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Spatial Scale of Agglomeration and Dispersion: Theoretical Foundations and Empirical Implications*

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Abstract

This paper revisits a wide variety of existing economic geography models in *a many-region setup*. It investigates the *spatial scale* of agglomeration and dispersion intrinsic to each model. In our unified analytical framework, these models reduce to *two canonical classes*: one with a *global dispersion force* and the other with a *local dispersion force*. Their formal distinction is that the former is dependent, whereas the latter is independent of the distance structure of the model. These classes exhibit two stark differences. The first difference concerns their response to transport costs: *Global* and *local* dispersion forces are triggered by *higher* and *lower* transport costs, respectively. Consequently, in a realistic model with both types of dispersion forces, a decrease in transport costs simultaneously causes both agglomeration at the global scale and dispersion at the local scale. The second difference concerns the agglomeration pattern: multiple agglomerations emerge and spread globally over the regions in the former, whereas agglomeration *always* takes the form of *a unimodal* regional distribution of mobile agents in the latter. Endogenous agglomeration mechanisms generally do not isolate the locations at which agglomerations grow or decline for a given change in transport costs. However, they offer predictions for the *global* spatial distribution of agglomerations as well as the *local* spatial extent of an individual agglomeration. This knowledge provides a consistent explanation for the set of seemingly unrelated empirical results from reduced-form regressions on regional agglomerations (e.g., Baum-Snow, 2007; Baum-Snow, Brandt, Henderson, Turner and Zhang, 2017; Duranton and Turner, 2012; Faber, 2014); it provides a new set of testable hypotheses. Moreover, our analytical framework provides formal predictions of treatment effects in the structural model-based approaches for regional agglomeration. Applications to the most standard formulations (e.g., Allen and Arkolakis, 2014; Redding and Sturm, 2008) are discussed.

Keywords: Agglomeration, Dispersion, Spatial scale, Multiple equilibria, Bifurcation

JEL Classification: R12, R13, F15, F22, C62

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1 Introduction

Empirical studies over the past few decades have let to the accumulation of ample evidence that agglomeration externalities are the major source of lumpy spatial distributions of economic activities (see, e.g., Rosenthal and Strange, 2004, for a survey). A wide variety of formal models have been proposed to investigate the underlying mechanisms (see, e.g., Duranton and Puga, 2004; Behrens and Robert-Nicoud, 2015, for surveys). For analytical tractability, most existing models rely on a location space that abstracts from the diversity of interregional distances inherent in the actual regional economies, where a typical approach assumes a location space comprising of just two regions.[1] Summarizing the spatial effects in a single interregional distance simplifies the analysis. However, this benefit comes at the cost of losing information on the *spatial scale* of agglomeration and dispersion.[2]

To see this, consider a model with any agglomeration force but without a dispersion force. In such a model, all the mobile agents will concentrate in one region. If some dispersion forces were added to the model, a fraction of mobile agents will deviate from the concentration. In a two-region economy, there is only one alternative region to head for. Hence, there is no variation in the spatial scale of dispersion. However, in a many-region economy wherein interregional distances are heterogeneous, the spatial scale of dispersion can vary depending on the nature of the dispersion force. Dispersion may occur *locally* to avoid crowding *inside* the agglomeration as in the case of an urban congestion externality, or it may occur *globally* through attraction from *outside* the agglomeration in the case of a distant, less crowded market.

This paper revisits a wide variety of existing economic geography models in *a many-region setup* with diverse interregional distances.[3] Characterizing their bifurcation behaviors behind the spontaneous formation of agglomerations *in a unified analytical framework*, we show that these models reduce to *two canonical classes*: (*i*) one with a *global dispersion force*[4] and (*ii*) the other with *local dispersion force*.[5] Formally, the two dispersion forces differ in that

[1]Another typical approach allows for the presence of many regions that are equidistant (often zero distance) from one another as in the system of cities model by Henderson (1974). See Tabuchi, Thisse and Zeng (2005) for a recent such application.

[2]Many extant empirical studies have abstracted from the space between locations and have focused on the *local interactions* between agglomeration size and location-specific factors (see, e.g., Combes and Gobillon, 2015, for a survey). The empirical studies discussed in Section 6 belong to the other strand of literature that account for global factors (such as interregional transport accessibility) on regional agglomerations.

[3]More specifically, we cover static many-region models with a single type of mobile agents (refer to footnotes 4, 5 and 6). We do not cover models with multiple types of mobile agents (e.g., urban models of Fujita and Ogawa, 1982; Ota and Fujita, 1993; Lucas and Rossi-Hansberg, 2002; Ahlfeldt, Redding, Sturm and Wolf, 2015; Owens, Rossi-Hansberg and Sarte, 2017), or dynamic models (e.g., Desmet and Rossi-Hansberg, 2009, 2014, 2015; Desmet, Nagy and Rossi-Hansberg, 2017; Nagy, 2017).

[4]For example, Krugman (1991); Puga (1999); Ottaviano, Tabuchi and Thisse (2002); Forslid and Ottaviano (2003); Pflüger (2004); Harris and Wilson (1978).

[5]For example, Beckmann (1976); Mossay and Picard (2011); Blanchet, Mossay and Santambrogio (2016);

the former is dependent, whereas the latter is independent of the distance structure of the model. The most realistic formulations incorporate both the forces.[6] **Table 1** presents the classification of the existing models.

These classes exhibit two stark differences. The first difference is with regard to the response to transport costs. *Global* dispersion (i.e., an increase in the number of agglomerations, a decrease in the spacing of agglomerations, and a decrease in the size of each individual agglomeration) is triggered by *higher* costs. In contrast, *local* dispersion (i.e., an enlargement of the spatial extent of an agglomeration) is triggered by *lower* costs. In a realistic model with both types of dispersion forces, *a decrease in transport costs simultaneously causes both agglomeration at the global scale and dispersion at the local scale*. The second difference is in terms of agglomeration patterns. In the former, multiple agglomerations emerge and spread globally over the regions; in the latter, the agglomeration *always* takes the form of *a unimodal* regional distribution of mobile agents. The typical location pattern can thus be described as *locally concentrated and globally dispersed* for the former and as *globally concentrated and locally dispersed* for the latter.

The notion of spatial scale of agglomeration and dispersion is not pervasive in the empirical literature of regional agglomeration. However, it is indispensable to understand the evolution of agglomeration patterns in reality. Consider the case of Japan from 1970 to date. The development of highways and high-speed railway networks in Japan was triggered by the Tokyo Olympics held in 1964. Between 1970 and 2010, the total highway (high-speed railway) length increased from 1,124 km (516 km) by more than 10 (4) times to 12,068 km (2,388 km). The 359 urban agglomerations that have survived throughout the forty-year period experienced a 34% increase in population size on average (controlling for the national population growth). This means that there was *a selective concentration* from all over the country, i.e., *at the global scale*.[7,8] However, this concentration at the global scale was associated with *a dispersion at the local scale*: there was a 107% increase in areal size on average with a 34% decrease in population density for individual agglomerations on average. These seemingly paradoxical evolutions of urban agglomerations in Japan turn out to be a standard outcome of a realistic model that combines both types of dispersion force (see Section 5.3).

Accordingly, our results provide novel perspectives for the three major strands of em-

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Helpman (1998); Redding and Sturm (2008); Murata and Thisse (2005); Allen and Arkolakis (2014).

[6]These models are called class (iii), where, for example, the studies by Tabuchi (1998); Pflüger and Südekum (2008); Takayama and Akamatsu (2011) are included.

[7]Each urban agglomeration is identified as the set of contiguous 1km-by-1km cells with population density at least $1000/\text{km}^2$ and total population at least 10,000. Population count data is obtained from Statistics Bureau, Ministry of Internal Affairs and Communications of Japan (1970, 2010). The transport network data can be obtained from the National Land Numerical Information Download Service of Japan at `http://nlftp.mlit.go.jp/ksj-e/gml/gml_datalist.html`. See Appendix A for deitals.

[8]The population size of each agglomeration is computed in terms of its share of national total population, and thus, the growth of the national population size is controlled.

pirical literature on regional agglomeration. One is on the measure of agglomeration (e.g., Ellison and Glaeser, 1997; Duranton and Overman, 2005; Brülhart and Traeger, 2005; Mori, Nishikimi and Smith, 2005). The other two are on the reduced-form regression approaches (see, e.g., Redding and Turner, 2015, §20.4, for a survey) and structural model-based approaches (see, e.g., Redding and Rossi-Hansberg, 2017, for a survey) to evaluate the impacts of exogenous changes, particularly, those of interregional transport access on regional agglomeration. Here, we highlight the basic issue in each context.

A scalar index has long been a natural choice for measuring agglomeration, reflecting that abstraction from interregional distances has been the rule in the formal analyses of agglomeration. On the premise of our theoretical results, when the dispersion force is effective at both global and local scales as in reality, agglomeration proceeds at the global scale when dispersion proceeds at the local scale and vice versa. Thus, the meaning of the net effect summarized by a scalar index is unclear. In Section 6.1, we argue for the necessity and utility of more disaggregated measures of agglomeration.

For the reduced-form regression exercises, consider, for example, a pair of contrasting studies on regional agglomeration (i.e., at a global scale) by Duranton and Turner (2012) and Faber (2014). The former focused on the growth of large metro areas in the US, while the latter focused on the growth of peripheral counties in China.[9] The former (latter) revealed a positive (negative) correlation between the size of agglomeration and interregional transport access in a given region. Notably, endogenous agglomeration mechanisms generally do not isolate which existing agglomerations grow or decline given an improvement in interregional transport access although the theory has a clear prediction on the overall spatial pattern of agglomerations in terms of their number and spacing. In light of class (i) models, these opposite responses may simply reflect the different sides of the same coin. That is, both the results may indicate the tendency of agglomeration at the global scale (toward larger regions) under the treatment, i.e., an improvement in interregional transport access (as in the case of Japan discussed above). Thus, one must carefully interpret the estimated treatment effect, since it is simply an average effect for the selected regions where the selections are not necessarily systematic. For the excluded but treated regions, the sign of the impacts may well be opposite. Section 6.2 provides a unified interpretation for a wider variety of empirical evidence on regional agglomeration in terms of our theoretical results. We propose a set of testable hypotheses on the spatial patterns of agglomerations and discuss the context in which these can actually be tested.

Finally, regarding structural model-based approaches for regional agglomeration, the two

[9]The amount of interregional highway linkages (e.g., the number and total length) within a given region is often interpreted as a measure of intra-urban transport infrastructure (e.g., Baum-Snow, 2007; Duranton and Turner, 2012). But, we suggest that it can also be interpreted as a measure of interregional transport infrastructure.

representative models by Redding and Sturm (2008) and Allen and Arkolakis (2014) belong to class (ii), i.e., they cannot explain endogenous formation of multiple agglomerations by construction. In other words, their basic premise is that the primary source of regional variation in agglomeration size is the heterogeneity in exogenous (or first-nature) regional advantages and that agglomeration externalities play only a secondary role. However, we demonstrate that even in this case, the comparative static outcome is still governed by agglomeration externalities and is specific to the model class. In fact, the signs of treatment effects on agglomeration typically reverse if multiple agglomerations are allowed to form endogenously, i.e., class (i) models were adopted instead.[10]

The remainder of the paper is organized as follows. Section 2 develops a general modeling framework for analyzing agglomeration patterns in a many-region economy and defines equilibria and their stability. Section 3 characterizes the nature of the dispersion force and provides a formal classification of the spatial patterns of agglomeration in terms of the spatial scale of dispersion force. Section 4 presents a mapping of existing models of economic geography to the classification. Section 5 outlines the impact of changes in transport costs on the stable equilibrium patterns of agglomeration under the representative models. Section 6 discusses the implications of our theoretical results to the empirical literature on regional agglomeration. Finally, Section 7 concludes and discusses future research agendas regarding models with richer and more realistic structures that are not treated in this paper.

2 A general modeling framework for spatial agglomerations

This section introduces a generic format of many-region spatial economic models, which we refer to as *economic geography models*, with agglomeration externalities and endogenous formation of spatial concentration. As essential preliminaries, technical aspects (stability and bifurcation of equilibria) and their economic interpretations are discussed.

Throughout our analyses, the term "region" indicates a discrete spatial unit wherein a mobile agent can locate. Whether the model is interpreted to be intraurban, interregional, or international is not essential for our results. A "region" may alternatively be termed as an urban zone, a municipality, a country, and so forth.[11]

2.1 General format of economic geography models

The economy compromises K discrete regions indexed from 0 as $i = 0, 1, \ldots, K-1$, and $\mathcal{K} \equiv \{0, 1, \ldots, K-1\}$ denotes the set of regions. There is a continuum of mobile agents of a single

[10]Refer to the discussions in Section 6.3 and the formal analysis in Appendix D.

[11]On assuming discrete space, also noted is that there are intrinsic difficulties with employing a continuous space in empirical analyses due to the discrete nature of the data as well as numerical computations.

type; an agent chooses a single region to locate within. We denote the spatial distribution of agents by $\boldsymbol{h} \equiv (h_i)_{i\in\mathcal{K}}$, where its ith element $h_i \geq 0$ is the mass of agents located in the region i. The total mass of mobile agents is exogenous constant H, i.e., $\sum_{i\in\mathcal{K}} h_i = H$. In concrete, the set of all possible spatial patterns is given by $\mathcal{D} \equiv \{\boldsymbol{h} \in \mathbb{R}^K \mid \sum_{k\in\mathcal{K}} h_k = H,\ h_k \geq 0\}$.

Given the spatial distribution $\boldsymbol{h}$ of agents, the payoff of choosing each region is determined. The payoff function is denoted by $\boldsymbol{v}(\boldsymbol{h}) \equiv (v_i(\boldsymbol{h}))_{i\in\mathcal{K}}$, where $v_i(\boldsymbol{h})$ denotes the payoff for an agent located in region $i \in \mathcal{K}$. Agents are mobile and are free to choose their locations to possibly improve their own payoffs. Thus, the equilibrium condition for the spatial distribution of agents is formulated as follows: $v^* = v_i(\boldsymbol{h})$ for all regions i such that $h_i > 0$, and $v^* \geq v_i(\boldsymbol{h})$ for any region i such that $h_i = 0$. Here, v^* is an equilibrium payoff level.

The indispensable feature of economic geography models is the presence of *space*: transportation costs are incurred by, e.g., shipment of goods between different regions or social interactions among agents in different locations. Therefore, there is a fundamental trade-off between *transportation costs* and *scale economies* associated with the spatial concentration of economic activities (Fujita and Thisse, 2013). Payoff functions of economic geography models include agglomeration and dispersion forces so that spatial equilibria are determined by a tense balance of the two opposing forces that depend on the interregional transportation costs. We assume that the spatial frictions between regions is summarized by a single *friction matrix* $\boldsymbol{D} = [d_{ij}]$, where $d_{ij} \in [0, 1)$ denotes the freeness of transport between the regions i, j.

Given the friction matrix $\boldsymbol{D}$ that encapsulates the role of the underlying geography, the microfoundations for the payoff function $\boldsymbol{v}(\boldsymbol{h})$ are typically provided by modeling the *short-run equilibrium* relating to spatial frictions between locations. Assuming that relocation of agents is sufficiently slow compared with that through market reactions, the short-run equilibrium conditions (e.g., factor and product markets clearing and trade balance) determine the payoff (utility or profit) in each region as a function of the spatial pattern of agents $\boldsymbol{h}$. We thus assume that the payoff function $\boldsymbol{v}(\boldsymbol{h})$ includes $\boldsymbol{D}$ as a parameter.

In sum, our analysis adheres to the most canonical form of spatial economic models: static models with a single type of mobile agent. In particular, the setup covers single-industry new economic geography (NEG) models because location incentives of firms and workers coincide. Thus, more involved models with multiple types of mobile agents,[12] sector-wise differentiated spatial frictions,[13] multiple types of increasing returns,[14] and dynamic models[15] are not covered. These directions are discussed in Section 7.

[12]For example, urban models of Fujita and Ogawa (1982); Ota and Fujita (1993); Lucas and Rossi-Hansberg (2002) as well as their recent applications, e.g., Ahlfeldt et al. (2015); Owens et al. (2017).

[13]For example, Fujita and Krugman (1995) and Mori (1997).

[14]For example, Fujita, Krugman and Mori (1999b); Tabuchi and Thisse (2011), and also Hsu (2012).

[15]Most notably, Desmet and Rossi-Hansberg (2009, 2014, 2015); Desmet et al. (2017); Nagy (2017).

2.2 Stability and bifurcation of equilibria

With positive externalities of spatial agglomeration, economic geography models often face a multiplicity of equilibria. A standard approach in the literature is to introduce equilibrium refinement based on *local stability* under myopic evolutionary dynamics, where the rate of change in the number of residents h_i in region i is modeled on the basis the spatial pattern of agents $\boldsymbol{h}$ and that of payoff $\boldsymbol{v}(\boldsymbol{h})$.[16] We denote the deterministic dynamic by $\dot{\boldsymbol{h}} = \boldsymbol{F}(\boldsymbol{h}, \boldsymbol{v}(\boldsymbol{h}))$, where the dot over $\boldsymbol{h}$ represents the time derivative. We assume (i) $\boldsymbol{F}$ satisfies differentiability with respect to both arguments in $\mathcal{D}$, (ii) agents relocate toward the direction of an increased aggregate payoff under $\boldsymbol{F}$, and (iii) the total mass of agents is preserved under $\boldsymbol{F}$.[17] Furthermore, we assume that any spatial equilibrium is a rest point of the dynamic.[18] Given an adjustment dynamic $\boldsymbol{F}$, the stability of an equilibrium is defined in terms of asymptotic stability under $\boldsymbol{F}$.

Stability of a given spatial equilibrium is parameter dependent. As emphasized by the NEG literature, changes in transportation technologies can trigger endogenous emergence of regional inequality. The basic core–periphery story after Krugman (1991) is that "Consider an economy with two regions that are ex-ante symmetric, where the regions have exactly same characteristics and mobile agents are uniformly distributed. When interregional transportation costs are high, the uniform distribution of mobile agents is a stable equilibrium. If the transportation cost falls below a certain threshold value, the pattern is no longer stable; the agglomeration toward one of the regions occurs, and the core–periphery pattern emerges by self-organization."

Although the intuitive story of the two-region economy backed by the rich interactions of economic forces has its own right, corresponding many-region studies are scarce in the literature. In particular, *what* spatial patterns emerge after an encountered destabilization in a many-region economy is far from obvious. One needs better methods to examine the stability of equilibrium patterns in a many-region economy.

Such an abrupt change in spatial patterns due to destabilization is an instance of *bifurcation*. Thus, bifurcation theory in general provides the canonical tools to tackle our problem. This paper builds on the following formal facts on stability and bifurcation of equilibria to examine the formation of spatial patterns in a many-region economy:[19]

[16] Another approach is *global* stability analysis based on perfect foresight dynamics (Oyama, 2009a,b).

[17] For (i), we assume differentiability of $\boldsymbol{F}(\boldsymbol{h}, \boldsymbol{v}(\boldsymbol{h}))$ as a whole on the tangent space of $\mathcal{D}$. The second, (ii), is called *positive correlation* (Sandholm, 2010) which is defined by $\sum_{i \in \mathcal{K}} v_i(\boldsymbol{h}) \cdot \dot{h}_i > 0$ for all $\boldsymbol{h} \in \mathcal{D}$. The last, (iii), requires that $\boldsymbol{F}(\boldsymbol{h}, \boldsymbol{v}(\boldsymbol{h}))$ live in the tangent cone of $\boldsymbol{D}$ for all $\boldsymbol{h} \in \mathcal{D}$. Furthermore, even though this paper focuses on homogeneous payoffs, one can analyze stability of spatial equilibria with idiosyncratic taste heterogeneity (e.g., Murata, 2003; Redding, 2016; Behrens, Mion, Murata and Südekum, 2017; Monte, Redding and Rossi-Hansberg, 2016) by *perturbed best response dynamics*.

[18] That is, if $\boldsymbol{h}^*$ is a spatial equilibrium, we have $\dot{\boldsymbol{h}} = \boldsymbol{F}(\boldsymbol{h}^*, \boldsymbol{v}(\boldsymbol{h}^*)) = \boldsymbol{0}$.

[19] For the rest of the paper, we had to sacrifice mathematical accuracy to reduce unnecessary burden for general readers. For rigorous and general textbook treatment of stability analysis of dynamical systems and

Fact 1. Consider a spatial equilibrium $\boldsymbol{h}^*$. Let $\boldsymbol{J} \equiv [\partial F_i(\boldsymbol{h}^*, \boldsymbol{v}(\boldsymbol{h}^*))/\partial h_j]$ be the Jacobian matrix of the dynamic $\boldsymbol{F}$ evaluated at $\boldsymbol{h}^*$. Let the eigenvalues of $\boldsymbol{J}$ be $\boldsymbol{g} = (g_k)_{k\in\mathcal{K}}$.[20] Then, $\boldsymbol{h}^*$ is stable if all the K eigenvalues have strictly negative real parts; it is unstable if any of the eigenvalues has a strictly positive real part.

Fact 2. Let $\boldsymbol{h}^*$ be a stable spatial equilibrium, i.e., an equilibrium at which every eigenvalue of $\boldsymbol{J}(\boldsymbol{h}^*)$ has strictly negative real parts. Suppose that any of the eigenvalues, say g_k, switches its sign because of a change in the value of an underlying model parameter. Then, a bifurcation occurs: $\boldsymbol{h}^*$ becomes unstable, and the spatial pattern moves toward the direction of $\boldsymbol{\eta}_k = (\eta_{k,i})_{i\in\mathcal{K}}$, which is the eigenvector associated to g_k; given a real number ϵ, a pattern that can be expressed as $\boldsymbol{h}^* + \epsilon\boldsymbol{\eta}_k$ emerges.

Note that when we employ Fact 2, we can focus on $\boldsymbol{\eta}_k$ with $\sum_{i\in\mathcal{K}} \eta_{k,i} = 0$ because we assume that the total number of mobile agents is preserved under $\boldsymbol{F}$.[21]

The two-region story is related to Facts 1 and 2 in the following way. Consider a two-region economy that comprises two regions 0 and 1 with completely homogeneous characteristics. The uniform pattern $\bar{\boldsymbol{h}} \equiv (h, h)$ is obviously a spatial equilibrium. The (two) eigenvectors of $\boldsymbol{J}$ are given by $\boldsymbol{\eta}_0 = (1, 1)$ and $\boldsymbol{\eta}_1 = (1, -1)$ with the associated eigenvalues g_0 and g_1, respectively. The former, $\boldsymbol{\eta}_0$, induces change in the total mass of mobile agents and is irrelevant in a closed economy. The latter, $\boldsymbol{\eta}_1$, expresses agglomeration of mobile agents toward one of the regions, say 0. The associated eigenvalue, g_1, then coincides with the differential of the payoff difference between the two regions $\Delta v(\boldsymbol{h}) \equiv v_0(\boldsymbol{h}) - v_1(\boldsymbol{h})$ up to a positive constant. If $g_1 < 0$, then a marginal increase in the population share of region 0 induces a relative *decrease* in payoff in region 0. Hence, no mobile agent hopes to leave region 1. If a decrease in transportation costs changes the sign of g_1 from negative to positive, then relocation becomes strictly beneficial for agents in region 1, i.e., $\bar{\boldsymbol{h}}$ become unstable, and agglomeration emerges.

2.3 Interpreting eigenvalues: Net agglomeration forces

By virtue of Facts 1 and 2, analyzing the eigenpairs (i.e., eigenvalues $\boldsymbol{g}$ and eigenvectors $\{\boldsymbol{\eta}_k\}$) of $\boldsymbol{J}(\boldsymbol{h}^*)$, one can examine *when* destabilization of a given equilibrium $\boldsymbol{h}^*$ occurs and *what* spatial pattern(s) emerge thereafter. Though seemingly mechanical, as one would expect from the above example of the two-region setup, $\boldsymbol{g}$ and $\{\boldsymbol{\eta}_k\}$ have rich economic meanings.

bifurcation theory, see, for example, Guckenheimer and Holmes (1983) and Kuznetsov (2004). An earlier attempt to apply bifurcation theory to spatial structural evolution can be found in Wilson (1981).

[20] Allowing a notational abuse, $\mathcal{K}$ denotes the K-dimensional index sets for the regions and the eigenvalues and eigenvectors of $\boldsymbol{J}$.

[21] To be precise, to examine the stability of a given (interior) equilibrium $\boldsymbol{h}^*$ it suffices to analyze the eigenvalues of the restricted linear map $\boldsymbol{J}(\boldsymbol{h}^*) : T\mathcal{D} \to T\mathcal{D}$ where $T\mathcal{D} \equiv \{\boldsymbol{\eta} \in \mathbb{R}^K \mid \boldsymbol{\eta} \cdot \mathbf{1} = 0\}$ is the tangent space of $\mathcal{D}$ (see Appendix B.4).

The sign of an eigenvalue g_k dictates whether $\boldsymbol{h}^*$ is stable in the direction of the associated eigenvector $\boldsymbol{\eta}_k$. We provide some intuitions. Given an interior equilibrium $\boldsymbol{h}^*$, consider a small variation in spatial pattern such that $\boldsymbol{h} = \boldsymbol{h}^* + \boldsymbol{\eta}_k$, where $\boldsymbol{\eta}_k \equiv (\eta_{k,i})_{i\in\mathcal{K}}$ is one of the eigenvectors of $\boldsymbol{J}(\boldsymbol{h}^*)$, whose associated eigenvalue is g_k. Then, under our assumptions on $\boldsymbol{F}$, one can show that[22]

$$\operatorname{sgn}[g_k] = \operatorname{sgn}[\delta V(\boldsymbol{\eta}_k)], \tag{2.1}$$

where $\delta V(\boldsymbol{\eta}_k)$ and $\delta V_i(\boldsymbol{\eta}_k)$ are respectively defined by

$$\delta V(\boldsymbol{\eta}_k) \equiv \sum_{i\in\mathcal{K}} \delta V_i(\boldsymbol{\eta}_k)\eta_{k,i} \quad \text{and} \quad \delta V_i(\boldsymbol{\eta}_k) \equiv \sum_{j\in\mathcal{K}} \frac{\partial v_i(\boldsymbol{h}^*)}{\partial h_j}\eta_{k,j}. \tag{2.2}$$

Note that $\eta_{k,i} = h_i - h_i^*$ is either positive or negative. Observe that $\delta V_i(\boldsymbol{\eta}_k)$ is the marginal increase in payoff in region i when the spatial pattern changed to $\boldsymbol{h} = \boldsymbol{h}^* + \boldsymbol{\eta}_k$. Accordingly, $\delta V(\boldsymbol{\eta}_k)$ is the weighted sum of the marginal increase in payoffs all over the regions.

If g_k is strictly negative (positive), $\delta V(\boldsymbol{\eta}_k)$ is strictly negative (positive). This implies that, if $g_k < 0$, the collateral deviation toward $\boldsymbol{\eta}_k$ direction is strictly undesirable for relocated agents. To see this, rewrite $\delta V(\boldsymbol{\eta}_k)$ as follows:

$$\delta V(\boldsymbol{\eta}_k) = \sum_{\eta_{k,i}>0} \delta V_i(\boldsymbol{\eta}_k)\left|\eta_{k,i}\right| - \sum_{\eta_{k,i}<0} \delta V_i(\boldsymbol{\eta}_k)\left|\eta_{k,i}\right|. \tag{2.3}$$

The first (second) term in the right hand side is the average payoff increase in the destination (origin) regions of migration; thus, the weighted sum $\delta V(\boldsymbol{\eta}_k)$ is the net increase of payoff experienced by the relocated agents. If all $\{g_k\}$ are strictly negative, for any direction there is no incentive to relocate and thus the equilibrium is stable. It is also intuitive to consider a single hypothetical agent who may want to relocate from region i to j; her payoff gain is given by $\delta V = \delta V_j - \delta V_i$. If all $\{g_k\}$ are strictly negative, it follows that $\delta V < 0$ and there is no incentive for such relocation. Conversely, if any of $\{g_k\}$ is positive, a collateral deviation toward the $\boldsymbol{\eta}_k$ direction is beneficial for any relocated agents and a snowball effect will kick the spatial pattern out of the equilibrium; that is, the equilibrium is unstable.

In the context of economic geography models, one can interpret each eigenvalue g_k as the *net force* in its associated direction of deviation $\boldsymbol{\eta}_k$ in the sense that g_k reflects the net effect of agglomeration and dispersion forces at work in the $\boldsymbol{\eta}_k$ direction. Depending on its sign, g_k expresses net agglomeration force (if positive) or net dispersion force (if negative). In particular, if only one of them happens to be positive, then the spatial pattern is unstable

[22]See Appendix B.4. The discussion here assumes that g_k and $\boldsymbol{\eta}_k$ are both real, as this property holds true throughout our analyses below.

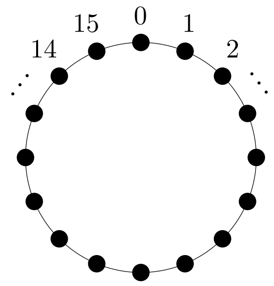


Figure 1: A racetrack economy ($K = 16$)

Note: The thin lines represent the transportation network and the black points represent the discrete regions where mobile agents can locate. The regions are sequentially numbered.

and agglomeration occurs in the direction of the associated eigenvector.

3 Spatial scale of endogenous agglomeration and dispersion

Although the general facts on local stability and bifurcation of equilibria are in principle applicable for any situation, in general geographical setups (i.e., assumed structures of $\boldsymbol{D}$), analytical results are difficult to obtain; thus, formal implications are limited. This section introduces a minimal and ideal geographical setup, namely, a *racetrack economy* that considerably simplifies local stability analysis of spatial equilibria in general economic geography models. Despite the technical simplification, the setup preserves heterogeneities in interregional distances—an indispensable feature to express *spatial scale of agglomeration and dispersion patterns*. Employing desirable properties of the geographical setup, we reveal the two distinct spatial scales of dispersion force that determine the spatial pattern of agglomerations. Concrete examples are discussed in Section 4.

3.1 Racetrack economy: Desired testbed

We assume a *racetrack economy* à la Krugman (1993) (**Figure 1**).[23] The K regions are equidistantly spread on a circle and are sequentially numbered from zero, with transportation possible only around the circumference. The circumferential length is normalized to unity. Furthermore, we assume that there are no region-fixed advantages in terms of, for instance, local amenities or productivity differences. The geographical setup provides an ideal "testbed" to analyze intrinsic properties of a many-region economic geography model for two reasons.

[23]Our approach to local stability analysis that utilizes a racetrack economy was developed by Akamatsu, Takayama and Ikeda (2012), and an application can be found in Osawa, Akamatsu and Takayama (2017); see Appendix B for a summary. As the approach focuses on *local bifurcations* from a given equilibrium, group-theoretic bifurcation theory combined with numerical analysis provide complementary insights into the *global bifurcation* behavior of equilibria. See Ikeda, Akamatsu and Kono (2012); Ikeda, Murota, Akamatsu, Kono and Takayama (2014); Ikeda, Murota and Takayama (2017a), as well as Ikeda and Murota (2014).

First, it allows us to focus on spatial patterns due to agglomeration externalities in a pure form. In particular, it abstracts from location-fixed advantages induced by the *shape* of the underlying transportation network. For instance, in a long narrow economy (e.g., Solow and Vickrey, 1971; Beckmann, 1976), the regions near the boundaries have fewer opportunities to access the other regions; the central portion is advantageous due to the shape of space. In our setup, in contrast, every region has the same level of accessibility to the other regions.[24]

Second, despite its simplicity, the setup incorporates heterogeneities in interregional distances. Let ℓ_{ij} denote the shortest path length from region i to j on the circumference; then, we have for example $\ell_{0,1} = \ell_{1,0} = 1/K$ and $\ell_{K-1,1} = \ell_{1,K-1} = 2/K$.[25] The heterogeneity in interregional distances makes *relative* location in space matter, which is not the case for the common two-region setup. Furthermore, the symmetric racetrack economy reduces to the two-region setup if $K = 2$; the former is thus a natural generalization of the latter.

In addition, in line with Krugman (1993), we assume that the spatial friction between each pair of two regions takes Samuelson's iceberg form, a standard choice for general equilibrium models.[26] In concrete terms, d_{ij} is given by $d_{ij} = \exp[-\tau\ell_{ij}]$ with a transport technology parameter $\tau \in (0, \infty)$. $\boldsymbol{D}$ is thus symmetric because $\ell_{ij} = \ell_{ji}$. Moreover, each d_{ij} is decreasing in τ. When we consider a steady improvement in transportation technology—that is, continued decrease of τ—the spatial frictions between the regions gradually vanish ($d_{ij} \to 1$ for all i and j as $\tau \to 0$).

3.2 Local and global forces and the basic roles of space

The first virtue of assuming a racetrack structure is that the uniform distribution is always an equilibrium when the payoff function is symmetric across the regions. For this reason, one can follow the extant theories where the spatial distribution of mobile agents is assumed to be initially uniform and study endogenous formation of spatial patterns due to pure economic forces. We denote the *flat-earth equilibrium* on the racetrack by $\bar{\boldsymbol{h}} \equiv (h, h, \ldots, h)$ with $h \equiv H/K$. Furthermore, it is typical that at the flat-earth equilibrium $\boldsymbol{J}$ and $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ are closely related. If we let $e_k(\tau)$ be the eigenvalues of $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$, we often have $g_k(\tau) = c e_k(\tau)$ with a positive constant c.[27] Thus, not only the sign but also the magnitude of $g_k(\tau)$ matters—in fact, the relative magnitude of $g_k(\tau)$ represents that of agglomeration and dispersion force

[24]In this sense, our setup has an intrinsic complementarity with the many-region analyses by Matsuyama (1999), who abstracted from endogenous positive feedbacks and focused on the role of *geography* itself.

[25]In concrete terms, $\ell_{ij} = \min\left\{|i-j|, K - |i-j|\right\}$.

[26]Some models, e.g., those by Ottaviano et al. (2002), Tabuchi et al. (2005), and Picard and Tabuchi (2013), have assumed noniceberg transport technology. In principle, our analytical approach is effective with respect to these models, albeit the analysis is far more tedious compared to the iceberg case; the models can be fit to either of classes (i) or (ii) [or (iii)] discussed in Section 1 and to be introduced below. We shall refrain from analyzing non-iceberg models to simplify our presentation.

[27]For instance, the *replicator dynamic* (Taylor and Jonker, 1978) satisfy $c = h$ (see Appendix B.4).

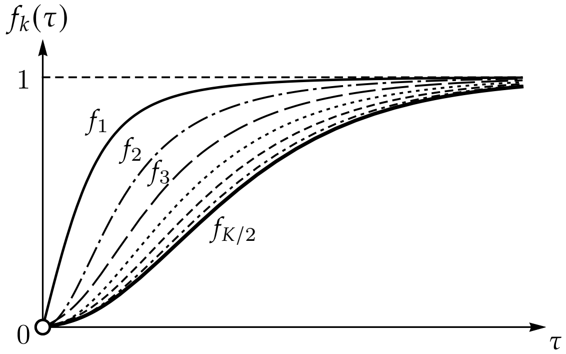


Figure 2: Eigenvalues of the friction matrix $\boldsymbol{D}$ for a racetrack economy with $K = 16$

Note: Every $f_k(\tau)$ for $1 \le k \le K$ is an increasing function of τ. Those for $1 \le k \le K/2$ are shown in the figure, because we have $f_k(\tau) = f_{K-k}(\tau)$ for $K/2 + 1 \le k \le K - 1$. Also, for each given level of τ, $f_k(\tau)$ is basically decreasing in k for $1 \le k \le K/2$ (see Appendix B).

toward the $\boldsymbol{\eta}_k$ direction.

The second and the most important utility of imposing a racetrack structure is that the role of transport cost in the net agglomeration force becomes transparent. To see this, the notion of *spatial scales* of agglomeration and dispersion forces is useful. Throughout this paper, we call an agglomeration or dispersion force *global* if it depends on the distance between regions (i.e., the friction structure $\boldsymbol{D}$), while that do not depend on the distance between regions are termed *local* agglomeration or dispersion force.

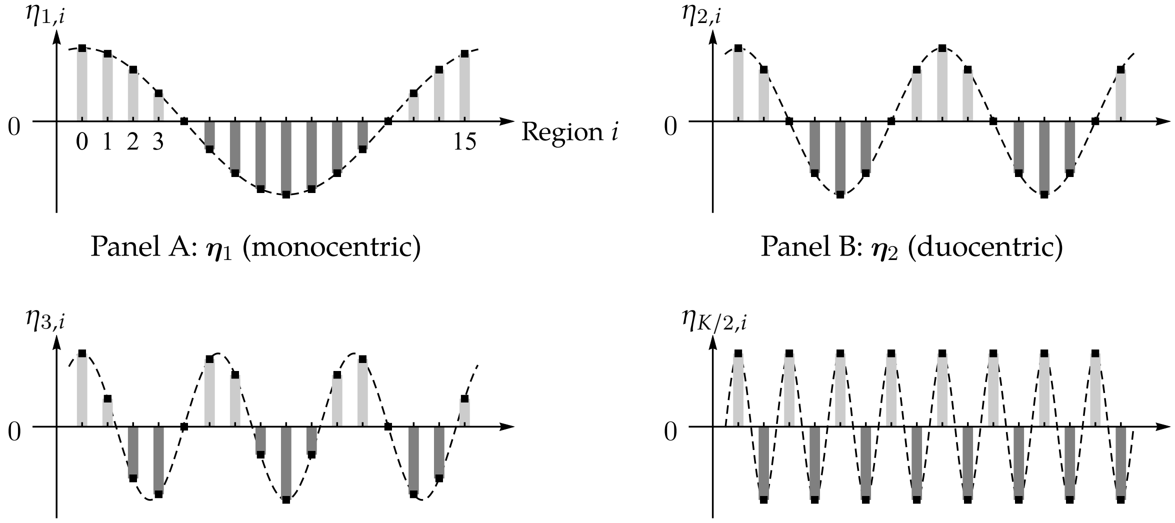
Below, we consider a toy model that reveals the intrinsic workings of a global force. Consider the following simplest reduced-form payoff specification that implements only a black-box positive externality of agglomeration but no dispersion force

$$\boldsymbol{v}(\boldsymbol{h}) = \boldsymbol{D}\boldsymbol{h}, \tag{3.1}$$

or, in the element-wise form, $v_i(\boldsymbol{h}) = \sum_{j\in\mathcal{K}} d_{ij}h_j$. This simple model is a canonical example of models with *global agglomeration force*—an agglomeration force that depends on interregional distances. It is evident that we have $\nabla\boldsymbol{v}(\boldsymbol{h}) = \boldsymbol{D}$ at the flat-earth equilibrium.

The net agglomeration forces $\{g_k(\tau)\}$, or the eigenvalues of $\boldsymbol{J}$, are thus given by $g_k(\tau) = hd(\tau)f_k(\tau)$ with $\{f_k(\tau)\}$ being the eigenvalues of the row-normalized version of the friction matrix $\bar{\boldsymbol{D}} \equiv \boldsymbol{D}/d(\tau)$, where $d(\tau) = \sum_{j\in\mathcal{K}} d_{ij}(\tau) > 0$ is the row-sum of $\boldsymbol{D}$.[28] In a racetrack economy, we have analytical expressions of the eigenvalues $\{f_k(\tau)\}$ as well as those of their associated eigenvectors $\{\boldsymbol{\eta}_k\}$ (see Appendix B). Consequently, the eigenvectors of $\boldsymbol{J}$ are also given by $\{\boldsymbol{\eta}_k\}$. The eigenvector associated with $g_k(\tau)$ is $\boldsymbol{\eta}_k = (\eta_{k,i}) = (\cos[\theta k i])$ with $\theta \equiv 2\pi/K$; we ignore g_0 in the following because $\boldsymbol{\eta}_0 = (1, 1, \ldots, 1)$ violates the conservation of the total mass of agents.

[28]We assume the replicator dynamic as the underlying dynamic $\boldsymbol{F}$ for illustration (see Example B.3 in Appendix B.4). Note also that the row-sum of $\boldsymbol{D}$ is row-independent in a racetrack economy.



Panel A: $\boldsymbol{\eta}_1$ (monocentric)

Panel B: $\boldsymbol{\eta}_2$ (duocentric)

Panel C: $\boldsymbol{\eta}_3$ (tricentric)

Panel D: $\boldsymbol{\eta}_{K/2}$ (alternative agglomeration)

Figure 3: Illustrations of the eigenvectors $\boldsymbol{\eta}_k$ ($K = 16$; $k = 1, 2, 3, K/2$)

Note: The negative (positive) elements of an eigenvector $\boldsymbol{\eta}_k$ indicate that if the flat-earth pattern is perturbed into the direction, so that the new spatial pattern is $\boldsymbol{h} = \bar{\boldsymbol{h}} + \epsilon\boldsymbol{\eta}_k$ with $\epsilon > 0$, such regions experience decrease (increase) in their population.

Figure 2 illustrates $\{f_k(\tau)\}_{k\geq 1}$ for $K = 16$. Each $f_k(\tau)$ ranges from 0 to 1 and decreases if τ decreases. When interregional transportation costs decline, the effects of the friction matrix vanish. Thus, we see that $g_k(\tau) > 0$ for all $k \geq 1$, and hence, $\bar{\boldsymbol{h}}$ is never stable (Fact 1). Because there is no dispersion force that can stabilize the flat-earth equilibrium, it is also natural that $\bar{\boldsymbol{h}}$ is unstable for any value of τ.

The relative magnitude of the net agglomeration forces $\{g_k(\tau)\}$ is of interest. To this end, for the toy model, one can see that $f_k(\tau)$ determines the relative strength between $\{g_k(\tau)\}$. Note that $f_k(\tau)$ is decreasing in k (see **Figure 2**), with the maximal $f_1(\tau)$ for all τ. Thus, the maximal among $g_k(\tau)$ is also $g_1(\tau)$. But why does this occur?

Looking at the eigenvectors $\{\boldsymbol{\eta}_k\}$ provides intuitions. Some examples of $\boldsymbol{\eta}_k$ with $K = 16$ are illustrated in **Figure 3** for $k = 1, 2, 3, K/2$.[29] A negative (positive) element $\eta_{k,i}$ in $\boldsymbol{\eta}_k$ indicates that if the spatial pattern slightly changed toward the $\boldsymbol{\eta}_k$ direction so that $\boldsymbol{h} = \bar{\boldsymbol{h}} + \epsilon\boldsymbol{\eta}_k$ with $\epsilon > 0$, the number of mobile agents decreases (increases) in the region. In a symmetric racetrack economy, the possible directions of change are characterized by the number of *peaks*, k, or, in other words, by the number of population concentrations (i.e., *agglomerations*). $\boldsymbol{\eta}_1$ (Panel A of **Figure 3**) is directed to a monopolar pattern with a single peak and hence expresses the emergence of a global concentration of mobile agents; $\boldsymbol{\eta}_2$ (Panel B) expresses the emergence of two major concentrations, while $\boldsymbol{\eta}_3$ (Panel C) expresses the emergence of three major concentrations; $\boldsymbol{\eta}_{K/2}$ (Panel D) expresses the emergence of the smallest possible

[29]To simplify the presentation, we assume that the number of regions K is a multiple of four. Qualitatively, the exact number of regions is inconsequential if it is sufficiently large.

agglomerations. In other words, $\boldsymbol{\eta}_1$ immediately pushes the flat-earth equilibrium to a unimodal agglomeration, while $\boldsymbol{\eta}_2$, $\boldsymbol{\eta}_3$, and $\boldsymbol{\eta}_{K/2}$ (as well as other $\boldsymbol{\eta}_k$ except for $\boldsymbol{\eta}_1$) pushes the flat-earth equilibrium to other multimodal patterns. As we assume a featureless space, the peaks are equidistantly spaced.

Given the knowledge of $\{\boldsymbol{\eta}_k\}$, the maximality of $g_1(\tau)$ now has clear economic meaning. We understand that the associated eigenvector $\boldsymbol{\eta}_1$ for $g_1(\tau)$ is a unimodal, monocentric agglomeration (Panel B of **Figure 3**). Since there are no negative effects of agglomeration in the model, a monocentric concentration is the most beneficial outcome for every agent. As the number of peaks in $\boldsymbol{\eta}_k$ increases, the size of a single agglomeration becomes smaller. It obviously reduces the magnitude of positive externalities and is less favorable. Also, $f_k(\tau)$ decreases as τ decreases because when the level of interregional transport costs is low there is less incentive toward agglomeration.

3.3 Endogenous formation of agglomeration out of uniformity

For canonical economic geography models in the literature, at the flat-earth equilibrium, $\boldsymbol{J}$ is related to the row-normalized version of the friction matrix, $\bar{\boldsymbol{D}}(\tau) = \boldsymbol{D}(\tau)/d(\tau)$, in the following form (see Appendices B and C):[30]

$$\boldsymbol{J} \simeq c_0\boldsymbol{I} + c_1\bar{\boldsymbol{D}}(\tau) + c_2\{\bar{\boldsymbol{D}}(\tau)\}^2, \tag{3.2}$$

where c_0, c_1, and c_2 are model-dependent (positive or negative) coefficients. Given the relation, the eigenvalues $\boldsymbol{g} = (g_k)_{k\in\mathcal{K}\setminus\{0\}}$ of $\boldsymbol{J}$ satisfy (see Appendix B)

$$\operatorname{sgn}[g_k(\tau)] = \operatorname{sgn}\left[G\big(f_k(\tau)\big)\right], \tag{3.3}$$

$$G(f) = c_0 + c_1 f + c_2 f^2, \tag{3.4}$$

where $(f_k(\tau))_{k\in\mathcal{K}\setminus\{0\}}$ are the eigenvalues of $\bar{\boldsymbol{D}}(\tau)$ (**Figure 2**). The eigenvector associated with each $g_k(\tau)$ is again $\boldsymbol{\eta}_k = (\eta_{k,i}) = (\cos[\theta k i])$ with $\theta \equiv 2\pi/K$ (**Figure 3**). Recall that one can ignore g_0 provided that the underlying dynamic $\boldsymbol{F}$ preserves the total mass of mobile agents.

Employing our definition of local and global forces, we see that c_0 summarizes the *local* agglomeration and dispersion forces in the model and c_1 and c_2 summarize the *global* ones. Usually, we have $c_0 < 0$, $c_1 > 0$, and $c_2 < 0$. For example, a crowding-out effect inside a region due to congestion or point-wise scarcity of land produces a *local* dispersion force, resulting in a negative constant term ($c_0 < 0$); a global social interaction (e.g., Beckmann, 1976) is

[30]The notation $\simeq$ for matrices means that the LHS coincides with the RHS multiplied by some real, symmetric, and circulant matrix $\boldsymbol{J}_0$ which is positive definite relative to $T\mathcal{D}$. For our purpose in this paper (i.e., local stability analysis of $\bar{\boldsymbol{h}}$), we can practically "ignore" $\boldsymbol{J}_0$ in our discussion. Also noted is that the convention is just for simplicity of presentation.

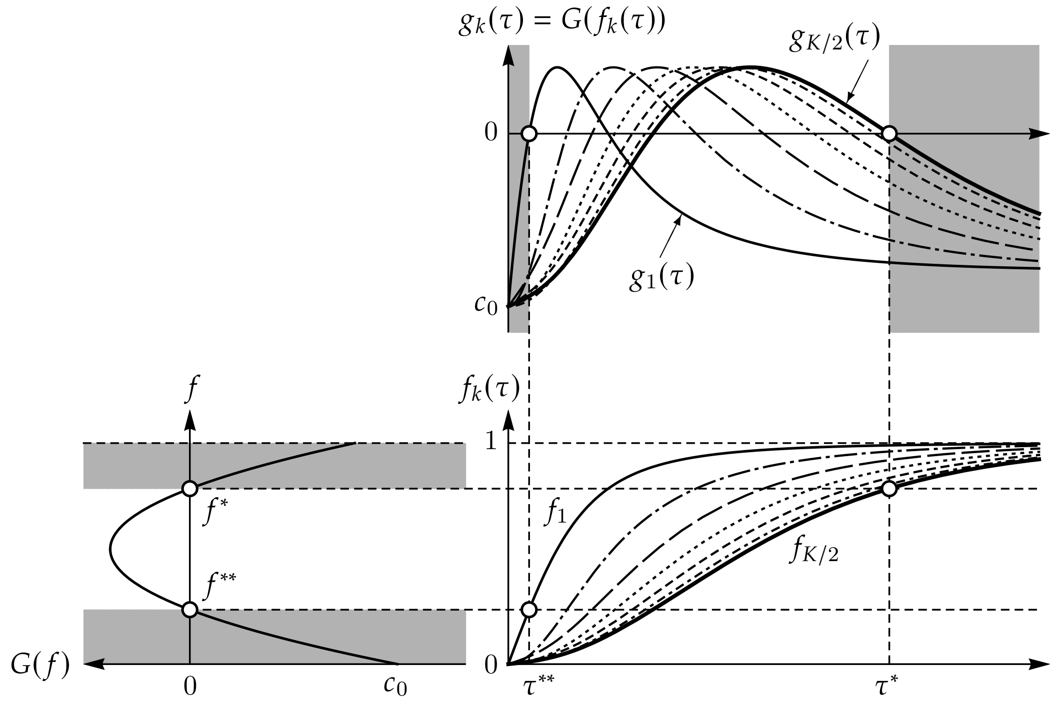


Figure 4: Net agglomeration forces and their model-dependent and -independent components

Note: Top: The net agglomeration forces $\{g_k(\tau)\}$. We consider the simplest case: $g_k(\tau) = G(f_k(\tau))$. Bottom left: An example of the model-dependent function $G(f)$. Bottom right: The eigenvalues $\{f_k(\tau)\}$ of $\bar{\boldsymbol{D}}$, which are model-independent. $\bar{\boldsymbol{h}}$ is stable in the dark gray regions of τ or f.

suggested by a positive first-order term ($c_1 > 0$); and goods demand from spatially dispersed consumers in other regions (e.g., Krugman, 1991) is indicated by a negative second-order term ($c_2 < 0$).[31]

In the following, we assume the most general case of $G(f)$ in the literature: $G(f)$ is given by $G(f) = c_0 + c_1 f + c_2 f^2$ with $c_0 < 0$, $c_1 > 0$, and $c_2 < 0$, with two roots f^* and f^{**} for $G(f) = 0$ in $(0, 1)$ such that $f^{**} < f^*$. The shape of $G(f)$ under the assumptions is shown in the bottom left panel of **Figure 4**. The functional form of $G(f)$ corresponds to a model with a local dispersion force, global agglomeration force, and global dispersion force.

The properties of $\{f_k(\tau)\}$ are completely *model independent*; because $\{f_k(\tau)\}$ are merely the eigenvalues of the (normalized version of) friction matrix $\bar{\boldsymbol{D}}(\tau)$, they are invariant regardless of the economic geography model (i.e., the payoff function $\boldsymbol{v}(\boldsymbol{h})$) one may assume. Instead, the function $G(f)$ in (3.4), or equivalently the matrix relation (3.2), encapsulates the net effects of economic interactions in the model and provides insights into the process of endogenous formation of spatial patterns.

The question posed is as follows: given such $G(f)$, what spatial pattern emerges after an encountered bifurcation? In particular, will it be a *unimodal pattern* or a *multipolar pattern*?

[31]In Appendix C, we present detailed analyses of how economic geography models are mapped to the coefficients $\{c_i\}$ taking models in the literature as concrete examples.

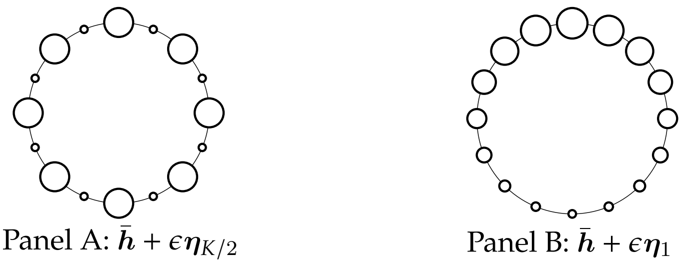


Figure 5: Schematic illustrations of the spatial patterns at τ^* and τ^{**} ($K = 16$)

Note: The size of a small white circle represents the number of mobile agent at the region. Panel A: bifurcation at τ^* (a locally concentrated and globally dispersed pattern); Panel B: bifurcation at τ^{**} (a globally concentrated and locally dispersed pattern).

Choose an appropriate value of τ so that $\bar{\boldsymbol{h}}$ is stable; that is, the net agglomeration forces $\{g_k(\tau)\}$ are strictly negative so that any deviation is strictly nonbeneficial. Consider a gradual change in τ. When any of the net agglomeration forces becomes positive, the flat-earth equilibrium stops being stable and agglomerations emerge. What one should observe here is the first $g_k(\tau)$ that changes its sign from negative to positive. Let $\tau^\star$ be a critical value at which this occurs. It is evident that $\tau^\star$, or the so-called *break point*, is a solution to the equation $\max_{k\in\mathcal{K}} \left\{g_k(\tau^\star)\right\} = 0$. Denote the index of the critical eigenvalue such that $g_k(\tau^\star) = \max_k g_k(\tau^\star)$ by $k^{\text{crit}}_\star$. Then, the spatial pattern at $\tau^\star$ is expressed in terms of the $k^{\text{crit}}_\star$th eigenvector as $\boldsymbol{h} = \bar{\boldsymbol{h}} + \epsilon\boldsymbol{\eta}_{k^{\text{crit}}_\star}$ where ϵ is a real number. Under our assumption of $G(f)$, the curves of $\{g_k(\tau)\}$ behave as in the top panel of **Figure 4**; the upper envelope of the curves represents $\max_{k\in\mathcal{K}} \left\{g_k(\tau^\star)\right\}$, and the critical points are found where the curve crosses the horizontal axis. There are two solutions, τ^* and τ^{**}, and we have $k^{\text{crit}}_* = K/2$ and $k^{\text{crit}}_{**} = 1$. See **Figure 5** for the spatial patterns that emerge at τ^* (Panel A) and τ^{**} (Panel B).

The stability of the flat-earth equilibrium for the *higher* level of τ is attributed to the *global* dispersion force, while that for the *lower* level of τ is attributed to the *local* dispersion force. As transport costs decline from high level, the flat-earth equilibrium collapses at τ^* because the global dispersion force declines (recall that $f_k(\tau)$ decreases as τ decreases). When τ decreases below another threshold, τ^{**}, it brings about a situation where the flat-earth equilibrium becomes stable again because the local dispersion force, which is always existent regardless of τ, overcomes the agglomeration force.

3.4 Rethinking redispersion

Panels A and B of **Figure 5** illustrate the two mutually distinct spatial patterns that emerge at τ^* and τ^{**}, respectively. Panel A illustrates the spatial pattern that emerges at τ^*, which is interpreted as a *locally concentrated and globally dispersed pattern*. It is characterized by the formation of many small agglomerations spread over the circumference. In the pattern, mobile agents are locally concentrated but the location of agglomerations are equidistantly spaced or globally dispersed.[32] Panel B illustrates the pattern at τ^{**}, which is interpreted as a *globally concentrated and locally dispersed pattern*. In this pattern, agents are globally concentrated to shape a unimodal distribution (a single agglomeration with a large spatial extent).

The two critical points τ^* and τ^{**} are customarily termed in the literature as "emergence of core and periphery" and "redispersion (revival of the periphery)," respectively, and the process as a whole is denoted as "bell-shaped development" (Fujita and Thisse, 2013). When transportation costs are very high ($\tau > \tau^*$), the symmetric equilibrium is stable. An the first stage of the decline of transportation costs, the destabilization of the symmetric equilibrium results in spatial inequality. In the later stage, once established, agglomeration is no longer sustainable and the symmetric configuration is stable again ($\tau < \tau^{**}$).

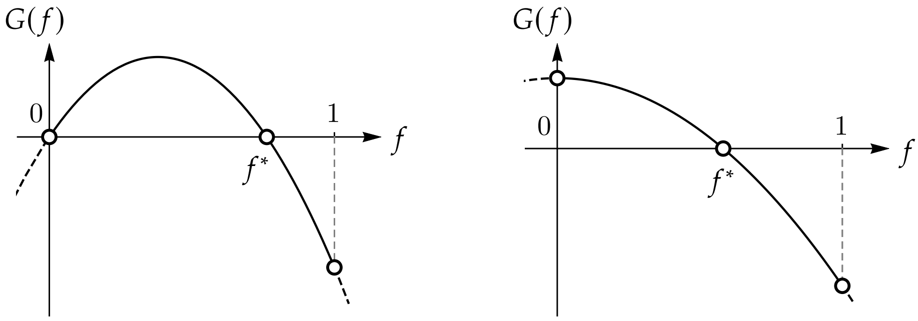
The redispersion process is considered to be just the reverse process of agglomeration. For any model with a single type of mobile agent, it is supposed that there is no essential difference in the spatial patterns at the two stages (around τ^* or τ^{**}).[33] Indeed, this is true in the two-region setup where the two relevant eigenvectors coincide: $\boldsymbol{\eta}_{K/2} = \boldsymbol{\eta}_1 = (1, -1)$. However, our analysis so far has shown that it is *not* the case in a many-region economy. The two bifurcations at τ^* and τ^{**} are of quite distinct nature: each represents the emergence of mutually distinct spatial patterns and are attributed to dispersion forces at different spatial scale, i.e., global and local.

4 Classification of models by the spatial scale of dispersion

The distinction between global and local dispersion forces allows one to reduce economic geography models to two canonical classes: (i) those with only a global dispersion force and (ii) those with only a local dispersion force. This section provides concrete examples of global and local dispersion forces employing selected models in the literature. For every model discussed in this section, $\boldsymbol{J}$ is expressed by up to the second-order term of $\bar{\boldsymbol{D}}$ as in (3.2) so that $G(f)$ is (at most) a quadratic of f as in (3.4). **Table 1** is the resultant classification. Detailed analyses of models in the table are relegated to Appendix C.

[32]Observe that the spatial pattern resembles those obtained by the numerical simulations by Krugman (1993) for $K = 12$. The spatial pattern is also similar to the pre-assumed spatial patterns in the study by Tabuchi and Thisse (2011).

[33]Takatsuka and Zeng (2009) analyzed redispersion behavior in the two-region economy new economic geography model with *multiple industries* with distinct returns to scale and indicated an asymmetry in the two processes: industrial composition at each region is different in the redispersion phase.



A: Krugman (1991) B: Harris and Wilson (1978)

Figure 6: $G(f)$ for the models by Krugman (1991) and Harris and Wilson (1978)

4.1 Class (i): Models with global dispersion force

Global dispersion forces are those that arise *outside* of a given agglomeration, typically implemented as spatially dispersed demand. Usually, a global dispersion force appears in $\boldsymbol{J}$ as a negative (second-order) term with respect to $\boldsymbol{D}$. For instance, the NEG models by Krugman (1991), Puga (1999), Forslid and Ottaviano (2003), and Pflüger (2004) satisfy $c_0 = 0$ and we have $G(f) = c_1 f + c_2 f^2$ with $c_1 > 0$ and $c_2 < 0$. Furthermore, the model by Harris and Wilson (1978), a vintage model of spatial self-organization proposed in the field of geography, satisfies $G(f) = c_0 + c_2 f^2$ with $c_0 > 0$ and $c_2 < 0$.[34]

Figure 6 illustrates $G(f)$ for Krugman (1991) and Harris and Wilson (1978). Because $G(f)$ is a concave quadratic with $G(0) \geq 0$, $G(f)$ has at most a single solution f^* in $(0, 1)$; it implies that a single critical value (break point) of transportation cost τ^* can exist.[35] As discussed in the previous section, at τ^*, the emergent pattern is *locally concentrated and globally dispersed* (or a *multimodal* pattern) in which multiple distinct agglomerations are endogenously formed (Panel A of **Figure 5**).

The seminal model of Krugman (1991) is considered as an example. Appendix C provides the omitted derivations of the indirect utility function and other formulae of this model, as well as the results under the other models discussed below. The payoff function (i.e., indirect utility function of mobile workers) is given by

$$v_i(\boldsymbol{h}) = w_i P_i^{-\mu} \tag{4.1}$$

[34]For the drawn cases, the underlying parameters satisfy the so-called "no-blackhole condition" that ensures the stability of the flat-earth pattern in the higher extreme of τ. Otherwise, we have $G(1) > 0$ and the flat-earth pattern is always unstable.

[35]One can show that an influential model by Ottaviano et al. (2002) also endogenously produces globally dispersed patterns; hence, this is a class (i) model. As the model assumes non-iceberg transportation technology, we do not discuss the model here to simplify our presentation (see also Footnote 26).

where w_i denotes the nominal wage of mobile workers and

$$P_i \equiv \left(\sum_{j \in \mathcal{K}} h_j w_j^{1-\sigma} d_{ji} \right)^{1/(1-\sigma)} \tag{4.2}$$

denotes the price index in region i. The parameters μ and σ are the expenditure share on the manufactured good and the elasticity of substitution between varieties, respectively. The wage is obtained as the (unique) solution of the so-called wage equation that reflects the short-run utility maximization of consumers, trade balance, and zero-profit condition for firms. In each region, there is an exogenous endowment of immobile workers.

For the model, one has

$$\boldsymbol{J} \simeq \mu \left(\frac{1}{\sigma - 1} + \frac{1}{\sigma} \right) \bar{\boldsymbol{D}} - \left(\frac{\mu^2}{\sigma - 1} + \frac{1}{\sigma} \right) \bar{\boldsymbol{D}}^2. \tag{4.3}$$

The exact mappings to the coefficients $c_1 > 0$ and $c_2 < 0$ are thus given by $c_1 = \mu(\bar{\kappa} + \kappa)$ and $c_2 = -(\mu^2 \bar{\kappa} + \kappa)$, where we let $\bar{\kappa} \equiv 1/(\sigma - 1)$ and $\kappa \equiv 1/\sigma$. The coefficients c_1 and c_2 captures the *net* effects of agglomeration and dispersion forces in the Krugman (1991) model, respectively. In particular, $\mu\bar{\kappa}$ in c_1 represents the so-called price-index effect, whereas $\mu\kappa$ represents a home-market effect. On the other hand, c_2 is the market-crowding effect: $\mu^2\bar{\kappa}$ in c_2 is due to firms' competition over demand from mobile agents and in κ is due to that from immobile agents. For more detailed discussions on interpretations of coefficients, refer to Remark C.2 in Appendix C.1.

4.2 Class (ii): Models with a local dispersion force

A local dispersion force acts *inside* each region and does not explicitly depend on the spatial distribution of mobile agents. Urban costs induced within each region (e.g., housing cost, congestion externality) are typical. Examples include the frameworks of Helpman (1998), Redding and Sturm (2008), Murata and Thisse (2005), as well as the perfectly competitive framework of Allen and Arkolakis (2014).[36] Furthermore, the model by Beckmann (1976) focusing on the internal structure of cities (Mossay and Picard, 2011; Blanchet et al., 2016) is another representative example.[37] At the flat-earth pattern, a local dispersion force appears

[36]Without exogenous location-fixed factors, the model of Redding and Rossi-Hansberg (2017) (§3) is equivalent to Redding and Sturm (2008). The model of Monte et al. (2016) also belongs to class (ii), albeit it adds an extra urban cost as well as taste heterogeneity; we note that an idiosyncratic utility shock is equivalent to a local dispersion force (see Appendix B for a brief discussion). It is also evident that Picard and Tabuchi (2013) is a class (ii) model.

[37]We here consider a discrete-space version of the Beckmann model as formulated in the study by Akamatsu, Fujishima and Takayama (2017). Akamatsu et al. (2017) showed that a discrete-space Beckmann model asymptotically converges to the continuous variant as the number of region increases.

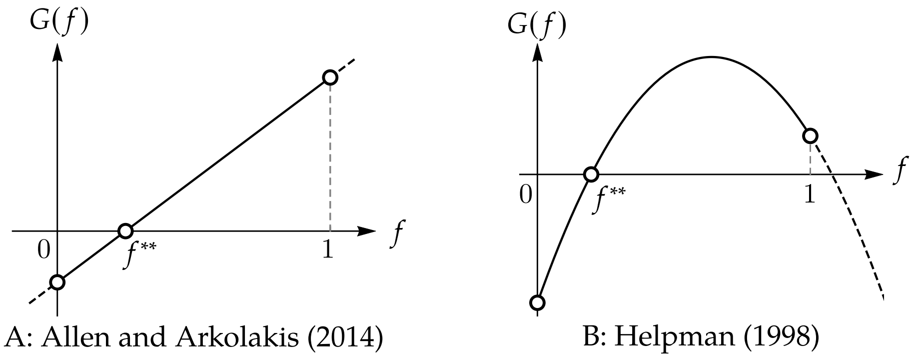


Figure 7: $G(f)$ for the models by Allen and Arkolakis (2014) and Helpman (1998)

in $\boldsymbol{J}$ as a negative *constant* term with respect to $\boldsymbol{D}$ (i.e., $c_0 < 0$).

Figure 7 illustrates $G(f)$ for the models by Allen and Arkolakis (2014) and Helpman (1998). For these models, there exists at most a single critical point of f^{**}. If the model parameters are set such that there is an endogenous formation of agglomeration, the flat-earth equilibrium is stable for the lower level of transport cost. At the only bifurcation point τ^{**}, a *globally concentrated and locally dispersed pattern* (or a *unimodal* pattern) with a single agglomeration is endogenously formed (Panel B of **Figure 5**). In this class of models, without location-fixed factors, the only possible spatial pattern associated with agglomeration is a globally concentrated and locally dispersed pattern.

The model by Allen and Arkolakis (2014) is a recent example. The indirect utility function of mobile workers is given by

$$v_i(\boldsymbol{h}) = h_i^{\beta} w_i P_i^{-1}, \tag{4.4}$$

where P_i denotes the price index for the model

$$P_i \equiv \left(\sum_{j \in \mathcal{K}} h_j^{\alpha(\sigma-1)} w_j^{1-\sigma} d_{ji} \right)^{1/(1-\sigma)} \tag{4.5}$$

The parameters $\alpha > 0$ and $\beta < 0$ are exponents for a reduced-form Marshallian externality and for a local congestion externality in amenities, respectively, and $w_i(\boldsymbol{h})$ is the market wage in the region i. For this model, the source of agglomeration is the reduced-form local positive externality represented by the parameter α. One has

$$\boldsymbol{J} \simeq -(\alpha + \beta - \gamma_0)\boldsymbol{I} + (\alpha + \beta + \gamma_1)\bar{\boldsymbol{D}}, \tag{4.6}$$

with $\gamma_0 \equiv (1+\alpha)/\sigma$ and $\gamma_1 \equiv (1-\beta)/\sigma$. We thus have $G(f) = c_0 + c_1 f$ with $c_0 = -(\alpha + \beta - \gamma_0)$ and $c_1 = \alpha + \beta + \gamma_1$. If $\alpha + \beta \leq 0$, there is *no agglomeration force* and the flat-earth equilibrium is stable for any value of τ. If $\alpha + \beta > 0$, there is a local positive agglomeration force; we have $c_0 < 0$ and $c_1 > 0$, as well as $G(1) > 0$. If the agglomeration force is strong ($0 < \alpha + \beta$), the model can express endogenous agglomeration. In the net form, as indicated in (4.6), the model does not have any global dispersion force. Thus, we conclude that the model produces only unimodal patterns. In fact, Figure VIII in the study by Allen and Arkolakis (2014) confirms this result. In the other contexts, the model by Beckmann (1976) (Mossay and Picard, 2011; Blanchet et al., 2016; Akamatsu et al., 2017) yields a similar linear functional form of $G(f)$ since the model incorporates a first-order global agglomeration force and a local dispersion force.

As discussed by Allen and Arkolakis (2014), their model is isomorphic to the Helpman (1998) model with local landownership (i.e., Redding and Sturm, 2008). One can show that the assumptions concerning landownership do not alter the above conclusion. For Helpman's original model, with public landownership, under appropriate normalizations, one obtains $\boldsymbol{J} \simeq c_0\boldsymbol{I} + c_1\bar{\boldsymbol{D}} + c_2\bar{\boldsymbol{D}}^2$ so that $G(f) = c_0 + c_1 f + c_2 f^2$ with $c_0 = -(1-\mu)$, $c_1 = \mu(\bar{\kappa} + \kappa)$, and $c_2 = -(\kappa + \mu^2\bar{\kappa}) + (1-\mu)$. Again, $\bar{\kappa} = 1/(\sigma - 1)$ and $\kappa = 1/\sigma$ where μ is the expenditure share on manufactured goods, and σ is the elasticity of substitution between manufactured goods. We have $c_0 < 0$, $c_1 > 0$, and $c_2 < 0$; for the model, the agglomeration force is derived from the second term in $\boldsymbol{J}$, whereas dispersion forces are derived from the others. Observe that c_1 is as per the model by Krugman (1991), meaning that the agglomeration force of the latter is isomorphic to that of the former. Panel B of **Figure 7** illustrates the shape of $G(f)$ for the model. It follows that whenever there is an endogenous agglomeration, we have $G(1) > 0$; thus, f^* does not exist; although $c_2 < 0$ and there seemingly exists a global dispersion force, it is not effective. The main dispersion force of the model is derived from the consumption of non-tradable housing stock that produces a negative pecuniary externality through the local housing market.

4.3 Classification of representative economic geography models

Table 1 classifies representative economic geography models in the literature according to the nature of their dispersion forces and resulting stable spatial patterns (including our toy model discussed in Section 3.2). The exact mapping to the coefficients of $G(f)$ is provided by **Table 2** at the end of Appendix C. As discussed, at the flat-earth equilibrium of a given model, one can characterize the fundamental trade-off between the centripetal and centrifugal forces by the coefficients $\{c_i\}$ or the shape of $G(f)$. In particular, one can clearly distinguish the spatial scale of the model's effective dispersion force.

There are two canonical model classes (i) and (ii). The former includes models with only a global dispersion force, while the latter includes models with only local dispersion force. The second column of **Table 1** summarizes characteristic spatial patterns for each of the two model classes. Although our analysis basically concerns the endogenous formation of

Table 1: Classification of economic geography models in the literature

Dispersion force	Spatial patterns	Economic geography models
None	Concentration to a *single* region (unimodal patterns)	The toy model defined by (3.1)
Only global [class (i)]	Locally concentrated and globally dispersed (multimodal patterns)	Krugman (1991) Puga (1999) Forslid and Ottaviano (2003) Pflüger (2004) Harris and Wilson (1978)
Only local [class (ii)]	Globally concentrated and locally dispersed (unimodal patterns)	Helpman (1998) Murata and Thisse (2005) Redding and Sturm (2008) Allen and Arkolakis (2014) Redding and Rossi-Hansberg (2017) (§3) Beckmann (1976) Mossay and Picard (2011) Blanchet et al. (2016)
Both [class (iii)]	Mixed characteristics of the classes (i) and (ii) (multimodal patterns)	Tabuchi (1998) Pflüger and Südekum (2008) Takayama and Akamatsu (2011)

Note: Appendix C provides detailed analyses of the models, with **Table 2** summarizing the exact mappings of each model to the coefficients of the corresponding model-dependent function $G(f) = c_0 + c_1 f + c_2 f^2$.

spatial patterns under a multiplicity of equilibria, class (i) and (ii) models have qualitatively different behavior and can yield mutually contradicting implications when employed for counterfactual exercises. This point is discussed in Section 6.3, and a formal analysis is provided in Appendix D.

In addition, there are a few models in the literature that have the two dispersion forces effectively at work. For instance, Tabuchi (1998), Pflüger and Südekum (2008), and Takayama and Akamatsu (2011) presented both local and global dispersion forces. We refer to them as *class* (*iii*) models. The models produce spatial patterns with mixed characteristics of global and local dispersion, which we discuss below by employing a numerical example. For this model class, $G(f)$ is a concave quadratic that has two roots in the $(0, 1)$ interval as in the bottom left panel of **Figure 4**. As we discussed in Section 3, the flat-earth equilibrium is stable for both high and low transport costs.

Notably, our classification seems to be backed by a more general principle. There is a large body of studies outside economics focused on spatial pattern formation, typically on the basis of reaction–diffusion systems (Kondo and Miura, 2010). In that literature, it is now widely accepted that the basic requirement to form multiple peaks in stationary spatial patterns (i.e., in our context, stable locally concentrated and globally dispersed patterns) is *a short-range positive feedback combined with a long-range negative feedback* with respect to a concentration of mobile factors (Meinhardt and Gierer, 2000). One would notice that a negative term (global dispersion force) of $\bar{\boldsymbol{D}}$ in $\boldsymbol{J}$ can be interpreted as a long-range negative feedback.

5 Numerical examples

In the many-region setup, the first bifurcation, or emergence of agglomeration, may not be the end of the story. The overall evolutionary path of spatial structure in line with monotonic change (e.g., decline) of transportation costs is of interest. Fortunately, intrinsic properties of the whole evolutionary process of the agglomeration patterns can be qualitatively predicted by the above results on the stability of the flat-earth pattern. This section provides some numerical illustrations. The models of Krugman (1991), Helpman (1998), and Pflüger and Südekum (2008) are chosen as representative examples for the models with only global dispersion force [class (i)], those with only local dispersion force [class (ii)], and those with both dispersion forces [class (iii)], respectively. The numerical examples in this section are conducted in an eight-region ($K = 8$) symmetric racetrack economy. Following the literature, the replicator dynamic (Taylor and Jonker, 1978) is employed as the underlying dynamic $\boldsymbol{F}$. The chosen parameters are described in Appendix C.

5.1 Class (i): Models with a global dispersion force

Figure 8 reports an evolutionary path of stable equilibrium patterns in the course of decreasing τ for the Krugman (1991) model. The black solid (dashed) curves depict stable (unstable) equilibrium values of population share $\boldsymbol{\lambda} = (\lambda_i)$ at each τ, where $\lambda_i \equiv h_i/H$. Consider a gradual decrease in τ from a sufficiently high level at which the flat-earth equilibrium is stable. The uniform distribution with no agglomerations is initially stable until τ reaches the break point τ^*. As discussed in the previous section, the bifurcation at τ^* pushes the spatial pattern toward the direction of $\boldsymbol{\eta}_{K/2} = (1, -1, 1, -1, 1, -1, 1, -1)$. It results in the formation of a globally dispersed pattern with four disjoint and equidistantly separated point-wise agglomerations. Further decrease in τ triggers the second and third bifurcations at τ^{**} and τ^{***}, respectively.[38] Observe that the bifurcations at τ^{**} and τ^{***} sequentially double the spacing between agglomerated regions, reducing their number as $4 \to 2 \to 1$. At the lower extreme of τ, a monopolar pattern emerges. Note that each agglomeration has no spatial extent at any level of transportation cost, since local dispersion force is absent. In the model, better interregional access (a smaller τ) makes the size of each agglomeration larger. Such effect

[38]In fact, one can analytically derive these critical values τ^{**}, τ^{***} (see Akamatsu et al., 2012; Osawa et al., 2017) and characterize spatial patterns that emerge at these points.

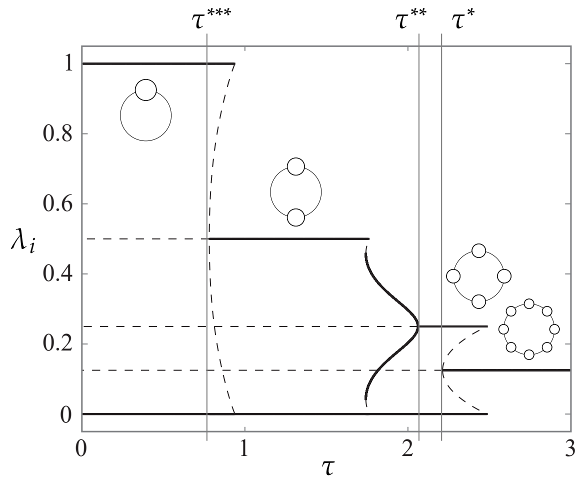


Figure 8: Bifurcation diagram of a class (i) model

Note: Krugman (1991) is taken as the underlying example model.

is, however, limited to the selected regions. Depending on the stage of spatial structural evolution, the impact of improvement of transportation on the size of *each* agglomeration can be either positive (for the selected regions) or negative (for the others). In this sense, there are no monotonic relationships between the level of transportation costs and the size of *each* agglomeration. In fact, this point is already apparent in the two-region models that explicitly incorporate agglomeration economies combined with interregional distance.

In our many-region setup, however, there comes another indeterminacy. As the spatial structure evolves, *once selected regions may decline* to form the hinterland of the currently selected ones—the *agglomeration shadow*.[39] Consider the fourth region at the six o'clock position in **Figure 8** as an example. The region is selected at the transitions at τ^* and τ^{**} so that the impact of a decrease in τ is positive. After τ^{***} is encountered, however, it immediately loses its population. For the region, a monotonic decrease in τ implies a win situation followed by a lose situation. It indicates that if empirical realities resemble this class of model, whether the impact of a further decline in transportation costs on a specific region is positive or negative is indeterminate *even when a monotonic relation for the region is supported by historical data*. At least in a symmetric racetrack economy, we do not have any clear implication for models in this class regarding the impact of a uniform reduction of transport cost on the population (or output) size of an *individual region* since whether the population share of a region grows or declines is *in principle* indeterminate a priori. Instead, possible predictions are focused on the *global spatial distribution* of agglomerations: the number of concentrations and the spacing between them which monotonically decreases and increases, respectively.

[39]The concept of agglomeration shadow was first introduced by Arthur (1994) and is formalized in the context of a general equilibrium model by Fujita and Krugman (1995).

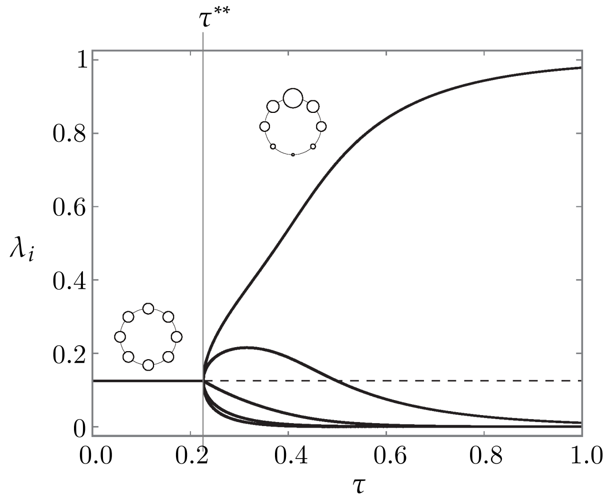


Figure 9: Bifurcation diagram of a class (ii) model

Note: Allen and Arkolakis (2014) is taken as the underlying example model.

5.2 Class (ii): Models with local dispersion force

Figure 9 is similar to **Figure 8** for the model by Allen and Arkolakis (2014). The model incorporates only a local dispersion force; the flat-earth equilibrium is stable for lower values of τ. At τ^{**} in **Figure 7**, a bifurcation in the direction of $\boldsymbol{\eta}_1$ leads to the emergence of a unimodal pattern. This is *the* bifurcation in the model; after the emergence of the unimodal pattern at τ^{**}, when τ increases further, the spatial pattern monotonically and smoothly converges to a monopolar pattern (i.e., the complete concentration of mobile agents at a single region) as τ approaches to infinity. Thus, if we define the number of agglomerations for the model by that of peaks (i.e., local maxima) in $\boldsymbol{h}$, it is at most one. The model does not allow locally concentrated and globally dispersed patterns to emerge; such models would be interpreted as expressing the evolution of the *spatial extent of a single agglomeration*.

Quantitative spatial models that employ class (ii) models (e.g., Redding and Sturm, 2008; Allen and Arkolakis, 2014) emphasize the uniqueness of equilibrium, through which calibrations and counterfactual analyses have determinate implications. The studies are conducted under parameter settings that ensure uniqueness of equilibrium *regardless* of the level of interregional transportation costs (Redding and Rossi-Hansberg, 2017). This is made possible because the local dispersion force in class (ii) models does *not* depend on the level of accessibility to the other regions; consequently, if a sufficiently strong local dispersion force is imposed, there is no endogenous agglomeration due to a decline in transportation costs. Notably, in our setup, since the uniform distribution of mobile agents across regions is always an equilibrium on a symmetric racetrack, uniqueness of equilibrium automatically implies that the flat-earth pattern $\bar{\boldsymbol{h}}$ is the only equilibrium and is stable. **Figure 10** indicates our classification of possible spatial patterns and their stability for the model by Allen and Arkolakis (2014) in a racetrack economy with arbitrary K. Their uniqueness condition is

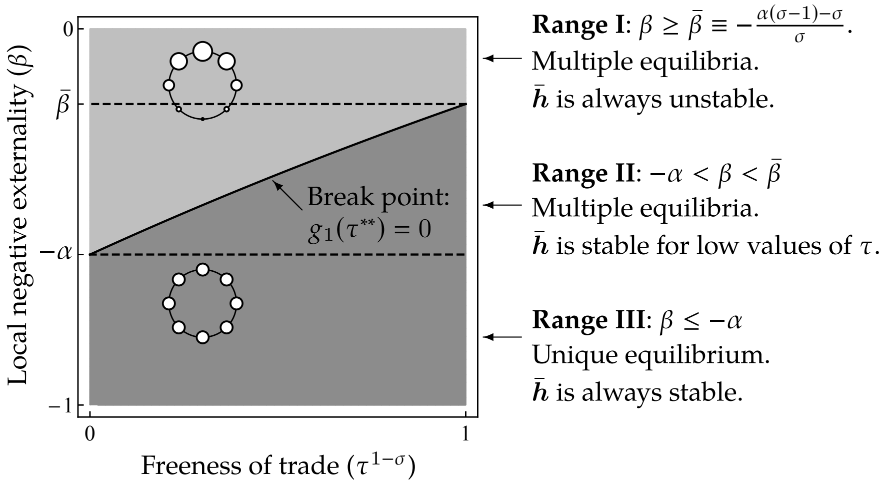


Figure 10: Classification of uniqueness and stability of equilibria for Allen and Arkolakis (2014)

Note: We let $\alpha = 0.6$ and $\sigma = 4$ for illustration. $\bar{\boldsymbol{h}}$ is stable in the dark gray region, while not in the light gray region. At τ^{**} such that $g_1(\tau^{**}) = 0$, a unimodal pattern emerges. $\alpha + \beta \leq 0$ is a sufficient condition for the stability of $\bar{\boldsymbol{h}}$ regardless of τ. A comparison with Figure I of Allen and Arkolakis (2014), where their classification corresponds to Ranges I, II, and III above, would be interesting.

$\beta \leq -\alpha$ (i.e., Range III in the figure).[40] One will observe that uniqueness directly implies that the uniform distribution is stable regardless of the level of τ.

5.3 Class (iii): Models with both dispersion forces

Tabuchi (1998), Pflüger and Südekum (2008), and Takayama and Akamatsu (2011) considered both local and global dispersion forces. In effect, these models exhibit rich and meaningful interplay between the *number and spacing of agglomerations* and *spatial extent of each agglomeration* without any location-fixed factors but only with pure economic forces.

One would expect that in these models, the evolutionary process of spatial agglomeration patterns in the course of monotonic change in τ is in some sense a combination of the two examples presented above. This is indeed the case. **Figure 11** depicts the evolution of the number of agglomerations in the course of decreasing τ for the model by Pflüger and Südekum (2008) in a symmetric eight-region racetrack economy. We define the number of agglomerations in a spatial distribution of mobile agents, $\boldsymbol{h}$, by that of the local maxima therein.[41] Comparing **Figure 11** with **Figure 8** and **Figure 9**, we observe that the former is basically a combination of the latter two, as expected. When τ gradually decreases from a very high level, the number of agglomerations evolves as $0 \to 4 \to 2 \to 1$ as in the class (i) models (**Figure 8**), while in the later stage $1 \to 0$ as per the class (ii) models (**Figure 9**).[42] The initial stage is governed by a decline of the global dispersion force, while the later stage is

[40] For the model by Helpman (1998), the condition for the uniqueness of equilibrium is given by $(1-\mu)\sigma > 1$.

[41] For example, for Pattern I in **Figure 12**, we evenly split the population of the region in the middle of the two peaks. Also, if two consecutive regions have the same population as in Pattern K in **Figure 12**, it is counted

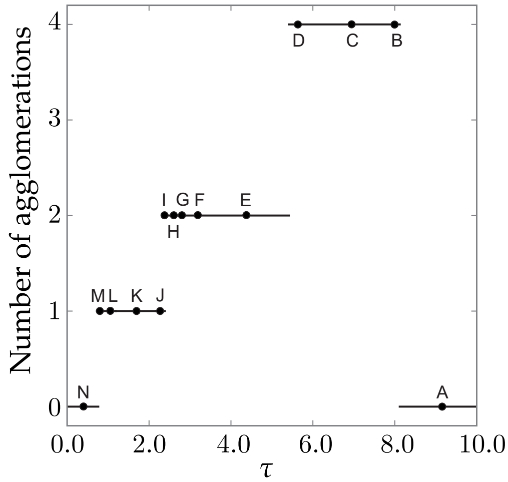


Figure 11: Evolution of the number of agglomerations in a class-(iii) model

Note: Pflüger and Südekum (2008) is taken as the underlying example model.

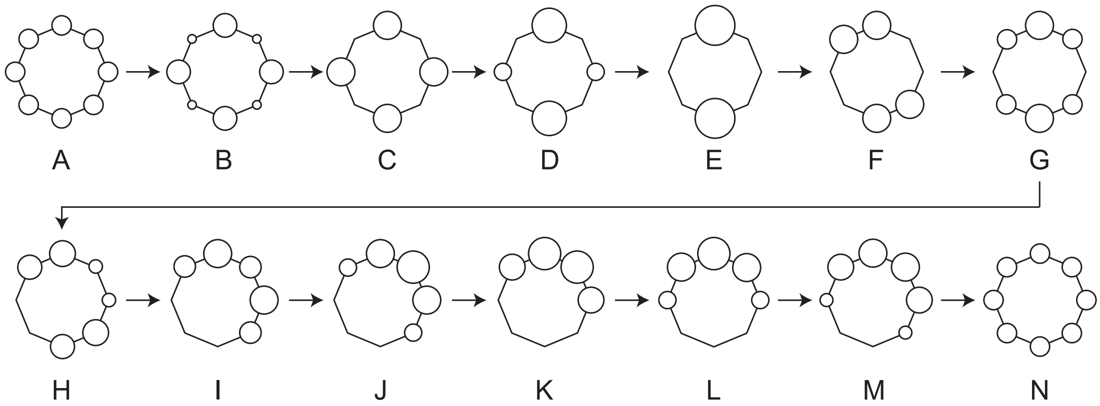


Figure 12: Evolution of the spatial pattern in **Figure 11**

Note: The alphabets below the spatial patterns correspond to those in **Figure 11**.

governed by a relative rise of the local dispersion force. As the importance of distance declines given improvements in transportation access, local congestion overcomes the agglomeration force and the so-called redispersion occurs.

The evolution of spatial patterns provides richer intuitions. **Figure 12** illustrates the spatial patterns associated with **Figure 11** (see also Section 3.2 to understand the figure). The flat-earth pattern is initially stable (Pattern A); the first bifurcation leads to a four-centric global dispersion (C), whereas the dispersion associated with the second bifurcation

as a single agglomeration.

[42]At this point, there is good reason to suspect that, even though seemingly identical, the flat-earth patterns at the higher and lower levels of τ are very distinct nature. Specifically, one would argue that the number of agglomerations must be eight (one), instead of zero, at large (low) τ. We shall refrain from these arguments, however, because the two stages of dispersion are indistinguishable by a mere observation of $\boldsymbol{h}$.

is two-centric (E). These transitions are wholly in line with those of Krugman (1991); they are governed by the gradual decline of the global dispersion force in the model. After these transitions, evolutionary behavior becomes more interesting; the decline of the global dispersion force increases the relative importance of the local dispersion force. The two-centric agglomerations formed in (E) gradually increase their spatial extent (F, G) due to local dispersion effects combined with relative decline of the global agglomeration force. Further decline in τ implies a triumph of the global agglomeration force against the global dispersion force since the latter declines faster than the former. Consequently, the two agglomerations gradually merge (H, I) to form a monopolar agglomeration (J, K), while maintaining their spatial extent due to the strong local dispersion force. As relative importance of the local dispersion force increases further, the gradual expansion of the single agglomeration occurs (L, M) followed by complete redispersion (N). The rich and intuitive interplay of global and local scales of agglomeration and dispersion can be studied only in many-region setups.

6 Implications for empirical studies

In the previous sections, we have argued that the explicit consideration of the presence of many regions, more specifically the diversity in interregional distances, is the key to explaining the actual spatial pattern of agglomeration. The understanding of agglomeration and dispersion mechanisms at different spatial scales helps us find an appropriate way to quantify the spatial patterns of agglomeration. It helps develop appropriate specifications for regression models as well as structural models to identify the causal mechanisms of regional agglomeration. This section highlights these points by reviewing selected empirical studies on the relation between transport costs and regional agglomerations.

6.1 Measures of agglomeration

There is a strand of literature concerning the measurement of industrial agglomeration. Unlike population agglomerations that have been identified in terms of distinct metropolitan areas or population clusters (e.g., Baum-Snow, 2007; Duranton and Turner, 2012; Rozenfeld, Rybski, Gabaix and Makse, 2011), industrial agglomerations have typically been measured by an aggregated scalar index (e.g., Ellison and Glaeser, 1997; Duranton and Overman, 2005; Brülhart and Traeger, 2005; Mori et al., 2005).

Of the two pioneering indices of industrial agglomeration, that proposed by Ellison and Glaeser (1997) controlled for the spatial concentration of employment that accrued from the distribution of employment among establishments, while the other by Duranton and Overman (2005) resolved spatial aggregation biases that arose from regional data by utilizing geo-coded micro data of establishments.

While these refinements may be reasonable, a major reservation about these scalar indices is that by construction they do not distinguish spatial scales of agglomeration and dispersion. With respect to cross-sectional comparisons among industries, this means that there is no way to distinguish spatial scales at which variations in the index value arise although the underlying agglomeration mechanisms qualitatively differ at each scale.[43] A consequence of the abstraction from spatial scales is that these indices inevitably neutralize the opposing responses of agglomeration at global and local scales to a given change in transport costs.

Distinguishing individual agglomerations on a map just like the case of population agglomerations is necessary to separate the effects at different spatial scales. Kerr and Kominers (2015) and Mori and Smith (2014) proposed clustering methods designed for economic agglomerations. Pelleg and Moore (2000), Ishioka (2000) and Brendan and Dueck (2007) proposed heuristic clustering techniques for general purposes. Our theory suggests that agglomeration at the global scale is reflected on a smaller number of agglomerations (as well as a larger spacing between agglomerations), whereas that at the local scale is reflected on a smaller spatial extent of each individual agglomeration. These spatial properties of agglomeration can be quantified using the identified clusters.

An advantage of such a clustering approach for industrial agglomerations is that unlike the case of population agglomeration, one can obtain variations in agglomeration patterns across industries. Using the clustering method proposed by Mori and Smith (2014), Mori and Smith (2015) indicated a wide variation in the the degrees of agglomeration both at global and local scales across three-digit manufacturing industries in Japan. The variations across industries in turn can be utilized to test the theoretical implications on the spatial patterns of agglomeration, for example, the causal relation among the number, size, and spatial extent of agglomerations and transport costs. One such application by Mori, Mun and Sakaguchi (2017) is discussed in the next section.

6.2 Reduced-form regression approaches

We have shown that endogenous agglomeration mechanisms generally do not isolate which existing agglomerations to grow or decline given an improvement in interregional transport access. This indeterminacy is due to the underlying second-nature advantage. Nonetheless, the theory offers a clear prediction regarding the global and local spatial pattern of agglomerations. The former prediction is that *there is agglomeration at the global scale*: the number of agglomerations decreases, the distance between neighboring agglomerations increases (reflecting the growing agglomeration shadow), and the sizes of the surviving individual agglomerations increase (refer to Sections 5.1 and 5.3). The latter prediction is that *there is*

[43] Duranton and Overman (2005) distinguished distances between establishments; however, they did not distinguish between intra- and inter-agglomeration distances.

dispersion at the local scale: The spatial extent of each individual agglomeration increases, for example, in the form of suburbanization (refer to Sections 5.2 and 5.3).

In Sections 6.2.1 and 6.2.2 below, we argue that these theoretical predictions are useful to understand the diverse results from extant empirical studies based on reduced-form regressions on regional agglomerations. Further, in Section 6.2.3, we discuss the context in which these predictions can be actually tested.

6.2.1 On the size of an agglomeration

A typical regression model to evaluate the impact of a new transport network on regional growth has the following form (see, e.g., Redding and Turner, 2015, §20.4, for a survey) :

$$\mathrm{SIZE}_{it} = C_0 + C_1 \mathrm{ACCESS}_{it} + C_2 x_i + \gamma_{it} + \eta_t + \epsilon_{it}\,, \tag{6.1}$$

where SIZE_{it} and ACCESS_{it} represent a measure of agglomeration size and a measure of interregional transport access, respectively, in region i at year t; x_i denotes the region-specific and year-invariant covariates, γ_{it} denotes the region- and year-specific unobserved effect, η_t denotes the year-specific unobserved effect, ϵ_{it} denotes the region- and year-specific error, and C_0, C_1 and C_2 are coefficients to be estimated, where C_1 is of interest here.

The existing literature concerning the relation between agglomeration size and interregional transport access in an individual region shows mixed results. We start from two studies drawing contrasting conclusions. Faber (2014) investigated the impact of the construction of the nation-wide highway network in China on the agglomerations in peripheral counties during 1997–2006. Duranton and Turner (2012) studied a similar situation in the US during 1983–2003; however, they focused on the impact on agglomerations in relatively large metro areas instead of peripheral alternatives. SIZE_{it} represents the changes in output measures such as the gross domestic product and gross value added as well as that of population size in a county in the former, while it is the change in metro-area population or employment in the latter. ACCESS_{it} represents the change in interregional highway accessibility in both cases.[44] Their results exhibited a stark difference: the former (latter) generally found a significantly negative (positive) estimate of C_1 in (6.1), even after controlling for the initial size of each region.

Yet there are other studies reporting indefinite results. For the Chinese data similar to that used by Faber (2014), Baum-Snow, Henderson, Turner, Zhang and Brandt (2016, Tables

[44]In the baseline specification of (6.1) in the study by Faber (2014), ACCESS_{it} represents a binary variable that takes the value 1 if a given region i is connected by the newly constructed highway at time t, while it is set to the initial sum of interstate highway length within a metro area (i.e., in 1983) in the study by Duranton and Turner (2012). In particular, Duranton and Turner (2012) considered ACCESS_{it} as the level of intra-urban (rather than inter-urban) transport infrastructure. But, we believe that the stock of interstate highways within a given metro area certainly reflects the level of inter-urban connectivity.

4 & 5) found insignificant estimates of C_1 when both large and small regions along the network were included in the regression. For the US data similar to that used by Duranton and Turner (2012), Baum-Snow (2017, Table 5) ran a variant of (6.1) to estimate the impact of interregional transport access on industry-specific employment in a metro area. However, the estimated coefficient of C_1 was insignificant for all but manufacturing employment; it was negatively significant for manufacturing employment.[45] Thus, the estimated impacts of interregional transport access on the size of an individual agglomeration vary widely, and there is no consensus even on the sign of the impacts.

From the knowledge of endogenous agglomeration mechanisms obtained in this paper, behind the incoherent regression results, we suspect the ignorance of the effects of interregional transport costs on the spatial distribution of agglomerations. What happened appears to be an agglomeration at the global scale toward a smaller number of larger regions. In the study by Faber (2014), the decline of peripheral regions is a mirror image of the growth of core regions excluded in his regression.[46] It is similarly expected that in the study by Duranton and Turner (2012), the growth of large metro areas is a mirror image of the decline in the peripheral areas excluded in their regression, although there is no explicit discussion on this aspect in their paper.

Although the regional variation in interregional transport access would certainly influence that in agglomeration size, this relation could be correctly identified only after controlling for the economy-wide effects of improved transport access on the overall spatial distribution of agglomerations. In class (i) models, an improvement of interregional accessibility at a given location does not necessarily result in a size growth or decline of agglomeration at that location (refer to Section 5.1). It is thus natural to obtain an insignificant average effect of improved transport access for both large and small regions along the new transport network as in Baum-Snow et al. (2016, Tables 4 & 5). Moreover, both Faber (2014, Table 6) and Duranton and Turner (2012, Table E2) found evidence for the agglomeration shadow, i.e., larger distance from the nearest major agglomeration tends to promote the growth of a region, which further suggests the relevance of class-(i) mechanisms.[47]

[45]Similar studies by Storeygard (2016) and Yamasaki (2017) established a positive relation between interregional transport access and regional agglomeration in the case of Sub-Saharan Africa for the 2002–2008 period and Japan for the 1885–1920 period, respectively. Since the the focus of these studies is the early stage of economic development, their results may not be directly comparable to those of Faber (2014) and Duranton and Turner (2012), as well as to our theoretical results. For example, the latter paper investigated the situation in which industrialization took place along with the introduction of railways in response to the spread of steam power in Japan. But, the decomposition of the causal relationship among industrialization, improvement in interregional transport access and urbanization is not obvious.

[46]It is also pointed out by Baum-Snow et al. (2016, Tables 4 & 5) that there was a significant increase in agglomeration size in large regions in the China case.

[47]In the study by Duranton and Turner (2012), the monotonic relationship between the size of a metro area and the level of inter-urban transport infrastructure in (6.1) is rationalized by assuming an open-city specification in the underlying theoretical model. But, the significant urban shadow effect among the included metro areas casts doubt on this justification.

If our reasoning were correct, the discrepancy between the actual SIZE_{it} and the estimated $\widehat{\text{SIZE}}_{it}$ in these regressions is likely to be systematic. In this case, the "clear" estimated impacts of interregional transport development based on a selected set of regions (e.g., relatively large or small regions or metro areas) could result in misleading policy implications.[48]

6.2.2 On the spatial extent of an agglomeration

The same specification (6.1) makes perfect sense when it comes to evaluating local dispersion. Baum-Snow (2007) and Baum-Snow et al. (2017) presented evidence for local dispersion as a consequence of improved interregional transport access in the cases of US metro areas for 1950–1990 and Chinese prefectures during 1990–2010, respectively.[49] In these studies, SIZE_{it} denotes the change in the population/production size of the central area within a larger region i (the metro area for the US and the province for China). They both reported a significantly negative estimated coefficient of C_1 given an improvement of interregional access after controlling for the growth of each region.[50,51] Their findings are consistent with our results on local dispersion (in sections 5.2 and 5.3).[52]

Recall the population agglomeration at the global scale and dispersion at the local scale in response to the development of the nation-wide highway and high-speed railway networks in Japan after 1970 discussed in Section 1 (and Appendix A). Empirical evidence suggests that China and the US experienced essentially the same phenomena.

6.2.3 On the spatial patterns of agglomeration

Finally, we explore the possibility of testing our theoretical predictions on the spatial patterns of agglomerations mentioned above rather than testing hypotheses about an individual agglomeration using the regression models of type (6.1). For population agglomerations, we typically have only a single set of agglomerations at a given point in time so that hy-

[48]It may be also important to include more extensive regional variations in interregional transport access to pin down the regions in which agglomerations grow. Among the possible asymmetries, those of particular importance are the locations of major terminals and modal intersections as well as their historical order of development that arise often from policy intensions rather than from geographical advantages. An explicit consideration of such exogenous evolution of transport network will make it possible to evaluate the causality between transport network development and population agglomeration.

[49]Garcia-López (2012) and Garcia-López, Holl and Viladecans-Marsal (2015) conducted similar studies using Spanish data.

[50]As discussed in these studies, their results of local dispersion can also be interpreted as suburbanization in response to improved intra-urban transport infrastructure in classical urban economic theory (e.g., Alonso, 1964).

[51]Faber (2014, Table 5 and Figure 4) showed a related evidence that agglomeration relatively proceeds in regions at a certain distance (around 100–150km) from the highways rather than those along the highways.

[52]Baum-Snow (2017) extended the work of Baum-Snow (2007) by replacing the outcome variable in (6.1) with the local dispersion (or suburbanization) of employment in each industry instead of population and showed that there are variations in the extent of local dispersion across industries.

potheses concerning their spatial distribution are not testable. However, such tests become possible by considering an individual industry as a unit of observation. If a distinct set of agglomerations can be identified for each industry, we have variations in the spatial patterns of agglomerations across industries at a given point in time. As a recent attempt, Mori et al. (2017) adopted the clustering framework developed by Mori and Smith (2014) and used the data on Japanese manufacturing three-digit industries during 1995–2015. Through this, they have shown that the number of agglomerations decreases (i.e., agglomeration at the global scale proceeds) and the average areal size of individual agglomerations increases (i.e., dispersion at the local scale proceeds) in response to a decrease in the industry-specific sensitivity to transport costs: the transport cost per unit distance and unit value of output.[53] In their study, by setting a unit of observation to an individual industry, the variation across industries made it possible to directly test the two predictions from our theoretical results mentioned above on the spatial patterns of agglomerations.

To sum up, knowledge of the behavior of general economic geography models brings together the seemingly unrelated pieces of empirical evidence on agglomeration patterns. Our interpretation of the results from the existing literature seems to suggest a strong relevance of endogenous agglomeration mechanisms to the observed regional variations in agglomeration size.

6.3 Structural model-based approaches

We now turn to the structural model-based approach to evaluate the causal effects of regional agglomerations summarized by Redding and Rossi-Hansberg (2017, §3). In this approach, perhaps one of the most popular approaches in quantitative spatial economics, the basic premise is that the primary source of the regional variation in agglomeration size is the heterogeneity in exogenous (or first-nature) regional advantages rather than endogenous (or second-nature) advantages considered in this paper. Thus, given the exogenous productivity or residential amenity difference across regions, a larger population of a given region is always associated with higher exogenous productivity or amenity in that region.

Redding and Sturm (2008) and Allen and Arkolakis (2014) employed the first nature-based approach to study regional agglomerations. A remarkable feature of these models is that they not only rely on exogenous advantages in explaining agglomeration patterns but also incorporate agglomeration externalities to the extent that the unique equilibrium is

[53]This definition of the sensitivity to transport costs is an empirical counterpart of the iceberg transport costs in our theoretical models.

guaranteed (so that the tractability of the model is preserved).[54,55] This subtle situation has been realized by adopting class (ii) models. As shown in Section 5.2, these models have a parameter range in which agglomeration diseconomies dominate agglomeration economies independently of the level of transport costs. This special property of class (ii) models is due to the independence of local dispersion forces on the distance structure of the model. In this context, the model parameters are calibrated to replicate the relevant regional variations (such as regional population sizes) in the absence (or presence) of a given treatment such as transport development; then the counterfactual regional variations are derived in the presence (or absence) of the treatment given these calibrated parameter values.

There are two caveats in understanding the implications obtained from the results of this approach.

First, although agglomeration externalities account for a part of the regional variations, most variations appear to be absorbed by the structural residuals. Consider the study by Redding and Sturm (2008) for example. Using a many-region extension of the Helpman (1998) model, they quantified the impact of the change in market accessibility in Germany before and after the division/reunification of the country after the war. To determine the time invariant set of parameter values, their model was calibrated to fit city size distributions in the prewar 1939 Germany. If the log of actual city size is regressed on the log of the unobserved amenity, i.e., the structural residuals, one can immediately find that most of the variation in city size is explained by these residuals, as shown in Figure 13, in which the dashed line indicates the fitted line by OLS:

$$\log(L_i/\overline{L}) = \underset{(0.210)}{-7.191} + \underset{(0.050)}{1.587}\log(\hat{A}_i)\,, \quad \text{adj.}\, R^2 = 0.896\,, \tag{6.2}$$

where L_i and $\overline{L}$ denote the population size of city i and the average city size in 1939 Germany, respectively, and $\hat{A}_i$ denotes the estimated unobserved amenity (i.e., the variation unexplained by the model) in city i. Standard errors are in the parentheses.[56] Thus, 90% of the variation in city size is unexplained by the model and fixed exogenously in their counterfactual exercises. While it is not surprising given that there is no possibility of endogenous agglomeration in this setup, this result indicates that the structure of their model itself plays only a small role (10% at maximum) in explaining the treatment effect.

Second, their predictions of treatment effects crucially depend on specific feature of the underlying economic geography models. In particular, in the class (ii) models, the reduction

[54]Redding (2016) and Monte et al. (2016) extended the work of Redding and Sturm (2008) by adding different sources of exogenous location-fixed factors.

[55]The majority of the structural model-based studies for a regional economy involve no agglomeration externalities (see, e.g., Donaldson and Hornbeck, 2016; Baum-Snow et al., 2016; Alder, 2016; Caliendo, Parro, Rossi-Hansberg and Sarte, 2016).

[56]The data for the regression are available from the online appendix for Redding and Sturm (2008).

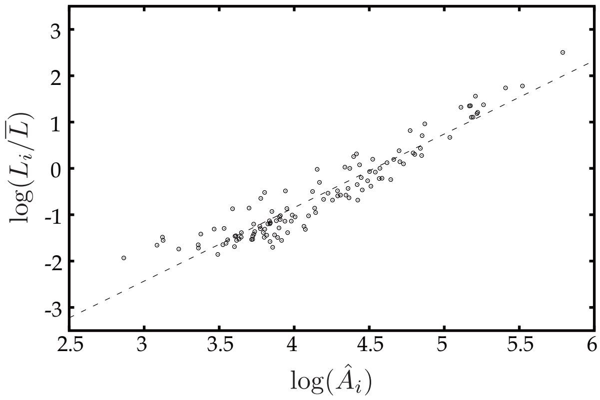


Figure 13: The relationship between city sizes and the estimated unobserved amenities in Redding and Sturm (2008)

of interregional transport costs reduces each regional agglomeration caused by the first-nature advantage of the region and promotes local dispersion. This causal relation has been demonstrated by a numerical counterfactual exercise in the study by Redding and Rossi-Hansberg (2017, §3.9). In fact, it can be formally shown (see Appendix D) that this scale-down effect is a general property of class (ii) models, although there are certain cases in which local landownership mitigates the effect by counteracting congestion externalities.

But, if class (i) models were used instead, even in the parameter range in which no endogenous agglomeration occurs (as in Redding and Rossi-Hansberg, 2017, §3), the sign of the treatment effect will reverse, i.e., agglomeration externalities would scale up the first-nature advantage (see Appendix D for a formal proof). The resulting implications thus become opposite depending on the choice of dispersion force to be included in the model.

Thus, even in the context in which first-nature heterogeneity plays a primary role, the knowledge of endogenous agglomeration and dispersion mechanisms is crucial to understand the logic and direction of the treatment effect. For that purpose, our stylized analytical framework appears to be useful.

More recent structural approaches have shifted to accommodate multiple equilibria with endogenous agglomerations (e.g., Ahlfeldt et al., 2015; Owens et al., 2017; Nagy, 2017). In that case, class (ii) models are no longer useful as they can endogenously generate at most a unimodal agglomeration. To explain both multiple agglomerations and local dispersion by endogenous mechanisms, one needs a model of either class (iii) or of a more general class that was not treated in this paper. We briefly touch on these issues in Section 7 in delineating future directions for research.

7 Concluding remarks

In this paper, we established a formal classification of economic geography models in terms of model-specific spatial patterns of agglomeration. By allowing for the presence of many regions, the spatial scale of agglomeration and dispersion is made explicit unlike the two-region setup or many-region setup without interregional space. In particular, two dispersions at high and low transport costs that look identical in a two-region setup turned out to differ and take place at global and local spatial scales, respectively. In fact, when dispersion proceeds at the local scale, typically, agglomeration proceeds at the same time but at the global scale and vice versa.

Our theoretical results indicated a new direction for future empirical research based on endogenous agglomerations. First, the contrasting agglomeration and dispersion behaviors at different spatial scales suggest the need for distinguishing individual agglomerations rather than measuring agglomeration by a scalar index. Second, endogenous agglomeration and dispersion mechanisms generally do not isolate the growth and decline of individual agglomerations and can only provide insights on their spatial patterns. Our new results regarding the impact of transport development on the spatial patterns of agglomeration could provide a unified interpretation of a variety of existing results from reduced-form regressions on regional agglomerations. Furthermore, we have shown that our analytical framework is useful for obtaining formal predictions of treatment effects in structural model-based analyses that aim to explain the agglomeration patterns.

But, the relatively simple classification of the spatial scale of agglomeration and dispersion in our paper owes to the simple structure of the models considered. Below, we discuss possible future directions of theoretical research to account for richer more realistic variations in agglomeration and dispersion across different spatial scales.

There are at least three major directions of research that could be pursued. The first possible extension is to distinguish location incentives between firms and consumers/workers as is traditionally done in the urban economics literature (e.g., Fujita and Ogawa, 1982; Ota and Fujita, 1993; Lucas and Rossi-Hansberg, 2002). In all the models considered in this paper, the location incentive of firms and that of consumers/workers coincide. This simplification may be justifiable when the global pattern of agglomeration (in particular, sizes and locations of cities) is the subject of the study. But, their distinction becomes crucial for explaining the location patterns within a city. There are recent attempts, for example, by Ahlfeldt et al. (2015) and Owens et al. (2017) in this direction. These models typically abstract from the inter-city/regional interactions in an *open-city* setup. But, possible equilibrium patterns and their stability properties in this class of models are still not well known and could thus represent a fruitful avenue for future research.

The second possibility is to consider different transport cost structures by industries.

For instance, Fujita and Krugman (1995) introduced transport costs for land-intensive rural (agricultural) goods along with that of urban goods. In the presence of rural goods that are costly to transport, the delivered price for such goods is lower in regions farther away from the agglomerations, which generates dispersion force. This is similar to the local dispersion force in that even a small deviation from an urban agglomeration will decrease the price of rural goods and increase the payoff of the deviant. But, the advantage of dispersion persists outside the agglomeration, i.e., it depends on the distance structure of the model. This type of dispersion force will lead to the formation of an *industrial belt*, a continuum of agglomeration associated with multiple atoms of agglomeration as demonstrated by simulations in the studies by Mori (1997) and Ikeda, Murota, Akamatsu and Takayama (2017b). But, the formal characterization of industrial belts remains to be done.

Finally, another possible extension is to incorporate diversity in increasing returns, leading to the diversity in spatial scale of agglomeration and dispersion and to the diversity in the size of agglomeration. In all the models with global dispersion force considered in this paper, the sizes of agglomerations in equilibrium are basically the same (refer to the simulation exercises in Section 5.1) since each model has only one type of increasing returns. This is counterfactual as actual city size distributions are quite diverse and are well known to roughly follow the power law.[57] Initial attempts to account for the diversity in increasing returns by introducing multiple increasing-returns industries have been made in the context of the NEG models by, e.g., Fujita et al. (1999b), Tabuchi and Thisse (2011), and Hsu (2012)'s spatial competition model. Alternatively, Desmet and Rossi-Hansberg (2009), Desmet and Rossi-Hansberg (2014), Desmet and Rossi-Hansberg (2015), Desmet et al. (2017), and Nagy (2017) incorporated dynamic externalities through endogenous innovation and spillover effects.

[57]The models considered in this paper are consistent with heterogeneously sized agglomerations in equilibrium; but, it is not possible to replicate the high diversity of city sizes seen in reality.

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A The development of transport network and urban agglomeration patterns in Japan for 1970-2010

To compare the urban agglomeration patterns in Japan, we define an *urban agglomeration* (*UA*) as the set of contiguous 1km-by-1km cells with population density at least $1000/\text{km}^2$ and the total population at least 10000.[58] (Basic results below remain the same for alternative threshold densities and population.) For the part of Japan that are contiguous by roads to at least one of the four major islands (Hokkaido, Honshu, Shikoku and Kyushu), 538 and 521 UAs are identified, and are depicted in Panels (a) and (b) of Figure 14 for years 1970 and 2010, respectively, where the warmer color indicates a larger population. They together account for 64% and 77% of the total population in 1970 and 2010, respectively. Thus, there is substantial 20% increase in the urban concentration in these forty years. Of the 538 UAs that existed in 1970, 359 have survived, while 59 disappeared and 120 have been integrated to other UAs by 2010. Of the 521 UAs that existed in 2010, 162 were newly formed after 1970 (including those split from the existing UAs).[59]

Panels (c) and (d) of Figure 14 show the highway and high-speed railway network in use as of 1970 and 2010, respectively. The comparison of these panels indicates an obvious substantial expansion of these networks during these forty years as mentioned in the text.

Panels (a), (b) and (c) of Figure 15 show the distributions of the rates of population growth, areal size growth and population density growth of individual UAs for the set of the 359 UAs that have survived throughout the forty year period. A UA experienced on average 34% (111%) of population growth, 107% (131%) of areal size growth and −36% (18%) of population density growth, respectively, where the numbers in parentheses are the standard deviations.

As a larger share of population have concentrated in a smaller number of UAs in 2010 than in 1970, the spatial size of an individual UA has doubled on average. But, these spatial expansions are not simply due to the shortage of available land in UAs. Note that the population density (after controlling for the growth of total population) decreased by one-third on average. We take this as an evidence of global concentration with local dispersion under the improvement of interregional transport access.

[58] Population count data are obtained from Statistics Bureau, Ministry of Internal Affairs and Communications of Japan (1970, 2010).

[59] A pair of UAs in two different time points are considered to be *the same* UA if the most populated cell of one of the UAs is contained in the other UA, and vice versa. If the most populated cell of a UA in 1970 is contained in a UA in 2010, but not vice versa, then the former UA is taken to be *integrated* into the latter UA. If the most populated cell of a UA in 2010 is contained in a UA in 1970, but not vice versa, then the former UA is taken to have *split* from the latter UA. If the most populated cell of a UA in 1970 is not contained in any UA in 2010, the former UA is taken to have *disappeared*. Similarly, if the most populated cell of a UA in 2010 is not contained in any UA in 1970, then the former UA is taken to be *newly formed*.

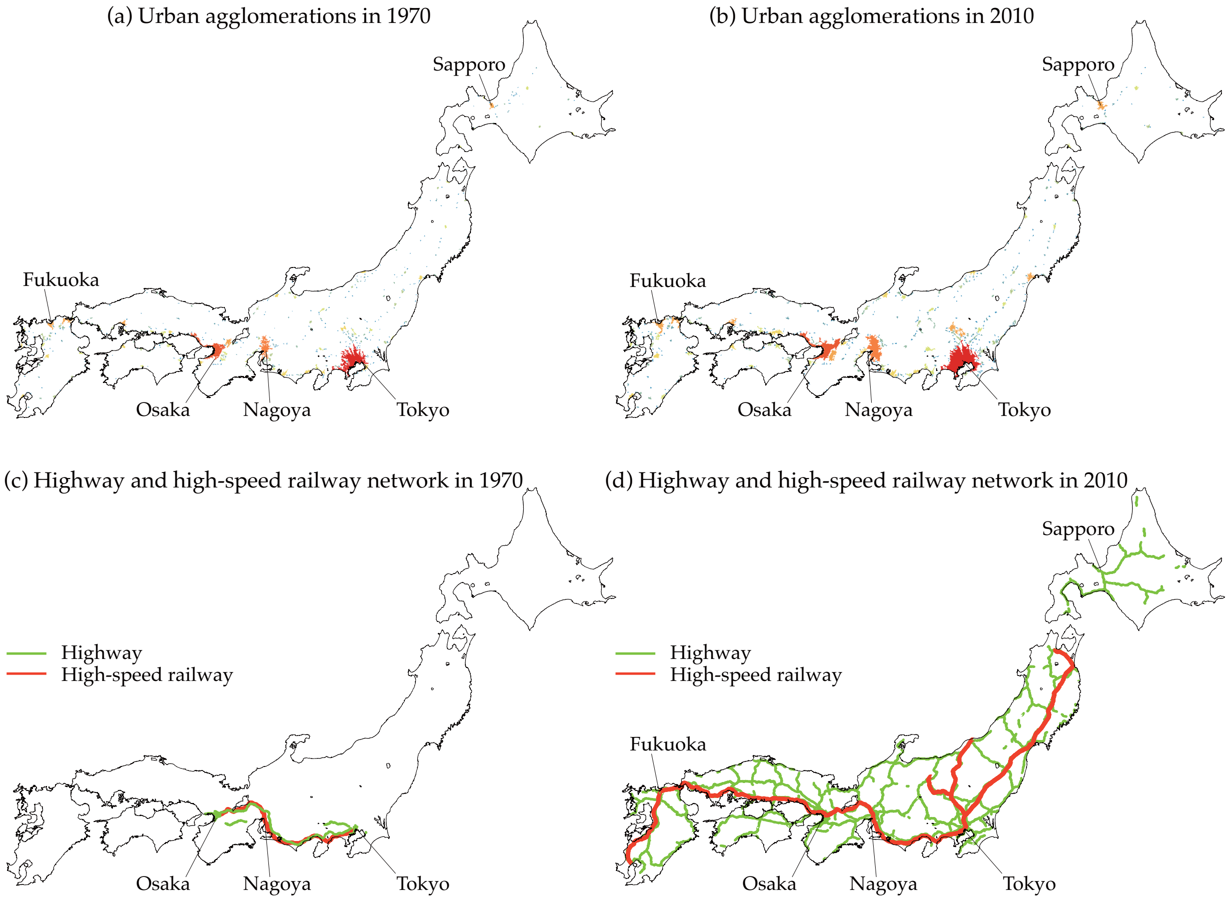


Figure 14: Urban agglomerations and transport network in Japan

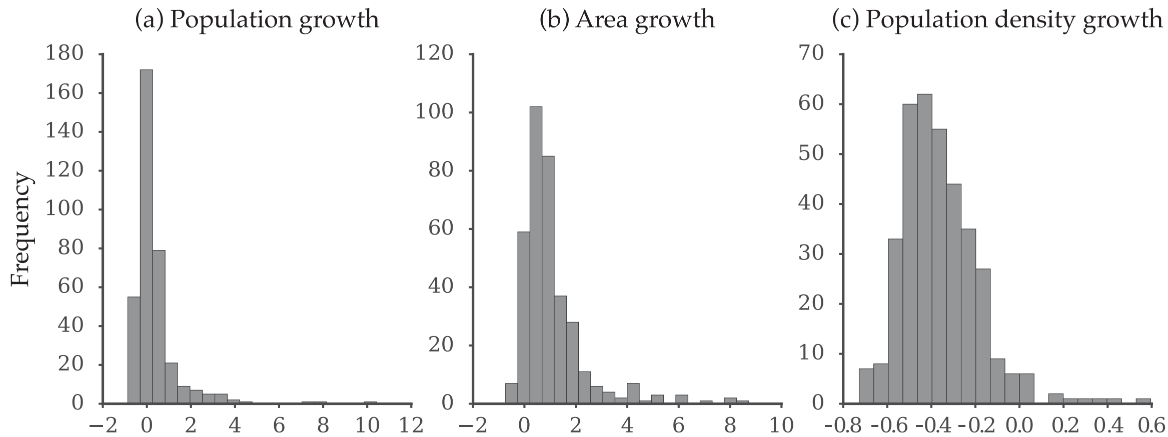


Figure 15: Growth rates of the sizes of urban agglomerations in Japan

B Racetrack economy and simplification of stability analysis

The friction matrix $\boldsymbol{D}$ for a racetrack economy is a *circulant matrix*. In this appendix, we first review some useful properties of circulant matrices. Then, we see how stability analysis of the flat-earth pattern on a symmetric racetrack economy is simplified by these properties.

B.1 Facts on properties of circulant matrices

A *circulant matrix* $\boldsymbol{C}$ of dimension K is defined as a K-by-K square matrix of the form

$$\boldsymbol{C} \equiv \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{K-2} & c_{K-1} \\ c_{K-1} & c_0 & c_1 & c_2 & \cdots & c_{K-2} \\ c_{K-2} & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & c_2 \\ c_2 & \cdots & c_{K-2} & c_{K-1} & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{K-2} & c_{K-1} & c_0 \end{bmatrix}. \tag{B.1.1}$$

The elements of each row of $\boldsymbol{C}$ are identical to those of the previous row, but are moved one position to the right and wrapped around. The whole circulant is evidently determined by the first row vector $\boldsymbol{c} = (c_i)_{i=0}^{K-1}$. Circulant matrices are known to satisfy the following properties (see, e.g., Horn and Johnson, 2012):

Lemma B.1. Let $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ be circulant matrices. Then, their sum $\boldsymbol{C}_1 + \boldsymbol{C}_2$ and product $\boldsymbol{C}_1\boldsymbol{C}_2$ also are circulants. Also, they are commutative, i.e., $\boldsymbol{C}_1\boldsymbol{C}_2 = \boldsymbol{C}_2\boldsymbol{C}_1$. Let $\boldsymbol{C}_3$ be a nonsingular circulant matrix. Then, its inverse $\boldsymbol{C}_3^{-1}$ also is a circulant.

Lemma B.2. Let $\boldsymbol{C}$ be a K-by-K circulant matrix. Let $\boldsymbol{Z} = [z_{jk}]$ be the discrete Fourier transformation (DFT) matrix where $z_{jk} = K^{-1/2}\exp[\mathrm{i}\theta jk]$ with $\theta \equiv 2\pi/K$ and $\mathrm{i} \equiv \sqrt{-1}$. Then, $\boldsymbol{C}$ is diagonalized by the similarity transformation by $\boldsymbol{Z}$ as $\boldsymbol{Z}^*\boldsymbol{C}\boldsymbol{Z} = \mathrm{diag}[\boldsymbol{\lambda}]$ where * denotes the conjugate transpose.

$\boldsymbol{\lambda} = (\lambda_i)$ are the eigenvalues of $\boldsymbol{C}$. The kth eigenvalue and the associated eigenvector of $\boldsymbol{C}$ are λ_k and the kth row of the DFT matrix $\boldsymbol{Z}$, respectively. Furthermore, $\boldsymbol{\lambda}$ is given by the DFT of the first row vector $\boldsymbol{c}$ of $\boldsymbol{C}$ as $\boldsymbol{\lambda} = K^{1/2}\boldsymbol{Z}\boldsymbol{c}^\top$.

Remark B.1. It follows that all K-by-K circulant matrices share the same eigenvectors.

Consider a matrix $\boldsymbol{A}$ that is defined by a matrix polynomial of a nonsingular circulant matrix $\boldsymbol{C}$:

$$\boldsymbol{A} = c_0\boldsymbol{I} + c_1\boldsymbol{C} + c_2\boldsymbol{C}^2 + \cdots. \tag{B.1.2}$$

Thanks to Lemma B.1 and B.2, one has that (i) $\boldsymbol{A}$ is a circulant matrix (note that $\boldsymbol{I}$ is also a circulant) and thus that (ii) its eigenvalues $\boldsymbol{\mu} \equiv (\mu_i)$ is given by those of $\boldsymbol{C}$, $\boldsymbol{\lambda}$, by the relation

$$\mathrm{diag}[\boldsymbol{\mu}] = \boldsymbol{Z}^*\boldsymbol{A}\boldsymbol{Z} = \boldsymbol{Z}^*(c_0\boldsymbol{I} + c_1\boldsymbol{C} + c_2\boldsymbol{C}^2 + \cdots)\boldsymbol{Z} = c_0\boldsymbol{I} + c_1\,\mathrm{diag}[\boldsymbol{\lambda}] + c_2\,\mathrm{diag}[\boldsymbol{\lambda}]^2 + \cdots, \tag{B.1.3}$$

or, in the element-wise manner,

$$\mu_k = c_0 + c_1\lambda_k + c_2\lambda_k^2 + \cdots . \tag{B.1.4}$$

Remark B.2. If $\boldsymbol{C}$ is in addition symmetric, $\boldsymbol{A}$ is also symmetric. It implies that the eigenvalues $\boldsymbol{\lambda} = (\lambda_k)$ and $\boldsymbol{\mu} = (\mu_k)$ as well as their associated eigenvectors are all real.

B.2 Eigenvalues and eigenvectors of the friction matrix

We derive eigenvalues and eigenvectors of the friction matrix for later use. To simplify notation, we define $r \in (0, 1)$ that represents the *freeness of transport* between two consecutive regions on the racetrack: $r(\tau) \equiv \exp[-\tau/K]$ where we rescale τ so that the circumferential length of the economy is fixed. From the definition of r, it is a monotonically decreasing function of the transportation cost (technology) parameter τ and hence r and τ are mutually interchangeable. We shall use r as the transport technology parameter in the present appendix. Using r, one has $d_{ij} = r^{\ell_{ij}}$.

For analyzing specific models, it is useful to derive the eigenvalues of the *row-normalized* friction matrix $\bar{\boldsymbol{D}} \equiv \boldsymbol{D}/d$ with $d \equiv \sum_{j\in\mathcal{K}} d_{0,j}$. We note that every row has exactly the same row sum because $\boldsymbol{D}$ is circulant. It turns out that on a racetrack economy, $\boldsymbol{D}$ is a circulant matrix since $d_{ij} = r^{\ell_{ij}} = r^{\ell_{i+1,j+1}} = d_{i+1,j+1}$ for all i, j ($\bmod K$ for indices). Furthermore, $\boldsymbol{D}$ is symmetric and real and hence all the eigenvalues and eigenvectors are real. The analytical expressions of the eigenvalues and eigenvectors of $\bar{\boldsymbol{D}}$ are available (Akamatsu et al., 2012):

Lemma B.3. Let $f_k(r)$ be the kth eigenvalue of the row-normalized friction matrix $\bar{\boldsymbol{D}}$ for a racetrack economy with K regions. Assume that K is a multiple of four. Define $\Psi_k(r) > 0$ and $\bar{\Psi}(r) > 0$ by

$$\Psi_k(r) \equiv \frac{1 - r^2}{1 - 2\{\cos[\theta k]\}r + r^2}, \ \bar{\Psi}(r) \equiv \frac{1 + r^{K/2}}{1 - r^{K/2}} \tag{B.2.1}$$

with $\theta = 2\pi/K$. Then, $\{f_k(r)\}$ is given by

$$f_k(r) = \begin{cases} \Psi_k(r)\Psi_{K/2}(r) & (k\text{: even}) \\ \Psi_k(r)\Psi_{K/2}(r)\bar{\Psi}(r) & (k\text{: odd}) \end{cases} \qquad k = 0, 1, 2, \ldots, K/2, \tag{B.2.2}$$

where $k = 1, 2, \ldots, K/2 - 1$ are of multiplicity two. The associated eigenvectors are

$$\boldsymbol{\eta}_0 = K^{1/2}\cos[0]) = K^{1/2}(1, 1, \ldots, 1), \tag{B.2.3}$$

$$\boldsymbol{\eta}_k^+ = K^{1/2}(\cos[\theta k i]), \ \boldsymbol{\eta}_k^- = K^{1/2}(\sin[\theta k i]) \qquad k = 1, 2, \ldots, K/2 - 1, \tag{B.2.4}$$

$$\boldsymbol{\eta}_{K/2} = K^{1/2}\cos[\pi i]) = K^{1/2}(1, -1, 1, -1, \ldots, 1, -1) \tag{B.2.5}$$

where $\boldsymbol{\eta}_k^+$ and $\boldsymbol{\eta}_k^-$ are the two eigenvectors associated to $f_k(r)$ ($k = 1, 2, \ldots, K/2 - 1$).

Remark B.3. For each k with $1 \le k \le K/2 - 1$, we can focus on a single eigenvector of the form $\boldsymbol{\eta}_k = \boldsymbol{\eta}_k^+ = K^{1/2}(\cos[\theta k i])$ since we do not distinguish rotationally symmetric patterns; any linear combination of $\boldsymbol{\eta}_k^+$ and $\boldsymbol{\eta}_k^-$ reduces to a single trigonometric curve with the same wave length as them.

See **Figure 2** and **Figure 3** for illustration. Note that we drop the constant $K^{1/2}$ in the main text for slight simplification. In particular, $f_0(r) = 1$, $f_1(r) = (1-r)/(1+r)$ and $f_{K/2}(r) = \{(1-r)/(1+r)\}^2$. Furthermore, employing the analytical expression of $\{f_k(r)\}$ in Lemma B.3, one shows:

Corollary B.1. $\{f_k(r)\}$ satisfy the following properties if K is a multiple of four.

1. Every $f_k(r)$ is a monotonically decreasing function of r except for $f_0(r) = 1$.
2. For all r, $\{f_k(r)\}$ with $k = 0, 1, 2, \ldots, K/2$ are ordered as

$$\begin{cases} 1 = f_0 > f_2 > \cdots > f_{2k} > \cdots > f_{K/2}, \\ 1 > f_1 > f_3 > \cdots > f_{2k+1} > \cdots > f_{K/2-1}. \end{cases} \tag{B.2.6}$$

   with $f_1(r) > f_2(r)$ and $f_{K/2-1}(r) > f_{K/2}(r)$.

The second property yields that $\min_k\{f_k(r)\} = f_{K/2}(r)$ and $\max_{k\geq 1}\{f_k(r)\} = f_1(r)$ for all r. We note that every $f_k(r(\tau))$ ($k \geq 1$) as a function of τ is monotonically increasing.

Example B.1. In a racetrack economy with $K = 4$, the friction matrix $\boldsymbol{D}$ is given by

$$\boldsymbol{D} = \begin{bmatrix} 1 & r & r^2 & r \\ & 1 & r & r^2 \\ & & 1 & r \\ Sym. & & & 1 \end{bmatrix}. \tag{B.2.7}$$

Its row sum is $d = 1 + r + r^2 + r = (1+r)^2$ and thus $\bar{\boldsymbol{D}} = \boldsymbol{D}/(1+r)^2$. The eigenvalues of $\bar{\boldsymbol{D}}$ are given by

$$f_0 = 1,\ f_1 = \frac{1-r}{1+r},\ f_2 = \left(\frac{1-r}{1+r}\right)^2. \tag{B.2.8}$$

The associated eigenvectors are $\boldsymbol{\eta}_0 = 2^{-1}(1,1,1,1)$, $\boldsymbol{\eta}_1^+ = 2^{-1}(1,0,-1,0)$, $\boldsymbol{\eta}_1^- = 2^{-1}(0,1,0,-1)$, and $\boldsymbol{\eta}_2 = 2^{-1}(1,-1,1,-1)$.

B.3 Representing the eigenvalues of $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ by those of $\bar{\boldsymbol{D}}$

We assume that the payoff function $\boldsymbol{v}$ is differentiable. For simplicity we assume that $\boldsymbol{v}$ is defined for the nonnegative orthant $\mathbb{R}_+^K$.

Assumption B.1. The payoff function $\boldsymbol{v} : \mathbb{R}_+^K \to \mathbb{R}^K$ is continuously differentiable.

Given $\boldsymbol{v}$, we define a spatial equilibrium by the following *variational inequality problem* (VIP):

$$\text{[VIP]} \quad \text{Find } \boldsymbol{h}^* \in \mathcal{D} \text{ such that } \ \boldsymbol{v}(\boldsymbol{h}^*) \cdot (\boldsymbol{h} - \boldsymbol{h}^*) \leq 0 \ \text{ for all } \boldsymbol{h} \in \mathcal{D}. \tag{B.3.1}$$

Definition B.1. A spatial equilibrium is a solution to [VIP].

Alternative equivalent definition of long-run equilibria is found in the main text.

The flat-earth equilibrium $\bar{\boldsymbol{h}} \equiv (h, h, \ldots, h)$ with $h \equiv H/K$ is obviously a spatial equilibrium; because $\boldsymbol{v}(\bar{\boldsymbol{h}}) = \bar{v}\mathbf{1}$ with the uniform level of payoff $\bar{v}$, we have $\boldsymbol{v}(\bar{\boldsymbol{h}}) \cdot (\boldsymbol{h} - \bar{\boldsymbol{h}}) = \bar{v}\mathbf{1} \cdot (\boldsymbol{h} - \bar{\boldsymbol{h}}) = \bar{v}(H - H) = 0$ for all $\boldsymbol{h} \in \mathcal{D}$. In preparation for Appendix B.4 below, we discuss the eigenvalues of the Jacobian matrix of the payoff function at the flat-earth equilibrium. Appendix C demonstrates that, at the flat-earth equilibrium in a racetrack economy, we can express the Jacobian matrix of the payoff function in the following way

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = G_0(\bar{\boldsymbol{D}}) G(\bar{\boldsymbol{D}}), \tag{B.3.2}$$

$$G(\bar{\boldsymbol{D}}) \equiv c_0 \boldsymbol{I} + c_1 \bar{\boldsymbol{D}} + c_2 \bar{\boldsymbol{D}}^2, \tag{B.3.3}$$

where $G_0(\cdot)$ and $G(\cdot)$ are interpreted as matrix polynomials. $G_0(\bar{\boldsymbol{D}})$ is a positive definite matrix defined by $\bar{\boldsymbol{D}}$. Since $\bar{\boldsymbol{D}}$ is circulant, $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ also is circulant. Thus, as discussed, we can express the kth eigenvalues of $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$, e_k, by that of $\bar{\boldsymbol{D}}$, f_k, in terms of a model-dependent functions $G_0(f)$ and $G(f)$

$$e_k = G_0(f_k) G(f_k) \tag{B.3.4}$$

where $G_0(f_k)$ and $G(f_k)$ are the kth eigenvalue of $G_0(\bar{\boldsymbol{D}})$ and $G(\bar{\boldsymbol{D}})$, respectively. The associated eigenvectors are the same as those of $\bar{\boldsymbol{D}}$. As we have $G_0(f_k) > 0$, for the purpose of examining the sign of e_k, we only need to check that of $G(f_k)$.

B.4 Stability analysis of the flat-earth equilibrium

Employing the above results, this section derives the results presented in Sections 2.3 and 3.

Notations. Let $\mathcal{D} \equiv \{\boldsymbol{h} \in \mathbb{R}^K \mid \boldsymbol{h} \cdot \mathbf{1} = H,\ h_i \geq 0\}$ denote the set of possible spatial patterns. Also, in relation to $\mathcal{D}$, let $T\mathcal{D}(\boldsymbol{h}) \equiv \{\boldsymbol{z} \in \mathbb{R}^K \mid \boldsymbol{z} = \alpha(\boldsymbol{y} - \boldsymbol{h})$ for some $\boldsymbol{y} \in \mathcal{D}$ and $\alpha \geq 0\}$ denote the tangent cone of $\mathcal{D}$ at $\boldsymbol{h} \in \mathcal{D}$, and $T\mathcal{D} \equiv \{\boldsymbol{z} \in \mathbb{R}^K \mid \boldsymbol{z} \cdot \mathbf{1} = 0\}$ denote the tangent space of $\mathcal{D}$. Note that for any $\boldsymbol{h} \in \operatorname{int} \mathcal{D}$, we have $T\mathcal{D}(\boldsymbol{h}) = T\mathcal{D}$ because $\mathcal{D}$ is a convex subset of a hyperplane.

B.4.1 Derivations for Section 2.3

We summarize our assumptions on the dynamic $\boldsymbol{F}$ as follows, where with a notational abuse we let $\boldsymbol{F}(\boldsymbol{h}) = \boldsymbol{F}(\boldsymbol{h}, \boldsymbol{v}(\boldsymbol{h}))$. We assume that $\boldsymbol{F}$ is defined for the nonnegative orthant $\mathbb{R}^K_+$.

Assumption B.2. The dynamic $\boldsymbol{F} : \mathbb{R}^K_+ \to \mathbb{R}^K$ satisfies the following properties.

1. (conservation) the total mass of mobile agents is invariant, i.e., $\boldsymbol{F}(\boldsymbol{h}) \in T\mathcal{D}(\boldsymbol{h})$ for all $\boldsymbol{h} \in \mathcal{D}$.
2. (differentiability) $\boldsymbol{F}(\boldsymbol{h})$ is continuously differentiable with respect to $\boldsymbol{h}$ and $\boldsymbol{v}(\boldsymbol{h})$ in $\mathcal{D}$.
3. (stationarity at spatial equilibria) if $\boldsymbol{h}^*$ is a spatial equilibrium, then $\boldsymbol{F}(\boldsymbol{h}^*) = \mathbf{0}$.
4. (positive correlation) $\boldsymbol{v}(\boldsymbol{h}) \cdot \boldsymbol{F}(\boldsymbol{h}) > 0$ for all $\boldsymbol{h} \in \mathcal{D}$ such that $\boldsymbol{F}(\boldsymbol{h}) \neq \mathbf{0}$.

Example B.2. The set of dynamics that satisfy Assumption B.2 includes the *replicator dynamic* (Taylor and Jonker, 1978), which is the leading instance of the general class of *imitative dynamics*, the Brown–von Neumann–Nash dynamic (Brown and von Neumann, 1950; Nash, 1951), which is an instance of *excess payoff dynamics*, and, for interior equilibria, the *projection dynamic* (Dupuis and Nagurney, 1993). For more examples, refer to Sandholm (2010).

Consider a small deviation $\boldsymbol{\eta} \in T\mathcal{D}(\boldsymbol{h}^*) = T\mathcal{D}$ at an interior equilibrium $\boldsymbol{h}^* \in \operatorname{int}\mathcal{D}$. By conservation we must have $\boldsymbol{F}(\boldsymbol{h}^* + \boldsymbol{\eta}) \in T\mathcal{D}$ for such $\boldsymbol{\eta}$; it follows that

$$\boldsymbol{F}(\boldsymbol{h}^* + \boldsymbol{\eta}) = \boldsymbol{F}(\boldsymbol{h}^*) + \nabla\boldsymbol{F}(\boldsymbol{h}^*)\boldsymbol{\eta} + o(\|\boldsymbol{\eta}\|) = \nabla\boldsymbol{F}(\boldsymbol{h}^*)\boldsymbol{\eta} + o(\|\boldsymbol{\eta}\|) \in T\mathcal{D}. \tag{B.4.1}$$

Since $\boldsymbol{J} \equiv \nabla\boldsymbol{F}(\boldsymbol{h}^*)$ maps all $\boldsymbol{\eta} \in T\mathcal{D}$ into $T\mathcal{D}$, $\boldsymbol{J}$ defines a linear map from $T\mathcal{D}$ to $T\mathcal{D}$. Thus, the stability analysis of an interior equilibrium $\boldsymbol{h}^*$ reduces to examining the eigenvalues of the restricted linear map $\boldsymbol{J} : T\mathcal{D} \to T\mathcal{D}$. We thus focus on deviations $\boldsymbol{\eta}$ that live in $T\mathcal{D}$ (i.e., $\boldsymbol{\eta}$ such that $\boldsymbol{\eta} \cdot \mathbf{1} = 0$). In effect, we can ignore g_0, which is the associated eigenvalue for $\boldsymbol{\eta}_0 \equiv (1, 1, \ldots, 1)$ because $\boldsymbol{\eta}_0$ is the basis for $T\mathcal{D}^\perp$ (the orthogonal space of $T\mathcal{D}$, which is one-dimensional).

For general isolated interior equilibria $\boldsymbol{h}^* \in \operatorname{int}\mathcal{D}$, we have $\boldsymbol{v}(\boldsymbol{h}^*) = \bar{v}\mathbf{1}$ with the uniform level of payoff $\bar{v}$ and $\boldsymbol{F}(\boldsymbol{h}^*) = \mathbf{0}$, implying that $\boldsymbol{v}(\boldsymbol{h}^*) \cdot \boldsymbol{F}(\boldsymbol{h}^*) = 0$. Because $\boldsymbol{h}^*$ is an isolated interior equilibrium, the positive correlation property of $\boldsymbol{F}$ requires that there is a neighborhood $O \subset \mathcal{D}$ of $\boldsymbol{h}^*$ such that $\boldsymbol{v}(\boldsymbol{h}) \cdot \boldsymbol{F}(\boldsymbol{h}) > 0$ for all $\boldsymbol{h} \in O \setminus \{\boldsymbol{h}^*\}$. Also, by differentiability of $\boldsymbol{v}$ and $\boldsymbol{F}$ in $\operatorname{int}\mathcal{D}$, we can expand $\boldsymbol{v}$ and $\boldsymbol{F}$ in the vicinity of the equilibrium; that is, for sufficiently small $\boldsymbol{\eta}$ such that $\boldsymbol{h}^* + \boldsymbol{\eta} \in \mathcal{D}$ (i.e., $\boldsymbol{\eta} \in T\mathcal{D}$), the positive correlation property is equivalent to the condition

$$(\bar{v}\mathbf{1} + \nabla\boldsymbol{v}(\boldsymbol{h}^*)\boldsymbol{\eta}) \cdot (\boldsymbol{F}(\boldsymbol{h}^*) + \boldsymbol{J}\boldsymbol{\eta}) = (\nabla\boldsymbol{v}(\boldsymbol{h}^*)\boldsymbol{\eta}) \cdot (\boldsymbol{J}\boldsymbol{\eta}) > 0. \tag{B.4.2}$$

Note that $(\boldsymbol{J}\boldsymbol{\eta}) \cdot \mathbf{1} = 0$ because $\boldsymbol{J}\boldsymbol{\eta} \in T\mathcal{D}$ for all $\boldsymbol{\eta} \in T\mathcal{D}$.

In (B.4.2), suppose that $\boldsymbol{\eta} = \boldsymbol{\eta}_k$, where $\boldsymbol{\eta}_k$ ($k \geq 1$) is the kth eigenvector of the restricted linear map $\boldsymbol{J}$ with associated eigenvalue being g_k. Then, with $\boldsymbol{h} = \boldsymbol{h}^* + \boldsymbol{\eta}_k$ we have

$$(\boldsymbol{J}\boldsymbol{\eta}_k) \cdot (\nabla\boldsymbol{v}(\boldsymbol{h}^*)\boldsymbol{\eta}_k) = (g_k\boldsymbol{\eta}_k) \cdot (\nabla\boldsymbol{v}(\boldsymbol{h}^*)\boldsymbol{\eta}_k) = g_k(\boldsymbol{\eta}_k^\top\nabla\boldsymbol{v}(\boldsymbol{h}^*)\boldsymbol{\eta}_k) > 0. \tag{B.4.3}$$

which shows that if g_k and $\boldsymbol{\eta}_k$ are real (in particular, if $\boldsymbol{J}$ is symmetric)

$$\operatorname{sgn}[g_k] = \operatorname{sgn}[\boldsymbol{\eta}_k^\top\nabla\boldsymbol{v}(\boldsymbol{h}^*)\boldsymbol{\eta}_k] = \operatorname{sgn}\left[\sum_{i\in\mathcal{K}} \delta V_i z_i\right] \quad \text{where} \quad \delta V_i \equiv \sum_{j\in\mathcal{K}} \frac{\partial v_i(\boldsymbol{h}^*)}{\partial h_j} z_j \tag{B.4.4}$$

as in (2.1), where for simplicity we let $z_i \equiv \eta_{k,i}$.

B.4.2 Derivations for Section 3

At the flat-earth equilibrium in a racetrack economy, we have stronger results. First, because $\boldsymbol{J}$ and $\nabla\boldsymbol{v}(\bar{\boldsymbol{h}})$ are both symmetric and circulant, the eigenvectors for the two matrices are both real and are the same (refer to Appendix B.1). Thus, letting e_k be the kth eigenvalue of $\nabla\boldsymbol{v}(\bar{\boldsymbol{h}})$ which is associated to the eigenvector $\boldsymbol{\eta}_k = (\eta_{k,i}) = (K^{1/2}\cos[\theta k i])$ (see Lemma B.3), (B.4.3) further implies that

$$g_k\left(\boldsymbol{\eta}_k^\top \nabla \boldsymbol{v}(\boldsymbol{h}^*)\boldsymbol{\eta}_k\right) = g_k\left(e_k \boldsymbol{\eta}_k^\top \boldsymbol{\eta}_k\right) = g_k e_k \|\boldsymbol{\eta}_k\|^2 = g_k e_k > 0. \tag{B.4.5}$$

Noting that g_k and e_k are real, at the flat-earth equilibrium in a racetrack economy, we have

$$\operatorname{sgn}[g_k] = \operatorname{sgn}[e_k] \tag{B.4.6}$$

for all $k \geq 1$. For convenience, we introduce a notation to describe the above situation.

Definition B.2. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two K-by-K symmetric circulant matrices. By $\boldsymbol{A} \simeq \boldsymbol{B}$, we denote that $\boldsymbol{A} = \boldsymbol{C}\boldsymbol{B}$ with a symmetric circulant matrix $\boldsymbol{C}$ that is positive definite relative to $T\mathcal{D}$.

Observe that if we have $\boldsymbol{J} \simeq \boldsymbol{B}$ for some symmetric circulant matrix $\boldsymbol{B}$, we may study eigenvalues of $\boldsymbol{B}$ instead of those of $\boldsymbol{J}$ to examine the stability of $\bar{\boldsymbol{h}}$. Because $\boldsymbol{J} = \boldsymbol{C}\boldsymbol{B}$ and $\boldsymbol{J}$, $\boldsymbol{C}$, and $\boldsymbol{B}$ are circulant, employing the properties of circulant matrices we have $g_k = c_k b_k$ with c_k and b_k being the eigenvalues of $\boldsymbol{C}$ and $\boldsymbol{B}$, respectively; also, because $\boldsymbol{J}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are symmetric, g_k, b_k, and c_k are real (Appendix B.1). Since $\boldsymbol{C}$ is symmetric, circulant, and positive definite relative to $T\mathcal{D}$, we have $c_k > 0$ for $k \geq 1$. In sum, it follows that $\operatorname{sgn}[g_k] = \operatorname{sgn}[b_k]$ for $k \geq 1$. We summarize as follows.

Lemma B.4. Assume that the dynamic $\boldsymbol{F}$ satisfy Assumption B.2 and consider the flat-earth equilibrium $\bar{\boldsymbol{h}}$ on a racetrack economy. Then, $\boldsymbol{J} \equiv \nabla \boldsymbol{F}(\bar{\boldsymbol{h}})$ and $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ are both symmetric and circulant. Furthermore, we have $\boldsymbol{J} \simeq \nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$.

Thus, for the stability analysis of $\bar{\boldsymbol{h}}$, we may study the signs of the eigenvalues e_k ($k \geq 1$) of $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ because we have $\operatorname{sgn}[g_k] = \operatorname{sgn}[e_k]$. In particular, using our notation, $\boldsymbol{J}$ satisfies

$$\boldsymbol{J} \simeq c_0 \boldsymbol{I} + c_1 \bar{\boldsymbol{D}} + c_2 \bar{\boldsymbol{D}}^2 \tag{B.4.7}$$

at the flat-earth equilibrium (see Appendix B.3). Thus, the stability of $\bar{\boldsymbol{h}}$ is governed by the model-dependent function $G(\cdot)$ in (B.3.3) because (B.4.7) implies

$$\operatorname{sgn}[g_k] = \operatorname{sgn}[G(f_k)], \tag{B.4.8}$$

$$G(f_k) \equiv c_0 + c_1 f_k + c_2 f_k^2. \tag{B.4.9}$$

Furthermore, not only the signs but also the magnitudes of the eigenvalues $\{g_k\}$ and $\{e_k\}$ of $\boldsymbol{J}$ and $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ are often related in a much stronger way.

Observation B.1. For canonical evolutionary dynamics in the literature, it often follows that $g_k = \bar{c} e_k$ for $k \geq 1$ with a common, positive constant $\bar{c}$.

Example B.3. The *replicator dynamic*, which is the de facto standard dynamic in the NEG literature, is defined by

$$F_i(\boldsymbol{h}) \equiv h_i\{v_i(\boldsymbol{h}) - \bar{v}(\boldsymbol{h})\}, \tag{B.4.10}$$

where $\bar{v}(\boldsymbol{h}) \equiv (1/H)\sum_{j\in\mathcal{K}} v_j(\boldsymbol{h})h_j$ is the average payoff all over the regions. One has

$$\nabla \boldsymbol{F}(\boldsymbol{h}) = \psi_0(\boldsymbol{h}) + \psi_1(\boldsymbol{h})\nabla \boldsymbol{v}(\boldsymbol{h}) \tag{B.4.11}$$

with $\psi_0(\boldsymbol{h})$ and $\psi_1(\boldsymbol{h})$ defined by $\psi_0(\boldsymbol{h}) \equiv \mathrm{diag}[\boldsymbol{v}(\boldsymbol{h}) - \bar{v}(\boldsymbol{h})\mathbf{1}] - (1/H)\boldsymbol{h}\boldsymbol{v}(\boldsymbol{h})^\top$ and $\psi_1(\boldsymbol{h}) \equiv \mathrm{diag}[\boldsymbol{h}](\boldsymbol{I} - (1/H)\mathbf{1}\boldsymbol{h}^\top)$, respectively. It follows that, at the flat-earth equilibrium, $\psi_0(\bar{\boldsymbol{h}}) = -\bar{v}\boldsymbol{E}$ and $\psi_1(\bar{\boldsymbol{h}}) = h\,(\boldsymbol{I} - \boldsymbol{E})$, where $\boldsymbol{E} \equiv (1/K)\mathbf{1}\mathbf{1}^\top$ is a K-by-K matrix whose elements are all $1/K$. It implies that

$$g_k = \begin{cases} -\bar{v} < 0 & \text{if } k = 0, \\ he_k & \text{if } 1 \le k \le K-1. \end{cases} \tag{B.4.12}$$

where $\{e_k\}$ are the eigenvalues of $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$. Therefore, $\boldsymbol{J} \simeq \nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ as well as $\bar{c} = h$.

B.4.3 Extension: Taste heterogeneities

Local stability of equilibria in models with idiosyncratic taste heterogeneity á la Murata (2003) and Redding (2016) can be analyzed by employing the associated *perturbed best response dynamics* as is done by Akamatsu et al. (2012) for logit equilibrium under the logit dynamic. To be precise, for models with randomized preference $\tilde{\boldsymbol{v}}(\boldsymbol{h})$ and the associated perturbed best response dynamic $\tilde{\boldsymbol{F}}$, we have $\boldsymbol{J} = \nabla\tilde{\boldsymbol{F}}(\bar{\boldsymbol{h}}, \tilde{\boldsymbol{v}}(\bar{\boldsymbol{h}})) \simeq \boldsymbol{\Phi}\nabla\boldsymbol{v}(\bar{\boldsymbol{h}}) - \eta\boldsymbol{I}$ where η is a positive constant that reflects magnitude of heterogeneity and $\boldsymbol{\Phi}$ is the projection matrix onto $T\mathcal{D}$. $\boldsymbol{v}(\boldsymbol{h})$ is interpreted to be the homogeneous part of the underlying payoff function $\tilde{\boldsymbol{v}}(\boldsymbol{h})$ (see Sandholm, 2010). Assuming idiosyncratic taste heterogeneity is thus mathematically equivalent to incorporating an *extra local dispersion force*.[60]

Example B.4 (Logit equilibrium)**.** Consider a logit equilibrium (an equivalent of an equilibrium under idiosyncratic multiplicative Fréchet shock in the payoff function) with the noise parameter η. It is standard that the equilibrium condition is

$$h_i = P_i(\boldsymbol{h})H, \quad \text{where } P_i(\boldsymbol{h}) \equiv \frac{\exp[-v_i(\boldsymbol{h})/\eta]}{\sum_{j\in\mathcal{K}}\exp[-v_j(\boldsymbol{h})/\eta]}. \tag{B.4.13}$$

The logit dynamic is defiled by

$$F_i(\boldsymbol{h}) = HP_i(\boldsymbol{h}) - h_i. \tag{B.4.14}$$

At $\bar{\boldsymbol{h}}$, for finite values of η we have

$$\boldsymbol{J} = \eta^{-1}\boldsymbol{\Phi}\nabla\boldsymbol{v}(\bar{\boldsymbol{h}}) - \boldsymbol{I} \simeq \boldsymbol{\Phi}\nabla\boldsymbol{v}(\bar{\boldsymbol{h}}) - \eta\boldsymbol{I}. \tag{B.4.15}$$

Observe that $\eta \to \infty$ implies $\boldsymbol{J} = -\boldsymbol{I}$, which indicates that $\bar{\boldsymbol{h}}$ is always stable; it is intuitive that under sufficient heterogeneity on the side of the preference of mobile agent, the equilibrium is unique.

[60] Interested readers should consult Chapter 8 of Sandholm (2010) for local stability analysis via linearization of evolutionary dynamics in population games as well as the consequences of assuming random utility models on the Jacobian matrix of the dynamic $\boldsymbol{J}$ at an equilibrium.

C Analyses of economic geography models

In this appendix, we derive the Jacobian matrix of the payoff function at the flat-earth equilibrium, $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$, for the models included in **Table 1**. As discussed in the main text and as in Appendix B above, this suffices for our purpose. **Table 2** at the end of this appendix summarizes the exact mappings from each model to the coefficients of a model-dependent function $G(f) = c_0 + c_1 f + c_2 f^2$. We note that as soon as one has analytical expression of $G(f)$, she will be able to derive break points with respect to relevant parameters and study implications of the model.

C.1 Krugman (1991) (Km) model

Following Fujita, Krugman and Venables (1999a), this section introduces a many-region version of Krugman (1991)'s seminal model in line with our context.

Assumptions. There are K discrete regions whose set is denoted by $\mathcal{K}$. There are two types of workers: unskilled and skilled. Each worker inelastically supplies one unit of labor. The total endowments of skilled and unskilled workers are H and L, respectively. The skilled workers are mobile across regions; $h_i \geq 0$ denotes their population in region i, whence $\boldsymbol{h} \equiv (h_i)_{i \in \mathcal{K}}$ is the spatial pattern of them across the regions. Throughout Appendix C, $\mathcal{D} \equiv \{\boldsymbol{h} \in \mathbb{R}^K \mid \boldsymbol{h} \cdot \mathbf{1} = H,\ h_i \geq 0\}$ denotes the set of all possible spatial distributions of mobile (skilled) workers. The unskilled workers are immobile; their population in region i is denoted by l_i.

There are two industrial sectors: agriculture (abbreviated as A) and manufacturing (abbreviated as M). The A-sector is perfectly competitive and a unit input of unskilled labor is required to produce one unit of goods. We choose A-sector goods as the numéraire. The M-sector is modeled by the Dixit–Stiglitz monopolistic competition. The M-sector goods are horizontally differentiated and produced under increasing returns to scale using skilled labor as input.

The goods of both sectors are transported. Transportation of A-sector goods is frictionless, while transportation of M-sector goods is of iceberg form. For each unit of M-sector goods transported from region i to j, only a fraction $1/\tau_{ij}$ arrives where $\tau_{ij} > 1$ for $i \neq j$ and $\tau_{ii} = 1$.

Preference. All workers share an identical preference over both M- and A-sector goods. The utility function U of a worker in region i is given by a two-tier form. The upper-tier is the following Cobb–Douglas function:

$$U(C_i^{\mathrm{M}}, C_i^{\mathrm{A}}) = \mu \ln C_i^{\mathrm{M}} + (1-\mu) \ln C_i^{\mathrm{A}} \quad (0 < \mu < 1), \tag{C.1.1}$$

where C_i^{A} is the consumption of A-sector goods in region i, C_i^{M} the lower-tier manufacturing aggregate in region i, and μ the constant expenditure share of manufactured goods. The lower tier, C_i^{M}, is defined by the following constant-elasticity-of-substitution (CES) aggregate:

$$C_i^{\mathrm{M}} \equiv \left(\sum_{j \in \mathcal{K}} \int_0^{n_j} q_{ji}(\xi)^{(\sigma-1)/\sigma} \mathrm{d}\xi \right)^{\sigma/(\sigma-1)}, \tag{C.1.2}$$

where n_j is the number of varieties produced in region j, $q_{ji}(\xi)$ is the consumption of variety $\xi \in [0, n_j]$, and σ is the constant elasticity of substitution between any two varieties. As we take A-sector goods as the numéraire, the budget constraint of a worker in region i is given by

$$C_i^{\mathrm{A}} + \sum_{j \in \mathcal{K}} \int_0^{n_j} p_{ji}(\xi) q_{ji}(\xi) \mathrm{d}\xi = y_i, \tag{C.1.3}$$

where $p_{ji}(\xi)$ denotes the delivered price in region i of the M-sector goods produced in region j and y_i denotes the income of the worker. The incomes (wages) of the skilled and the unskilled workers are represented by w_i and w_i^{u}, respectively.

Demand. Utility maximization yields the following demand:

$$C_i^{\mathrm{M}} = \mu \frac{y_i}{P_i}, \; C_i^{\mathrm{A}} = (1 - \mu) y_i, \; q_{ji}(\xi) = \frac{\{p_{ji}(\xi)\}^{-\sigma}}{P_i^{-\sigma}} C_i^{\mathrm{M}}, \tag{C.1.4}$$

where P_i denotes the price index of the differentiated product in region i:

$$P_i \equiv \left(\sum_{j \in \mathcal{K}} \int_0^{n_j} p_{ji}(\xi)^{1-\sigma} \mathrm{d}\xi \right)^{1/(1-\sigma)}. \tag{C.1.5}$$

Since the total income in region i is given by $Y_i = w_i h_i + w_i^{\mathrm{u}} l_i$, we have the following total demand $Q_{ji}(\xi)$ for the variety ξ produced in j:

$$Q_{ji}(\xi) = \frac{\mu \{p_{ji}(\xi)\}^{-\sigma}}{P_i^{1-\sigma}} (w_i h_i + w_i^{\mathrm{u}} l_i). \tag{C.1.6}$$

The total supply $x_i(\xi)$ of the differentiated variety ξ in region i should meet the total demand from all regions including transport costs incurred by shipments:

$$x_i(\xi) = \sum_{j \in \mathcal{K}} \tau_{ij} Q_{ij}(\xi). \tag{C.1.7}$$

Firm behavior. With free trade in the A-sector, the wage of the unskilled worker w_i^{u} is equalized. As A-sector goods is the numéraire, we have $w_i^{\mathrm{u}} = 1$. In the M-sector, to produce x_i unit of differentiated product a firm requires $\alpha + \beta x_i$ unit of skilled labor. With increasing returns, every firm specializes to a single variety. The cost function of a firm in region i producing the variety ξ is thus given by

$$C_i(x_i(\xi)) \equiv w_i \{\alpha + \beta x_i(\xi)\}. \tag{C.1.8}$$

Therefore, a M-sector firm located in region i specialized to the variety ξ faces the following profit:

$$\Pi_i(\xi) = \sum_{j \in \mathcal{K}} p_{ij}(\xi) Q_{ij}(\xi) - C_i(x_i(\xi)). \tag{C.1.9}$$

Since we have a continuum of firms, each one is negligible in the sense that its action has no impact on the market (i.e., the price indices). It is standard that profit maximization of firms yields

$$p_{ij}(\xi) = \frac{\sigma\beta}{\sigma - 1} w_i \tau_{ij} \tag{C.1.10}$$

and that $p_{ij}(\xi)$ is independent of ξ. This fact in turn implies that $Q_{ij}(\xi)$ and $x_i(\xi)$ also do not depend on ξ. We thus omit ξ in the following.

Short-run equilibrium. In the short run, the spatial distribution $\boldsymbol{h} = (h_i)_{i\in\mathcal{K}}$ of skilled workers is fixed. Given $\boldsymbol{h}$, we determine the short-run equilibrium wage $\boldsymbol{w} \equiv (w_i)_{i\in\mathcal{K}}$ by the M-sector product market clearing condition (PMCC), the zero-profit condition (ZPC), the skilled labor market clearing condition (LMCC). First, the ZPC for every M-sector firm dictates that $x_i = \alpha(\sigma - 1)/\beta$, so that the required skilled-labor input being $\alpha\sigma$. Then, the skilled labor-market clearing yields $\alpha\sigma n_i = h_i$. We note that using $n_i = h_i/(\alpha\sigma)$, we have

$$P_i = \frac{\sigma\beta}{\sigma - 1}\left(\frac{1}{\alpha\sigma}\sum_{j\in\mathcal{K}} h_j (w_j \tau_{ji})^{1-\sigma}\right)^{1/(1-\sigma)}, \tag{C.1.11}$$

with $d_{ij} \equiv \tau_{ij}^{1-\sigma}$; $\boldsymbol{D} = [d_{ij}] = [\tau_{ij}^{1-\sigma}]$ is the friction matrix. Employing the formula up to here, the M-sector PMCC (C.1.7) implies that

$$w_i h_i = \mu \sum_{j\in\mathcal{K}} \frac{h_i w_i^{1-\sigma} d_{ij}}{\sum_{k\in\mathcal{K}} h_k w_k^{1-\sigma} d_{kj}} (w_j h_j + l_j), \tag{C.1.12}$$

which is the so-called *wage equation*. Adding up (C.1.12), we obtain

$$\sum_{i\in\mathcal{K}} w_i h_i = \frac{\mu}{1-\mu} L \tag{C.1.13}$$

which constrains $\boldsymbol{w}$ at any configuration $\boldsymbol{h}$. The existence and uniqueness of solution for the wage equation under a fixed $\boldsymbol{h}$ and the normalization constraint (C.1.13) follows by a standard nonlinear-complementarity-problem arguments (Facchinei and Pang, 2007) and thus we shall omit. Given the solution $\boldsymbol{w}(\boldsymbol{h})$ of (C.1.12), we have the following indirect utility function of skilled workers:

$$v_i(\boldsymbol{h}) = \bar{\kappa} \ln[\Delta_i] + \ln[w_i] \tag{C.1.14}$$

where $\bar{\kappa} \equiv \mu/(\sigma - 1)$ and $\Delta_i \equiv \sum_{k\in\mathcal{K}} h_k w_k^{1-\sigma} d_{ki}$. Note that we omitted the constant terms as they do not affect properties of equilibrium spatial patterns. We shall follow this convention for the rest of Appendix C. Long-run equilibria are defined by [VIP] in Appendix B.3 based on the payoff function (C.1.14).

Jacobian matrix at the flat-earth equilibrium. Assume a racetrack economy (i.e., $d_{ij} = \tau_{ij}^{1-\sigma} = \exp[-\tau\ell_{ij}]$ with $\tau > 0$; see Section 3.1) with uniform unskilled labor endowment (i.e., $l_i = l \equiv L/K$ for all $i \in \mathcal{K}$). Then, it is trivial that the flat-earth pattern is a long-run equilibrium. As we must evaluate $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$, we shall first derive $\nabla \boldsymbol{v}(\boldsymbol{h}) = [\partial v_i(\boldsymbol{h})/\partial h_j]$ at any interior solution $\boldsymbol{h}$. We have

$$\frac{\partial v_i(\boldsymbol{h})}{\partial h_j} = \frac{\bar{\kappa}}{\Delta_i}\left(\frac{\partial \Delta_i}{\partial h_j} + \sum_{k\in\mathcal{K}} \frac{\partial \Delta_i}{\partial w_k}\frac{\partial w_k}{\partial h_j}\right) + \frac{1}{w_i}\frac{\partial w_i}{\partial h_j} \tag{C.1.15}$$

$$= \bar{\kappa}\left(\frac{1}{\Delta_i} w_j^{1-\sigma} d_{ji} + (1-\sigma)\sum_{k\in\mathcal{K}} \frac{1}{\Delta_i} h_k w_k^{-\sigma} d_{ki}\frac{\partial w_k}{\partial h_j}\right) + \frac{1}{w_i}\frac{\partial w_i}{\partial h_j} \tag{C.1.16}$$

$$= \bar{\kappa}\left(\frac{1}{h_j} m_{ji} + (1-\sigma)\sum_{k\in\mathcal{K}} m_{ki}\frac{1}{w_k}\frac{\partial w_k}{\partial h_j}\right) + \frac{1}{w_i}\frac{\partial w_i}{\partial h_j} \tag{C.1.17}$$

where $m_{ij} \equiv h_i w_i^{1-\sigma} d_{ij}/\Delta_j$, or in the vector–matrix form $\boldsymbol{M} = [m_{ij}] = \mathrm{diag}[\boldsymbol{w}^{1-\sigma} \circ \boldsymbol{h}]\boldsymbol{D}(\mathrm{diag}[\boldsymbol{\Delta}])^{-1}$ with $\boldsymbol{\Delta} = [\Delta_i] = \boldsymbol{D}^\top \mathrm{diag}[\boldsymbol{w}]^{1-\sigma}\boldsymbol{h}$. We let $\boldsymbol{x}^a \equiv [x_i^a]$ and $\boldsymbol{x} \circ \boldsymbol{y} \equiv [x_i y_i]$. Noting that $\bar{\kappa}(1-\sigma) = -\mu$, we have

$$\nabla \boldsymbol{v}(\boldsymbol{h}) = \bar{\kappa}\boldsymbol{M}^\top \mathrm{diag}[\boldsymbol{h}]^{-1} - \mu \boldsymbol{M}^\top \mathrm{diag}[\boldsymbol{w}]^{-1}\nabla \boldsymbol{w}(\boldsymbol{h}) + \mathrm{diag}[\boldsymbol{w}]^{-1}\nabla \boldsymbol{w}(\boldsymbol{h}) \tag{C.1.18}$$

$$= \bar{\kappa}\boldsymbol{M}^\top \mathrm{diag}[\boldsymbol{h}]^{-1} + (\boldsymbol{I} - \mu\boldsymbol{M}^\top)\,\mathrm{diag}[\boldsymbol{w}]^{-1}\nabla \boldsymbol{w}(\boldsymbol{h}) \tag{C.1.19}$$

where $\nabla \boldsymbol{w}(\boldsymbol{h}) \equiv [\partial w_i(\boldsymbol{h})/\partial h_j]$ is yet to be known. Letting

$$W_i(\boldsymbol{h}, \boldsymbol{w}) \equiv w_i h_i - \mu \sum_{k\in\mathcal{K}} m_{ik}(w_k h_k + l), \tag{C.1.20}$$

the wage equation is equivalent to $\boldsymbol{W}(\boldsymbol{h}, \boldsymbol{w}) = \boldsymbol{0}$. Thanks to the implicit function theorem, it can be shown that $\nabla \boldsymbol{w}(\boldsymbol{h}) = -(\nabla_{\boldsymbol{w}}\boldsymbol{W})^{-1}(\nabla \boldsymbol{W})$ where $\nabla_{\boldsymbol{w}}\boldsymbol{W} \equiv [\partial W_i/\partial w_j]$ and $\nabla \boldsymbol{W} \equiv [\partial W_i/\partial h_j]$ are given by

$$\frac{\partial W_i}{\partial w_j} = \delta_{ij} h_i - \mu\sum_{k\in\mathcal{K}} \frac{\partial m_{ik}}{\partial w_j}(w_k h_k + l) - \mu m_{ij} h_j \tag{C.1.21}$$

$$= \delta_{ij} h_i - \mu(1-\sigma)\frac{1}{w_j}\left(\delta_{ij}\sum_{k\in\mathcal{K}} m_{ik}(w_k h_k + l) - \sum_{k\in\mathcal{K}} m_{ik} m_{jk}(w_k h_k + l)\right) - \mu m_{ij} h_j, \tag{C.1.22}$$

$$\frac{\partial W_i}{\partial h_j} = \delta_{ij} w_i - \mu\sum_{k\in\mathcal{K}} \frac{\partial m_{ik}}{\partial h_j}(w_k h_k + l) - \mu m_{ij} w_j \tag{C.1.23}$$

$$= \delta_{ij} w_i - \mu\frac{1}{h_j}\left(\delta_{ij}\sum_{k\in\mathcal{K}} m_{ik}(w_j h_j + l) - \sum_{k\in\mathcal{K}} m_{ik} m_{jk}(w_k h_k + l)\right) - \mu m_{ij} w_j, \tag{C.1.24}$$

with δ_{ij} being the Kronecker's delta. In the vector–matrix form, we have

$$\nabla_{\boldsymbol{w}}\boldsymbol{W} = \mathrm{diag}[\boldsymbol{h}] - \mu(1-\sigma)(\mathrm{diag}[\boldsymbol{M}\boldsymbol{Y}] - \boldsymbol{M}\,\mathrm{diag}[\boldsymbol{Y}]\boldsymbol{M}^\top)\,\mathrm{diag}[\boldsymbol{w}]^{-1} - \mu\boldsymbol{M}\,\mathrm{diag}[\boldsymbol{h}] \tag{C.1.25}$$

$$\nabla \boldsymbol{W} = \mathrm{diag}[\boldsymbol{w}] - \mu(\mathrm{diag}[\boldsymbol{M}\boldsymbol{Y}] - \boldsymbol{M}\boldsymbol{Y}\boldsymbol{M}^\top)\,\mathrm{diag}[\boldsymbol{h}]^{-1} - \mu\boldsymbol{M}\,\mathrm{diag}[\boldsymbol{w}] \tag{C.1.26}$$

where $\boldsymbol{Y} = [Y_i] \equiv [w_i h_i + l]$ is the vector of the regional total income.

Assume the flat-earth equilibrium where $\boldsymbol{h} = h\boldsymbol{1}$ with $h \equiv H/K$. Then, we know that the (uniform level of) equilibrium wage is given by $\bar{w} \equiv \mu/(1-\mu) \cdot L/H$ and the (uniform level of) total income of a region $\bar{Y} \equiv l/(1-\mu) = 1/(1-\mu) \cdot L/K$, whence we have $\bar{Y}/\bar{w} = h/\mu$ and $\bar{Y}/h = \bar{w}/\mu$. We also have

$\boldsymbol{M} = \bar{\boldsymbol{D}} = \boldsymbol{D}/d$ where d is the row sum of $\boldsymbol{D}$. It then follows that

$$\nabla \boldsymbol{w}(\bar{\boldsymbol{h}}) = \frac{\bar{w}}{h}\left[\sigma \boldsymbol{I} - \mu \bar{\boldsymbol{D}} - (\sigma - 1)\bar{\boldsymbol{D}}^2\right]^{-1} \bar{\boldsymbol{D}}\left(\mu \boldsymbol{I} - \bar{\boldsymbol{D}}\right) \tag{C.1.27}$$

and thus that

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = \frac{1}{h}\left[\boldsymbol{I} - \kappa \bar{\boldsymbol{D}} - \rho \bar{\boldsymbol{D}}^2\right]^{-1}[(\kappa + \bar{\kappa})\bar{\boldsymbol{D}} - (\mu\bar{\kappa} + \sigma^{-1})\bar{\boldsymbol{D}}^2] \tag{C.1.28}$$

where $\kappa \equiv \mu/\sigma$ and $\rho \equiv (\sigma - 1)/\sigma \in (0, 1)$. We recall that circulant matrices commute (Lemma B.1). It thus follows that for the Km model, we have $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) \simeq c_1 \bar{\boldsymbol{D}} + c_2 \bar{\boldsymbol{D}}^2$ with $c_1 = \kappa + \bar{\kappa}$ and $c_2 = -(\mu\bar{\kappa} + \sigma^{-1})$.

Remark C.1. A comparison with the literature would be useful for providing some insights. Letting $\{e_k\}$ be the eigenvalues of $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$, we have

$$e_k = \frac{1}{h} f_k \frac{(\kappa + \bar{\kappa}) - (\mu\bar{\kappa} + \sigma^{-1}) f_k}{1 - \kappa f_k - \rho f_k^2} = \frac{K}{H}\left(\frac{1-\rho}{\rho}\right) f_k \left[\frac{\mu(1+\rho) - (\mu^2 + \rho) f_k}{1 - \mu(1-\rho) f_k - \rho f_k^2}\right]. \tag{C.1.29}$$

But then one will immediately notice that this expression is a generalized version of the equation (5.27) of Fujita et al. (1999a) for the Km model in the symmetric two-region setting (with a rearrangement):

$$\frac{1}{P_0^{-\mu}} \frac{\mathrm{d}\omega}{\mathrm{d}\lambda} = \frac{2}{\lambda + (1-\lambda)} \left(\frac{1-\rho}{\rho}\right) Z \left[\frac{\mu(1+\rho) - (\mu^2 + \rho) Z}{1 - \mu(1-\rho) Z - \rho Z^2}\right] \tag{C.1.30}$$

which expresses the change in the real wage $\omega \equiv w_0 P_0^{-\mu}$ at region 0 when its share of skilled worker λ slightly increased. Here, Z is "an index of trade barriers" defined by the equation (5.25), ibid:

$$Z \equiv \frac{1 - T^{1-\sigma}}{1 + T^{1-\sigma}} \tag{C.1.31}$$

where $T > 1$ is the iceberg transport cost parameter between the regions. We thus see that *the "real wage differential" exercise often conducted in the literature is a special case of our approach*. In fact, if we assume $K = 2$, the only possible deviation direction is $\boldsymbol{\eta}_1 = (1, -1)$, which corresponds to agglomeration toward one of the regions. Given the freeness of transport r between the two regions, its associated eigenvalue of $\bar{\boldsymbol{D}}$ is given by

$$f_1 = \frac{1 - r}{1 + r}, \tag{C.1.32}$$

which precisely coincides with the above Z—since $r = T^{1-\sigma}$ for this case. In the two-region economy, there is only a single possible deviation direction: agglomeration. We thus only have to investigate the sign of $\mathrm{d}\omega/\mathrm{d}\lambda$. In a many-region racetrack economy, however, there are multiple possible deviation directions and thus stability of the flat-earth pattern depends on the signs of all $(e_k)_{k=1}^{K-1}$.

Remark C.2. As emphasized by the literature, the coefficients of $G(f)$ have clear economic interpretations. The first, c_1, represents demand externality through price index ($\bar{\kappa}$) and a home-market effect (κ). For the former, $\bar{\kappa}$, observe that $1/(\sigma - 1)$ in $\bar{\kappa}$ is the markup of firms or the magnitude of product differentiation; an agglomeration, by improving proximity of mobile agents to production location of firms, increases payoff of agglomerated regions. The latter, κ, is a home market effect. Note that $1/\sigma$ in κ is the share of fixed cost (wage of a mobile agent which is required to operate) in a firm's production cost. The second, c_2, on the other hand, represents the dispersion force. The centrifugal force of the model is due to increased market competition caused by concentration of firms (the so-called "market crowding effect"). Since there is spatially dispersed demand (immobile agents), firms in a region of agglomeration may hope to relocate to other, less crowded regions (σ^{-1} in c_2). In addition, the price-index effect by reducing firm's market share and hence wage of mobile agents produces another global dispersion force ($\mu\bar{\kappa}$ in c_2). The effect produces a dispersion force from *outside* of a region.

Numerical simulation. **Figure 8** assumes the Km model. Parameters are set as $\mu = 0.4$, $\sigma = 10$, $L = 8$, and $H = 1$.

C.2 Forslid and Ottaviano (2003) (FO) model

The FO model is a slightly simplified version of Krugman (1991)'s NEG model. The model is sometimes called the *footloose-entrepreneur model*, since a unit of skilled (mobile) labor is required as the fixed input of a manufacturing firm. The only difference is that variable input of M-sector firms in the Km model is now replaced by unskilled labor. Specifically, in order to produce $x_i(\xi)$ unit of product ξ, a M-sector firm now requires α unit of skilled labor and $\beta x_i(\xi)$ unit of unskilled labor. Therefore, for the FO model, the total cost of production for a firm in region i is

$$C_i(x(\xi)) = \alpha w_i + \beta x_i(\xi) w_i^{\mathrm{u}}. \tag{C.2.1}$$

Wage equalization of the A-sector ($w_i^{\mathrm{u}} = 1$ for all $i \in \mathcal{K}$) then implies that

$$p_{ij}(\xi) = \frac{\sigma\beta}{\sigma - 1}\tau_{ij} \tag{C.2.2}$$

provided that A-sector goods are produced at every region (we assume $\beta x_i n_i < l_i$ for all $i \in \mathcal{K}$). Again, we drop ξ in what it follows.

Short-run equilibrium. The short-run equilibrium conditions are again the PMCC, the LMCC, and the ZPC. First, since a firm requires α unit of skilled labor, the LMCC implies that $\alpha n_i = h_i$, which in turn yields the price index P_i for the FO model:

$$P_i = \frac{\sigma\beta}{\sigma - 1}\left(\frac{1}{\alpha}\sum_{j\in\mathcal{K}} h_j d_{ji}\right)^{1/(1-\sigma)} \tag{C.2.3}$$

where $d_{ji} \equiv \tau_{ji}^{1-\sigma}$ is the trade friction between the regions i and j. Note that, unlike the Km model, P_i does not depend on the wage $\boldsymbol{w} = (w_i)_{i\in\mathcal{K}}$. The ZPC implies that the operating profit of a firm is entirely absorbed by the wage bills:

$$w_i = \left(\sum_{j \in \mathcal{K}} p_{ij} Q_{ij} - \beta x_i \right). \tag{C.2.4}$$

Together with the PMCC, we have the following wage equation for the model:

$$w_i = \frac{\mu}{\sigma} \sum_{j \in \mathcal{K}} \frac{d_{ij}}{\sum_{k \in \mathcal{K}} d_{kj} h_k} (w_j h_j + l_j) \tag{C.2.5}$$

The equation is analytically solvable. Specifically, in the vector–matrix form, we have

$$\boldsymbol{w} = \kappa \left[\boldsymbol{I} - \kappa \boldsymbol{M} \, \mathrm{diag}[\boldsymbol{h}] \right]^{-1} \boldsymbol{M} \boldsymbol{l} \tag{C.2.6}$$

where $\kappa \equiv \mu/\sigma$, $\boldsymbol{l} \equiv (l_i)$, and $\boldsymbol{M} \equiv [m_{ij}] = [d_{ij}/\Delta_j] = \boldsymbol{D}\{\mathrm{diag}[\boldsymbol{\Delta}]\}^{-1}$ with $\Delta_i = \sum_{j \in \mathcal{K}} d_{ji} h_j$ so that $\boldsymbol{\Delta} = [\Delta_i] = \boldsymbol{D}^\top \boldsymbol{h}$. The indirect utility $\boldsymbol{v}(\boldsymbol{h})$ of each of the many-region FO model is expressed as

$$v_i(\boldsymbol{h}) = \bar{\kappa} \ln[\Delta_i] + \ln[w_i] \tag{C.2.7}$$

where $\bar{\kappa} \equiv \mu/(\sigma - 1)$. We again ignore the constant terms. Long-run equilibria are defined by (B.3.1) with respect to the above (C.2.7).

Jacobian matrix at the flat-earth equilibrium. In a racetrack economy, following completely the same line of logic as in the Km model, we obtain

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = \frac{1}{h} \left[\boldsymbol{I} - \kappa \bar{\boldsymbol{D}} \right]^{-1} \left[(\bar{\kappa} + \kappa) \bar{\boldsymbol{D}} - (\bar{\kappa}\kappa + 1) \bar{\boldsymbol{D}}^2 \right] \tag{C.2.8}$$

where $\bar{\boldsymbol{D}} \equiv \boldsymbol{D}/d$. We thus conclude that $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) \simeq c_1 \bar{\boldsymbol{D}} + c_2 \bar{\boldsymbol{D}}^2$ with $c_1 = \bar{\kappa} + \kappa$ and $c_2 = -(\bar{\kappa}\kappa + 1)$.

C.3 Pflüger (2004) (Pf) model

The Pf model is a further simplified version of the FO model (and hence the Km model) in which we assume a quasi-linear utility function for the upper-tier as follows:

$$U(C_i^{\mathrm{M}}, C_i^{\mathrm{A}}) = C_i^{\mathrm{A}} + \mu \ln C_i^{\mathrm{M}}. \tag{C.3.1}$$

Taking the A-sector goods as numéraire, it is standard that utility maximization yields the following demand, where the income effect in C_i^{M} is lost compared to the Km and FO models:

$$C_i^{\mathrm{A}} = y_i - \mu, \; C_i^{\mathrm{M}} = \mu \frac{1}{P_i}. \tag{C.3.2}$$

where the price index P_i is the same as the FO model. Thus, replacing the total income of a region $Y_i = w_i h_i + l_i$ in (C.2.5) by the total number of workers $h_i + l_i$, we obtain the following "wage equation"

$$w_i = \frac{\mu}{\sigma} \sum_{j \in \mathcal{K}} \frac{d_{ij}}{\sum_{k \in \mathcal{K}} d_{kj} h_k} (h_j + l_j) \tag{C.3.3}$$

which is, actually, already solved. The indirect utility is given by

$$v_i(\boldsymbol{h}) = \bar{\kappa} \ln[\Delta_i] + w_i \tag{C.3.4}$$

where $\Delta_i \equiv \sum_{j \in \mathcal{K}} d_{ji} h_j$. Long-run equilibria are defined by (B.3.1) with respect to the above (C.2.7).

Jacobian matrix at the flat-earth equilibrium. We show

$$\nabla \boldsymbol{v}(\boldsymbol{h}) = \bar{\kappa} \boldsymbol{M}^\top + \kappa \left(\boldsymbol{M} - \boldsymbol{M} \operatorname{diag}[\boldsymbol{H}] \boldsymbol{M}^\top \right) \tag{C.3.5}$$

with $\boldsymbol{H} = [H_i] \equiv [h_i + l_i]$ and $\boldsymbol{M} = [m_{ij}] \equiv [d_{ij}/\Delta_j]$ and thus that

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = \frac{1}{h} \left[(\bar{\kappa} + \kappa) \bar{\boldsymbol{D}} - \kappa (1 + \epsilon) \bar{\boldsymbol{D}}^2 \right] \tag{C.3.6}$$

with $\epsilon = L/H$ being the ratio of the number of unskilled worker to that of skilled. We thus see that $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) \simeq c_1 \bar{\boldsymbol{D}} + c_2 \bar{\boldsymbol{D}}^2$ with $c_1 = \bar{\kappa} + \kappa$ and $c_2 = -\kappa(1 + \epsilon)$.

C.4 Helpman (1998) (Hm) model

Helpman (1998) removed the A-sector in Krugman (1991) and thereby assumes that all workers are mobile. Instead of the A-sector, the Hm model introduces the housing (abbreviated as H) sector and each region i is endowed with a fixed stock A_i of housing.

Preference. The utility function of a worker in region i is given by

$$U(C_i^{\mathrm{M}}, C_i^{\mathrm{H}}) = \mu \ln C_i^{\mathrm{M}} + \gamma \ln C_i^{\mathrm{H}} \tag{C.4.1}$$

where C_i^{H} is the consumption of H-sector goods in region i and γ is the constant expenditure share on it ($\gamma + \mu = 1$). The budget constraint of a worker located at i is represented by

$$p_i^{\mathrm{H}} C_i^{\mathrm{H}} + \sum_j \int_0^{n_j} p_{ji}(\xi) q_{ji}(\xi) \mathrm{d}\xi = y_i, \tag{C.4.2}$$

where p_i^{H} is the price of H-sector goods in region i. Utility maximization leads to the following demand for the H-sector goods:

$$C_i^{\mathrm{H}} = \gamma \frac{y_i}{p_i^{\mathrm{H}}}. \tag{C.4.3}$$

Housing market clearing. In the H-sector, the total demand $h_i C_i^{\mathrm{H}}$ in region i cannot be greater than the maximum supply A_i. If the demand in region i is less than the supply, the price p_i^H should be the lower boundary (i.e., zero), otherwise positive. Thus, we have the following housing market clearing condition:

$$\begin{cases} h_i C_i^{\mathrm{H}} = A_i & \text{if} \quad p_i^{\mathrm{H}} > 0, \\ h_i C_i^{\mathrm{H}} \leq A_i & \text{if} \quad p_i^{\mathrm{H}} = 0, \end{cases} \quad \forall i. \tag{C.4.4}$$

From (C.4.3), $p_i^{\mathrm{H}} \neq 0$ for any long-run equilibrium; because $C_i^{\mathrm{A}} \to \infty$ and thus $U \to \infty$ as $p_i^{\mathrm{H}} \to 0$ and such a spatial pattern is never sustainable. We thus conclude that

$$C_i^{\mathrm{H}} = \frac{A_i}{h_i}, \; p_i^{\mathrm{H}} = \gamma \frac{y_i h_i}{A_i} \tag{C.4.5}$$

and that $h_i > 0$ at any long-run equilibrium.

Landownership. We here consider two types of assumptions on landownership: *public landownership* (abbreviated as PL) and *local landownership* (LL). In the original formulation, the housing stocks are equally owned by all workers (i.e., PL). In this way, the income of a worker in region i is the sum of wage and devident of rental revenue, $y_i = w_i + \bar{w}^{\mathrm{H}}$, where

$$\bar{w}^{\mathrm{H}} = \frac{1}{H} \sum_{i \in \mathcal{K}} p_i^{\mathrm{H}} C_i^{\mathrm{H}} h_i = \frac{\gamma}{H} \sum_{i \in \mathcal{K}} y_i h_i \tag{C.4.6}$$

so that rearrangement yields

$$\bar{w}^{\mathrm{H}} = \frac{\gamma}{(1-\gamma)H} \sum_{i \in \mathcal{K}} w_i h_i. \tag{C.4.7}$$

We shall set $\bar{w}^{\mathrm{H}} = 1$ so that we can normalize w_i to satisfy $\sum_{i \in \mathcal{K}} w_i h_i = (\mu/\gamma)H$. On the other hand, Ottaviano et al. (2002), Murata and Thisse (2005), and Redding and Sturm (2008) assume that the housing stocks are locally owned (i.e., LL). It is, $y_i = w_i + w_i^{\mathrm{H}}$ where $\bar{w}_i = p_i^{\mathrm{H}} C_i^{\mathrm{H}} = \gamma y_i$, which in turn yields that $y_i = w_i/\mu$. Also for this case, analogous to the public landownership case, we shall constrain $\boldsymbol{w}$ by the condition $\sum_{i \in \mathcal{K}} w_i h_i = (\mu/\gamma)H$ for normalization.

Short-run equilibrium. Regarding the short-run equilibrium conditions, the only difference from the Km model is the total expenditure at each region, which is now

$$Y_i = \begin{cases} (w_i + 1) h_i, & \text{(for PL)}, \\ w_i h_i / \mu, & \text{(for LL)}. \end{cases} \tag{C.4.8}$$

The short-run equilibrium wage equation is thus given by:

$$[\mathrm{PL}] \quad w_i h_i = \mu \sum_{j \in \mathcal{K}} \frac{d_{ij} w_i^{1-\sigma} h_i}{\sum_{k \in \mathcal{K}} d_{kj} w_k^{1-\sigma} h_k} (w_j + 1) h_j, \tag{C.4.9}$$

$$\text{[LL]} \quad w_i h_i = \sum_{j\in\mathcal{K}} \frac{d_{ij} w_i^{1-\sigma} h_i}{\sum_{k\in\mathcal{K}} d_{kj} w_k^{1-\sigma} h_k} w_j h_j. \tag{C.4.10}$$

Given the solution $\boldsymbol{w}$ for (C.4.9) or (C.4.9), the indirect utility $\boldsymbol{v}(\boldsymbol{h})$ is expressed as

$$v_i(\boldsymbol{h}) = \begin{cases} \bar{\kappa} \ln[\Delta_i] + \mu \ln[w_i + 1] - \gamma\big(\ln[h_i] - \ln[A_i]\big), & \text{(for PL)}, \\ \bar{\kappa} \ln[\Delta_i] + \mu \ln[w_i] - \gamma\big(\ln[h_i] - \ln[A_i]\big), & \text{(for LL)} \end{cases} \tag{C.4.11}$$

where $\Delta_i \equiv \sum_{j\in\mathcal{K}} h_j w_j^{1-\sigma} d_{ji}$.

Jacobian matrix at the flat-earth equilibrium. Let $A_i = A$ for all regions to abstract from location-fixed exogenous effects. For the PL case, one can show that

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = \frac{1}{h}\left\{\bar{\kappa}\bar{\boldsymbol{D}} + \mu(\mu\boldsymbol{I} - \bar{\boldsymbol{D}})\left[\sigma\boldsymbol{I} - \mu\bar{\boldsymbol{D}} - (\sigma-1)\bar{\boldsymbol{D}}^2\right]^{-1}\bar{\boldsymbol{D}}(\boldsymbol{I} - \bar{\boldsymbol{D}}) - \gamma\boldsymbol{I}\right\} \tag{C.4.12}$$

$$= \frac{\sigma}{h}\left[\sigma\boldsymbol{I} - \mu\bar{\boldsymbol{D}} - (\sigma-1)\bar{\boldsymbol{D}}^2\right]^{-1}\left\{-\gamma\boldsymbol{I} + (\bar{\kappa} + \kappa)\bar{\boldsymbol{D}} + \left\{\gamma - \left(\mu\bar{\kappa} + \frac{1}{\sigma}\right)\right\}\bar{\boldsymbol{D}}^2\right\} \tag{C.4.13}$$

so that $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) \simeq c_0\boldsymbol{I} + c_1\bar{\boldsymbol{D}} + c_2\bar{\boldsymbol{D}}^2$ with $c_0 = -\gamma$, $c_1 = \mu\left(\frac{1}{\sigma-1} + \frac{1}{\sigma}\right)$, and $c_2 = \gamma - \frac{1}{\sigma} - \frac{\mu^2}{\sigma-1}$. We recall that γ is the expenditure share on housing good, whence one can understand that the dispersion force expressed by $c_0 < 0$ solely arises from local housing. For the LL case one can show that

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = \frac{1}{h}\left\{\bar{\kappa}\bar{\boldsymbol{D}} + \mu(\boldsymbol{I} - \bar{\boldsymbol{D}})\left[\sigma\boldsymbol{I} + (\sigma-1)\bar{\boldsymbol{D}}\right]^{-1}\bar{\boldsymbol{D}} - \gamma\boldsymbol{I}\right\} \tag{C.4.14}$$

$$= \frac{\sigma}{h}\left[\sigma\boldsymbol{I} + (\sigma-1)\bar{\boldsymbol{D}}\right]^{-1}\left\{-\gamma\boldsymbol{I} + \left(\frac{\mu}{\sigma-1} + \frac{\mu}{\sigma} - \gamma\frac{\sigma-1}{\sigma}\right)\bar{\boldsymbol{D}}\right\}. \tag{C.4.15}$$

From this, we conclude that for the LL case $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) \simeq c_0\boldsymbol{I} + c_1\bar{\boldsymbol{D}}$ with $c_0 = -\gamma$ and $c_1 = \frac{\mu}{\sigma-1} + \frac{\mu}{\sigma} - \gamma\frac{\sigma-1}{\sigma}$.

Remark C.3. For the model, the condition for the uniqueness of equilibrium is given by $\gamma\sigma = (1-\mu)\sigma > 1$ (Redding and Sturm, 2008). One can show that if $\gamma\sigma > 1$ is satisfied, regardless of the assumption on landownership, the flat-earth eqilibrium is stable.

Remark C.4. The regional model formulated in §3 of Redding and Rossi-Hansberg (2017) is an enhanced version of the Hm model with local landownership, in which variable input of skilled labor is allowed to depend on region i (i.e., productivity differs across regions). That is, the cost function of firms in region i becomes

$$\mathrm{C}_i(x_i(\xi)) = w_i(\alpha + \beta_i x_i(\xi)). \tag{C.4.16}$$

It then implies that the short-run equilibrium price and price index in region i becomes

$$p_{ij}(\xi) = \frac{\sigma\beta_i}{\sigma-1}\tau_{ij} w_i, \tag{C.4.17}$$

$$P_i = \frac{\sigma}{\sigma-1}\left(\frac{1}{\alpha\sigma}\sum_{j\in\mathcal{K}} h_j(\beta_j w_j \tau_{ji})^{1-\sigma}\right)^{1/(1-\sigma)}, \tag{C.4.18}$$

respectively. As the model assumes local landownership, the wage equation for the model is

$$w_i h_i = \sum_{j\in\mathcal{K}} \frac{h_i A_i w_i^{1-\sigma} d_{ij}}{\sum_{k\in\mathcal{K}} h_k A_k w_k^{1-\sigma} d_{kj}} w_j h_j, \tag{C.4.19}$$

where we define $A_i \equiv \beta_i^{1-\sigma}$. Thus, abstracting from first natures by setting $A_i = \bar{A}$, the model reduces to the Hm model under LL.

C.5 Puga (1999) (Pg) model

Puga (1999) generalized the Km model in two directions: namely, (i) intersector mobility of workers between agriculture (A) and manufacture (M) (without immobile workers but land) and (ii) intermediate inputs in the M-sector, both as in Krugman and Venables (1995).

Assumptions. There are only a mass H of mobile workers, with h_i denoting the number of workers located in region i. We denote by h_i^{M} and h_i^{A} the number of workers engaged in the M- and A-sectors, respectively ($h_i = h_i^{\mathrm{M}} + h_i^{\mathrm{A}}$). The homogeneous preference of consumers is the same as Km model, with the expenditure share on M-sector good μ and the elasticity of substitution between manufactured varieties σ. Each region is endowed with A_i unit of land, which is owned by immobile landlords that have the same preference as the workers. We assume that, if a worker should relocate, then she enters in the M-sector of the destination region in the first place. The stability of the spatial pattern $\boldsymbol{h}$ is then reduced to the study of $\boldsymbol{h}^{\mathrm{M}} \equiv [h_i^{\mathrm{M}}]$.

A-sector. The A-sector is perfectly competitive and produces a homogeneous output using labor and land under constant returns to scale. The A-sector goods are costless to trade and are set to the numéraire. Let X_i^{A} be the gross regional product of the A-sector. In line with the original paper, we shall specify a Cobb–Douglas production function with labor share $\bar{\mu}$; in concrete terms, we have $X_i^{\mathrm{A}} = (h_i^{\mathrm{A}})^{\bar{\mu}} A_i^{1-\bar{\mu}}$. It implies that the total cost of A-sector firms on labor is given by $\bar{\mu} X_i^{\mathrm{A}} = w_i h_i^{\mathrm{A}}$ as well as that on land (= the total rental revenue of landlords) by $(1-\bar{\mu}) X_i^{\mathrm{A}} = \frac{1-\bar{\mu}}{\bar{\mu}} w_i h_i^{\mathrm{A}}$. In particular, the labor demand in this sector is given by a function of the wage $h_i^{\mathrm{A}} = A_i (w_i/\bar{\mu})^{1/(\bar{\mu}-1)}$, because $w_i = \bar{\mu}(h_i^{\mathrm{A}}/A_i)^{\bar{\mu}-1}$. Let $h_i^{\mathrm{A}} = \epsilon_i h_i^{\mathrm{M}}$, so that $h_i = (1+\epsilon_i) h_i^{\mathrm{M}}$; we here consider the case $h_i^{\mathrm{M}} \neq 0$, because we are interested in the stability of complete dispersion. We also have $\epsilon_i \equiv (A_i/h_i^{\mathrm{M}})(w_i/\bar{\mu})^{1/(\bar{\mu}-1)}$. The regional rental revenue from land, R_i in terms of h_i^{M} is:

$$R_i \equiv \frac{1-\bar{\mu}}{\bar{\mu}} \epsilon_i w_i h_i^{\mathrm{M}}. \tag{C.5.1}$$

We also note that, employing the above formulae, the elasticity ν_i of a region's labor supply to the M-sector with respect to wage is

$$\nu_i \equiv \frac{w_i}{h_i^{\mathrm{M}}} \frac{\partial h_i^{\mathrm{M}}}{\partial w_i} = \frac{h_i^{\mathrm{A}}}{h_i^{\mathrm{M}}} \frac{1}{1-\bar{\mu}} = \epsilon_i \frac{1}{1-\bar{\mu}}. \tag{C.5.2}$$

It is also noted that if $\bar{\mu} = 0$, we have $X_i^{\mathrm{A}} = A_i$ as well as $R_i = A_i$ and $\epsilon_i = 0$.

M-sector. Considering the simplest possible model of intermediate inputs as in Krugman and Venables (1995), the minimum cost function of the M-sector is replaced by

$$C(x_i(\xi)) = P_i^{\hat{\mu}} w_i^{1-\hat{\mu}} (\alpha + \beta x_i(\xi)) \tag{C.5.3}$$

where P_i is the price index of M-sector goods in region i and $\hat{\mu}$ the share of intermediates in firms' costs. The profit-maximizing price is given by

$$p_{ij}(\xi) = \frac{\sigma\beta}{\sigma - 1} P_i^{\hat{\mu}} w_i^{1-\hat{\mu}} \tau_{ij}, \tag{C.5.4}$$

which, together with the definition of P_i, implies that we should solve a system of nonlinear equations to obtain P_i. In concrete terms, the price indices $\{P_i\}$ should satisfy

$$P_i = \frac{\sigma\beta}{\sigma - 1} \left(\sum_{j \in \mathcal{K}} n_j \left(P_j^{\hat{\mu}} w_j^{1-\hat{\mu}} \right)^{1-\sigma} d_{ji} \right)^{1/(1-\sigma)} \tag{C.5.5}$$

where $d_{ij} \equiv \tau_{ij}^{1-\sigma}$. We must solve (C.5.5) along with the wage equation to be defined below.

In line of the Km model, the ZPC of firms implies $x_i(\xi) = \alpha(\sigma - 1)/\beta$. Firms' minimized production cost in region i is then given by $C_i = (\alpha\sigma) P_i^{\hat{\mu}} w_i^{1-\hat{\mu}}$, so that labor demand in the M-sector of region i is

$$h_i^{\mathrm{M}} = (1 - \hat{\mu}) \frac{C_i}{w_i} n_i = \alpha\sigma(1 - \hat{\mu}) P_i^{\hat{\mu}} w_i^{-\hat{\mu}} n_i. \tag{C.5.6}$$

The mass of varieties produced in region i is thus given as follows:

$$n_i = \frac{1}{\alpha\sigma(1 - \hat{\mu})} P_i^{-\hat{\mu}} w_i^{\hat{\mu}} h_i^{\mathrm{M}}. \tag{C.5.7}$$

For simplicity, in the following, as in the original paper we shall normalize constants such that $\alpha = 1/\sigma$ and $\beta = (\sigma - 1)/\sigma$. Then, plugging (C.5.7) to (C.5.5), we have

$$P_i^{1-\sigma} = \frac{1}{1 - \hat{\mu}} \sum_{j \in \mathcal{K}} h_j^{\mathrm{M}} P_j^{-\hat{\mu}\sigma} w_j^{1-\sigma+\hat{\mu}\sigma} d_{ji}. \tag{C.5.8}$$

Land is locally owned by immobile landlords that share the same preference as mobile workers; the regional expenditure on M-sector goods from them is given by μR_i. Also, the regional expenditure of firms on intermediates is given by

$$\hat{\mu} C_i n_i = \frac{\hat{\mu}}{1 - \hat{\mu}} w_i h_i^{\mathrm{M}}. \tag{C.5.9}$$

The total expenditure in region i on M-sector goods is $Y_i = \mu w_i h_i + \mu R_i + \hat{\mu} C_i n_i$. Using (C.5.1) as well as $h_i = (1+\epsilon_i)h_i^{\mathrm{M}}$, it is simplified to

$$Y_i = \mu(1+\epsilon_i)w_i h_i^{\mathrm{M}} + \mu R_i + \frac{\hat{\mu}}{1-\hat{\mu}} w_i h_i^{\mathrm{M}} = \left[\mu\left(1+\frac{\epsilon_i}{\bar{\mu}}\right) + \frac{\hat{\mu}}{1-\hat{\mu}}\right] w_i h_i^{\mathrm{M}}. \tag{C.5.10}$$

From the ZPC of firms, the wage equation for the model is given by[61]

$$\underbrace{\frac{1}{1-\hat{\mu}} w_i h_i^{\mathrm{M}}}_{\text{M-sector firms' total cost}} = \underbrace{\sum_{j\in\mathcal{K}} \frac{h_i^{\mathrm{M}} P_i^{-\hat{\mu}\sigma} w_i^{1-\sigma+\hat{\mu}\sigma} d_{ij}}{\sum_{k\in\mathcal{K}} h_k^{\mathrm{M}} P_k^{-\hat{\mu}\sigma} w_k^{1-\sigma+\hat{\mu}\sigma} d_{kj}} \left[\mu\left(1+\frac{\epsilon_j}{\bar{\mu}}\right) + \frac{\hat{\mu}}{1-\hat{\mu}}\right] w_j h_j^{\mathrm{M}}}_{\text{M-sector firms' total revenue}}. \tag{C.5.11}$$

The short-run wage $\boldsymbol{w} = (w_i)$ and price index $\boldsymbol{P} = (P_i)$ are obtained as the solution for the system of nonlinear equations (C.5.8) and (C.5.11). It is noted that we must require $\hat{\mu} < \frac{\sigma-1}{\sigma}$ so that $\boldsymbol{P}$ and $\boldsymbol{w}$ are uniquely determined for any value of transportation cost.

Given $\boldsymbol{P}$ and $\boldsymbol{w}$, the indirect utility function is

$$v_i(\boldsymbol{h}) = \frac{\mu}{\sigma-1}\ln[\Delta_i] + \ln[w_i] \tag{C.5.12}$$

with $\Delta_i = \sum_{j\in\mathcal{K}} h_j^{\mathrm{M}} P_j^{-\hat{\mu}\sigma} w_j^{1-\sigma+\hat{\mu}\sigma} d_{ji}$.

Jacobian matrix of the payoff function at the flat-earth equilibrium. Let $A_i = A$ for all i and consider the flat-earth equilibrium. Let $h \equiv H/K$ be the uniform number of mobile agents; let also h^{M} and h^{A} be the number of mobile agents engaged in the M- and A-sector, respectively. Also, let $\bar{Y}$, $\bar{P}$, $\bar{w}$, $\bar{\Omega}$ and $\bar{\epsilon}$ be the uniform level of regional expenditure, the price index, the wage, Ω_i, and the ratio ϵ_i of h_i^{A} to h_i^{M}, at the flat-earth equilibrium, respectively. Adding up the wage equations (C.5.11) at the flat-earth equilibrium, we show

$$\bar{\epsilon} = \frac{h^{\mathrm{A}}}{h^{\mathrm{M}}} = \bar{\mu}\frac{1-\mu}{\mu}. \tag{C.5.13}$$

A larger $\bar{\mu}$ (μ) implies a larger (smaller) $\bar{\epsilon} = h^{\mathrm{A}}/h^{\mathrm{M}}$, which is intuitive. The explicit formula of $\bar{\epsilon}$ yields

$$\bar{Y} = \frac{\bar{w}}{1-\hat{\mu}}\frac{\bar{\epsilon}}{1+\bar{\epsilon}}h, \ \bar{w} = \bar{\mu}\left(\frac{h}{A}\frac{\bar{\epsilon}}{1+\bar{\epsilon}}\right)^{\bar{\mu}-1}. \tag{C.5.14}$$

Observe that, together with the fact $\bar{\epsilon}/\bar{\mu} = \frac{\mu}{1-\mu}$, $\bar{\mu}$ does not affect stability of $\bar{\boldsymbol{h}}$ but scales the total amount of the world income. We also have $\bar{P} = \rho\bar{w}$ where $\rho \equiv \{(h^{\mathrm{M}}d)(1-\hat{\mu})\}^{1/(1-\sigma+\hat{\mu}\sigma)}$ with d being the row-sum of $\boldsymbol{D}$ and that $\Delta_i = \bar{\Delta} \equiv (1-\hat{\mu})\bar{P}^{1-\sigma}$. Note that at the equilibrium, we have

$$\frac{1}{\bar{\mu}}\frac{\partial\epsilon_i}{\partial h_i^{\mathrm{M}}} = -\frac{1}{\bar{\mu}}\frac{\bar{\epsilon}}{h^{\mathrm{M}}} = -\frac{1}{h^{\mathrm{M}}}\frac{1-\mu}{\mu} \tag{C.5.15}$$

[61] The original analyses in Puga (1999) allow possibilities of positive profit of firms. In this appendix, we shall adhere to zero-profit so that comparisons with the other models become possible.

Also, with $\bar{\nu} \equiv \frac{\bar{\mu}}{1-\bar{\mu}} \frac{1-\mu}{\mu}$ being the elasticity of labor supply from A- to the M-sector with respect to w_i,

$$\frac{\partial h_i^{\mathrm{M}}}{\partial w_i} = \bar{\nu} \frac{h^{\mathrm{M}}}{w_i}, \ \frac{1}{\bar{\mu}} \frac{\partial \epsilon_i}{\partial w_i} = \frac{1}{\bar{\mu}} \frac{\partial \epsilon_i}{\partial h^{\mathrm{M}}} \frac{\partial h_i^{\mathrm{M}}}{\partial w_i} = -\bar{\nu} \frac{1}{w_i} \frac{1-\mu}{\mu} \tag{C.5.16}$$

Also, we assume that $\partial h_i / \partial h_i^{\mathrm{M}} = \partial h_i^{\mathrm{M}} / \partial h_i = 1$ as discussed. It follows that, at $\bar{\boldsymbol{h}}$,

$$\frac{\partial Y_i}{\partial h_i^{\mathrm{M}}} = \left(\frac{\hat{\mu}}{1-\hat{\mu}} + \mu \right) \bar{w}, \ \frac{\partial Y_i}{\partial w_i} = \left(\frac{1}{1-\hat{\mu}} + (1-\mu)\bar{\nu} \right) h^{\mathrm{M}}. \tag{C.5.17}$$

The Jacobian matrix of the payoff function is computed as

$$\frac{\partial v_i}{\partial h_j} = \frac{\mu}{\sigma - 1} \frac{1}{\Delta_i} \left(\frac{\partial \Delta_i}{\partial h_j} + \sum_{k \in \mathcal{K}} \frac{\partial \Delta_i}{\partial P_k} \frac{\partial P_k}{\partial h_j} + \sum_{k \in \mathcal{K}} \frac{\partial \Delta_i}{\partial w_k} \frac{\partial w_k}{\partial h_j} \right) + \delta_{i,j} \frac{1}{w_i} \frac{\partial w_i}{\partial h_j}, \tag{C.5.18}$$

where $\delta_{i,j}$ is Kronecker's delta; below, we shall evaluate $\nabla \boldsymbol{\Delta} \equiv [\partial \Delta_i / \partial h_j]$, $\nabla_{\boldsymbol{P}} \boldsymbol{\Delta} \equiv [\partial \Delta_i / \partial P_j]$, $\nabla_{\boldsymbol{w}} \boldsymbol{\Delta} \equiv [\partial \Delta_i / \partial w_j]$, $\nabla \boldsymbol{P} \equiv [\partial P_i / \partial h_j]$, and $\nabla \boldsymbol{w} \equiv [\partial w_i / \partial h_j]$.

For $\nabla \boldsymbol{\Delta}$, $\nabla_{\boldsymbol{P}} \boldsymbol{\Delta}$, and $\nabla_{\boldsymbol{w}} \boldsymbol{\Delta}$, we compute as follows: $\nabla \boldsymbol{\Delta} = \bar{\Delta} (h^{\mathrm{M}})^{-1} \bar{\boldsymbol{D}}$, $\nabla_{\boldsymbol{w}} \boldsymbol{\Delta} = \bar{\Delta} \bar{w}^{-1} a \bar{\boldsymbol{D}}$, as well as $\nabla_{\boldsymbol{P}} \boldsymbol{\Delta} = \bar{\Delta} \bar{P}^{-1} b \bar{\boldsymbol{D}}$, with $a \equiv 1 - \sigma + \hat{\mu}\sigma$ and $b \equiv -\hat{\mu}\sigma$. Thus, at the flat-earth pattern, $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ is

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = \frac{\mu}{\sigma - 1} \left(\frac{1}{h^{\mathrm{M}}} \bar{\boldsymbol{D}} + \frac{1}{\bar{P}} b \bar{\boldsymbol{D}} \nabla \boldsymbol{P} \right) + \frac{1}{\bar{w}} \left(\boldsymbol{I} + \frac{\mu a}{\sigma - 1} \bar{\boldsymbol{D}} \right) \nabla \boldsymbol{w} \tag{C.5.19}$$

The remaining task is to evaluate $\nabla \boldsymbol{P}$ and $\nabla \boldsymbol{w}$. First, totally differentiating the definition of price index (C.5.8), we have $\nabla \boldsymbol{Q} \mathrm{d} \boldsymbol{h}^{\mathrm{M}} + \nabla_{\boldsymbol{w}} \boldsymbol{Q} \mathrm{d} \boldsymbol{w} + \nabla_{\boldsymbol{P}} \boldsymbol{Q} \mathrm{d} \boldsymbol{P} = \boldsymbol{0}$ with

$$\nabla_{\boldsymbol{P}} \boldsymbol{Q} \equiv [(\sigma - 1)\boldsymbol{I} + b\bar{\boldsymbol{D}}], \ \nabla_{\boldsymbol{h}} \boldsymbol{Q} \equiv \frac{\bar{P}}{h^{\mathrm{M}}} \bar{\boldsymbol{D}}, \ \nabla_{\boldsymbol{w}} \boldsymbol{Q} \equiv a \frac{\bar{P}}{\bar{w}} \bar{\boldsymbol{D}}. \tag{C.5.20}$$

Also, total differentiation of the wage equation implies $\nabla \boldsymbol{W} \mathrm{d} \boldsymbol{h}^{\mathrm{M}} + \nabla_{\boldsymbol{w}} \boldsymbol{W} \mathrm{d} \boldsymbol{w} + \nabla_{\boldsymbol{P}} \boldsymbol{W} \mathrm{d} \boldsymbol{P} = \boldsymbol{0}$ with

$$\nabla_{\boldsymbol{w}} \boldsymbol{W} \equiv h^{\mathrm{M}} \boldsymbol{I} - (1 - \hat{\mu}) \left[a \frac{1}{\bar{w}} \bar{Y} (\boldsymbol{I} - \bar{\boldsymbol{D}}^2) + \frac{\partial \bar{Y}}{\partial w} \bar{\boldsymbol{D}} \right], \tag{C.5.21}$$

$$\nabla_{\boldsymbol{h}} \boldsymbol{W} \equiv \bar{w} \boldsymbol{I} - (1 - \hat{\mu}) \left[\frac{1}{h^{\mathrm{M}}} \bar{Y} (\boldsymbol{I} - \bar{\boldsymbol{D}}^2) + \frac{\partial \bar{Y}}{\partial h^{\mathrm{M}}} \bar{\boldsymbol{D}} \right], \tag{C.5.22}$$

$$\nabla_{\boldsymbol{P}} \boldsymbol{W} \equiv -(1 - \hat{\mu}) b \frac{1}{\bar{P}} \bar{Y} (\boldsymbol{I} - \bar{\boldsymbol{D}}^2) \nabla \boldsymbol{P}. \tag{C.5.23}$$

We already computed $\bar{Y}$, $\partial \bar{Y} / \partial w$, and $\partial \bar{Y} / \partial h^{\mathrm{M}}$. These relations yields the analytical expressions for the Jacobian matrices $\nabla \boldsymbol{P}$ and $\nabla \boldsymbol{w}$:

$$\nabla \boldsymbol{P} = -[\nabla_{\boldsymbol{w}} \boldsymbol{Q} \nabla_{\boldsymbol{P}} \boldsymbol{W} - \nabla_{\boldsymbol{P}} \boldsymbol{Q} \nabla_{\boldsymbol{w}} \boldsymbol{W}]^{-1} [\nabla_{\boldsymbol{w}} \boldsymbol{Q} \nabla_{\boldsymbol{h}} \boldsymbol{W} - \nabla_{\boldsymbol{h}} \boldsymbol{Q} \nabla_{\boldsymbol{w}} \boldsymbol{W}], \tag{C.5.24}$$

$$\nabla \boldsymbol{w} = [\nabla_{\boldsymbol{w}} \boldsymbol{Q} \nabla_{\boldsymbol{P}} \boldsymbol{W} - \nabla_{\boldsymbol{P}} \boldsymbol{Q} \nabla_{\boldsymbol{w}} \boldsymbol{W}]^{-1} [\nabla_{\boldsymbol{P}} \boldsymbol{Q} \nabla_{\boldsymbol{h}} \boldsymbol{W} - \nabla_{\boldsymbol{h}} \boldsymbol{Q} \nabla_{\boldsymbol{P}} \boldsymbol{W}]. \tag{C.5.25}$$

Summing up the computations up to here, a patient computation yields

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = \boldsymbol{J}_0^{-1}\left[\check{\mu}\left(\frac{1}{\sigma-1}+\frac{1}{\sigma}\right)\bar{\boldsymbol{D}} - \left(\frac{\check{\mu}^2}{\sigma-1}+\frac{1}{\sigma}+\omega\right)\bar{\boldsymbol{D}}^2\right] \tag{C.5.26}$$

where $\boldsymbol{J}_0$ is a positive definite matrix defined by $\bar{\boldsymbol{D}}$, $\check{\mu} \equiv \hat{\mu} + \mu(1-\hat{\mu})$ which is loosely interpreted as the aggregate expenditure share on M-sector goods, and $\omega \equiv \frac{\mu(1-\check{\mu})}{\sigma(\sigma-1)}(1-\bar{\nu})$ is a constant that summarizes the effects of labor mobility between A- and M-sectors at $\bar{\boldsymbol{h}}$. Thus, $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) \simeq c_1\bar{\boldsymbol{D}} + c_2\bar{\boldsymbol{D}}^2$ with

$$c_1 = \check{\mu}\left(\frac{1}{\sigma-1}+\frac{1}{\sigma}\right) > 0, \tag{C.5.27}$$

$$c_2 = -\left(\frac{\check{\mu}^2}{\sigma-1}+\frac{1}{\sigma}+\omega\right) < 0. \tag{C.5.28}$$

C.6 Tabuchi (1998) (Tb) model

The Tb model introduces internal structure of regions to the Km model. The main thrust of the model is, unlike majority of regional models, the city boundary in each region is endogenously determined by a full-fledged monocentric city model of Alonso–Muth–Mills. This produces a rich structure of urban costs, because the trade-off between commuting cost and land rent is made explicit.

In this model, there are all of the three sectors of M, H, A in the Km and Hm models. Internal structure of each region is featureless, except that it is endowed with a single central business district (CBD) with negligible spatial extent. In each region, locations are indexed by the distance from the CBD, $x \geq 0$. At any point, the land endowment density is assumed to be unity. The total numbers of skilled and unskilled workers are given by H and L, respectively. The number of skilled workers in region i is denoted by h_i, whereas the spatial distribution (density) in that region is, allowing a notational abuse, denoted by $h_i(x)$. Thus, we have

$$\int_0^{\bar{x}_i} h_i(x)\mathrm{d}x = h_i \tag{C.6.1}$$

where $\bar{x}_i \geq 0$ is the city boundary in region i that is endogenously determined. Unskilled workers are employed by the A-sector and do not commute to the CBD, whereas those skilled do. A skilled worker at distance x from the CBD incur generalized cost of commuting $T(x)$ which is measured by the numéraire.

Preference. The utility of a representative worker living in region i and located at x is given by

$$U(C_i^{\mathrm{M}}, C_i^{\mathrm{H}}, C_i^{\mathrm{A}}) = \mu \ln C_i^{\mathrm{M}} + \gamma \ln C_i^{\mathrm{H}} + (1-\mu-\gamma)\ln C_i^{\mathrm{A}} \tag{C.6.2}$$

where μ and γ with $\mu + \gamma < 1$ are the constant expenditure shares for manufactured goods and housing goods, respectively; C_i^{M} is the CES aggregate of the M-sector goods defined by (C.1.2), C_i^{H} the consumption of housing space (the H-sector goods), C_i^{A} the consumption of agricultural products (the A-sector goods) in region i. The M-sector goods are subject to iceberg transport cost, whereas those of the A-sector not for both intra- and interregional transportation. The H-sector goods are local and nontradable. Choosing the A-sector goods as the numéraire, the budget constraint of a skilled worker at location x in region i is

$$C_i^{\mathrm{A}} + r_i(x)C_i^{\mathrm{H}}(x) + \sum_{j\in\mathcal{K}}\int_0^{n_j} p_{ji}(\xi)q_{ji}(\xi)\mathrm{d}\xi + T_i(x) = y_i, \tag{C.6.3}$$

where $r_i(x)$ is the land rent prevailing at location x in the region i, $T(x)$ the generalized cost of commuting from location x to the CBD, and y_i the income of the worker. We assume that $T(x)$ is differentiable and increasing in x with $T(0) = 0$. Note that $T(x)$ is independent of its population and is homogeneous among the regions. Given price including land rent profile $\{r_i(x)\}$, utility maximization yields

$$C_i^{\mathrm{M}}(x) = \mu\frac{y_i(x)}{P_i},\ C_i^{\mathrm{H}}(x) = \gamma\frac{y_i(x)}{r_i(x)},\ C_i^{\mathrm{A}}(x) = (1-\mu-\gamma)y_i(x),\ q_{ji}(\xi) = \frac{\{p_{ji}(\xi)\}^{-\sigma}}{P_i^{-\sigma}}C_i^{\mathrm{M}} \tag{C.6.4}$$

where $y_i(x) = y_i - T(x)$ is the net income of a worker residing at x in region i. Following the tradition of urban economics, the model assumes absentee landowners who keep the rental revenue of housing so that we have $y_i = w_i$ for every skilled worker. Unskilled workers live outside the city and does not commute to the CBD. Thus, they face the agricultural land rent $r^{\mathrm{A}} > 0$ and zero commuting cost, as well as $y_i = 1$. For simplicity, we assume that r^{A} is the same across the regions. We also assume that intracity transportation of M-sector products is costless, so that both unskilled and skilled workers face the same M-sector product price.

Internal structure of each region. As discussed, the difference compared to the Km model is that the internal structure of each region is now explicitly modeled by a monocentric city model. The standard first-order condition for equilibrium spatial pattern is that

$$C_i^{\mathrm{H}}(x)\frac{\mathrm{d}r_i(x)}{\mathrm{d}x} + \frac{\mathrm{d}T(x)}{\mathrm{d}x} = 0 \tag{C.6.5}$$

for $0 \le x \le \bar{x}$ with the boundary condition being $r_i(\bar{x}) = r^{\mathrm{A}}$. In the following, we focus on a single region given fixed values of w_i and h_i. For simplicity, we omit index i unless otherwise noted. Combining $C_i^{\mathrm{H}}(x)$ in (C.6.4), we obtain the land rent profile $r(x)$ given w:

$$r(x) = \hat{r}\{1 - T(x)/w\}^{1/\gamma} \tag{C.6.6}$$

with $r(\bar{x}) = r^{\mathrm{A}}$ at the city boundary $\bar{x}$ of the region. Thus, $\hat{r}$, the land rent at the CBD ($x = 0$) when the city boundary is at $\bar{x}$ and wage rate is w, is determined as

$$\hat{r}(\bar{x}, w) = \frac{r^{\mathrm{A}}}{\{1 - T(\bar{x})/w\}^{1/\gamma}}. \tag{C.6.7}$$

We observe that $\hat{r} = r^{\mathrm{A}}$ when $h_i = 0$ because $\bar{x} = 0$ and $T(0) = 0$. With a notational abuse, the population density function $h(x)$ in the region for given $\bar{x}$ and w becomes

$$h(x) = \frac{a(x)}{C^{\mathrm{H}}(x)} = \frac{a(x)r(x)}{\gamma y(x)} = \frac{\hat{r}(\bar{x}, w)}{\gamma w} a(x)\{1 - T(x)/w\}^{1/\gamma - 1} \tag{C.6.8}$$

where $a(x)$ is land endowment at distance x. We here note that as $r(\bar{x}) = r^{\mathrm{A}}$,

$$h(\bar{x}) = \frac{a(\bar{x})r^{\mathrm{A}}}{\gamma(w - T(\bar{x}))}. \tag{C.6.9}$$

In Tabuchi (1998), it is assumed that $a(x) = 2\pi x$ so that the city is disk-shaped.

Comparative statistics for the internal structure of a region. Before studying stability of the flat-earth equilibrium in the regional scale, we shall first investigate how changes in h_i and w_i affect the internal structure of a single region. We note that the population density function $h(x)$ satisfies

$$\frac{\partial h(x)}{\partial \bar{x}} = \frac{T'(\bar{x})}{\gamma(w - T(\bar{x}))} h(x) > 0 \tag{C.6.10}$$

$$\frac{\partial h(x)}{\partial w} = \left(\frac{1 - \gamma}{\gamma(w - T(x))} - \frac{1}{\gamma(w - T(\bar{x}))}\right) h(x) < 0 \tag{C.6.11}$$

provided that $w - T(\bar{x}) > 0$, which must be the case because otherwise the utility of an agent at $\bar{x}$ becomes negative infinity. The latter inequality says that, as it is standard in the literature, population density decreases as the income increases. Define a function $H(\bar{x}, w)$ that returns the population in the interval $[0, \bar{x})$ by

$$H(\bar{x}, w) = \int_0^{\bar{x}} h(x)\mathrm{d}x. \tag{C.6.12}$$

Then, the location of the city boundary $\bar{x}$ for given h and w is determined by the equation

$$h = H(\bar{x}, w), \tag{C.6.13}$$

whence $\bar{x}$ becomes a function of h and w. For later use, we shall investigate the effects of h and w on $\bar{x}$. Applying the implicit function theorem to the equation $H(\bar{x}, w) - h = 0$, we have

$$\frac{\partial \bar{x}}{\partial h} = \frac{1}{h(\bar{x})} \text{ and } \frac{\partial \bar{x}}{\partial w} = -\frac{1}{h(\bar{x})}\frac{\partial H}{\partial w}, \tag{C.6.14}$$

where we assume that w is determined in the region-scale trade balance so that w and h are the independent variables. Note that $\partial H / \partial \bar{x} = h(\bar{x})$. From (C.6.11), we have

$$\frac{\partial H}{\partial w} = \int_0^{\bar{x}} \frac{\partial h(x)}{\partial w}\mathrm{d}x < 0 \tag{C.6.15}$$

which says that the population in the interval $[0, \bar{x})$ decreases when the income increases. Then, from (C.6.14) we conclude that (i) the city boundary $\bar{x}$ is increasing in w_i, (ii) $\bar{x}$ is increasing in h_i. Thus, we see that $\bar{x}$ is increasing in both h_i and w_i, which is standard. Also, define the total (generalized) costs incurred by commuting in the region by

$$T_i = \int_0^{\bar{x}} T(x)h(x)\mathrm{d}x. \tag{C.6.16}$$

Then, one can show that

$$\frac{\partial T_i}{\partial h} = T(\bar{x}) + \frac{v}{h(\bar{x})}T_i > 0, \; \frac{\partial T_i}{\partial w} = -\frac{\partial T_i}{\partial h}\frac{\partial H}{\partial w} > 0 \tag{C.6.17}$$

where v is the elasticity of land rent at the city boundary $\bar{x}$:

$$v \equiv -\frac{r'(\bar{x})}{r(\bar{x})} = \frac{T'(\bar{x})}{\gamma(w - T(\bar{x}))}. \tag{C.6.18}$$

Thus, the total commuting cost increases in both h_i and w_i ceteris paribus, which is also standard.

Short-run equilibrium. Consider the regional scale and recover region indices. Given $\bar{x}_i$, the total expenditure in region i net of commuting cost is given by $Y_i = w_i h_i - T_i + l_i$. The wage equation for the model is given by

$$w_i h_i = \mu \sum_{j\in\mathcal{K}} \frac{h_i w_i^{1-\sigma} d_{ij}}{\sum_{k\in\mathcal{K}} h_k w_k^{1-\sigma} d_{kj}} (w_j h_j - T_j + l_j). \tag{C.6.19}$$

We impose the following constraint on $\boldsymbol{w}$ for normalization

$$\sum_{i\in\mathcal{K}} (w_i h_i - T_i) = \frac{\mu}{1-\mu} L, \tag{C.6.20}$$

where we note that T_i depends on both h_i and w_i. Given the short-run wage, the indirect utility for region i is obtained by evaluating it at the CBD ($x = 0$) since utility is equalized in each region:

$$v_i(\boldsymbol{h}) = \bar{\kappa} \ln[\Delta_i] + \ln[y_i(\bar{x}_i)] \tag{C.6.21}$$

where $\Delta_i = \sum_{j\in\mathcal{K}} h_j w_j^{1-\sigma} d_{ji}$ and $y_i(\bar{x}_i) = w_i - T(\bar{x}_i)$.

Jacobian matrix at the flat-earth equilibrium. We compute as follows:

$$\nabla \boldsymbol{v}(\boldsymbol{h}) = \bar{\kappa} \boldsymbol{M}^\top \operatorname{diag}[\boldsymbol{h}]^{-1} - \mu \boldsymbol{M}^\top \nabla \boldsymbol{w}(\boldsymbol{h}) \operatorname{diag}[\boldsymbol{w}]^{-1} + \operatorname{diag}[y_i(\bar{x}_i)]^{-1} \nabla[y_i(\bar{x}_i)] \tag{C.6.22}$$

with $\boldsymbol{M}$ defined in line with the Km model and $\nabla[y_i(\bar{x}_i)] = \nabla \boldsymbol{w}(\boldsymbol{h}) - \nabla \operatorname{diag}[T(\bar{x}_i)]$, where we note that

$$\nabla \operatorname{diag}[T(\bar{x}_i)] = \operatorname{diag}[T'(\bar{x}_i)] \nabla[\bar{x}_i(h_i, w_i)] = \operatorname{diag}[T'(\bar{x}_i)] \big\{ \operatorname{diag}[\partial \bar{x}_i / \partial h_i] + \operatorname{diag}[\partial \bar{x}_i / \partial w_i] \nabla \boldsymbol{w}(\boldsymbol{h}) \big\}.$$

Thus, letting $\boldsymbol{\Psi}_0 \equiv \operatorname{diag}[T'(\bar{x}_i)\partial \bar{x}_i / \partial h_i]$ and $\boldsymbol{\Psi}_1 \equiv \operatorname{diag}[T'(\bar{x}_i)\partial \bar{x}_i / \partial w_i]$, we have

$$\nabla[y_i(\bar{x}_i)] = \nabla \boldsymbol{w}(\boldsymbol{h}) - (\boldsymbol{\Psi}_0 + \boldsymbol{\Psi}_1 \nabla \boldsymbol{w}(\boldsymbol{h})) = -\boldsymbol{\Psi}_0 + (\boldsymbol{I} - \boldsymbol{\Psi}_1) \nabla \boldsymbol{w}(\boldsymbol{h}) \tag{C.6.23}$$

As in the Km model, $\nabla_{\boldsymbol{w}} = [\partial/\partial w_i]$. For $\nabla \boldsymbol{w}(\boldsymbol{h})$, we have $\nabla \boldsymbol{w}(\boldsymbol{h}) = -(\nabla_{\boldsymbol{w}} \boldsymbol{W})^{-1}(\nabla \boldsymbol{W})$ with

$$\nabla_{\boldsymbol{w}} \boldsymbol{W} = \mathrm{diag}[\boldsymbol{h}] + \mu(\sigma - 1)(\mathrm{diag}[\boldsymbol{M}\boldsymbol{Y}] - \boldsymbol{M}\,\mathrm{diag}[\boldsymbol{Y}]\boldsymbol{M}^\top)\,\mathrm{diag}[\boldsymbol{w}]^{-1} - \mu \boldsymbol{M}\nabla_{\boldsymbol{w}}\boldsymbol{Y}, \tag{C.6.24}$$

$$\nabla \boldsymbol{W} = \mathrm{diag}[\boldsymbol{w}] - \mu(\mathrm{diag}[\boldsymbol{M}\boldsymbol{Y}] - \boldsymbol{M}\boldsymbol{Y}\boldsymbol{M}^\top)\,\mathrm{diag}[\boldsymbol{h}]^{-1} - \mu \boldsymbol{M}\nabla \boldsymbol{Y} \tag{C.6.25}$$

where $\boldsymbol{Y} = [Y_i] = [w_i h_i - T_i + l_i]$, $\nabla_{\boldsymbol{w}} \boldsymbol{Y} = \mathrm{diag}[\boldsymbol{h}] - \nabla_{\boldsymbol{w}} \boldsymbol{T}$, and $\nabla \boldsymbol{Y} = \mathrm{diag}[\boldsymbol{w}] - \nabla \boldsymbol{T}$.

Consider the flat-earth equilibrium in a symmetric racetrack economy with $l_i = l$. Let $\bar{w}$ and $\bar{T}$ be the uniform level of nominal wage rate and the total commuting cost in each region. Note that $\bar{T}$ is a function of $\bar{w}$ and $\bar{x}$. Given the commuting cost function $T(x)$ and the location of city boundary and the wage $(\bar{x}, \bar{w})$, at the flat-earth equilibrium we shall require

$$\bar{w}h - \bar{T}(\bar{x}, \bar{w}) = \frac{\mu}{1-\mu} l \tag{C.6.26}$$

so that wage is normalized. Then, one can show that there exists a unique positive solution $(\bar{x}^*, \bar{w}^*)$ such that $\bar{w}^* - T(\bar{x}^*) > 0$ for the system of nonlinear equations defined by (C.6.13) and (C.6.26) for given h. Employing the solution $(\bar{x}^*, \bar{w}^*)$, the total income in Y is given by $\bar{Y} = l/(1-\mu)$. Define the ratios ϕ of the regional total of disposable wage of skilled worker, and $\hat{\phi}$ of the regional total expenditure to the total nominal wage:

$$\phi \equiv \frac{\bar{w}h - \bar{T}}{\bar{w}h}, \ \hat{\phi} \equiv \frac{\bar{Y}}{\bar{w}h}. \tag{C.6.27}$$

The latter implies that $\bar{Y}/\bar{w} = \hat{\phi}h$ and $\bar{Y}/h = \hat{\phi}\bar{w}$. Given $(\bar{x}^*, w^*)$, we define $T(x)$-dependent positive constants ψ_0, ψ_1, ρ_0, and ρ_1 such that $\boldsymbol{\Psi}_0 = \psi_0 \boldsymbol{I}$, $\boldsymbol{\Psi}_1 = \psi_1 \boldsymbol{I}$, $\nabla \boldsymbol{Y} = \rho_0 \bar{w} \boldsymbol{I}$, and $\nabla_{\boldsymbol{w}} \boldsymbol{Y} = \rho_1 h \boldsymbol{I}$. Then, we can calculate the Jacobian matrix of the payoff function at the flat-earth equilibrium as follows:

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = h^{-1}\bar{\kappa}\bar{\boldsymbol{D}} - \bar{w}^{-1}\mu\bar{\boldsymbol{D}}\nabla \boldsymbol{w}(\bar{\boldsymbol{h}}) - \bar{y}^{-1}\psi_0 \boldsymbol{I} + \bar{y}^{-1}(1-\psi_1)\nabla \boldsymbol{w}(\bar{\boldsymbol{h}}), \tag{C.6.28}$$

$$= h^{-1}\bar{\kappa}\bar{\boldsymbol{D}} - \bar{y}^{-1}\psi_0 \boldsymbol{I} + \{\bar{y}^{-1}(1-\psi_1)\boldsymbol{I} - \bar{w}^{-1}\mu\bar{\boldsymbol{D}}\}\nabla \boldsymbol{w}(\bar{\boldsymbol{h}}) \tag{C.6.29}$$

where $\bar{y} \equiv y(\bar{x}^*) = \bar{w}^* - T(\bar{x}^*)$ is the net wage at $\bar{x}^*$ and $\nabla \boldsymbol{w}(\bar{\boldsymbol{h}}) = -(\nabla_{\boldsymbol{w}} \boldsymbol{W})^{-1}(\nabla \boldsymbol{W})$ with

$$\nabla \boldsymbol{W} = -\bar{w}\big[-(1-\hat{\phi}\mu)\boldsymbol{I} + \rho_0\mu\bar{\boldsymbol{D}} - \hat{\phi}\mu\bar{\boldsymbol{D}}^2\big],, \tag{C.6.30}$$

$$\nabla_{\boldsymbol{w}} \boldsymbol{W} = h\big[\big\{\hat{\phi}\mu(\sigma-1)+1\big\}\boldsymbol{I} - \rho_1\mu\bar{\boldsymbol{D}} - \hat{\phi}\mu(\sigma-1)\bar{\boldsymbol{D}}^2\big]. \tag{C.6.31}$$

Illustration. Following Tabuchi (1998), we shall investigate the simplest case where the commuting cost function is linear with respect to distance: $T(x) = tx$. We shall also simplify the analysis by assuming the internal structure of each region is one-dimensional and extends symmetrically about the CBD over the interval $[-\bar{x}, \bar{x}]$ á la Murata and Thisse (2005). Although this change strengthens the role of urban costs in each region, it does not affect intrinsic properties of the model. For this case, letting $a(x) = 1$, we obtain

$$\bar{x} = \frac{1}{t}(1 - \epsilon^{\gamma})\bar{w}. \tag{C.6.32}$$

where the non-dimensional constant $\epsilon \in (0,1)$ is defined by $\epsilon \equiv (1+\hat{t}h)^{-1}$. The parameter $\hat{t} \equiv (t/2)/r^{\mathrm{A}}$ is interpreted as a measure of relative magnitude of commuting cost to land rent. As expected, $\bar{x}$ is decreasing in the generalized commuting cost per distance t. Then, solving (C.6.26) implies that

$$\bar{w} = \frac{1}{\phi} \cdot \frac{\mu}{1-\mu} \cdot \frac{L}{H} \quad \text{and} \quad \phi = \frac{1}{1+\gamma} \cdot \frac{1-\epsilon^{1+\gamma}}{1-\epsilon} \tag{C.6.33}$$

as well as $\bar{y} = \epsilon^{\gamma}\bar{w}$, $\bar{Y} = l/(1-\mu)$, and $\hat{\phi} = \phi/\mu$. Then, we also have

$$\psi_0 = T'(\bar{x})\frac{\partial \bar{x}}{\partial h} = h^{-1}\bar{y}\gamma(1-\epsilon),\ \psi_1 = T'(\bar{x})\frac{\partial \bar{x}}{\partial w} = 1-\epsilon^{\gamma}, \tag{C.6.34}$$

$$\rho_0 = \frac{1}{\bar{w}}\frac{\partial Y_i}{\partial h_i} = \frac{1}{\bar{w}}\left(w_i - \frac{\partial T_i}{\partial h_i}\right) = 1-\gamma(1-\epsilon)\phi,\ \rho_1 = \frac{1}{h}\frac{\partial Y_i}{\partial w_i} = \frac{1}{h}\left(h_i - \frac{\partial T_i}{\partial w_i}\right) = \phi. \tag{C.6.35}$$

Summarizing computations up to here yields analytical expression of $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ as follows

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = h^{-1}\bar{\kappa}\bar{\boldsymbol{D}} + (\boldsymbol{I} - \mu\bar{\boldsymbol{D}})\bar{w}^{-1}\nabla \boldsymbol{w}(\bar{\boldsymbol{h}}) - h^{-1}\hat{\gamma}\boldsymbol{I}, \tag{C.6.36}$$

$$\nabla \boldsymbol{w}(\bar{\boldsymbol{h}}) = \bar{w}h^{-1}\left[\hat{c}_0\boldsymbol{I} + \hat{c}_1\bar{\boldsymbol{D}} + \hat{c}_2\bar{\boldsymbol{D}}^2\right]^{-1}\left[\bar{c}_0\boldsymbol{I} + \bar{c}_1\bar{\boldsymbol{D}} + \bar{c}_2\bar{\boldsymbol{D}}^2\right] \tag{C.6.37}$$

with the coefficients being

$$\begin{cases} \hat{c}_0 \equiv 1 + (\sigma-1)\phi > 0, \\ \hat{c}_1 \equiv -\mu\phi < 0, \\ \hat{c}_2 \equiv -(\sigma-1)\phi < 0, \end{cases} \qquad \begin{cases} \bar{c}_0 \equiv -(1-\phi) < 0, \\ \bar{c}_1 \equiv \mu(1-\hat{\gamma}\phi) > 0, \\ \bar{c}_2 \equiv -\phi < 0 \end{cases} \tag{C.6.38}$$

where $\hat{\gamma} \equiv \gamma(1-\epsilon)$. Note that ϕ and $\hat{\gamma}$ toghether summarize the net effects of the two types of urban costs; ϕ and $\hat{\gamma}$ would represent those from commuting and nontradable land, respectively. As a consequence, we have $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) \simeq c_0\boldsymbol{I} + c_1\bar{\boldsymbol{D}} + c_2\bar{\boldsymbol{D}}^2$ with

$$c_0 = -\hat{\gamma}\left(\frac{1}{\sigma} + \frac{\sigma-1}{\sigma}\phi\right) < 0, \tag{C.6.39}$$

$$c_1 = \mu\left(\frac{1}{\sigma-1} + \frac{1}{\sigma}\right) > 0, \tag{C.6.40}$$

$$c_2 = -\left[\frac{\mu^2}{\sigma-1}\underbrace{\left(\frac{\phi}{\sigma} + \frac{\sigma-1}{\sigma}(1-\hat{\gamma}\phi)\right)}_{=:\omega_0} + \underbrace{\frac{1}{\sigma}\phi - \hat{\gamma}\phi\frac{\sigma-1}{\sigma}}_{=:\omega_1/\sigma}\right] \equiv -\left(\frac{\mu^2}{\sigma-1}\omega_0 + \frac{1}{\sigma}\omega_1\right) \tag{C.6.41}$$

Remark C.5. Observe that if $\hat{t}$ and γ are both infinitesimally small so that there are virtually no urban costs, we have $\hat{\gamma} = \gamma(1-\epsilon) \approx 0(1-1) = 0$ and $\phi \approx (1+\gamma)^{-1} \approx 1$. Then, the coefficients c_0, c_1, and c_2 reduces to those of the Km model, which is intuitive. We note that for general cases, the sign of c_2 is ambiguous. In particular, if γ is large relative to μ and in addition $1-\epsilon$ is small ($\hat{t}$ or h is small), c_2 can be positive. It is because while housing is important relative to manufactured goods, commuting cost is quite low; this implies that a concentration of skilled workers is beneficial despite higher market competition on the side of firms.

C.7 Pflüger and Südekum (2008) (PS) model

The PS model builds on Pflüger (2004), with only difference being that it introduces housing sector (denoted by H) which prodece a local dispersion force.

Preference. The homogeneous preference of skilled workers is given by the following quasilinear form with respect to the A-sector good (numéraire):

$$U(C_i^{\mathrm{M}}, C_i^{\mathrm{H}}, C_i^{\mathrm{A}}) = \mu \ln C_i^{\mathrm{A}} + \gamma \ln C_i^{\mathrm{H}} + C_i^{\mathrm{A}} \tag{C.7.1}$$

where C_i^{M}, C_i^{H}, and C_i^{A} are again consumption of manufacturing aggregate, C_i^{H} the consumption of housing good, and C_i^{A} the agricultural good, respectively. Then, indirect utility of a skilled worker in region i is obtained as

$$v_i(\boldsymbol{h}) = \bar{\kappa} \ln[\Delta_i] - \gamma(\ln[h_i + l_i] - \ln A_i) + w_i \tag{C.7.2}$$

where $\Delta_i = \sum_{j \in \mathcal{K}} d_{ji} h_j$ and l_i and A_i denote the number of unskilled worker and the amount of housing stock in region i, respectively, and the nominal wage in region i is given by

$$w_i = \frac{\mu}{\sigma} \sum_{j \in \mathcal{K}} \frac{d_{ij}}{\Delta_j}(h_j + l_j) \tag{C.7.3}$$

as in the Pf model.

Jacobian matrix. At the flat-earth equilibrium with $l_i = l$ and $A_i = A$ for all i, we can show

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = h^{-1} \left[-\gamma(1+\epsilon)^{-1} \boldsymbol{I} + (\bar{\kappa} + \kappa) \bar{\boldsymbol{D}} - \kappa(1+\epsilon) \bar{\boldsymbol{D}}^2 \right] \tag{C.7.4}$$

where $\epsilon \equiv L/H$ being the ratio of the total number unskilled worker to that of skilled. We thus conclude that $c_0 = -\gamma(1+\epsilon)^{-1} < 0$, $c_1 = \bar{\kappa} + \kappa > 0$, and $c_2 = -\kappa(1+\epsilon) < 0$.

Numerical simulation. **Figure 11** and **Figure 12** assumes Pflüger and Südekum (2008)'s model. The parameters are set to $\mu = 0.4$, $\sigma = 2.5$, $L = 4$, $H = 1$, $\gamma = 0.5$, and $A = 1$.

C.8 Murata and Thisse (2005) (MT) model

Similar to the Tb model, Murata and Thisse (2005) studies the interplay between commuting costs and interregional transport costs employing a simplified yet reasonable specification. The internal structure of each region is assumed to be one-dimensional and featureless except that there is a given CBD; the city expands symmetrically about the origin. There are only skilled and mobile workers that choose their own residential region i and location $x \geq 0$ in that region, where the CBD is located at $x = 0$. The total number of skilled workers is fixed and assumed to be H.

The internal structure of a region. Land endowment equals unity at everywhere in a region and the workers are assumed to inelastically consume one unit of land. The opportunity cost of land is normalized to zero in every region. Then, the city is spread in the interval $\mathcal{X}_i \equiv [-\bar{x}_i, \bar{x}_i]$ where $\bar{x}_i \equiv h_i/2$ denotes the city boundary. Commuting cost takes an iceberg form. Specifically, a worker located at x supplies

$$s(x) = 1 - 4\theta|x| \qquad\qquad x \in \mathcal{X}_i \tag{C.8.1}$$

unit of labor where we require $\theta \in [0, 1/(2H))$ so that we have $s(x) \geq 0$ for all $x \in \mathcal{X}$ and for all region i at any configuration. Then, the total effective labor supply at the CBD of region i is given by

$$S_i = \int_{\mathcal{X}_i} s(x)\mathrm{d}x = h_i(1 - \theta h_i). \tag{C.8.2}$$

Note that in particular $S_i = h_i$ when commuting is costless: $\theta = 0$. Letting $r_i(x)$ the land rent profile, at an equilibrium it must satisfy

$$s(x)w_i - r_i(x) = \bar{w}_i, \qquad\qquad \forall x \in \mathcal{X}_i, \tag{C.8.3}$$

where $\bar{w}_i \equiv s(\bar{x}_i)w_i - r_i(\bar{x}_i) = s(\bar{x}_i)w_i = s(-\bar{x}_i)w_i = (1 - 2\theta h_i)w_i$ is the disposable wage level of an worker located at the boundary of the city. We thus have

$$r_i(x) = 2\theta(h_i - 2|x|)w_i, \qquad\qquad \forall x \in \mathcal{X}_i \tag{C.8.4}$$

so that the aggregate land rent in region i is

$$R_i \equiv \int_{\mathcal{X}_i} r_i(x)\mathrm{d}x = \theta w_i h_i^2. \tag{C.8.5}$$

Land is locally owned, so that the income of a worker in region i and any location x is

$$y_i = s(x)w_i - r_i(x) + \frac{R_i}{h_i} = \bar{w}_i + \theta w_i h_i = (1 - \theta h_i)w_i. \tag{C.8.6}$$

Preference. The homogeneous preference of skilled workers in region i is given by

$$U(C_i^{\mathrm{M}}) = \ln C_i^{\mathrm{M}} \tag{C.8.7}$$

where, as usual, C_i^{M} is the consumption of the CES aggregate defined by (C.1.2). The budget constraint of a mobile worker becomes

$$\sum_{j \in \mathcal{K}} \int_0^{n_j} p_{ji}(\xi)q_{ji}(\xi)\mathrm{d}\xi = y_i, \tag{C.8.8}$$

where y_i denotes income of the worker. It is immediate that given y_i, utility maximization yields

$$C_i^{\mathrm{M}} = \frac{y_i}{P_i},\ q_{ji}(\xi) = \frac{\{p_{ji}(\xi)\}^{-\sigma}}{P_i^{-\sigma}} C_i^{\mathrm{M}} \tag{C.8.9}$$

where P_i is the price index in region i.

Firms. Manufacturing firms are assumed to the same as in the Km model. Specifically, to produce x_i units of good a firm requires $\alpha + \beta x_i$ unit of skilled labor. Thus, the cost function that a firm in region i faces is given by $C_i(x_i) = w_i(\alpha + \beta x_i)$. Profit maximization yields $p_{ij}(\xi)$ as in the Km model (C.1.10), which of course does not depend on ξ. Noting that the number of firms n_i in region i is given by $n_i = S_i/x_i^* = (\alpha\sigma)^{-1}S_i$, the price index in region i is given as

$$P_i = \frac{\alpha\sigma}{\sigma - 1}\left(\frac{1}{\alpha\sigma}\sum_{j\in\mathcal{K}} S_j w_j^{1-\sigma} d_{ji}\right)^{1/(1-\sigma)} \tag{C.8.10}$$

with $d_{ij} = \tau_{ij}^{1-\sigma}$ and $S_i = (1-\theta h_i)h_i$.

Short-run equilibrium. Noting that the aggregate income in region i is given by $Y_i = w_i S_i$, the wage equation for the MT model becomes

$$w_i S_i = \sum_{j\in\mathcal{K}} \frac{S_i w_i^{1-\sigma} d_{ij}}{\sum_{k\in\mathcal{K}} S_k w_k^{1-\sigma} d_{kj}} w_j S_j. \tag{C.8.11}$$

To normalize $\boldsymbol{w}$ we shall assume that $\sum_{i\in\mathcal{K}} w_i S_i = W > 0$. Given the solution $\boldsymbol{w}$ to the equation, the indirect utility of workers in region i is obtained as

$$v_i(\boldsymbol{h}) = \bar{\kappa}\ln[\Delta_i] + \ln[w_i] + \ln[1-\theta h_i] \tag{C.8.12}$$

where $\bar{\kappa} = 1/(\sigma-1)$ and $\Delta_i \equiv \sum_{k\in\mathcal{K}} h_i(1-\theta h_i)w_k^{1-\sigma}d_{ki}$.

Jacobian matrix at the flat-earth equilibrium. We compute as follows:

$$\nabla\boldsymbol{v}(\boldsymbol{h}) = \bar{\kappa}\boldsymbol{M}^\top \operatorname{diag}[\boldsymbol{S}]^{-1}\operatorname{diag}[1-2\theta h_i] + (\boldsymbol{I}-\boldsymbol{M})\operatorname{diag}[\boldsymbol{w}]^{-1}\nabla\boldsymbol{w}(\boldsymbol{h}) - \theta\operatorname{diag}[1-\theta h_i]^{-1} \tag{C.8.13}$$

where $\nabla\boldsymbol{w}(\boldsymbol{h}) = -(\nabla_{\boldsymbol{w}}\boldsymbol{W})^{-1}(\nabla\boldsymbol{W})$ with

$$\nabla_{\boldsymbol{w}}\boldsymbol{W} = \operatorname{diag}[\boldsymbol{S}] + (\sigma-1)(\operatorname{diag}[\boldsymbol{M}\boldsymbol{Y}] - \boldsymbol{M}\operatorname{diag}[\boldsymbol{Y}]\boldsymbol{M}^\top)\operatorname{diag}[\boldsymbol{w}]^{-1} - \boldsymbol{M}\operatorname{diag}[\boldsymbol{S}] \tag{C.8.14}$$

$$\nabla\boldsymbol{W} = \left[\operatorname{diag}[\boldsymbol{w}] - (\operatorname{diag}[\boldsymbol{M}\boldsymbol{Y}] - \boldsymbol{M}\boldsymbol{Y}\boldsymbol{M}^\top)\operatorname{diag}[\boldsymbol{S}]^{-1} - \boldsymbol{M}\operatorname{diag}[\boldsymbol{w}]\right]\operatorname{diag}[1-2\theta h_i] \tag{C.8.15}$$

with $\boldsymbol{Y} = [Y_i] = [w_i(1-\theta h_i)h_i]$ and $\boldsymbol{S} = [S_i] = [(1-\theta h_i)h_i]$. Note that $Y_i = w_i S_i$. Assume a symmetric racetrack economy. We have

$$\nabla_{\boldsymbol{w}}\boldsymbol{W} = (1-\theta h)h\left[\sigma\boldsymbol{I} + (\sigma-1)\bar{\boldsymbol{D}}\right]\left[\boldsymbol{I}-\bar{\boldsymbol{D}}\right], \tag{C.8.16}$$

$$\nabla\boldsymbol{W} = -\bar{w}(1-2\theta h)\bar{\boldsymbol{D}}\left[\boldsymbol{I}-\bar{\boldsymbol{D}}\right], \tag{C.8.17}$$

which in turn yield

$$\nabla \boldsymbol{v}(\boldsymbol{h}) = \frac{1-2\theta h}{(1-\theta h)h}\left(\left[\sigma \boldsymbol{I} + (\sigma-1)\bar{\boldsymbol{D}}\right]^{-1}\left(\frac{1}{\sigma-1}+\frac{1}{\sigma}\right)\bar{\boldsymbol{D}} - \frac{\theta h}{1-2\theta h}\boldsymbol{I}\right) \tag{C.8.18}$$

As a consequence, we obtain $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) \simeq c_0\boldsymbol{I} + c_1\bar{\boldsymbol{D}}$ where, with $\hat{\theta} \equiv (\theta h)/(1-2\theta h)$,

$$c_0 = -\hat{\theta},\ c_1 = (1-\hat{\theta})\left(\frac{1}{\sigma-1}+\frac{1}{\sigma}\right) - \hat{\theta}\frac{\sigma-1}{\sigma}. \tag{C.8.19}$$

Remark C.6. We must require that $0 \leq \hat{\theta} < 1/\{2(K-1)\} < 1$ to ensure that S_i is positive for all region i. In particular, when $H = 1$ and $K = 2$ so that $h = 1/2$ as in the original paper, we have $\hat{\theta} = \theta/\{2(1-\theta)\}$ and $\hat{\theta} \in (0, 1/2)$. Also, by letting $\gamma \equiv \hat{\theta}$ and $\mu \equiv 1-\hat{\theta}$, the model is isomorphic to Helpman (1998) model with local landownership, albeit there is a restriction on γ.

C.9 Harris and Wilson (1978) (HW) model

The HW model is an archetypal economic geography model formulated in the field of geography well before mainstream economists Krugman start to emphasize self-organization of spatial allocation of ecnomic activity. The model has fruitful applications in urban planning. Detailed analysis of the model can be found in Osawa et al. (2017). The model can be also interpreted to be a spatial conpetition model with discrete locations but continuum of firms.

Assumptions. We consider a city that is discretized into K *zones* and associated centroids. There is a continuum of retailing firms in each zone that operates a shop. The number of firms at zone i is denoted by $h_i \geq 0$; $\boldsymbol{h}$ denotes the spatial distribution of retailers. There is a fixed portion of consumers residing at each zone. Consumers are assumed to inelastically buy retail goods from some shop located in the city. The total per capita consumer demand for shopping activity at zone i is a constant O_i. The consumers' shopping behavior is captured by a set of origin-constrained gravity equations. For any given $\boldsymbol{h}$, consumer demand $S_{ij}(\boldsymbol{h})$ from zone i to j, measured as a cash flow, is given by

$$S_{ij}(\boldsymbol{h}) = \frac{h_j^\alpha \exp[-\beta t_{ij}]}{\sum_{k\in\mathcal{K}} h_k^\alpha \exp[-\beta t_{ik}]}O_i \tag{C.9.1}$$

where t_{ij} is the travel cost from zone i to j. The parameters $\alpha, \beta > 0$ are exogenous constants. The term h_i^α is the attractiveness of the retailers in the zone i where α determines the economy of scale. We assume $\alpha > 1$ and hence there is an increasing return to scale. β dictates how fast the demand decreases with travel cost t_{ij} (respecting the original formulation, this section uses β instead of τ). Note that one may recast the demand function into the context of spatial competition by interpreting α^{-1} as the magnitude of product differentiation.

Payoff. The payoff (profit) of a retailer at zone i is defined as follows:

$$\Pi_i(\boldsymbol{h}) = \frac{\sum_{j\in\mathcal{K}} S_{ji}(\boldsymbol{h})}{h_i} - \kappa_i, \tag{C.9.2}$$

where κ_i is the fixed cost of entry. Assume that $O_i = 1$ and that $\kappa_i = \kappa$ for all i. Then, we have

$$\mathbf{\Pi}(\boldsymbol{h}) = \boldsymbol{M}^\top - \kappa\mathbf{1} \tag{C.9.3}$$

where $\boldsymbol{M} \equiv \mathrm{diag}[\boldsymbol{D}\,\mathrm{diag}[\boldsymbol{h}]^\alpha\mathbf{1}]^{-1}\boldsymbol{D}\,\mathrm{diag}[\boldsymbol{h}]^{\alpha-1}$ with $d_{ij} \equiv \exp[-\beta t_{ij}]$.

Long-run equilibrium. The HW model is an open-city model. The total number of retailers at an equilibrium is thus determined from the following equilibrium condition itself: $h_i\Pi_i(\boldsymbol{h}) = 0,\ h_i \geq 0,\ \Pi_i(\boldsymbol{h}) \leq 0$. However, at any equilibrium we have $\sum_{i\in\mathcal{K}} \kappa_i h_i = \sum_{i\in\mathcal{K}} O_i$; the set $\mathcal{D} \equiv \{\boldsymbol{h} \in \mathbb{R}^K \mid \sum_{i\in\mathcal{K}} \kappa_i h_i = \sum_{i\in\mathcal{K}} O_i,\ h_i \geq 0\}$ is globally attracting.

Dynamics. Harris and Wilson (1978) assumes that the spatial pattern $\boldsymbol{h}$ gradually evolves in proportion to the profit $\mathbf{\Pi}(\boldsymbol{h})$ and the state $\boldsymbol{h}$. Specifically, we define $\dot{\boldsymbol{h}} = \boldsymbol{F}(\boldsymbol{h}) \equiv \mathrm{diag}[\boldsymbol{h}] \cdot \mathbf{\Pi}(\boldsymbol{h}) = [S_i(\boldsymbol{h}) - \kappa_i h_i]$

Jacobian matrix at the flat-earth equilibrium. It is immediate that $\boldsymbol{J} = \nabla\boldsymbol{F}(\bar{\boldsymbol{h}})$ is given by

$$\boldsymbol{J} = \kappa\left\{(\alpha-1)\boldsymbol{I} - \alpha\bar{\boldsymbol{D}}^2\right\} \tag{C.9.4}$$

where $\boldsymbol{I}$ is the identity matrix and $\bar{\boldsymbol{D}} \equiv \boldsymbol{D}/d$ with $d \equiv \sum_{j\in\mathcal{K}} d_{0,j}$. We see that $\boldsymbol{J} \simeq c_0\boldsymbol{I} + c_2\bar{\boldsymbol{D}}^2$ with

$$c_0 = 1 - \frac{1}{\alpha},\ c_2 = -1. \tag{C.9.5}$$

It is clear that c_0 reflects magnitude of *local* increasing return. c_0 is positive as long as $\alpha > 1$; $\alpha < 1$ yields that the flat-earth equilibrium is always stable. $c_2 = -1$ represents, analogous to the FO model, firms' competition over demand from immobile consumers.

C.10 Beckmann (1976) (Bm) model

We formulate a discrete-space version of Beckmann (1976)'s spatial model of social interactions. Since the original formulation of Beckmann (1976) uses linear communication cost, we shall introduce suitable modifications. Yet, as long as every consumer communicate with every other consumers, our modification does not alter the intrinsic properties of agglomeration and dispersion. In particuar, whether possible equilibria are of unimodal or multimodal does not change. We shall also avoid unnecessary complication and stick to the simplest possible specification.

Assumptions. Consider a city that is discretized into K areas. Each area i is endowed with fixed amount A_i of housing stock. Housing stocks are owned by absentee landlords. The city is endowed with H homogeneous consumers that can choose his/hers residential location and consumes land and composite good. The income of consumers is a fixed constant Y, which is sufficiently large.

Preference. In addition to land and composite good, every consumer draws a social utility due to communication with others. Specifically, everyone at area i draws the following social utility

$$S_i(\boldsymbol{h}) = \log[\Delta_i], \tag{C.10.1}$$

where $\Delta_i \equiv \sum_{j\in\mathcal{K}} d_{ij}h_j$ with $d_{ij} \equiv \exp[-\tau\ell_{ij}]$. Note that Δ_i is exponential accessibility function à la

Fujita and Ogawa (1982). Given the spatial distribution of consumers $\boldsymbol{h}$, the utility of residing area i takes the following quasilinear form

$$U_i(z_i, s_i; \boldsymbol{h}) = z_i + \gamma \log[s_i] + S_i(\boldsymbol{h}) \tag{C.10.2}$$

where z_i and s_i is the consumption of the composite and housing goods, respectively, and γ is an exogenous constant. We set the composite good to the numéraire and the budget constraint of a worker in area i is

$$Y = z_i + r_i s_i, \tag{C.10.3}$$

whence utility maximization yields $s_i = A_i/h_i$, $r_i = \alpha h_i/A_i$, and $z_i = Y - \gamma$. Then, assuming $A_i = 1$ at every area and removing constants, the indirect utility at area i is given by

$$v_i(\boldsymbol{h}) = \log[\Delta_i] - \gamma \log[h_i]. \tag{C.10.4}$$

Jacobian matrix at the flat-earth equilibrium. Assuming a racetrack economy, it is immediate that the Jacobian matrix at the flat-earth equilibrium is given by

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = h^{-1}\big[-\gamma \boldsymbol{I} + \bar{\boldsymbol{D}}\big]. \tag{C.10.5}$$

We thus see that $c_0 = -\gamma$ and $c_1 = 1$ for the model. Without any location-fixed factors, Mossay and Picard (2011) and Blanchet et al. (2016) are essentially the same model as the one presented here.

C.11 Takayama and Akamatsu (2011) (TA) model

Takayama and Akamatsu (2011) is a reduced-form partial equilibrium model that introduce a spatial competition effect à la Harris and Wilson (1978) into the Bm model. Specifically, in essence, they introduced firms that sell goods at a fixed price to spatially immobile consumers. The consumers in the Bm model are now workers; each worker inelastically provides a single unit of labor.

Immobile consumers. In each area, there are l_i immobile consumers with $\sum_i l_i = L$ that demand a single unit of goods produced by firms; the immobile consumers are assumed to engage in jobs in other industries. Given spatial distribution $\boldsymbol{n} = (n_i)_{i \in \mathcal{K}}$ of firms, the demand from area j to i is given by the following origin-constrained gravity equation

$$q_{ji} = \frac{\hat{d}_{ji}}{\sum_{k \in \mathcal{K}} \hat{d}_{jk} n_k} l_j \tag{C.11.1}$$

with $\hat{d}_{ij} \equiv \exp[-\hat{\tau}\ell_{ij}]$, whose microfoundation can be found at, of course, a CES preference or alternatively some taste heterogeneity.

Firms. A manufacturing firm produces a single unit of manufactured good with fixed price μ using a single unit of labor of mobile consumers. Thus, we must have $n_i = h_i$. The profit function of firm at i is given by

$$\Pi_i(\boldsymbol{h}) = \mu \sum_{j\in\mathcal{K}} \frac{\hat{d}_{ji}}{\sum_{k\in\mathcal{K}} \hat{d}_{jk} h_k} l_j - w_i. \tag{C.11.2}$$

For simplicity, we force zero profit for firms and abstract from commuting between different areas. Then, wage of a mobile worker at area i equals

$$w_i(\boldsymbol{h}) = \mu \sum_{j\in\mathcal{K}} \frac{\hat{d}_{ji}}{\sum_{k\in\mathcal{K}} \hat{d}_{jk} h_k} l_j \tag{C.11.3}$$

so that the indirect utility of the worker becomes

$$v_i(\boldsymbol{h}) = w_i(\boldsymbol{h}) + \log[\Delta_i] - \gamma \log[h_i]. \tag{C.11.4}$$

Jacobian matrix at the flat-earth equilibrium. Let $l_i = L/K$ for all i and assume that $d_{ij} = \hat{d}_{ij}$ for all i and j (i.e., $\tau = \hat{\tau}$). Then, we compute as follows:

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = h^{-1}\big[-\gamma \boldsymbol{I} + \bar{\boldsymbol{D}} - \mu\epsilon \bar{\boldsymbol{D}}^2 \big] \tag{C.11.5}$$

where $\epsilon \equiv L/H$, whence we see that $c_0 = -\gamma$, $c_1 = 1$, and $c_2 = -\mu\epsilon$.

C.12 Allen and Arkolakis (2014) (AA) model

The AA model is formulated as a perfectly competitive Armington (1969)-based framework with positive (production) and negative (congestion) reduced-form *local* agglomeration externalities. We introduce a discrete-space version of the AA model, instead of the continuous-space version of the original paper, to fit it in our context.

Assumptions. There are fixed number H of mobile consumers that choose residents. We denote the spatial pattern of consumers by $\boldsymbol{h}$. At each region i, a unique differentiated variety of good is produced as Armington (1969). Production is assumed to be perfectly competitive and labor is the only factor of production. Each mobile consumer inelastically supplies a single unit of labor. As usual, we do not consider commuting of workers between two different regions. We denote the wage of workers by $\boldsymbol{w}$. The transportation of goods between regions takes iceberg form; the firms at i must export $\tau_{ij} > 0$ unit of good to meet a single unit of demand at region j.

At each region, total factor productivities (TFP) and amenity at each region are directly affected by the number of its inhabitants, h_i. These externalities are *local* in the sense that it does not depend on distance between regions. The number of consumers at different region does not affect neither of TFP or amenity at a region; it is exclusively enjoyed by the agents located at each region. As the analysis in the present section will demonstrate, such assumption turns out to be insufficient for endogenously producing polycentricity of spatial agglomeration patterns.

Preference. The utility function of a consumer in region i is defined as the following CES function:

$$u_i(\{q_{ji}\}) = a_i \cdot \left(\sum_{j \in \mathcal{K}} q_{ji}^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)}, \tag{C.12.1}$$

where q_{ji} is the quantity of the good variety that is produced at region $j \in \mathcal{K}$ and consumed in region i. The constant $\sigma > 1$ is the elasticity of substitution between varieties, and $a_i(h_i)$ is the local amenity. The local amenity deteriorates as population h_i at i increase; it is defined by the following power function that produces a congestion effect:

$$a_i(h_i) = \bar{a}_i h_i^{-\beta}, \tag{C.12.2}$$

where $\bar{a}_i > 0$, $\beta \geq 0$ is exogenously given constants. In particular, $\bar{a}_i$ represents the *unobserved amenity* in region i. When $\beta = 0$, there is no congestion effect and the local amenity is exogenous constant $\bar{a}_i$.

The income of consumers comes only from the wage from production firms. We denote the price of the variety that is produced at j and consumed at i is denoted by p_{ji}. The wage in region i is denoted by $w_i \geq 0$. Then, the budget constraint of a consumer who locates at i is given by the following equation:

$$w_i = \sum_{j \in \mathcal{K}} p_{ji} q_{ji}. \tag{C.12.3}$$

To normalize wage, we impose a constraint $\sum_{i \in \mathcal{K}} w_i h_i = W$, which means that the total income in the economy always equals to the fixed constant W.

Utility maximization of consumers under a given price system $\boldsymbol{p}$ yields

$$q_{ji} = \frac{p_{ji}^{-\sigma}}{P_i^{1-\sigma}} w_i, \tag{C.12.4}$$

where P_i is the price index of the good in region i

$$P_i \equiv \left(\sum_{k \in \mathcal{K}} p_{ki}^{1-\sigma} \right)^{1/(1-\sigma)}. \tag{C.12.5}$$

Production. Firms at region $i \in \mathcal{K}$ produce goods under perfect competition. As a result, the final price of the good that is produced at i and sold at j, which we denote p_{ij}, equals to

$$p_{ij} = \frac{w_i}{m_i} \tau_{ij}. \tag{C.12.6}$$

where m_i denotes the TFP in region i. To model Marshallian agglomeration economy (Marshall, 1989) in a reduced-form, the TFP in region i is assumed to be an increasing power function of its population

$$m_i(h_i) = \bar{m}_i h_i^{\alpha} \tag{C.12.7}$$

with $\bar{m}_i > 0,\ \alpha \geq 0$ being exogenous constants. If $\alpha = 0$, the TFP in region i is a given constant $\bar{m}_i$.

Short-run equilibrium. In the following, we set $\bar{m}_i = 1,\ \bar{a}_i = 1$ for all i to abstract from any first-nature advantages. In the short run, consumers are immobile across regions. We determine the short-run indirect utility as a function of $\boldsymbol{h}$ by general equilibrium conditions, which consist of the product market clearing and the zero-profit condition of firms. First, plugging (C.12.6) and (C.12.7) into (C.12.5), with $d_{ki} \equiv \tau_{ki}^{1-\sigma}$ we obtain

$$P_i = \left(\sum_{k\in\mathcal{K}} w_k^{1-\sigma} h_k^{\alpha(\sigma-1)} d_{ki} \right)^{1/(1-\sigma)}. \tag{C.12.8}$$

The zero-profit condition of firms requires that the total revenue in region i is exhausted. This yields the wage equation for the model:

$$w_i h_i = \sum_{j\in\mathcal{K}} \frac{w_i^{1-\sigma} h_i^{\alpha(\sigma-1)} d_{ij}}{\sum_{k\in\mathcal{K}} w_k^{1-\sigma} h_k^{\alpha(\sigma-1)} d_{kj}} w_j h_j. \tag{C.12.9}$$

Given the short-run equilibrium wage $\boldsymbol{w}$, the indirect utility function is given by

$$v_i(\boldsymbol{h}) = \frac{h_i^{-\beta} w_i}{P_i}. \tag{C.12.10}$$

Jacobian matrix at the flat-earth equilibrium. A direct computation shows that the Jacobian matrix of the payoff function $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ is given by

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = \left[\sigma \boldsymbol{I} - \bar{\boldsymbol{D}} - (\sigma-1)\bar{\boldsymbol{D}}^2\right]^{-1} \left[-(\alpha+\beta-\gamma_0)\boldsymbol{I} + (\alpha+\beta+\gamma_1)\bar{\boldsymbol{D}}\right] \tag{C.12.11}$$

where $\gamma_0 \equiv \frac{1+\alpha}{\sigma}$ and $\gamma_1 \equiv \frac{1-\beta}{\sigma}$. Thus, one concludes that

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) \simeq c_0 \boldsymbol{I} + c_1 \bar{\boldsymbol{D}}, \tag{C.12.12}$$

with $c_0 = -(\alpha+\beta-\gamma_0)$ and $c_1 = \alpha+\beta+\gamma_1$.

Numerical example. **Figure 9** assumes Allen and Arkolakis (2014)'s model. The parameters are set to $\alpha = 0.5$, $\beta = 0.3$, $\sigma = 6$, and $H = 10$.

Table 2: Exact mappings of economic geography models to the coefficients of $G(f) = c_0 + c_1 f + c_2 f^2$

Model class	**Specific model**	**Local force**	**Global forces**	
		c_0	c_1	c_2
Class (i)	Krugman (1991)	0	$\mu\left(\frac{1}{\sigma-1}+\frac{1}{\sigma}\right)$	$-\left(\frac{\mu^2}{\sigma-1}+\frac{1}{\sigma}\right)$
	Puga (1999)	0	$\check{\mu}\left(\frac{1}{\sigma-1}+\frac{1}{\sigma}\right)$	$-\left(\frac{\check{\mu}^2}{\sigma-1}+\frac{1}{\sigma}+\omega\right)$
	Forslid and Ottaviano (2003)	0	$\mu\left(\frac{1}{\sigma-1}+\frac{1}{\sigma}\right)$	$-\left(\frac{\mu^2}{\sigma(\sigma-1)}+1\right)$
	Pflüger (2004)	0	$\mu\left(\frac{1}{\sigma-1}+\frac{1}{\sigma}\right)$	$-\frac{\mu}{\sigma}\frac{L+H}{H}$
	Harris and Wilson (1978)	$1-\frac{1}{\alpha}$	0	-1
Class (ii)	Helpman (1998)	$-\gamma$	$\mu\left(\frac{1}{\sigma-1}+\frac{1}{\sigma}\right)$	$-\left(\frac{\mu^2}{\sigma-1}+\frac{1}{\sigma}\right)+\gamma$
	Redding and Sturm (2008)	$-\gamma$	$\mu\left(\frac{1}{\sigma-1}+\frac{1}{\sigma}\right)-\gamma\frac{\sigma-1}{\sigma}$	0
	Murata and Thisse (2005)	$-\hat{\theta}$	$(1-\hat{\theta})\left(\frac{1}{\sigma-1}+\frac{1}{\sigma}\right)-\hat{\theta}\frac{\sigma-1}{\sigma}$	0
	Allen and Arkolakis (2014)	$-(\alpha+\beta)+\frac{1+\alpha}{\sigma}$	$(\alpha+\beta)+\frac{1-\beta}{\sigma}$	0
	Beckmann (1976)	$-\gamma$	1	0
Class (iii)	Tabuchi (1998)	$-\hat{\gamma}\left(\frac{1}{\sigma}+\frac{\sigma-1}{\sigma}\phi\right)$	$\mu\left(\frac{1}{\sigma-1}+\frac{1}{\sigma}\right)$	$-\left(\frac{\mu^2}{\sigma-1}\omega_0+\frac{1}{\sigma}\omega_1\right)$
	Pflüger and Südekum (2008)	$-\gamma\frac{H}{L+H}$	$\mu\left(\frac{1}{\sigma-1}+\frac{1}{\sigma}\right)$	$-\frac{\mu}{\sigma}\frac{L+H}{H}$
	Takayama and Akamatsu (2011)	$-\gamma$	1	$-\mu\frac{L}{H}$

Note: The positive (negative) coefficients indicate agglomeration (dispersion) forces. Observe that the class (i) models incorporate global dispersion force, the class-(ii) local one, and the class-(iii) both ones. Refer to the analyses above for the derivations and the definitions of the parameters. The model of Mossay and Picard (2011) (and hence Blanchet et al. (2016)) is an equivalent to that of Beckmann (1976).

D Comparative statics: Role of local factors

The majority of structural exercises in the current stream of quantitative spatial economics employs local unobserved factors (i.e., heterogeneities in local amenities or productivity of firms) in replicating the actual data, often under conditions where the uniqueness of equilibrium is ensured (Redding and Rossi-Hansberg, 2017). For example, in the simplest form, structural residuals under fixed values of the main exogenous parameters of the model (e.g., the expenditure share on manufactured goods μ or the elasticity of substitution σ) are given broad interpretations such as recovered "local amenities" and then utilized as exogenous parameters to conduct counterfactual analyses. In this section, we explore implications of such approaches by simple comparative static analyses.

D.1 Structure of equilibrium spatial pattern with location-fixed factors

The payoff function of an economic geography model would be written as $v_i(\boldsymbol{h}, \boldsymbol{A})$, where $\boldsymbol{A} \equiv (A_i)_{i\in\mathcal{K}}$ is the vector of location-fixed factors. There are two canonical examples of how location-fixed factors are modeled in the literature.

The first and perhaps the simplest example is a location-fixed factor in the payoff function

$$v_i(\boldsymbol{h}, A_i) = \hat{v}_i(\boldsymbol{h}) + A_i \tag{D.1.1}$$

where $\hat{v}_i(\boldsymbol{h})$ is the $\boldsymbol{A}$-independent component of $v_i(\boldsymbol{h}, A_i)$, which we term a *local* heterogeneity. We note that the specification (D.1.1) includes many models with location-fixed factors that *directly* affect the (indirect) utility of mobile workers. For instance, one will show that, by taking logarithm, the indirect utility function of Allen and Arkolakis (2014)'s model that incorporates location-fixed amenities reduces to (D.1.1). Such effects also arise from local nontradable goods, where a representative example being Helpman (1998). As it is evident from (C.4.11), when we let $A_i := (1-\mu)\log[A_i]$, the model reduces to (D.1.1).

The second and more involved example is location-fixed factors that affect interregional trade flows, which we term a *global* heterogeneity. The regional model of Redding and Rossi-Hansberg (2017), §3, is an example. Due to heterogeneities in local productivity of firms A_i, prices of manufactured goods differ across regions; then, trade balance implies that the wage in region i depends on the whole pattern of $\boldsymbol{A}$. Thus, $v_i(\boldsymbol{h}, \boldsymbol{A})$ is (with slight notational abuse)

$$v_i(\boldsymbol{h}, \boldsymbol{A}) = v_i(\boldsymbol{h}, \boldsymbol{w}(\boldsymbol{h}, \boldsymbol{A})) \tag{D.1.2}$$

where $\boldsymbol{w}(\boldsymbol{h}, \boldsymbol{A}) = (w_i(\boldsymbol{h}, \boldsymbol{A}))$ denotes the wage vector. Krugman (1991)'s model also is an example, where one may interpret that A_i represents the number of immobile workers in region i or, alternatively, the region-specific productivity (as in Redding and Rossi-Hansberg (2017), §3).

We have seen that, assuming a racetrack economy and abstracting from first-nature advantages by letting $\boldsymbol{A} = \bar{\boldsymbol{A}} \equiv \bar{A}\mathbf{1}$, the flat-earth equilibrium $\bar{\boldsymbol{h}} \equiv h\mathbf{1}$ is always an equilibrium. The question asked in the present appendix is that: *What happens when we consider variation in the spatial pattern of* *location-fixed factors? Does our classification obtained under no heterogeneities still matter?*

Suppose that $\bar{\boldsymbol{h}}$ is the unique stable equilibrium. Then, we may view that the equilibrium spatial pattern is a function of $\boldsymbol{A}$ so that $\boldsymbol{h} = \boldsymbol{h}(\boldsymbol{A})$. In the vicinity of $\bar{\boldsymbol{h}}$, we have

$$\boldsymbol{h}(\boldsymbol{A}) = \boldsymbol{h}(\bar{\boldsymbol{A}} + \boldsymbol{\delta}) \approx \bar{\boldsymbol{h}} + \boldsymbol{J}^{\mathrm{A}}\boldsymbol{\delta}, \tag{D.1.3}$$

where $\boldsymbol{\delta} = (\delta_i) \equiv \boldsymbol{A} - \bar{\boldsymbol{A}} = (A_i - \bar{A})$ is variation in $\boldsymbol{A}$ and $\boldsymbol{J}^{\mathrm{A}} \equiv [\partial h_i/\partial A_j]$ is the Jacobian matrix of the spatial pattern of mobile agents with respect to $\boldsymbol{A}$ evaluated at $\bar{\boldsymbol{A}}$. Also, we define ϵ by

$$\epsilon \equiv \boldsymbol{\delta}^{\top}(\boldsymbol{h} - \bar{\boldsymbol{h}}) = \boldsymbol{\delta}^{\top}\boldsymbol{J}^{\mathrm{A}}\boldsymbol{\delta}. \tag{D.1.4}$$

Observe that if $\epsilon = \sum_{i\in\mathcal{K}} \delta_i(h_i - \bar{h}) \geq 0$ for *any* imposed nonzero variation $\boldsymbol{\delta}$ in location-fixed factors, we have $\delta_i(h_i - \bar{h}) = (A_i - \bar{A})(h_i - \bar{h}) \geq 0$ for all $i \in \mathcal{K}$. The fact implies the following lemma:

Lemma D.1. Assume that $\boldsymbol{J}^{\mathrm{A}} = [\partial h_i/\partial A_i]$ is positive definite at $\boldsymbol{A} = \bar{\boldsymbol{A}}$ and consider a small variation $\boldsymbol{\delta} = (\delta_i) \neq \boldsymbol{0}$ in $\boldsymbol{A}$ such that $\boldsymbol{A} = \bar{\boldsymbol{A}} + \boldsymbol{\delta}$. Then, the sign of the variation in the location-fixed factor of region i, $\delta_i = A_i - \bar{A}$, and that of the marginal increase in its population, $h_i - h$, coincides.

The above lemma provides a sufficient condition *for any economic geography model* under which an increase of location-fixed factor A_i implies population growth in region i and vice versa.

To employ Lemma D.1, we should evaluate $\boldsymbol{J}^{\mathrm{A}}$. Below, we shall show that it is represented by the Jacobian matrix of the payoff function. First, we recall that an interior equilibrium with $h_i > 0$ for all i must be a solution to the following system of nonlinear equations:

$$\boldsymbol{v}(\boldsymbol{h}, \boldsymbol{A}) - \bar{v}(\boldsymbol{h}, \boldsymbol{A})\boldsymbol{1} = \boldsymbol{0}, \tag{D.1.5}$$

where $\bar{v}(\boldsymbol{h}, \boldsymbol{A}) \equiv H^{-1}\sum_{i\in\mathcal{K}} v_i(\boldsymbol{h}, \boldsymbol{A})h_i$ denotes the average payoff. The implicit function theorem regarding the equilibrium equation (D.1.5) implies that, at $(\bar{\boldsymbol{h}}, \bar{\boldsymbol{A}})$, $\boldsymbol{J}^{\mathrm{A}}$ is evaluated as follows

$$\boldsymbol{J}^{\mathrm{A}} = [c\boldsymbol{E} - (\boldsymbol{I} - \boldsymbol{E})\boldsymbol{J}]^{-1}[\boldsymbol{I} - \boldsymbol{E}]\hat{\boldsymbol{J}} \tag{D.1.6}$$

where $c \equiv h^{-1}\bar{v}$, $\boldsymbol{E} \equiv K^{-1}\boldsymbol{1}\boldsymbol{1}^{\top}$ is a matrix whose elements are all $1/K$, $\boldsymbol{J} \equiv [\partial v_i/\partial h_i]$, and $\hat{\boldsymbol{J}} \equiv [\partial v_i/\partial A_i]$. All matrices are evaluated at the flat-earth pattern $(\bar{\boldsymbol{h}}, \bar{\boldsymbol{A}})$.

Since $\boldsymbol{J}^{\mathrm{A}}$ is symmetric at the flat-earth equilibrium, it is positive definite if and only if its eigenvalues are all positive. But, because $\boldsymbol{J}^{\mathrm{A}}$ is circulant, its eigenvalues is computable by the same procedure as our stability analysis (Lemma B.2). We conclude that the eigenvalues a_k of $\boldsymbol{J}^{\mathrm{A}}$ are given by[62]

$$a_k = \begin{cases} 0, & k = 0, \\ -e_k^{-1}\hat{e}_k, & k = 1, 2, \ldots, K-1, \end{cases} \tag{D.1.7}$$

[62] We note that $\boldsymbol{I} - \boldsymbol{E}$ and $\boldsymbol{E}$ represent the projections onto the subspace of $\mathbb{R}^K$ defined by $\sum_{i\in\mathcal{K}} x_i = 0$ and its orthogonal subspace, respectively, and their eigenvalues are $(0, 1, 1, \ldots, 1)$ and $(1, 0, 0, \ldots, 0)$.

with e_k and $\hat{e}_k$ being the kth eigenvalue of $\boldsymbol{J}$ and $\hat{\boldsymbol{J}}$, respectively, where we assume that $e_k \neq 0$. Moreover, the eigenvectors of $\boldsymbol{J}^{\mathrm{A}}$ are again $\{\boldsymbol{\eta}_k\}$ with $\boldsymbol{\eta}_k = (\cos[\theta k i])$ with $k = 0, 1, \ldots, K-1$. Note that we have $a_0 = 0$. It is intuitive because it says that a uniform increase of A_i across the regions does not affect the spatial pattern—in other words, what matters is the *relative* variation in location-fixed factors. Thus, without loss of generality, we shall rewrite $\boldsymbol{\delta} = \sum_{k\in\mathcal{K}} C_k \boldsymbol{\eta}_k$ and assume $C_0 = 0$ so that $\boldsymbol{\delta} \cdot \mathbf{1} = 0$. We then have $\boldsymbol{h} - \bar{\boldsymbol{h}} = \sum_{k\in\mathcal{K}} C_k a_k \boldsymbol{\eta}_k$ and

$$\epsilon = \boldsymbol{\delta}^{\top}(\boldsymbol{h} - \bar{\boldsymbol{h}}) = \sum_{k\in\mathcal{K}} C_k^2 a_k, . \tag{D.1.8}$$

If $a_k > 0$ for all $k \geq 1$, we have $\epsilon > 0$. Each a_k is an amplifying factor in the direction of $\boldsymbol{\eta}_k$ in the sense that if $\boldsymbol{\delta} = \boldsymbol{\eta}_k$ we obtain $\boldsymbol{h} - \bar{\boldsymbol{h}} = a_k \boldsymbol{\eta}_k$.

That said, we have two questions regarding the properties of a_k. The first is obvious:

Question 1. *Is $a_k > 0$ for all $k \geq 1$?*

If true, from Lemma D.1, it implies that a relative advantage of a region implies a relative increase of its population and vice versa. As we will see below, generally this is the case.

The second is of importance: *What happens on $\{a_k\}$ if we face change (in particular, a decrease) in transportation costs?* Put another way, does an increase of the trade freeness r (see Section B.2) imply a strengthened role of first natures—or converse? In concrete terms:

Question 2. *Is $\mathrm{d}a_k/\mathrm{d}r$ positive (or negative) for all $k \geq 1$?*

We see that, because

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}r} = \sum_{i\in\mathcal{K}} C_k^2 \frac{\mathrm{d}a_k}{\mathrm{d}r}, \tag{D.1.9}$$

if $\mathrm{d}a_k/\mathrm{d}r$ happened to be positive for all $k \geq 1$, it means that as r increases (τ decreases), the location-fixed factors matters more; converse is also true.

D.2 Role of location-fixed factors: Model class matters

For simplicity, consider the simplest case (*local* heterogeneity), as in (D.1.1). We note that for (D.1.1), we have $\hat{\boldsymbol{J}} = \boldsymbol{I}$ and thus $\hat{e}_k = 1$, which in turn implies that $a_k = -e_k^{-1}$. Recalling that if the flat-earth equilibrium is stable, we have $e_k < 0$ for all k, we see that $a_k > 0$. Thus, it must be that $\epsilon > 0$ for *any* relative variation $\boldsymbol{\delta}$ in $\boldsymbol{A}$. Thus, the answer to the first question is "yes": any relative first-nature (dis)advantage in terms of location-fixed amenities increases (decreases) local population when the flat-earth equilibrium is stable—it is, of course, hardly a surprise.

We shall turn our attention to the second question. As we will see, asking the question reveals a major watershed between the model classes (i) and (ii): *when the economy faces a decrease in transportation costs, the effects of location-fixed advantages are typically in the opposite direction for class-(i) and (ii).*

For class (i) models in the literature, there is a determinate implication regarding the effects of a decline in interregional transport costs on first-nature advantages. So long as the flat-earth pattern is stable, we have[63]

$$\frac{\mathrm{d}a_k}{\mathrm{d}r} > 0. \tag{D.2.1}$$

Thus, *the positive effects of relative location-fixed advantages increase according to the decrease of interregional transportation costs*. Under the stability of the flat-earth equilibrium, a decrease in interregional transportation cost fosters more agglomeration at the regions with relative advantages in amenity. In fact, this leads to instability of the flat-earth equilibrium because at the first break point we have $e_k = 0$ for some k and hence $a_k = \infty$ for that k. Thus, the model leads to regional divergence, even in the range of transportation costs where the flat-earth equilibrium is stable.

For class (ii) models, *a decrease in transportation costs has the opposite implication* compared with class (i) models. We illustrate it using Helpman (1998)'s model. For the original model with public landownership, we have

$$\frac{\mathrm{d}a_k}{\mathrm{d}r} < 0 \tag{D.2.2}$$

whenever stability of the flat-earth equilibrium is ensured regardless of the level of r, by the condition $\sigma(1-\mu) > 1$. Thus, the regions once flourished by first-nature advantages due to larger endowments of housing space will decline if interregional transportation cost decrease. Assuming different specification of local factors as in (D.1.2) does not alter the result. In fact, as we will see, if we consider a variant model where $\boldsymbol{A}$ is interpreted to heterogeneities in local productivity as in the regional model of Redding and Rossi-Hansberg (2017), §3 (see Section C.4), we have the same result: $a_k > 0$ and that $\mathrm{d}a_k/\mathrm{d}r < 0$; the result also is consistent with the numerical exercise conducted by the paper. In short, in class (ii) models, the role of initial heterogeneity declines in line with decreasing transportation costs.

The interpretation of the behavior of class (ii) models is straightforward. As the role of interregional transportation costs declines, local dispersion force dominates. Then, an agglomeration that is formed solely by its local advantages must face relative *second-nature* disadvantage due to local congestion compared to those formerly behind, leading to a relative decline of such a region.

In light of this, assumptions on landownership can have impacts on the sign of $\mathrm{d}a_k/\mathrm{d}r$. In particular, local landownership, by redistribution of local rental revenue, can relax the magnitude of the second-nature disadvantage at the regions where housing rent is high. If expenditure share on housing good is sufficiently high, via redistribution, it can overcome relative second-nature disadvantage, so that $\mathrm{d}a_k/\mathrm{d}r > 0$. If we assume local landownership in Helpman (1998) as in Redding and Sturm

[63]For all class (i) models in the literature, we have $e_k = G(f_k(r))/\phi(f_k(r))$ with strictly positive and decreasing function $\phi(f)$ (see Appendix C). Noting that $\mathrm{d}f_k/\mathrm{d}r < 0$, it then implies that

$$\frac{\mathrm{d}a_k}{\mathrm{d}r} = -\frac{\mathrm{d}e_k^{-1}}{\mathrm{d}r} = -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\phi(f_k(r))}{G(f_k(r))}\right) = -\frac{\phi'(f_k)G(f_k) - \phi(f_k)G'(f_k)}{\{G(f_k)\}^2}\frac{\mathrm{d}f_k}{\mathrm{d}r} > 0$$

where we note that $\phi'(f_k)G(f_k) - \phi(f_k)G'(f_k)$ is strictly positive since $\phi'(f_k) < 0$, $\phi(f_k) > 0$, and because the flat-earth equilibrium is stable $G(f_k) < 0$ and $G'(f_k) < 0$.

(2008), we obtain

$$\frac{\mathrm{d}a_k}{\mathrm{d}r} \begin{cases} < 0, & \text{if} \quad \bar{\mu} < \mu < 1, \\ > 0, & \text{if} \quad 0 < \mu < \bar{\mu}, \end{cases} \tag{D.2.3}$$

with $\bar{\mu} \equiv \frac{2(\sigma-1)^2}{2\sigma^2-2\sigma+1} < \frac{\sigma-1}{\sigma}$, which confirms the above speculation. The result illustrates the basic role of a local dispersion force and typically less featured assumptions on landownership.

Thus, whether the second-nature causation of an economic geography model boosts first-nature advantages in line of decreasing transportation costs or not depends on the model class it belongs.

Below, in addition to the simplest case (D.1.1), we provide examples of *global* heterogeneity where the payoff is given by (D.1.2). For this case, we have $\nabla_{\boldsymbol{A}}\boldsymbol{v} = \nabla_{\boldsymbol{w}}\boldsymbol{v}\nabla_{\boldsymbol{A}}\boldsymbol{w}$. Because a_k $(k \geq 1)$ is the kth eigenvalue of $(\nabla_{\boldsymbol{h}}\boldsymbol{v})^{-1}(\nabla_{\boldsymbol{A}}\boldsymbol{v})$, we first evaluate the two matrices and then their product. We note that, given any wage equation $\boldsymbol{W}(\boldsymbol{h}, \boldsymbol{w}, \boldsymbol{A}) = \boldsymbol{0}$ that incorporates local factors $\boldsymbol{A}$, we have the following computation:

$$\nabla_{\boldsymbol{h}}\boldsymbol{v} = \{\phi(\bar{\boldsymbol{D}})\}^{-1}G_{\mathrm{H}}(\bar{\boldsymbol{D}}),\ \nabla_{\boldsymbol{A}}\boldsymbol{v} = \{\phi(\bar{\boldsymbol{D}})\}^{-1}G_{\mathrm{A}}(\bar{\boldsymbol{D}}), \tag{D.2.4}$$

where we define matrix polynomials ϕ, G_{H}, and G_{A} of $\bar{\boldsymbol{D}}$ by

$$\phi(\bar{\boldsymbol{D}}) \equiv (\nabla_{\boldsymbol{w}}\boldsymbol{W})^{-1},\ G_{\mathrm{H}}(\bar{\boldsymbol{D}}) \equiv \nabla_{\boldsymbol{h}}\boldsymbol{v}\nabla_{\boldsymbol{w}}\boldsymbol{W} - \nabla_{\boldsymbol{w}}\boldsymbol{v}\nabla_{\boldsymbol{h}}\boldsymbol{W},\ G_{\mathrm{A}}(\bar{\boldsymbol{D}}) \equiv \nabla_{\boldsymbol{A}}\boldsymbol{v}\nabla_{\boldsymbol{w}}\boldsymbol{W} - \nabla_{\boldsymbol{w}}\boldsymbol{v}\nabla_{\boldsymbol{A}}\boldsymbol{W}. \tag{D.2.5}$$

Employing these formula, we see $e_k = G_{\mathrm{H}}(f_k)/\phi(f_k)$ and $\hat{e}_k = G_{\mathrm{A}}(f_k)/\phi(f_k)$, so that we have $a_0 = 0$, and, for $k \geq 1$,

$$a_k = -\frac{G_{\mathrm{A}}(f_k)}{G_{\mathrm{H}}(f_k)}. \tag{D.2.6}$$

This in turn implies

$$\frac{\mathrm{d}a_k}{\mathrm{d}r} = -\frac{G_{\mathrm{A}}'(f_k)G_{\mathrm{H}}(f_k) - G_{\mathrm{A}}(f_k)G_{\mathrm{H}}'(f_k)}{\{G_{\mathrm{H}}(f_k)\}^2}\frac{\mathrm{d}f_k}{\mathrm{d}r}. \tag{D.2.7}$$

But, since we have $\mathrm{d}f_k/\mathrm{d}r < 0$, we conclude

$$\operatorname{sgn}\frac{\mathrm{d}a_k}{\mathrm{d}r} = \operatorname{sgn}\left[G_{\mathrm{A}}'(f_k)G_{\mathrm{H}}(f_k) - G_{\mathrm{A}}(f_k)G_{\mathrm{H}}'(f_k)\right]. \tag{D.2.8}$$

Basically, location-fixed factors that affect trade flows can be modeled by employing either of the two form in the following examples. The following two examples demonstrates that the above implication, namely *model class matters even when uniqueness is the case*, holds true for the cases where the level of location-fixed factor in a region affects the nominal wages in the other regions.

Example D.1 (Heterogeneous local productivity (Redding and Rossi-Hansberg, 2017, §3))**.** Productivity of firms differ across regions and thus affect the regional *share* in trade flows. The wage equation for the model is defined by (C.4.19):

$$W_i(\boldsymbol{h}, \boldsymbol{w}, \boldsymbol{A}) = w_i h_i - \sum_{j\in\mathcal{K}} \frac{h_i A_i w_i^{1-\sigma} d_{ij}}{\sum_{k\in\mathcal{K}} h_k A_k w_k^{1-\sigma} d_{kj}} w_j h_j = 0. \tag{D.2.9}$$

Without heterogeneities in the per capita housing space, the indirect utility function is

$$v_i = \frac{\mu}{\sigma - 1} \ln[\Delta_i] + \mu \ln[w_i] - (1-\mu)\ln[h_i] \tag{D.2.10}$$

with $\Delta_i \equiv \sum_{k\in\mathcal{K}} h_j A_i w_j^{1-\sigma} d_{ji}$. Employing these formulae, we compute as follows:

$$\nabla_{\boldsymbol{h}} \boldsymbol{v} = \frac{1}{h}\left(\frac{\mu}{\sigma-1}\bar{\boldsymbol{D}} - (1-\mu)\boldsymbol{I}\right),\ \nabla_{\boldsymbol{w}} \boldsymbol{v} = \frac{1}{\bar{w}}\mu(\boldsymbol{I} - \bar{\boldsymbol{D}}),\ \nabla_{\boldsymbol{A}} \boldsymbol{v} = \frac{1}{A}\frac{\mu}{\sigma-1}\bar{\boldsymbol{D}}, \tag{D.2.11}$$

$$\nabla_{\boldsymbol{h}} \boldsymbol{W} = -\bar{w}\bar{\boldsymbol{D}}(\boldsymbol{I} - \bar{\boldsymbol{D}}),\ \nabla_{\boldsymbol{w}} \boldsymbol{W} = h\{\sigma\boldsymbol{I} - (\sigma-1)\bar{\boldsymbol{D}}\}(\boldsymbol{I} + \bar{\boldsymbol{D}}),\ \nabla_{\boldsymbol{A}} \boldsymbol{W} = -\frac{1}{A}\bar{w}h(\boldsymbol{I} - \bar{\boldsymbol{D}})(\boldsymbol{I} + \bar{\boldsymbol{D}}). \tag{D.2.12}$$

These formulae implies that, without heterogeneities in the per capita housing space, we have:

$$G_{\mathrm{H}}(f) = \frac{1}{\sigma}(1-f)\left[-(1-\mu) + \left(\frac{\mu\sigma}{\sigma-1} - \frac{\sigma-1}{\sigma}\right)f\right] \tag{D.2.13}$$

$$G_{\mathrm{A}}(f) = -\frac{h}{A}\frac{\mu}{\sigma-1}(1-f)\left[(\sigma-1) + \sigma f\right] < 0 \tag{D.2.14}$$

Employing the formulae, one can show that, whenever equilibrium is unique ($\sigma(1-\mu) > 1$), we have $G_{\mathrm{H}}(f) < 0$ and thus $a_k \geq 0$ for all k. Also, it follows that $\mathrm{d}a_k/\mathrm{d}r < 0$ for all $k \geq 1$. We also note that $G_{\mathrm{H}}(f) < 0$ implies stability of $\bar{\boldsymbol{h}}$. We note that, if there are no exogenous heterogeneities in $\boldsymbol{A}$, the model is isomorphic to Redding and Sturm (2008) and Allen and Arkolakis (2014) regarding second-nature mechanism.

Example D.2 (Heterogeneous local market size (Krugman, 1991))**.** Consider Krugman (1991)'s model. Assuming there are first-nature heterogeneities in local endowments of immobile agents, one can model heterogeneities in *market size*. For the model, the wage equation is

$$W_i(\boldsymbol{h}, \boldsymbol{w}, \boldsymbol{A}) = w_i h_i - \mu \sum_{j\in\mathcal{K}} \frac{h_i w_i^{1-\sigma} d_{ij}}{\sum_{k\in\mathcal{K}} h_k w_k^{1-\sigma} d_{kj}} (w_i h_i + A_i) = 0 \tag{D.2.15}$$

where A_i as the number of immobile workers in region i. We compute as follows:

$$\nabla_{\boldsymbol{h}} \boldsymbol{v} = \frac{1}{h}\frac{\mu}{\sigma-1}\bar{\boldsymbol{D}},\ \nabla_{\boldsymbol{w}} \boldsymbol{v} = \frac{1}{\bar{w}}(\boldsymbol{I} - \mu\bar{\boldsymbol{D}}),\ \nabla_{\boldsymbol{A}} \boldsymbol{v} = \boldsymbol{0}, \tag{D.2.16}$$

$$\nabla_{\boldsymbol{h}} \boldsymbol{W} = -\bar{w}\bar{\boldsymbol{D}}(\mu\boldsymbol{I} - \bar{\boldsymbol{D}}),\ \nabla_{\boldsymbol{w}} \boldsymbol{W} = h\{\sigma\boldsymbol{I} - \mu\bar{\boldsymbol{D}} - (\sigma-1)\bar{\boldsymbol{D}}^2\},\ \nabla_{\boldsymbol{A}} \boldsymbol{W} = -\mu\bar{\boldsymbol{D}}. \tag{D.2.17}$$

Then, we have

$$G_{\mathrm{H}}(f) = \frac{1}{\sigma}\left[\left(\frac{\mu}{\sigma-1} + \frac{\mu}{\sigma}\right)f - \left(\frac{\mu^2}{\sigma-1} + \frac{1}{\sigma}\right)f^2\right], \tag{D.2.18}$$

$$G_{\mathrm{A}}(f) = \frac{\mu}{\overline{w}} f(1 - \mu f) > 0. \tag{D.2.19}$$

Employing these formulae, one can show that $a_k \geq 0$ for all k and that $\mathrm{d}a_k/\mathrm{d}r > 0$ for all $k \geq 1$ whenever the flat-earth equilibrium is stable (i.e., $G_{\mathrm{H}}(f) < 0$).

Remark D.1. There are models, e.g., Redding and Turner (2015), §20.3, that employ both local and global heterogeneities such that

$$v_i(\boldsymbol{h}, \boldsymbol{A}, \boldsymbol{B}) = v_i(\boldsymbol{h}, \boldsymbol{w}(\boldsymbol{h}, \boldsymbol{A})) + B_i \tag{D.2.20}$$

where $\boldsymbol{A} = (A_i)$ and $\boldsymbol{B} = (B_i)$ are exogenous constants that reflect global and local heterogeneities, respectively. Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are not related to each other, the Jacobian matrix with respect to these two heterogeneities is given by a block-diagonal form and the effects of each heterogeneity can be studied separately.

D.3 Numerical examples

This section provides numerical examples to complement the above formal analysis which is focused on infinitesimally small variations in $\boldsymbol{A}$. Below, focusing on the most canonical form of location-fixed factors as in (D.1.1), we add an extra positive constant term A_0 to the indirect utility of region 0, so that the region has an exogenous advantage. Our numerical results suggest that the drawn formal conclusions correctly predict tendency in agglomeration patterns even when a strong location-fixed effect is imposed.

Figure 16 and **Figure 17** report the results of our numerical experiments under three representative setting, namely, a class (ii) model under uniqueness of equilibrium, class-(i) and (ii) models under a multiplicity of equilibria. In line with the numerical examples discussed in Section 5 (**Figure 8** and **Figure 9**), Krugman (1991) and Allen and Arkolakis (2014) are employed for the examples for class (i) and (ii), respectively. We note that the latter is isomorphic to Helpman (1998) with local landownership (i.e., Redding and Sturm, 2008; Redding and Rossi-Hansberg, 2017).

The figures show the population share of region 0 at stable equilibria, $\lambda_0 \equiv h_i/H$, against τ for four different settings of A_0 in $\{0, 0.001, 0.005, 0.01\}$. $A_0 = 0$ is the baseline case with no location-fixed advantage. Under our parameter setting, A_0 accounts for $0.5{\sim}100\%$ of the indirect utility of region 0 and hence has significant effects on equilibrium patterns.

Figure 16 reports evolutionary paths of λ_0 for the model by Allen and Arkolakis (2014) [class (ii)] under *uniqueness of equilibrium*. The parameters are the same as **Figure 9** except that we let $\beta = 0.6$. This implies $\alpha + \beta \leq 0$ and hence equilibrium is unique regardless of the level of transportation costs (refer to Section 5.2). Compared to the baseline case $A_0 = 0$, λ_0 is larger for the other cases ($A_0 = 0.001, 0.005, 0.01$); this corresponds to the condition $a_k > 0$. Also, λ_0 is increasing in A_0, which is intuitive. Furthermore, λ_0 decreases in line with τ, which is consistent with $\mathrm{d}a_k/\mathrm{d}r < 0$.

Figure 17 reports evolutionary paths of λ_0 for the models by Krugman (1991) [class (i)] and Allen and Arkolakis (2014) [class (ii)] under a *multiplicity of equilibria*. The basic model parameters other

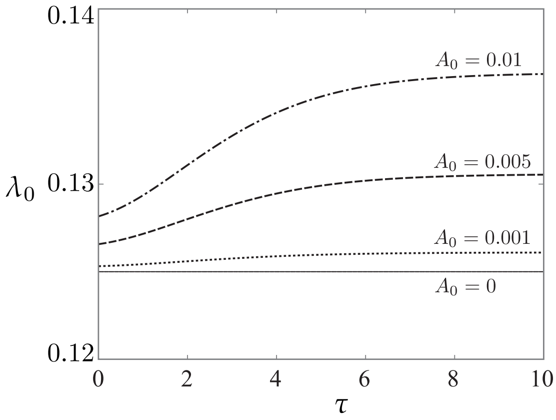
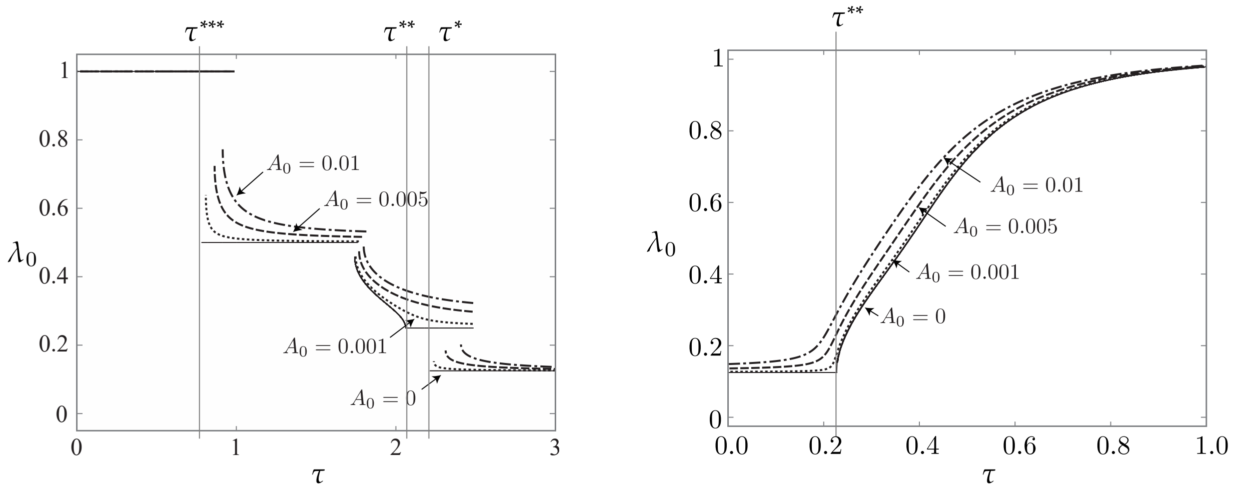


Figure 16: Population share of region 0 under uniqueness of equilibrium [Allen and Arkolakis (2014)'s model]



Panel A: Krugman (1991)'s model

Panel B: Allen and Arkolakis (2014)'s model

Figure 17: Population share of region 0 and under multiplicity of equilibria

than A_0 are the same as **Figure 8** and **Figure 9**. One confirms that the figures also exhibit consistency with our predictions: that (a) $a_k > 0$ and that (b) $\mathrm{d}a_k/\mathrm{d}r > 0$ for class-(i) and $\mathrm{d}a_k/\mathrm{d}r < 0$ for class (ii) model, provided that the $\bar{\boldsymbol{h}}$ is stable. For all $A_0 = 0.001, 0.005, 0.01$, λ_0 is greater than that for $A_0 = 0$, which confirms $a_k > 0$. Also, focusing on the ranges $\tau \in (\tau^*, \infty)$ (for Panel A) and $\tau \in (0, \tau^{**})$ (for Panel B), the curves confirms (b). It is interesting to observe that, even though our predictions do not cover $\tau \in (0, \tau^*)$ for Panel A, a similar relation robustly holds true: so long as the global structure of the spatial pattern is unchanged (i.e., bifurcation is not encountered), a monotonic decrease of τ imply a greater role of location-fixed advantage of region 0.