j))$ is given by (2.6), (3.2) is concave as a function of the r-parameters $\{r_j, r_{ij}\}$, or as a function of $\{a_1, a_2, \ldots, a_m\}$.

□

4. Parameter Estimation by Maximum Likelihood

In this section, we propose a parameter estimation algorithm by maximum likelihood for models (2.4A) and (2.6). Note that models (1.4) and (1.5) are the special cases for models (2.4A) and (2.6) where loan credit quality score is dropped.

As commonly observed in practice, loan default intensity increases for the first few terms, then decreases, and becomes flat in the long-run. To best capture portfolio default risk for the credit index, we fit $\{a_1, a_2, ..., a_m\}$ by using model (1.4), dropping the loan specific score x_0 and using only the data over the first few terms. We thus divide the fitting process into two parts:

(1) Fit the coefficients $\{a_1, a_2, ..., a_m\}$ for the credit index by model (1.4) using the data for the first term.
(2) When the credit index is determined, fit for the intercept parameters for model (2.4A), and the sensitivity parameters for models (2.4A) and (2.6).

A. Fitting for credit index

Parameter initialization: Initially, all $\{r_{11}, r_{21}, ..., r_{k1}\}$ in (1.4) are set to 1. We estimate the parameters $\{a_1, a_2, ..., a_m\}$ and $\{b_{11}, b_{21}, ..., b_{k1}\}$ by maximizing the log-likelihood FL_1 of (3.1). Recall that (3.1) is concave as a function of $\{a_1, a_2, ..., a_m\}$ and $\{b_{11}, b_{21}, ..., b_{k1}\}$ by Proposition 3.2 (a), therefore global maximum estimates are granted. We rescale the a-parameter estimates by a scalar $\rho > 0$ to make sum squared of $(a_1, a_2, ..., a_m)$ equal to 1, and then set each component of $\{r_{11}, r_{21}, ..., r_{k1}\}$ to $1/\rho$. This completes the parameter initialization.

Step 1. Given $\{a_1, a_2, ..., a_m\}$ and $\{b_{i1}\}$, we estimate the sensitivity parameters $\{r_{i1}\}$ in model (1.4) by maximizing the log-likelihood FL_1 in (3.1).

Step 2. Given sensitivity parameters $\{r_{i1}\}$, we estimate the intercept parameters $\{b_{i1}\}$ and macroeconomic coefficients $\{a_1, a_2, ..., a_m\}$ together by maximizing the log-likelihood FL_1 in (3.1) for all initial ratings. We rescale the new estimates for $\{a_1, a_2, ..., a_m\}$ by an appropriate ρ to make the sum squared of the vector equal to 1, and rescale $\{r_{i1}\}$ by the scalar $1/\rho$.

Step 3. We repeat the above two steps until a convergence is reached, i.e., the maximum deviation of estimates between two consecutive iterations is less than, for example, 10^{-4} for all parameters.

B. Fitting for other parameters

At this stage, the credit index is known. For model (2.4A), we are required only to fit for the sensitivity and the intercept parameters for each term. We perform steps 4 and 5 below for each forward term j, until a convergence is reached:

Step 4. Fit $\{r_j, r_{ij}\}$ for all risk ratings and a fixed j given $\{b_{ij}\}$ (initialized appropriately) and the credit index, by maximizing the log-likelihood FL_j in (3.1) with $\tilde{s}_{ij}(z(t_j))$ being given by (2.4A). To avoid over fitting, we impose for each forward term j a monotonicity constraint

$$r_{1j} \le r_{2j} \le r_{3j} \le ... \le r_{k-1\,j} \tag{4.1}$$

Step 5. Given $\{r_j, r_{ij}\}$ and the credit index, fit $\{b_{ij}\}$ by maximizing the log-likelihood FL_j in (3.1) with $\tilde{s}_{ij}(z(t_j))$ being given by (2.4A). Similarly, we impose for each forward term j a monotonicity constraint

$$b_{1j} \le b_{2j} \le b_{3j} \le \ldots \le b_{k-1\,j} \qquad (4.2)$$

For model (2.6), we fit for each term j the sensitivity parameters $\{r_j, r_{ij}\}$ by maximizing the log-likelihood FL_j in (3.1) with $\tilde{s}_{ij}(z(t_j))$ being given by (2.6), using the threshold values calculated from the historical long-run average forward term survival rate. Monotonicity constraint (4.1) is imposed.

The above process can be implemented by using, for example, SAS procedure PROC NLMIXED ([7]).

5. An Empirical Example: The Point-in-Time PD Term Structure for a Commercial Portfolio

The sample is created synthetically by an appropriate proportion re-sampling from a historical dataset of a commercial portfolio containing quarterly rating level default frequency (the default rate does not represent the original default rate). A loan level behaviour score summarizing the loan credit quality at the beginning of each quarter is available. There are 21 ratings with rating R_1 as the best quality rating and R_{21} the default rating. Higher index ratings carry higher default risk.

We use two macroeconomic variables and one credit quality score as described below:

(a) 3-month treasury bill interest rate (lagged by one quarter)
(b) Unemployment rate
(c) The change in the credit score (score at current quarter minus the score two quarters ago)

We fit for the following two forward survival probability models:

FSPM1-The forward survival probability model (1.5) using only the above two macroeconomic variables.
FSPM2 -The forward survival probability model (2.6) using the change of credit score in addition to the same two macroeconomic variables used by the previous model.

First, we follow the algorithm (steps 1-3) proposed in section 4 to fit for the credit index (Note that both models have the same credit index). The table below shows the estimates for two macroeconomic variable coefficients (here constraint (2.5B) is imposed).

Table 1. Credit index parameters

v1	v2	p1	p2
0.4548	0.5452	<0.0001	<0.0001

Given the credit index, we then fit for the sensitivity parameters for models (2.6) and (1.5) with monotonicity constraint (4.1) being imposed (see section 4). To reduce the number of sensitivity parameters, we fit only for the yearly sensitivity, i.e., we assume that the sensitivity parameter is constant for all quarters within each year for a total of four years. The table below shows the estimates for these sensitivities for all 20 ratings and for each of these four years. For example, the column labelled as r_1 stores four sensitivities for each of two models, while the column labelled as r_0 stores for model FSPM2 those four sensitivities with respect to the loan credit quality score x_0.

Table 2. Sensitivity parameters

Model	yr	r0	r1	r2	r3	r4	r5	r6	r7	r8	r9	r10	r11	r12	r13	r14	r15	r16	r17	r18	r19	r20
FSPM1	1		0.2610	0.2610	0.2610	0.2610	0.2610	0.2610	0.2610	0.2610	0.2610	0.2610	0.2610	0.2610	0.2610	0.2611	0.2611	0.2611	0.2611	0.3026	0.8255	0.8256
	2		0.1928	0.1928	0.1928	0.1928	0.1928	0.1928	0.1928	0.1928	0.1928	0.1929	0.1929	0.1929	0.2481	0.2481	0.4516	0.4516	0.4884	0.5134	0.5134	0.5134
	3		0.0959	0.0959	0.0959	0.0959	0.0959	0.0959	0.0959	0.0959	0.1273	0.1273	0.1273	0.1683	0.3165	0.3664	0.5928	0.6476	0.6476	0.6476	0.6476	0.7198
	4		0.2908	0.2908	0.2908	0.2908	0.2908	0.2908	0.2908	0.2908	0.2908	0.2908	0.2908	0.2908	0.2908	0.3000	0.8951	0.8952	0.8952	0.8952	0.8952	0.8952
FSPM2	1	0.1737	0.1183	0.1183	0.1183	0.1183	0.1183	0.1183	0.1183	0.1183	0.1183	0.1183	0.1183	0.1246	0.1348	0.1349	0.1445	0.1511	0.1662	0.1756	0.1991	0.1992
	2	0.1895	0.1288	0.1288	0.1288	0.1288	0.1288	0.1288	0.1288	0.1288	0.1288	0.1289	0.1289	0.1290	0.1554	0.1555	0.2166	0.2167	0.2511	0.2708	0.2708	0.2709
	3	0.0955	0.1103	0.1103	0.1103	0.1103	0.1103	0.1103	0.1103	0.1103	0.1187	0.1187	0.1200	0.1308	0.1466	0.1548	0.1743	0.1798	0.1799	0.1799	0.1799	0.1864
	4	0.1292	0.1173	0.1173	0.1173	0.1173	0.1173	0.1173	0.1173	0.1173	0.1173	0.1173	0.1173	0.1174	0.1175	0.1287	0.3813	0.3814	0.3814	0.3814	0.3814	0.3815

The table below shows the back-tested R-Squared for predicting cumulative PD at portfolio level for 1-4 years. These empirical results show, the model with a loan specific credit quality score, outperforms significantly its counterpart without the credit quality score.

Table 3. Back-test RSQ for portfolio level cumulative PD

Model	1 Quar	1 Year	2 Years	3 Years	4 Years
FSPM1	0.38	0.51	0.50	0.58	0.61
FSPM2	0.56	0.59	0.65	0.70	0.73

Conclusion. Most point-in-time PD term structure models used in industry for stress testing and IFRS9 expected loss estimation apply only to macroeconomic scenarios. Loan level credit quality is not a factor. In practice, loan credit quality plays an important role in the performance of the loan during its lifetime, and is an essential factor for loan ECL assessment. The point-in-time PD term model proposed in this paper extends the forward-looking point-in-time PD term structure model proposed in [8] by including a loan specific credit quality score known at initial time. The model can be derived under the Merton model framework. Empirical results show, adding a loan specific credit quality score improves model performance significantly.

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