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Abstract

This paper analyzes all-pay auctions where the bidders have affiliated values for the object for sale and where the signals take binary values. Since signals are correlated, high signals indicate a high degree of competition in the auction and since even losing bidders must pay their bid, non-monotonic equilibria arise.

We show that the game has a unique symmetric equilibrium, and that whenever the equilibrium is non-monotonic the contestants earn no rents. All-pay auctions result in low expected rents to the bidders, but also induce inefficient allocations in models with affiliated private values. With two bidders, the effect on rent extraction dominates, and all-pay auction outperforms standard auctions in terms of expected revenue. With many bidders, this revenue ranking is reversed for some parameter values and the inefficient allocations persist even in large auctions.

1. Introduction

In an all-pay auction, bidders compete for a fixed prize by submitting simultaneous bids under the rule that the highest bidder wins and all the bidders must pay their bid regardless of whether they win or not. Even though all-pay auctions are seldom conducted

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in the real world, the format has been extensively studied because of its theoretical connection to winner-takes-all contests where bidders take the role of contestants expending resources to win a fixed prize.[1]

Very little is known about the all-pay auction and contest models when the bidders' valuations are correlated, even though such correlation is often quite natural. For example, when lobbying for a policy with uncertain economic effects, when undertaking R&D to obtain a patent, or when competing for a rent-generating position, it is natural that players' estimated values from winning are correlated.[2] The key implication of correlation is that a higher valuation implies high valuations for other contestants and hence a higher perceived probability of losing at a fixed bid if bidders with high signals submit higher bids. Hence a high signal carries two different messages: the value of the object and the level of competition are *both* likely to be high. With all-pay rules, the importance of competition is highlighted since also losing bids (or sunk efforts in the contest model) must be paid. When added competition is more important than the good news on the value of the object, the monotonicity of bidding strategies (i.e. the requirement that bidders with high valuations always win over bidders with low valuations) may fail. We show that the failure of monotonicity results in qualitative changes in the outcomes.

In this paper, we consider the simplest possible informational model with affiliated binary signals and interdependent valuations, and thereby tackle the potential non-monotonicity of bidding strategies. Each participant has one of two possible signals (high or low) on the value of the object, and her payoff depends on the entire vector of signals. Since all the bids are forfeited, the all-pay auction cannot have symmetric equilibria with a positive probability of ties for highest bids, or in other words, symmetric equilibria must be in atomless mixed strategies. Our framework is general enough to accommodate the mineral rights model, the model with affiliated private values, and cases in between. We show that the correlation in the signals calls for a re-evaluation of the previous results on rent dissipation and the efficiency of symmetric equilibria.

Our main findings are twofold. First, all-pay auctions are effective in dissipating the bidders' information rents. The unique symmetric equilibrium of the model features full rent dissipation whenever the equilibrium is non-monotonic. In other words, optimistic contestants are held to the same expected payoff as the pessimistic ones. Second, unless we are in the case of pure common values (where the identity of the winner does not

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[1]The early literature of all-pay auctions has generally focused on environment where bidders have complete information about each bidder's value of the object and cost of bidding. Examples of such papers include Hillman and Riley (1989), Baye *et al.* (1993) and Che and Gale (1998). Siegel (2009) provides a definitive treatment of this model by allowing heterogeneity on the bidder's characteristics. The recent survey paper by Kaplan and Zamir (2015) gives a comprehensive picture of recent developments in the all-pay auction and contest theory.

[2]The same effects arise if the private effort costs of the contestants are correlated.

matter for efficiency), all-pay auctions feature allocative inefficiencies. We show that these inefficiencies remain significant even when the number of players increases.

To appreciate the role of correlation, consider two alternative information structures: complete information and independent types. With common values and complete information, the payoffs from winning are the same for all bidders, and as a result, all bidders have the same symmetric equilibrium strategies. With private values and complete information, only the high-type bidders submit positive bids as long as there are at least two high-type bidders. With independent types, our results in this paper imply that the symmetric equilibria are monotone. In all of these cases, bidders of both types agree on the distribution of competing bids. With correlation, this is no longer the case. Bidders with a high signal perceive the correlation differently from those with a low signal. This asymmetric information on the degree of competition gives rise to our new insights.

These results have implications for less structured contest settings as well. Whenever a single leading candidate is picked in a field of contestants and the selection stage is preceded by a sunk investment (or prior effort) by the contestants, the issues that we highlight in this paper arise. While the rents are (at least approximately) dissipated in contests with large numbers of potential participants as expected, it may come as a surprise that the allocation may be inefficient. We show that the associated efficiency losses may be quite large in comparison to the total surplus generated.[3]

To grasp a better idea why full rent dissipation might hold in equilibrium, suppose for starters that bidding is in monotone strategies and therefore bidders with a high signal always win against bidders with a low signal. By affiliation, the high-type bidders believe that they are more likely to face a competitor who observed a high signal. Hence there are two counteracting effects of having a high signal: a *valuation effect* (the high type is more optimistic about the value of the object) and a *competition effect* (the high type expects to face a more aggressive competition than the low type).

Up to this point in the discussion, we have not considered the auction format at all and hence the reasoning above applies to standard auction formats as well. To understand why monotonicity fails under all-pay rules but not under standard rules, consider a standard first-price auction. As explained in Wang (1991), the low-type bidders bid the value of the object conditional on all bidders having low signals. In this case, a bidder makes a payment only if she wins the auction. As a result, the high-type bidders can safely outbid the low-type bidders without a fear of losses and this leads to an equilibrium where bidders with high valuations bid above bidders with low valuations. With

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[3]It is also worth mentioning that the same method of analysis allows us to compute the equilibria for contests where success requires an effort above a given threshold (or equivalently in auctions with a minimal bid). With this modification, the symmetric equilibria of the model display a random number of participants in the sense of supplying a level of effort that exceeds the minimum required.

all-pay rules, any bid winning all low signal bids but losing to high signal bids results in a loss if another high valuation bidder exists. The losses are particularly likely for a high signal bidder if the signals are strongly correlated. In this case, it is better to avoid those losses by submitting a bid of zero and as a result, zero is in the support of the bid distribution of the high signal bidders. Since zero is also in the support of the low signal bidders, this implies that equilibrium rents are fully dissipated. Such an equilibrium is non-monotonic in the sense that a low valuation bidder wins against a high valuation bidder with a positive probability.

Since different auction formats result in different allocations in symmetric equilibrium, the revenue across auction formats cannot be compared based on the linkage principle. By constructing the symmetric equilibria in the different cases, we can directly compare the total surplus generated and its division between seller's and buyers' rents by investigating carefully the supports of the equilibrium bid functions. We show that whenever a monotone strategy equilibrium exists in the all-pay auction, the expected revenue in the all-pay auction exceeds the revenue in standard auction formats as in Krishna and Morgan (1997). When equilibria are non-monotonic, the revenue comparison is more subtle. All-pay auctions induce two countervailing effects on the revenue, which are absent from standard auctions. First, rents to bidders are diminished and often completely eliminated, which increases revenue. Second, inefficient allocation may reduce the total surplus, which results in lower revenue.

We show that the information rent received by the high valuation bidders is always smaller in the all-pay auction than in standard auction formats (first-price and second-price auctions). In the case of pure common values the total surplus is independent of the allocation decision (i.e. whether a high signal bidder or a low signal bidder gets the object), and hence in that case the expected revenue in the all-pay auction is always weakly higher than in standard auctions.

With affiliated private values, the revenue comparison is more interesting. In contrast to the common values model, non-monotonic equilibria introduce allocational inefficiencies. In order to obtain a revenue comparison between the different auction formats, we must therefore compare the rent reduction with the inefficiency. This equilibrium trade-off between rent extraction and efficiency has not been shown in the prior literature.

We show that with two bidders, rent reduction dominates inefficiency and all-pay auctions result in higher expected revenues than standard auctions. With more bidders, this result may be reversed. Not surprisingly, rent reduction is not important as the number of players increase since the increased competition drives down the bidders' rents regardless of the auction format. More surprisingly, even large all-pay auctions may have inefficient allocations, and hence the revenue comparison tilts to the favor of the standard auctions.

In order to exposit this trade-off in the clearest manner, we analyze a two-state special case of the model and show that even in the limit where the number of bidders increases towards infinity, the surplus loss due to inefficient allocation may remain significant. One may find it surprising that a bidder with a low private valuation ends up winning the auction with a non-negligible probability even though it is commonly understood that there is a large number of high valuation bidders in both states of the world.

Previous work on all-pay auctions has concentrated on models with monotone equilibria. An early contribution by Krishna and Morgan (1997) derives sufficient conditions for the existence of a symmetric pure strategy equilibrium in monotone strategies. Unfortunately, the conditions are very strong and furthermore not easily verified in terms of the primitives of the model. More recently Siegel (2014) analyzes a model with a finite set of possible signals on the value of the prize, and derives conditions for the existence of a monotonic mixed strategy equilibrium. Another recent paper Rentschler and Turocy (2016) goes beyond monotonic equilibria in an affiliated all pay auction with a general discrete signal structure (but only two bidders) and provides an algorithm for finding symmetric non-monotonic equilibria.[4] In contrast to that paper, we provide a full characterization of the symmetric equilibria for a subclass of models assuming binary signal structure, and analyze this subclass for an arbitrary number of bidders.

Our paper is also related to auctions with entry costs. A recent paper by Murto and Välimäki (2017) compares the expected revenue in first- and second-price common value auctions when prior to the auction stage, the bidders make a costly entry decision. The connection to the current non-standard auction forms comes from the observation that the total payment by losing bidders is positive even in these standard auction formats once we account for the entry cost.

2. The Model

A single indivisible object is sold in an all-pay auction to one of N potential risk-neutral bidders. We assume that each bidder i observes privately a binary signal (or type) $t_i \in \{L, H\}$. We order the signals $H > L$ with the idea that H is good news about the value of the object for sale. The signals are assumed to be affiliated with another random variable $\theta \in \Theta = \{\theta_0, \theta_1, \cdots, \theta_{M-1}\}$, which we call the state of the world and order with $\theta_{m-1} < \theta_m$. Denoting by $p(\theta, \mathbf{t})$ the joint probability distribution of the state and the signal vector $\mathbf{t} = (t_1, \cdots, t_N)$, we require p to be symmetric over $\mathbf{t}$ and log-supermodular in $(\theta, \mathbf{t})$. This implies the monotone likelihood ratio property for all bidders' signals, and each signal

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[4]Subsequent to our working paper (Chi *et al.* (2015)), Liu and Chen (2016) extends the analysis of non-monotonic equilibria in the model with two players to cover the case of negatively correlated signals.

and state separately.

The prior on the state is denoted by $q(\theta) \in \Delta(\Theta)$. We assume that the signals are identically and independently distributed given θ. Due to its binary structure, the distribution of t_i can be represented by $\alpha_m := \Pr(t_i = H | \theta = \theta_m)$ for $m \in \{0, \cdots, M-1\}$. Our assumption of log-supermodular $p(\theta, \mathbf{t})$ then translates into the requirement that $\alpha_{m-1} \leq \alpha_m$. We denote by $p(\theta | \mathbf{t})$ the posterior distribution of the state given a vector of signals.

Bidder i's value of the object is given by $v_i(\theta, \mathbf{t}) > 0$. We assume further that the players are symmetric, and that the bidder i's valuation depends only on θ and t_i. With this assumption, we can write each bidder's valuation as

$$v_i(\theta, \mathbf{t}) = v(\theta, t_i).$$

The environment is therefore a binary signal version of the general symmetric affiliated model formulated in Milgrom and Weber (1982). The most important special cases of our model are the *mineral rights model* where $v(\theta, t_i) = v(\theta)$, and the *affiliated private values model* where $v(\theta, t_i) = v_{t_i}$. Finally, we assume that $v(\theta, t_i)$ is log-supermodular and increasing in each argument.[5]

In this framework, the payoff-relevant information to bidder i is contained in the statistic (t_i, Y_i), where Y_i indicates the number of other bidders $j \neq i$ observing the high signal. By the symmetry of the model, we can write the expected value of the object conditional on bidder i's information as

$$V_k(n) := \mathbb{E}_\theta[v(\theta, t_i) | Y_i = n, t_i = k],$$

which is increasing in k and n by the monotonicity of v and the log-supermodularity of p. Also, we denote by $p_k(n)$ the probability of the event $Y_i = n$ conditional on $t_i = k$.

In the all-pay auction, all bidders submit nonnegative bids simultaneously and the highest bidder receives the object while all bidders pay their bid. In case of multiple highest bidders, any arbitrary tie-breaking rule can be adopted to allocate the object between them. We represent the (mixed) strategy of bidder i by $\mathbf{F}_i = (F_i^L, F_i^H)$, where F_i^k for each $k = L, H$ is a distribution function on nonnegative real numbers. We use $\text{supp}[F_i^k]$ to denote the support of F_i^k for each type. In line with the symmetry assumptions that we have imposed, we concentrate on equilibria in symmetric strategies, i.e. $\mathbf{F}_i = \mathbf{F}_*$ for all i.

Suppose that bidder i observes $t_i = k$ and makes a nonnegative bid b, and that his opponents employ a symmetric strategy $\mathbf{F}_* = (F_*^L, F_*^H)$. Suppressing the index i, the

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[5]Our results remain valid as long as $v(\theta, H) - v(\theta, L)h(\theta)$ satisfies the single-crossing property in θ for all $h : \Theta \to \Re_+$ decreasing in θ.

expected equilibrium payoff $u\left(b, k | \mathbf{F}_*\right)$ to the bidder can be written as

$$u\left(b, k | \mathbf{F}_*\right) := -b + \sum_{n=0}^{N-1} V_k(n) p_k(n) \left(F_*^H(b_-)\right)^n \left(F_*^L(b_-)\right)^{N-n-1} + \pi\left(b, k | \mathbf{F}_*\right), \quad (1)$$

where the second term $\pi\left(b, k | \mathbf{F}_*\right)$ denotes the expected value of the object conditional on tying with (at least) one highest bidder, and $F_*^k(b_-)$ indicates the limit from the left at b of F_*^k. We show in the beginning of the next section that all symmetric equilibria are in atomless strategies and as a result, the second term π vanishes in the equilibrium analysis and $F_*^k(b_-) = F_*^k(b)$.

To interpret the payoff formula, $V_k(n)$ represents the expected value of the object conditional on winning when there are n high types among bidder i's opponents, and the term $\left[F_*^H(b_-)\right]^n \left[F_*^L(b_-)\right]^{N-n-1}$ indicates the corresponding winning probability when bidding b and facing n opponents with high signals.

A symmetric Bayes-Nash equilibrium of the all-pay auction is a pair of distributions $\mathbf{F}_* = \left(F_*^L, F_*^H\right)$ such that for each $k = L, H$,

$$\text{if } b \in \text{supp}[F_*^k], \text{ then } u\left(b, k | \mathbf{F}_*\right) \geq u\left(b', k | \mathbf{F}_*\right) \quad \text{for all } b' \geq 0.$$

3. Existence and Uniqueness of Symmetric Equilibria

We begin our analysis by establishing some basic facts for symmetric equilibria. Our first lemma shows that in every symmetric equilibrium, bidders employ an atomless bidding strategy and the union of the supports is a connected interval. As a consequence of this lemma, the expected payoff $u(b, k | \mathbf{F}_*)$ in (1) is continuous in b for all $\mathbf{F}_*$ and the tie-breaking term π in the expression is redundant. Furthermore, the continuity of the equilibrium payoff function in bids implies that any two distinctive bids in $\text{supp}[F_*^k]$ must yield the same expected payoff to a type-k bidder. This indifference condition serves as a key analytic tool in what follows.

Lemma 1. *In every symmetric equilibrium of the all-pay auction the following properties hold:*

1. *For each k, F_*^k is continuous, i.e., neither distribution has mass points.*

2. *The union of two supports, $\text{supp}[F_*^L] \cup \text{supp}[F_*^H]$, is a connected interval that includes zero.*

Proof. See Appendix A.1. □

In games of incomplete information, monotone strategies play a prominent role in characterizations and existence proofs of Bayes-Nash equilibria (see Athey (2001)). Since our equilibrium is not in pure strategies, the definition of monotonicity is not obvious. We call a symmetric equilibrium monotonic if no bidder with a high signal ever loses to another bidder with a low signal: for every b^H and b^L with $b^H \in \text{supp}[F_*^H]$ and $b^L \in \text{supp}[F_*^L]$, we have $b^H \geq b^L$. In the light of Lemma 1, an equilibrium is monotonic only if the bid supports of the two types are connected non-overlapping intervals.

The existing analysis of all-pay auctions has mostly concentrated on monotonic equilibria where the high signal is unambiguously a good news to bidders compared to the low signal. Our main goal in this section is to establish existence and uniqueness of symmetric equilibrium and to provide a necessary and sufficient condition under which the equilibrium is in monotone strategies. For this purpose, we define the following function for $n \in \{0, \cdots, N-1\}$:

$$\psi(n) := V_H(n)p_H(n) - V_L(n)p_L(n).$$

We will show that under our assumptions on the model, the function $\psi(n)$ changes its sign at most once from negative to positive. This single-crossing property is important for our equilibrium characterization. As we show formally in the proof of Proposition 1, this property will guarantee that the equilibrium bidding support for each of the two types must be a connected interval.

The sign of the function $\psi(n)$ at $n = 0$ is the key determinant of whether the equilibrium is monotonic. To understand why this is the case, consider a hypothetical monotonic equilibrium, where the low type bidding support $\text{supp}[F_*^L]$ is a connected interval containing zero. For this to be an equilibrium, the high type must make a non-negative profit by bidding at $\max\left(\text{supp}[F_*^L]\right)$. To see when this can hold, let us first derive $\max\left(\text{supp}[F_*^L]\right)$ by utilizing the indifference of the low-type within her bidding support. By bidding zero, she never wins and hence her expected payoff is zero. By bidding $\max\left(\text{supp}[F_*^L]\right)$, she wins with probability $p_L(0)$ (that is, when all of her opponents are of low type) and receives a payoff $V_L(0)$ so that her expected gain is $V_L(0)p_L(0)$. Equating this gain with her bid so as to guarantee expected payoff of zero gives

$$\max\left(\text{supp}[F_*^L]\right) = V_L(0)p_L(0).$$

Consider next a high-type bidder, who bids $\max\left(\text{supp}[F_*^L]\right) = V_L(0)p_L(0)$. By making this bid, she wins with probability $p_H(0)$ and receives a payoff $V_H(0)$. Therefore, this bid gives a non-negative payoff if $V_H(0)p_H(0) \geq V_L(0)p_L(0)$. This shows that $\psi(0) \geq 0$ is a necessary condition for a monotonic equilibrium. Proposition 1 below states that it is also

a sufficient condition.

To examine how the sign of $\psi(\cdot)$ varies over n, recall that $V_k(n)$ is increasing in each variable and thus $V_H(n) \geq V_L(n)$ for every n. Affiliation (or the monotone likelihood ratio property) guarantees that $p_H(N-1) > p_L(N-1)$, and as a result, ψ takes on a positive value at $n = N-1$ at least. In case of two bidders, therefore, the function ψ is single-crossing automatically.[6] For an arbitrary number of bidders, because the likelihood ratio $p_H(n)/p_L(n)$ increases with n, the function ψ would be single-crossing unless $V_H(n)/V_L(n)$ decreases too rapidly over n.[7] Our assumption that $v(\theta, t_i)$ is log-supermodular in fact guarantees that the ratio $V_H(n)/V_L(n)$ is increasing in n and so the desired property follows. We record this in the following lemma:

Lemma 2. *If the valuation function $v(\theta, t_i)$ is log-supermodular, then $\psi(n)$ is single-crossing.*

Proof. See Appendix A.1. □

We are now ready to state our main characterization result.

Proposition 1. *The all-pay auction has a unique symmetric equilibrium, which is monotonic if and only if $\psi(0) \geq 0$. Specifically,*

- *If $\psi(0) \geq 0$, then $\operatorname{supp}[F_*^L] = [0, V_L(0)p_L(0)]$ and $\operatorname{supp}[F_*^H] = [V_L(0)p_L(0), \overline{B}_H]$ for some $\overline{B}_H > V_L(0)p_L(0)$.*

- *If $\psi(0) < 0$, then $\operatorname{supp}[F_*^L] = [0, \overline{B}_L]$ and $\operatorname{supp}[F_*^H] = [0, \overline{B}_H]$ for some $0 < \overline{B}_L < \overline{B}_H$.*

The low-type bidders earn a zero expected rent, whereas the high-type bidders earn a rent of $\max\{0, \psi(0)\}$.

Proof. See Appendix A.2. □

Proposition 1 shows that all the bidders' rents are fully dissipated in the non-monotonic equilibrium. This is in sharp contrast with the standard result in allocation problems under asymmetric information that due to informational advantages (and privacy), the arrival of good news leaves a positive rent to an agent.

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[6]For the analysis of two-bidder case, we can dispense with the assumption of conditionally i.i.d. types and the log-supermodularity of $v(\theta, t_i)$.

[7]To see one example where ψ does not satisfy the single-crossing property, consider $V_k(n) = \alpha \mathbb{1}_{\{k=H\}} + (1-\alpha)n + \epsilon$, where $\epsilon > 0$ is a sufficiently small constant and $\alpha \geq \frac{1}{2}$ and $\mathbb{1}_{\{k=H\}}$ is the indicator function of the event that $t_i = k$. Then the ratio $V_H(n)/V_L(n)$ drops drastically when n increases from 0 to 1. As a result, if $p_H(0)/p_L(0) > \epsilon/(\alpha + \epsilon)$ but $p_H(1)/p_L(1) < (1-\alpha+\epsilon)/(1+\epsilon)$, then we have $\psi(0) > 0$ but $\psi(1) < 0$.

Figure 1 illustrates the different types of equilibria for the case of two bidders. The left panel displays the monotonic equilibrium which is qualitatively similar to the equilibrium in an all-pay auction with independent private values. In this equilibrium, each bidder competes against bidders of her own type and a bid of zero is in the support of the low type. As a result, the low type makes no rent but the high type may earn a positive rent. Since the cost of increasing a bid by db is constant within the support, the benefit (i.e. the increased probability of winning) must also be constant. Hence the symmetric bid distributions must be uniform on the support of each type of bidder.

The right panel displays a non-monotonic equilibrium where the density of the high type equilibrium bid distribution must be constant on the part of the support that does not overlap with the low type bid support by the same logic as for monotonic equilibria. In the interior of the overlapping part of the supports, both types must be indifferent between increasing their bid by db and remaining at the current bid. Since winning has a different value for the two types, the constant densities (denoted f_*^L and f_*^H) for the equilibrium bid distributions in this region solve the following pair of equations that equalize the gains and losses from a higher bid for each type:

$$V_H(0)p_H(0)f_*^L db + V_H(1)p_H(1)f_*^H db = db,$$
$$V_L(0)p_L(0)f_*^L db + V_L(1)p_L(1)f_*^H db = db.$$

With multiple bidders, the qualitative picture remains the same but each bid distribution is no longer uniform over the support since the relevant endogenous variable for determining the expected gains is the highest order statistic amongst competing bidders. If the equilibrium remains monotonic even when a large number of bidders compete, which is the case e.g. with independent signals, we see from the first part of Proposition 1 that the bidding support of the low type would shrink to zero length as the number of bidders is increased (since $p_L(0)$ approaches zero), and thus in the limit only a high-type bidder may submit a strictly positive bid. However, it turns out that this is not the case if the equilibrium morphs into a non-monotonic one as N increases. In Section 5, we return to this issue and discuss the bid distributions in a large all-pay auction in more detail.

We end this section by discussing in more detail the conditions under which we should expect a monotonic or a non-monotonic equilibrium. For this purpose, rewrite the condition $\psi(0) \geq 0$ as

$$\frac{V_H(0)}{V_L(0)} \geq \frac{p_L(0)}{p_H(0)}. \tag{2}$$

Observe first that the ratio on the right-hand side of (2) increases as the correlation between signals increases. With independent signals, we see immediately that the ratio

(a) Monotonic Equilibrium when $\psi(0) \geq 0$ (b) Non-monotonic Equilibrium when $\psi(0) < 0$
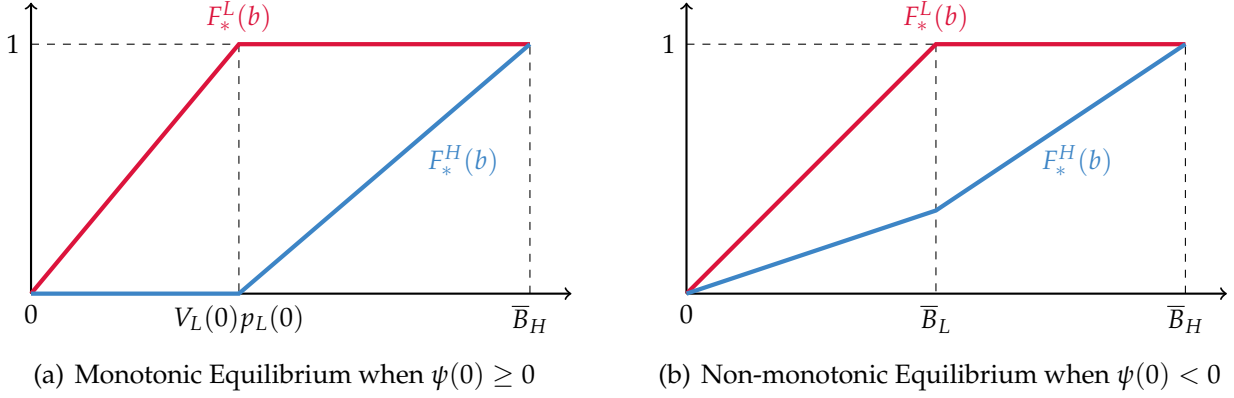
Figure 1: Symmetric Equilibrium in Case of Two Bidders

is equal to unity and hence the unique equilibrium is always monotonic. On the other hand, if the payoff difference between the types is small enough so that the left-hand side is close to one, only a slight degree of positive correlation is needed to kill the monotone equilibria.

Finally, let us examine in more detail the nature of equilibrium as the number of players N increases. In the mineral rights model, the effect of an individual bidder's signal on the value of the object would naturally diminish as N grows, and hence the left-hand side of (2) converges to one. On the other hand, the right-hand side of (2) converges to $\frac{p(\theta_0|t_i=L)}{p(\theta_0|t_i=H)}$, which represents the likelihood ratio of the lowest possible state across the two signals. Intuitively, only the lowest state θ_0 matters for the ratio $\frac{p_L(0)}{p_H(0)}$ as $N \to \infty$, since the likelihood of $Y_i = 0$ declines to zero at a higher rate in all the other states than in state θ_0. Moreover, because the likelihood ratio of θ_0 is larger than one due to affiliation, we conclude that the equilibrium must always be in non-monotone strategies for large N.

In the affiliated private value model, the left-hand side of (2) is constant at $\frac{v_H}{v_L} > 1$ whereas the right-hand side is increasing in N as we formally show in the proof of Proposition 2. Intuitively, the news that an additional bidder turns out to observe a low signal comes as a great surprise to the high type rather than to the low type. Hence the denominator $p_H(0)$ decreases at a higher rate than the nominator $p_L(0)$, as N is increased. The ratio $\frac{p_L(0)}{p_H(0)}$ therefore converges monotonically from below to $\frac{p(\theta_0|t_i=L)}{p(\theta_0|t_i=H)} > 1$ as N increases. Consequently, if $\frac{p(\theta_0|t_i=L)}{p(\theta_0|t_i=H)} < \frac{v_H}{v_L}$, the equilibrium remains monotonic for all N, while otherwise the equilibrium converts into a non-monotonic one at least for large enough N. The following proposition summarizes our discussion.

Proposition 2. *As the number of players N is increased, the symmetric equilibrium satisfies:*

1. *In the mineral rights model, there is a $\overline{N} < \infty$ such that for all $N > \overline{N}$, the symmetric equilibrium of the model is non-monotonic.*

11

2. *In the affiliated private value model, if $\frac{v_H}{v_L} > \frac{p(\theta_0|t_i=L)}{p(\theta_0|t_i=H)}$, then the equilibrium is monotonic for all N. If $\frac{v_H}{v_L} < \frac{p(\theta_0|t_i=L)}{p(\theta_0|t_i=H)}$, then there exists a $\overline{N}$ such that the equilibrium is non-monotonic for all $N > \overline{N}$.*

Proof. See Appendix A.3. □

4. Revenue and Efficiency Properties

We now turn to the revenue and efficiency properties of the equilibrium. We want to contrast the allocation and expected total payment in the unique equilibrium of the all-pay auction to those in the two standard auction formats, specifically the first- and second-price auctions. To begin, we prove that like the all-pay auction, the standard auction formats have a unique symmetric equilibrium in our framework, and that this equilibrium is always in monotone strategies. Furthermore, the two auction formats are payoff equivalent.[8]

Proposition 3. *Both standard auction formats, the first and second price auction, have a unique symmetric equilibrium, which is monotonic. In both formats, the low-type bidders earn no rent but the high-type bidders earn a positive rent of $p_H(0)(V_H(0) - V_L(0))$.*

Proof. See Appendix A.4. □

Propositions 1 and 3 characterize the unique symmetric equilibrium in all-pay auctions and standard auctions, respectively. We see that in both cases the low type bidders get no rent, as expected. Since by affiliation $p_L(0) \geq p_H(0)$, we have

$$p_H(0)(V_H(0) - V_L(0)) \geq p_H(0) V_H(0) - p_L(0) V_L(0) = \psi(0),$$

which means that the high-type gets a higher rent in the standard auctions than in the all-pay auction. We have therefore an unambiguous ranking of the auction formats according to the bidders' rents:

Remark 1. *The expected rent of bidders is higher in the standard auction formats than in the all-pay auction.*

Let us next turn to the comparison of the allocation across the auction formats. Since the equilibrium in the standard auctions is monotonic, a high type, whenever present,

---

[8]The payoff equivalence of the two standard auction formats is specific to the binary signal structure. When we allow a richer signal space, we know from the linkage principle (Milgrom and Weber (1982)) that the second-price auction is revenue-superior to the first-price auction.

always wins against a low type and as a result the allocation is efficient. In contrast, in the all-pay auction the equilibrium may be non-monotonic in the sense that the bidding supports of the two types overlap. In such a case, there is a positive probability that a low type wins even when high-type bidders are present, leading to an inefficient allocation.

Nevertheless, there are two situations where the all-pay auction achieves allocative efficiency. First, when $\psi(0) \geq 0$, the equilibrium is monotonic by Proposition 1. As seen in (2), this is the case when $\frac{V_H(0)}{V_L(0)}$ is large in comparison to $\frac{p_L(0)}{p_H(0)}$, in other words, when the effect of own signal on (estimated) value is large in comparison to the affiliation effect. Second, when the identity of the winner does not matter for the efficiency, even a non-monotonic equilibrium leads to efficient allocation. This is the case in the mineral-rights model.

Whenever the equilibrium allocation is efficient, the revenue comparison across the auctions is simple. The revenue is the total surplus minus the bidders' rents, and therefore the revenue increases whenever bidders' rent share decreases. Remark 1 leads directly to the following result:

Proposition 4. *If the allocation is efficient in the all-pay auction, then the revenue to the seller is higher in the all-pay auction than in the standard auctions. This is the case if, either:*

- $v(\theta, t_i) = v(\theta)$ *(Mineral rights model), or if*

- $\psi(0) \geq 0$ *(Monotonic equilibrium).*

Note that the second case in the proposition corresponds to the result obtained in Krishna and Morgan (1997), which analyzes the corresponding model under a continuum signal space but under a parameter restriction that rules out non-monotonic cases.[9]

The revenue comparison is more interesting when the all-pay auction features allocative inefficiency. This is the case when $v(\theta, t_i)$ depends on t_i and $\psi(0) < 0$. We see from Proposition 1 that whenever $\psi(0) < 0$, the bidders' rents are fully dissipated. In this case, therefore, the revenue comparison boils down to comparing the revenue loss due to inefficient allocation in the all-pay auction and the revenue loss due to bidders' rents in the standard auctions.

To make sense of this comparison, consider first the special case where there are only two bidders. To compute the revenue loss in the all-pay auction, note that an inefficient allocation may occur only when there is one high type bidder and one low type bidder present. Denote by $\mathbb{P}(1)$ the probability of this event. The inefficient allocation indeed

---

[9]More precisely, Krishna and Morgan (1997) compares the first-price auction with the all-pay auction in terms of the expected revenue in the (unique) monotone symmetric equilibrium, and shows that the all-pay auction outperforms. The revenue ranking between the all-pay auction and the second-price auction is ambiguous.

occurs when the low type outbids the high type, which takes place with some strictly positive probability $\Pr(b_L > b_H)$ in the non-monotonic equilibrium, resulting in a reduction of the total surplus by $V_H(0) - V_L(1)$. The expected revenue loss due to the inefficiency in the all-pay auction can therefore be written as

$$\mathbb{P}(1)\big(V_H(0) - V_L(1)\big)\Pr(b_L > b_H).$$

On the other hand, it follows from Proposition 3 that the revenue loss in the standard auctions arising from information rents given up to the high type amounts to

$$\mathbb{P}(1)\big(V_H(0) - V_L(0)\big).$$

This is clearly strictly larger than the revenue loss in the all-pay auction because $V_L(1) \geq V_L(0)$ and $\Pr(b_L > b_H) < 1$. For this reason, the all-pay auction outperforms the standard auctions even in the case when monotonicity fails. We summarize the above discussion in the following proposition.

Proposition 5. *With two bidders, the all pay auction generates a higher expected revenue than the standard auction formats.*

When there are more than two bidders, the revenue comparison is less straightforward. We will examine this in the next section in the context of a version of the model where the number of bidders grows large. We will see that as N increases, information rents vanish due to increased competition irrespective of the auction format, but the inefficiency loss of the all-pay auction survives and remains significant in some cases even when N goes to infinity. This will reverse the revenue ranking result of Proposition 5.

5. Many Bidders and a Binary State

In this section, we let the number of bidders N increase. In order to get the sharpest results, we consider the special case of our model where the state of the world is also binary, and study the limiting behavior of the model as $N \to \infty$. We start with the affiliated private values model and then consider the mineral rights model with common values. As already pointed out in Proposition 2, the failure of monotonicity is typical for models with large numbers of players. The main insight in this section is that with affiliated private values, this implies that the probability of misallocating the object to a low type bidder remains considerable even in the limit where the number of both types of bidders grows without bound, irrespective of the true state.

5.1. Affiliated Private Values

There are N bidders, two states $\theta \in \{\theta_0, \theta_1\}$, and signals $t_i \in \{L, H\}$ are conditionally i.i.d. given the state. Let $q \in (0,1)$ denote the prior belief on the event $\{\theta = \theta_1\}$ and parameterize the distribution of signals by

$$
\begin{aligned}
\alpha_1 &:= \Pr(t_i = H \,|\, \theta_1), \\
\alpha_0 &:= \Pr(t_i = H \,|\, \theta_0).
\end{aligned}
$$

By Bayes' rule, we can write the posterior beliefs on the state as

$$
\begin{aligned}
q_H &:= \Pr(\theta_1 \,|\, t_i = H) = \frac{q\alpha_1}{q\alpha_1 + (1 - q)\,\alpha_0}, \\
q_L &:= \Pr(\theta_1 \,|\, t_i = L) = \frac{q(1 - \alpha_1)}{q(1 - \alpha_1) + (1 - q)(1 - \alpha_0)}.
\end{aligned}
\tag{3}
$$

For the analysis of the limiting behavior, it is useful to consider the objective probabilities of winning *given* state θ at bid b, rather than those given the number of high-type opponents. In state θ and the symmetric equilibrium $\mathbf{F}_*$, the probability that an arbitrary bidder submits a bid below b is given by

$$
\alpha_\theta F_*^H(b) + (1 - \alpha_\theta) F_*^L(b).
$$

Since there are $N - 1$ other bidders, the probability of winning at bid b given θ is then given by

$$
x_m(b) := \Pr(\text{win by bidding } b \,|\, \theta_m) = \left[\alpha_m F_*^H(b) + (1 - \alpha_m) F_*^L(b)\right]^{N-1}, \quad m = 0, 1. \tag{4}
$$

Write each bidder's private value as v_{t_i} with $v_H > v_L$. We start the analysis by assuming that we are in the case of non-monotonic equilibria, so that the bid $b = 0$ is in the support of the symmetric equilibrium bid distribution for both types of bidders. Then for every bid b in the overlapping region of the two supports of the bidding distributions, we can write the indifference condition between bidding b and zero to each type as

$$
\begin{aligned}
q_H x_1(b) v_H + (1 - q_H) x_0(b) v_H &= b, \\
q_L x_1(b) v_L + (1 - q_L) x_0(b) v_L &= b.
\end{aligned}
$$

The left-hand side of the above equation makes use of equation (3) to express the expected gain to the bidder of each type when she makes a bid of b, as a weighted average of her private value by the winning probabilities given states. Observe from the form of

the equations that both $x_0(b)$ and $x_1(b)$ must be linear in b. This is consistent with the standard all-pay auction logic that due to the unconditional payment rule, the marginal increase in the winning probability by an increment of bid should be constant across the bidding supports.

Solving for the two winning probabilities gives

$$x_1(b) = \frac{b}{q_H - q_L}\left(\frac{1 - q_L}{v_H} - \frac{1 - q_H}{v_L}\right) = b\gamma_1,$$
$$x_0(b) = \frac{b}{q_H - q_L}\left(\frac{q_H}{v_L} - \frac{q_L}{v_H}\right) = b\gamma_0, \tag{5}$$

where we have denoted

$$\gamma_1 = \frac{\frac{1 - q_L}{v_H} - \frac{1 - q_H}{v_L}}{q_H - q_L}, \quad \gamma_0 = \frac{\frac{q_H}{v_L} - \frac{q_L}{v_H}}{q_H - q_L}.$$

Since $v_H > v_L$ and $q_H \geq q_L$, we have $\gamma_0 > 0$ and $\gamma_1 \leq \gamma_0$ for all parameter values. On the other hand, γ_1 is nonnegative if and only if

$$\frac{1 - q_L}{1 - q_H} \geq \frac{v_H}{v_L}.$$

Note that with the binary state structure, the left-hand side is equal to $\frac{p(\theta_0|t_i=L)}{p(\theta_0|t_i=H)}$, and therefore this condition is in line with the result we established in Proposition 2.

Notice that the binary-state model enables us to derive the winning probabilities directly from the indifference conditions of the two types of bidders. This turns out to be extremely useful, since we have an alternative way of expressing the winning probabilities in terms of the bid distributions as is displayed in equation (4). Accordingly, we can find the equilibrium bid distributions (F_*^H, F_*^L) by solving the following pair of equations obtained by combining (4) and (5):

$$\left[\alpha_1 F_*^H(b) + (1 - \alpha_1) F_*^L(b)\right]^{N-1} = b\gamma_1,$$
$$\left[\alpha_0 F_*^H(b) + (1 - \alpha_0) F_*^L(b)\right]^{N-1} = b\gamma_0. \tag{6}$$

Using this system of equations, we analyze the limiting behavior of the symmetric equilibrium as the number of bidders N grows. The system looks very simple at first sight, but note that the bid functions $F_*^H(b)$ and $F_*^L(b)$ themselves depend on the number of bidders.

Clearly, both $F_*^H(b)$ and $F_*^L(b)$ must converge to one as N increases, otherwise the

left-hand side of (6) would converge to zero. To understand the limit behavior of the model, it is more useful to work with the limiting values of $\left[F_*^H(b)\right]^{N-1}$ and $\left[F_*^L(b)\right]^{N-1}$ as N grows. If there exists a solution to (6), it must be the case that for a fixed b, both $\left[F_*^H(b)\right]^{N-1}$ and $\left[F_*^L(b)\right]^{N-1}$ converge to some values in $(0,1)$. Let us therefore denote those limiting values by

$$
\begin{aligned}
G_*^H(b) \;\; &: \;\; = \lim_{N\to\infty}\left(F_*^H(b)\right)^{N-1}, \\
G_*^L(b) \;\; &: \;\; = \lim_{N\to\infty}\left(F_*^L(b)\right)^{N-1}.
\end{aligned}
$$

Using this notation, it is straightforward to verify that[10]

$$
\left[\alpha_m F_*^H(b) + (1-\alpha_m)F_*^L(b)\right]^{N-1} \to \left(G_*^H(b)\right)^{\alpha_m}\cdot\left(G_*^L(b)\right)^{1-\alpha_m}.
$$

We can then write the indifference conditions for the two types in the limit $N\to\infty$ as:

$$
\begin{aligned}
\left(G_*^H(b)\right)^{\alpha_1}\cdot\left(G_*^L(b)\right)^{1-\alpha_1} &= b\gamma_1, \\
\left(G_*^H(b)\right)^{\alpha_0}\cdot\left(G_*^L(b)\right)^{1-\alpha_0} &= b\gamma_0,
\end{aligned}
\tag{7}
$$

where $\left(G_*^H(b)\right)^{\alpha_m}$ is the limit probability that the highest bid by a high-type is below b, and $\left(G_*^L(b)\right)^{1-\alpha_1}$ is the limit probability that the highest bid by a low-type is below b, conditional on state θ_m. Solving (7) for $G_*^H(b)$ and $G_*^L(b)$, we get

$$
\begin{aligned}
G_*^H(b) &= b\,(\gamma_1)^{\frac{1-\alpha_0}{\alpha_1-\alpha_0}}\,(\gamma_0)^{\frac{\alpha_1-1}{\alpha_1-\alpha_0}}, \\
G_*^L(b) &= b\,(\gamma_1)^{\frac{-\alpha_0}{\alpha_1-\alpha_0}}\,(\gamma_0)^{\frac{\alpha_1}{\alpha_1-\alpha_0}}.
\end{aligned}
$$

We can then compute explicitly the probability distribution $\Gamma_k(b;\theta)$ for the highest bids of each type $k = H, L$ conditional on θ_m as:

$$
\begin{aligned}
\Gamma_H(b;\theta_m) &= \Pr\left(\text{highest bid of type } H \text{ below } b\,|\theta_m\right) \\
&= \left(G_*^H(b)\right)^{\alpha_m} = b^{\alpha_m}\,(\gamma_1)^{\frac{\alpha_m(1-\alpha_0)}{\alpha_1-\alpha_0}}\,(\gamma_0)^{\frac{\alpha_m(\alpha_1-1)}{\alpha_1-\alpha_0}},
\end{aligned}
$$

---

[10]To see this, note that for any positive numbers γ_H and γ_L, we have

$$
\left(\alpha_m(\gamma_H)^{\frac{1}{N}} + (1-\alpha_m)(\gamma_L)^{\frac{1}{N}}\right)^N \to (\gamma_H)^{\alpha_m}(\gamma_L)^{1-\alpha_m},
$$

which can be verified by taking the logarithm and using L'Hôpital's rule. This limit argument is familiar from the connection between CES and Cobb-Douglas functions.

$$\Gamma_L(b;\theta_m) = \text{Pr}(\text{highest bid of type } L \text{ below } b \,|\theta_m)$$

$$= \left(G_*^L(b)\right)^{1-\alpha_m} = b^{1-\alpha_m}(\gamma_1)^{\frac{-(1-\alpha_m)\alpha_0}{\alpha_1-\alpha_0}}(\gamma_0)^{\frac{(1-\alpha_m)\alpha_1}{\alpha_1-\alpha_0}}.$$

Let $\overline{B}_L$ denote the highest bid where the two supports overlap, i.e. where $G_*^L\left(\overline{B}_L\right) = 1$:

$$\overline{B}_L = (\gamma_1)^{\frac{\alpha_0}{\alpha_1-\alpha_0}}(\gamma_0)^{\frac{-\alpha_1}{\alpha_1-\alpha_0}}. \tag{8}$$

It is self-evident from the last formula that the common support $[0, \overline{B}_L]$ does not shrink even though we let the number of bidders grow in the auction. Together with the fact that the winning probabilities are linear in b, this suggests that the probability of inefficient allocation does not vanish even with a large number of bidders.

We can verify this result by computing the probability of inefficient allocation conditional on state θ_m:

$$\text{Pr}(\text{low type wins} \,|\theta_m) = \int_0^{\overline{B}_L} \frac{\partial \Gamma_L(b;\theta_m)}{\partial b} \cdot \Gamma_H(b;\theta_m)\, db$$

$$= \overline{B}_L(1-\alpha_m)(\gamma_1)^{\frac{\alpha_m-\alpha_0}{\alpha_1-\alpha_0}}(\gamma_0)^{\frac{\alpha_1-\alpha_m}{\alpha_1-\alpha_0}}$$

Substituting (8) into $\overline{B}_L$ above, we can simplify the desired probability into

$$\text{Pr}(\text{low type wins} \,|\theta_m) = (1-\alpha_m)\left(\frac{\gamma_1}{\gamma_0}\right)^{\frac{\alpha_m}{\alpha_1-\alpha_0}}.$$

The ex-ante probability of misallocation is therefore

$$\text{Pr}(\text{low type wins}) = q(1-\alpha_1)\left(\frac{\gamma_1}{\gamma_0}\right)^{\frac{\alpha_1}{\alpha_1-\alpha_0}} + (1-q)(1-\alpha_0)\left(\frac{\gamma_1}{\gamma_0}\right)^{\frac{\alpha_0}{\alpha_1-\alpha_0}}.$$

This calculation confirms that the probability of misallocation is bounded away from zero in the all-pay auction with a large number of bidders. In the limit as $N \to \infty$, there would be a large number of high-type bidders in both states. Nevertheless they do not bid aggressively enough to win over the low-type bidders. To understand how this can happen, recall that the high type is more likely to perceive the unknown state as high and the number of high-type bidders is larger in the high state. As a consequence, the high degree of competition forces the high-type bidders to bid relatively cautiously because all bids are forfeited regardless of whether they win or not. The low types, on the other hand, assign a lower probability to the high state by affiliation. Hence they would anticipate less fierce competition compared to the high types, and this makes them bid relatively

aggressively leading to the possibility of misallocation.

To appreciate this finding, contrast the situation to a slightly modified version of our model, where only the high-type bidders exist and their number depends on the state. In that version of the model, the high types compete only against each other, and the allocation is always efficient. When the number of bidders is increased, all the rents are eliminated in this version as well, and hence the equilibrium revenue and total social surplus is higher than in the original model. In other words, if one could prevent the low-type bidders from participating, the total surplus and the revenue accruing to the seller would increase.

It should be pointed out that in the limit as $N \to \infty$, the standard auctions are both efficient and result in a very low rent to the high type. Hence the expected payment received by the auctioneer is smaller in the all-pay auction than in standard auction formats, in contrast to the result with only two bidders (Proposition 5).

It is perhaps also worthwhile to interpret the model in terms of total effort expended in a contest model. The affiliated private values case with a large number of potential bidders can be taken to reflect heterogeneity in the valuation of the prize or idiosyncratic (but correlated) differences in the cost of effort across contestants. One might guess that a competitive model such as this will result in efficient allocations in the sense that the contestants with a high valuation or low cost of effort will dissipate the entire rent. Our analysis shows that the intuition concerning the rent dissipation is indeed correct: in the limit as $N \to \infty$, no participant earns a strictly positive rent. Importantly, however, the correlation in the contestants' valuations or costs often makes it impossible to achieve efficient allocation. The inefficiency in the context of contest means that the total effort is inefficiently low, or looking from a different perspective, the total cost of achieving a given equilibrium effort level is inefficiently high.

We end this subsection with some numerical comparative statics. The results demonstrate that the magnitude of the surplus loss due to misallocation can be substantial. Adopting the auction interpretation of the model, the measure of efficiency is the total surplus generated, which we denote by Π:

$$\Pi = \Pr(\text{high type wins}) \cdot v_H + \Pr(\text{ low type wins}) \cdot v_L.$$

We normalize $v_H = 1$ so that the total surplus under efficient allocation in a large auction is $\Pi = 1$. Letting $\alpha := \alpha_1 = 1 - \alpha_0 \geq 0.5$, Figure 2 plots the total surplus as a function of two key parameters, α and v_L (the only remaining parameter is q, the prior on state $\theta = \theta_1$, which we fix here as $q = 0.5$).

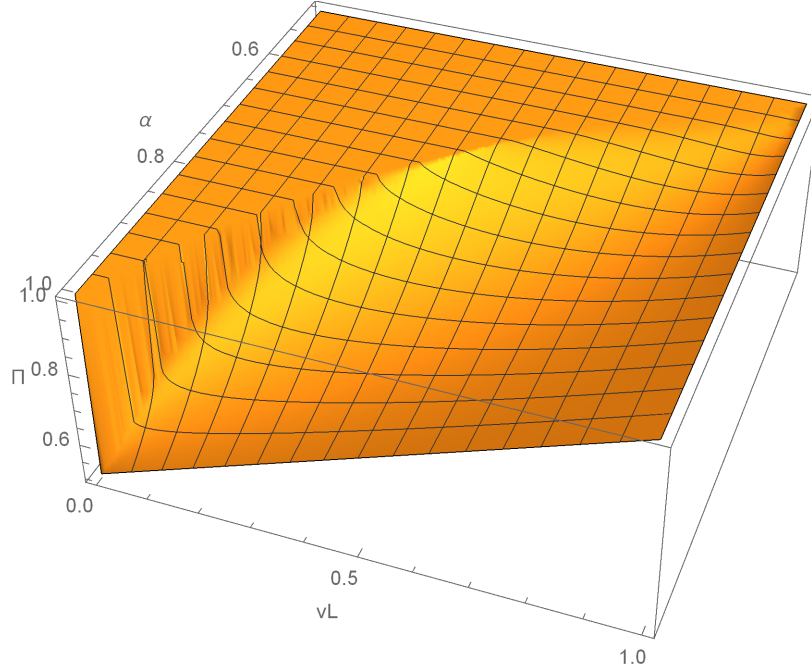We see that for low values of v_L and/or α, the allocation is efficient as $\Pi = 1$. This is

Figure 2: Total Surplus Π in a large all-pay auction as a function of α and v_L. The other parameters are $q = 0.5$ and $v_H = 1$.

the region in the parameter space where equilibrium is monotonic, i.e.

$$\frac{1 - q_L}{1 - q_H} < \frac{v_H}{v_L}.$$

Increasing α and/or v_L the non-monotonic equilibria emerge as indicated by surplus reducing below 1. The higher the correlation in the signals, the more pronounced the effect on misallocation becomes, as seen by Π decreasing monotonically in α. The effect of v_L is more subtle. After passing the threshold of non-monotonic equilibrium, an increase in v_L reduces surplus sharply. But as v_L further increases towards $v_H = 1$, the effect of misallocation on total surplus weakens despite its increasing probability with v_L, and as a result the total surplus is U-shaped as a function of v_L. To understand better the driving force behind this observation, note that when v_L is sufficiently low, the low valuation bidders have little incentive to win the auction. This results in a monotonic equilibrium. As v_L is increased, however, the auction becomes more attractive to low-type bidders and they start bidding more aggressively. When the competition effect starts dominating the valuation effect, the equilibrium becomes non-monotonic and incurs the misallocation. It is quite striking how steeply the total surplus reduces, once v_L crosses the threshold of monotonicity.

It is also interesting to note how the prior $q = \Pr(\theta = \theta_1)$ affects the total surplus.
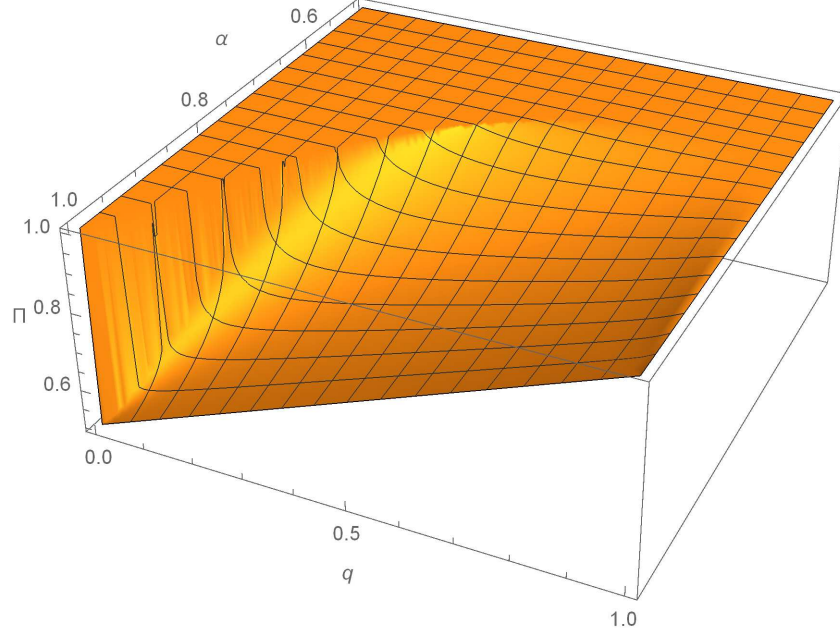
Figure 3: Total Surplus Π in a large all-pay auction as a function of α and q. The other parameters are $v_L = 0.5$ and $v_H = 1$.

Figure 3 plots Π as a function of q and α. It may come as a surprise that an increase in q impairs the efficiency in some parts of the parameter space. To understand this, fix the correlation parameter α and start with a low value of q. As seen in the figure, the equilibrium is monotonic and the allocation is efficient. When q is increased, it becomes more likely that there are a lot of high-type bidders. This induces them to bid more cautiously, which in turn renders the auction more attractive to the low types (who perceive it less likely that there are many high-type bidders). At some point the equilibrium becomes non-monotonic, and the possibility of inefficient allocation sharply reduces the total surplus. As q is increased towards 1, the uncertainty about the number of bidders vanishes and thus the equilibrium becomes efficient again.

Lastly, we examine the effect of the number of bidders on the total surplus by reformulating Π as a function of $N < \infty$. We show in Appendix A.5 that the probability of an inefficient allocation given state θ_m and a fixed number of bidders can be written as:

$$
\Pr\left(\text{low type wins} \mid \theta_m\right)
$$
$$
= \sum_{n=0}^{N} \binom{N}{n} (\alpha_m)^n (1 - \alpha_m)^{N-n} \frac{N-n}{N} \left[\frac{(1-\alpha_0)(\gamma_1)^{\frac{1}{N-1}} - (1-\alpha_1)(\gamma_0)^{\frac{1}{N-1}}}{\alpha_1 (\gamma_0)^{\frac{1}{N-1}} - \alpha_0 (\gamma_1)^{\frac{1}{N-1}}} \right]^n .
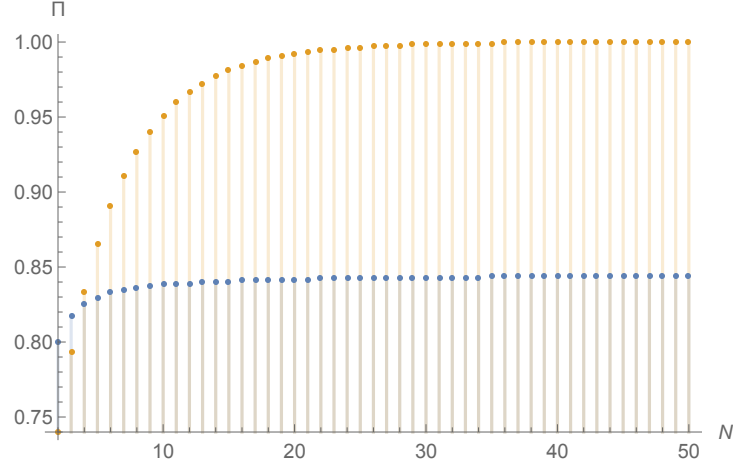$$

21

Figure 4: Total Surplus Π in an all-pay auction with N bidders. Blue: $q = 0.4$. Yellow: $q = 0.2$. The other parameters are $\alpha = 0.8$ and $v_L = 0.5$.

Accordingly, we can explicitly compute the ex-ante probability of misallocation as

$$\Pr(\text{low type wins}) = q \Pr(\text{low type wins} | \theta_1) + (1-q) \Pr(\text{low type wins} | \theta_0).$$

Figure 4 displays the total surplus as a function of N under two different prior probabilities, $q = 0.4$ and $q = 0.2$ (the other parameters are $\alpha = 0.8$ and $v_L = 0.5$). We can see that in both cases the surplus increases in N as expected. For small N, the total surplus is higher when $q = 0.4$. This is simply because the probability that (at least) one high type exists in the first place is substantially higher under that prior. However, the ranking is reversed for large N. The reason can be found in Figure 3, which shows that the two different priors lead to different equilibrium configurations for large N. With $q = 0.2$, equilibrium remains monotonic for all N, while with $q = 0.4$, equilibrium is non-monotonic for large N (in fact, the equilibrium is monotonic only for $N = 2$ but non-monotonic for all $N \geq 3$ in this case). This means that there is a substantial surplus loss from inefficiency even for large N, which appears in Figure 4 as a much weaker effect of the number of bidders on total surplus.

5.2. Mineral Rights

Here the setting is otherwise identical to the previous subsection, but we assume now that $v(\theta, t) = v(\theta)$. The analysis is now somewhat easier since the solution to the system

$$q_H x_H(b) v(\theta_1) + (1 - q_H) x_L(b) v(\theta_0) = b,$$
$$q_L x_H(b) v(\theta_1) + (1 - q_L) x_L(b) v(\theta_0) = b$$

22

is easily seen to be

$$x_H(b) = \frac{b}{v(\theta_1)}, \quad x_L(b) = \frac{b}{v(\theta_0)}.$$

Hence we see that a positive solution for winning probabilities resulting in indifference to both types exists for all parameter values, as expected in the light of Proposition 2. This is the key difference from the model with affiliated private values, where the equilibrium remains monotonic even for large N if $\frac{p(\theta_0|t_i=L)}{p(\theta_0|t_i=H)} < \frac{v_H}{v_L}$.

Obviously the economic consequences of the failure of monotonicity are less dramatic in the model with common values, since the allocation is always efficient. At any rate, it can be shown that all-pay auctions yield a high expected revenue to the seller, since all the rents are dissipated in the non-monotonic equilibrium.

We can again solve the symmetric equilibrium distributions for the limit $N \to \infty$ as in the affiliated private values case. Since the analysis is analogous to the previous case, we omit the details here.

6. Conclusion

Correlation in signals causes problems for the existence of monotone equilibria in all-pay auctions. This limits seriously the scope of the traditional analysis based on auction theoretic arguments. In a simple model with two types of bidders, we show that the non-existence of monotone equilibria has significant implications for the efficiency of allocations. We show that even if we let the number of players be arbitrarily large, the allocation may be inefficient when the bidders have affiliated private values.

We hope our findings in this simple setting inspires further work in related models. In addition to exploring richer informational models, further research should address contests with multiple prizes as well as contests with less extreme outcome functions and study efficiency and information aggregation in such environments.

A. Appendix

A.1. Proof of Lemmas

PROOF OF LEMMA 1: To see that there cannot be any atoms, assume to the contrary that b is an atom for F_*^k. If all bidders have signal $t_j = k$, then a bid of b ties for the highest bid with a positive probability. Since we have assumed that $v(\theta, t_i) > 0$, bidding $b + \varepsilon$ for ε small enough is a profitable deviation.

Consider any nonnegative bid $b \notin \text{supp}[F_*^L] \cup \text{supp}[F_*^H]$ such that there is a $b' > b$ with $b' \in \text{supp}[F_*^L] \cup \text{supp}[F_*^H]$. Since the union of supports is a closed set, we can take b' to be the minimal such bid. By the first part of the proof, no distributions has an atom at b'. Then the winning probabilities at b and b' are identical, and thus b is a profitable deviation from b'. This shows that the union of the supports is a connected set that includes zero. $\square$

PROOF OF LEMMA 2: To show that $\psi(n) = V_H(n)p_H(n) - V_L(n)p_L(n)$ is single-crossing, it is sufficient to establish the log-supermodularity of $V_k(n)p_k(n)$ in k and n. We know that $p_k(n)$ is log-supermodular by the monotone likelihood ratio proerty. Hence we need to show only that $V_k(n)$ is log-supermodular. Since

$$V_k(n) = \mathbb{E}_\theta\left[v(\theta, t_i) \,\middle|\, Y_i = n, \, t_i = k\right],$$

and since we have assumed the log-supermodularity of both $v(\theta, t_i)$ and $p(\theta, \mathbf{t})$, the result follows from the fact that log-supermodularity is preserved by integration and multiplication (See Karlin and Rinott (1980)). $\square$

A.2. Proof of Proposition 1

We shall prove this result through a series of lemmas. The first lemma shows that in any equilibrium, the low-type bidder earns zero expected payoff, which results from the fact that the function ψ takes a positive value at $N-1$.

Lemma A.1. *In any equilibrium* $\mathbf{F}_*$, $0 \in \text{supp}[F_*^L]$ *and as a consequence the low type bidders earn a zero expected rent.*

Proof. Suppose to the contrary that $\underline{B}_L \equiv \min\left(\text{supp}[F_*^L]\right) > 0$. Then by Lemma 1, it follows that $F_*^H(\underline{B}_L) > 0$, and that zero and $\underline{B}_L$ must belong to $\text{supp}[F_*^H]$. Noticing that a zero bid yields a payoff of zero, indifference between these bids gives:

$$u(\underline{B}_L, H|\mathbf{F}_*) = 0 \implies \underline{B}_L = V_H(N-1)p_H(N-1)\left(F_*^H(\underline{B}_L)\right)^{N-1},$$

where we used $F_*^L(\underline{B}_L) = 0$ and the fact that by bidding $\underline{B}_L$ the high type wins only if there is no low-type bidder. Using the above alternative expression of $\underline{B}_L$, we compute the expected payoff from bidding $\underline{B}_L$ to the low type, to obtain

$$u(\underline{B}_L, L|\mathbf{F}_*) = -\underline{B}_L + V_L(N-1)p_L(N-1)\left(F_*^H(\underline{B}_L)\right)^{N-1}$$

$$= -\psi(N-1)\left(F_*^H(\underline{B}_L)\right)^{N-1},$$

which is strictly negative because $\psi(N-1) > 0$ and $F_*^H(\underline{B}_L) > 0$. Therefore, the low type has a profitable deviation to bidding zero. □

The next lemma presents a sufficient condition for the existence of a monotone strategy equilibrium.

Lemma A.2. *If $\psi(0) \geq 0$, then the all-pay auction has a unique symmetric BNE in monotone strategies.*

Proof. Since $\psi(n)$ satisfies the single-crossing property, $\psi(0) \geq 0$ implies $\psi(n) > 0$ for all $n = 1, \cdots, N-1$. This implies that the effect of a marginal increase in b is increasing in the signal:

$$\frac{\partial}{\partial b}u(b, H|F_*) - \frac{\partial}{\partial b}u(b, L|\mathbf{F}_*) = \frac{\partial}{\partial b}\sum_{n=0}^{N-1}\psi(n)\left(F_*^H(b)\right)^n\left(F_*^L(b)\right)^{N-n-1} > 0.$$

This implies that the bidder's expected payoff function is supermodular in $(b; t_i)$, so any symmetric equilibrium must be in monotone strategies: for every $b_L \in \text{supp}[F_*^L]$ and $b_H \in \text{supp}[F_*^H]$, we must have $b_L \leq b_H$.[11] Since $0 \in \text{supp}[F_*^L]$ by the previous lemma, and since the low type has a chance to win only if his opponents are all high types in this monotonic equilibrium, we have $\text{supp}[F_*^L] = [0, \overline{B}_L]$ where $\overline{B}_L := \max\left(\text{supp}[F_*^L]\right) = \underline{B}_H := \min\left(\text{supp}[F_*^H]\right) = V_L(0)\,p_L(0)$. Furthermore, we can characterize F_*^L from the indifference condition between bidding zero and every $b \in \text{supp}[F_*^L]$ to the low type:

$$b = V_L(0)\,p_L(0)\left(F_*^L(b)\right)^{N-1},$$

for each $b \in [0, V_L(0)\,p_L(0)]$. As its right-hand side is strictly increasing in $F_*^L(b)$, the equilibrium bid distribution function of the low type is unique.

For the high type bidders, indifference at all $b \in \text{supp}[F_*^H]$ holds if and only if

$$\sum_{n=0}^{N-1}p_H(n)\,V_H(n)\left(F_*^H(b)\right)^n - b = V_H(0)\,p_H(0) - V_L(0)\,p_L(0) > 0,$$

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[11]As $u(b, k|\mathbf{F}_*)$ is supermodular, we have $\text{supp}[F_*^L] \leq \text{supp}[F_*^H]$ in the strong set order (See Milgrom and Shannon (1994)). This has two implications on the equilibrium support. First, each support must be a connected interval, for otherwise there exists a pair of bids $b_H < b_L$ such that $b_L \in \text{supp}[F_*^L]$ and $b_H \in \text{supp}[F_*^H]$. Second, the supports must be disjoint, because the strict supermodularity implies that if two bids, say b_1 and $b_2 > b_1$, are indifferent to the low type, then b_2 must be strictly preferred to b_1 by the high type.

where the expression on the right-hand side indicates the expected payoff of the high type from bidding $\underline{B}_H$, namely the information rent. From the last equation, we see the uniqueness of $F_*^H(b)$ for $b \in \text{supp}[F_*^H]$, and using $F_*^H(\overline{B}_H) = 1$, we get

$$\overline{B}_H := \max\left(\text{supp}[F_*^H]\right) = \sum_{n=0}^{N-1} p_H(n) V_H(n) - V_L(0) p_L(0).$$

$\square$

We examine next the other case of $\psi(0) < 0$. The next lemma shows that in this case the high type bidders also earn zero rent in equilibrium. Together with Lemma A.1 and A.2, the lemma tells us that there is full rent dissipation if and only if $\psi(0) < 0$.

Lemma A.3. *If $\psi(0) < 0$, then $0 \in \text{supp}[F_*^H]$ in every equilibrium $\mathbf{F}_*$, and as a result, the expected payoff to the high type bidder is zero.*

Proof. Suppose $\min\left(\text{supp}[F_*^H]\right) = \underline{B}_H > 0$. Since the union of the supports is a connected interval by Lemma 1, we must have $F_*^L(\underline{B}_H) > 0$ in this case. Also, since bidding $\underline{B}_H$ gives the same payoff as bidding zero for the low type by Lemma A.1, we have $\underline{B}_H = V_L(0)p_L(0)\left(F_*^L(\underline{B}_H)\right)^{N-1}$. The expected payoff to high-type bidders from bidding $\underline{B}_H$ is then

$$u(\underline{B}_H, H|\mathbf{F}_*) = -\underline{B}_H + V_H(0)p_H(0)\left(F_*^L(\underline{B}_H)\right)^{N-1} = \psi(0)\left(F_*^L(\underline{B}_H)\right)^{N-1} < 0.$$

Because every bidder has an option of bidding zero, any bids in the support must result in a nonnegative expected payoff. As a result, we conclude that $\underline{B}_H > 0$ $\square$

To fully characterize the equilibrium supports, we show in next lemma that each support is a connected interval and that $\text{supp}[F_*^L] \subset \text{supp}[F_*^H]$.

Lemma A.4. *Suppose $\psi(0) < 0$. In any symmetric BNE $\mathbf{F}_*$, both $\text{supp}[F_*^L]$ and $\text{supp}[F_*^H]$ are connected intervals.*

Proof. Suppose to the contrary that there is an open interval $(b_1', b_2') \subset [0, \overline{B}_L]$ such that $\text{supp}[F_*^L] \cap (b_1', b_2') = \emptyset$. Let (b_1, b_2) be the maximal (in the sense of set inclusion) open interval such that

$$(b_1', b_2') \subset (b_1, b_2) \text{ and } \text{supp}[F_*^L] \cap (b_1, b_2) = \emptyset$$

Then b_1 and b_2 must belong to $\text{supp}[F_*^L] \cap \text{supp}[F_*^H]$, and $F_*^H(b_2) > F_*^H(b_1)$ by Lemma 1.

Using Lemma A.1, we first obtain an alternative expression for b_1 and b_2:

$$b_1 = \sum_{n=0}^{N-1} V_L(n) p_L(n) \left(F_*^L(b_1) \right)^{N-1-n} \left(F_*^H(b_1) \right)^n$$

$$b_2 = \sum_{n=0}^{N-1} V_L(n) p_L(n) \left(F_*^L(b_1) \right)^{N-1-n} \left(F_*^H(b_2) \right)^n,$$

where we used $F_*^L(b_1) = F_*^L(b_2)$ to derive the expression of b_2. Also, it follows from Lemma 1 and A.3 that $b_1 \in \text{supp}[F_*^H]$ and the expected payoff from making b_1 must be zero to the high type, namely

$$u(b_1, H | \mathbf{F}_*) = -b_1 + \sum_{n=0}^{N-1} V_H(n) p_H(n) \left(F_*^L(b_1) \right)^{N-1-n} \left(F_*^H(b_1) \right)^n$$

$$= \sum_{n=0}^{N-1} \psi(n) \left(F_*^L(b_1) \right)^{N-1-n} \left(F_*^H(b_1) \right)^n = 0.$$

Observe that the expression $\psi(n) \left(F_*^L(b_1) \right)^{N-1-n} \left(F_*^H(b_1) \right)^n$ also satisfies the single-crossing property in n. This implies that

$$u(b_2, H | \mathbf{F}_*) = \sum_{n=0}^{N-1} \psi(n) \left(F_*^L(b_1) \right)^{N-1-n} \left(F_*^H(b_2) \right)^n$$

$$= \sum_{n=0}^{N-1} \psi(n) \left(F_*^L(b_1) \right)^{N-1-n} \left(F_*^H(b_1) \right)^n \cdot \left(\frac{F_*^H(b_2)}{F_*^H(b_1)} \right)^n > 0,$$

where the inequality comes from the (discrete version) Folk single-crossing lemma and the fact that $\left(\frac{F_*^H(b_2)}{F_*^H(b_1)} \right)^n$ is a strictly increasing function in n.[12] Therefore, the high type is strictly better off by bidding b_2 rather than b_1, which contradicts with the proposition that the two bids are indifferent. A similar argument can be used to establish that $\text{supp}[F_*^H]$ is also connected. $\qquad\square$

The proof of existence and uniqueness for the case $\psi(0) < 0$ is based on following lemma:

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[12] For a discrete domain N, the single-crossing lemma states that if $f : N \to \Re$ satisfies the (strict) single-crossing property and $\sum_{n \in N} f(n) = 0$, then $\sum_{n \in N} f(n) g(n) \geq (>) 0$ for an (strictly) increasing function $g : N \to \Re$. Note that the given properties of f imply $\sum_{n \geq k} f(n) \geq 0$ for every k. Hence the lemma follows from the fact that every increasing function can be approximated by $\sum_i \gamma_i \mathbb{1}_{\{n \geq k_i\}}$.

Lemma A.5. *Define a function* $G : [0,1] \times [0,1] \to \Re$ *as*

$$G(x,y) = \sum_{n=0}^{N-1} g(n) \, x^n \, y^{N-n-1},$$

where the function $g : \{0,1,\cdots,N-1\} \to \Re$ *is single-crossing,* $g(0) < 0$, *and* $\sum_{n=0}^{N-1} g(n) > 0$. *Then there exists a unique mapping* $\xi : (0,1] \to (0,1)$ *such that* $G(\xi(y),y) = 0$ *for every* $y \in (0,1]$. *Furthermore, the mapping* ξ *is continuous, strictly increasing, and*

$$\lim_{y \downarrow 0} \xi(y) = 0.$$

Proof. We start with some properties of function G. First, G is clearly continuous, and it is easy to check that for any $x > 0$ there is some $\delta_x > 0$ such that $G(x,y) > 0$ for $y \in (0,\delta_x)$, and for any $y > 0$ there is some $\delta_y > 0$ such that $G(x,y) < 0$ for $x \in (0,\delta_y)$. In particular, $G(1,y) > 0$ for every $y \in (0,1)$. This follows from the fact that $g(n)$ is single crossing in n, that $\sum_{n=1}^{N-1} g(n) > 0$, and that y^{N-n-1} is strictly increasing in n for $y \in (0,1)$. Consequently, there exists a pair of $(x,y) \in (0,1) \times (0,1)$ at which $G(x,y) = 0$.

The following pairwise strict single-crossing property of G is the key to the Lemma: if $G(x,y) = 0$ for some (x,y), then

$$G(x',y) \begin{cases} > 0 \text{ for } x' \in (x,1) \\ < 0 \text{ for } x' \in (0,x) \end{cases} \tag{A.1}$$

$$G(x,y') \begin{cases} < 0 \text{ for } y' \in (y,1) \\ > 0 \text{ for } y' \in (0,y) \end{cases} \tag{A.2}$$

To prove (A.1), fix $x \in (0,1)$ and $y \in (0,1)$ such that $G(x,y) = \sum_{n=0}^{N-1} g(n) \, x^n \, y^{N-n-1} = 0$. Then for any $x' \neq x$, we have

$$G(x',y) = \sum_{n=0}^{N-1} g(n) (x')^n \, y^{N-n-1} = \sum_{n=0}^{N-1} g(n) \, x^n y^{N-n-1} \left(\frac{x'}{x}\right)^n.$$

Since $g(n)$ is assumed to satisfy the single-crossing property in n, so does $g(n) \, y^{N-n-1} x^n$ because both y^{N-n-1} and x^n are positive (i.e., sign-preserving) functions of n. For $x' > (<) x$, the fraction $\left(\frac{x'}{x}\right)^n$ is strictly increasing (decreasing) in n, and thus we have

$$\sum_{n=0}^{N-1} g(n) \, x^n \, y^{N-n-1} \left(\frac{x'}{x}\right)^n > (<) \, 0.$$

The proof for (A.2) is completely analogous so is omitted.

Using the properties of G just established, we can now prove the lemma. Fix $y \in (0, 1]$. Since $G(x, y) < 0$ for x sufficiently small and $G(1, y) > 0$, there is some x such that $G(x, y) = 0$ by continuity of G. Because $G(x, y)$ as a function of x satisfies the single-crossing property by (A.1), this sign-changing point is unique, and hence defines a unique $\xi(y)$ for which $G(\xi(y), y) = 0$.

To see that $\lim_{y \downarrow 0} \xi(y) = 0$, recall that for every $x > 0$, there exists a $\delta_x > 0$ such that $G(x, y) > 0$ for all $y \in (0, \delta_x)$. This means that for all $y < \delta_x$, we have $\xi(y) < x$. Since x can be chosen arbitrarily low, it follows that $\lim_{y \downarrow 0} \xi(y) = 0$. Finally, continuity and strict monotonicity of $\xi(y)$ follow in a straightforward manner from the continuity and pairwise strict single-crossing property of $G(x, y)$. The proof is now complete. $\qquad \square$

The existence and the uniqueness of equilibrium can be established as follows.

Proof. By Lemmas A.1, A.3 and A.4, when $\psi(0) < 0$, in any equilibrium we must have $0 \in \text{supp}[F_*^L] \cap \text{supp}[F_*^H]$ and both supports are connected intervals. Consequently, there must be some interval $[0, \overline{B}_L]$ where the two supports overlap. Note that every bid $b \in [0, \overline{B}_L]$ must yield zero expected payoff (i.e., the same payoff) to both types. Below we demonstrate that there exists only one pair of (F_*^L, F_*^H) satisfying this property.

For each $y \in (0, 1]$, we define $\xi : (0, 1] \to (0, 1)$ as the solution to the equation:

$$\sum_{n=0}^{N-1} \Big[\underbrace{V_H(n) p_H(n) - V_L(n) p_L(n)}_{=\psi(n)} \Big] (\xi(y))^n y^{N-n-1} = 0. \tag{A.3}$$

As the function $\psi(n)$ satisfies all the given properties for function g in Lemma A.5, we know from that lemma that there exists a unique continuous and strictly increasing mapping satisfying (A.3) and $\lim_{y \to 0} \xi(y) = 0$.

Given this function ξ, we define $\overline{B}_L$ as

$$\sum_{n=0}^{N-1} V_H(n) p_H(n) (\xi(1))^n = \overline{B}_L.$$

For each $b < \overline{B}_L$, let $F_*^L(b) \in [0, 1)$ denote the unique value of y that solves the equation

$$\sum_{n=0}^{N-1} V_H(n) p_H(n) (\xi(y))^n y^{N-n-1} = b.$$

As the expression on the left-hand side of the equation is strictly increasing in y and $b < \overline{B}_L$, the solution $F_*^L(b)$ exists and is unique. Furthermore, it is easy to check that the

solution $F_*^L(b)$ retains all the necessary properties of a distribution function: it is strictly increasing and continuous in b, $F_*^L(0) = 0$, and $F_*^L(\overline{B}_L) = 1$ (by definition of $\overline{B}_L$). Label $\xi(F_*^L(b)) = F_*^H(b)$. Then $F_*^H(b)$ is strictly increasing in b and $F_*^H(0) = 0$. This process characterizes the symmetric equilibrium on $\text{supp}[F_*^L] \cap \text{supp}[F_*^H]$.

To characterize the equilibrium on $\text{supp}[F_*^H] \cap \{\text{supp}[F_*^L]\}^c$, let

$$\overline{B}_H = \sum_{n=1}^{N-1} V_H(n) \, p_H(n).$$

For each $b \in (\overline{B}_L, \overline{B}_H)$, set $F_*^L(b) = 1$ and define $F_*^H(b)$ as the solution to the equation

$$\sum_{n=0}^{N-1} V_H(n) \, p_H(n) \, x^n = b.$$

Since the left-hand side is strictly increasing in x, $F_*^H(b)$ is uniquely determined. $\qquad\square$

A.3. Proof of Proposition 2

The result we established in Proposition 1 tells us that the unique symmetric equilibrium is monotonic if and only if

$$\frac{V_H(0)}{V_L(0)} \geq \frac{p_L(0)}{p_H(0)}. \tag{A.4}$$

For the proof of Proposition 2, we need therefore investigate the limiting behavior of each side of (A.4) as the number of bidders N increases.

Part 1 - Mineral Rights Model

We first show that the ratio $\frac{V_H(0)}{V_L(0)}$ on the left-hand side converges to one as $N \to \infty$. To keep our notations simple, let $\mathbf{t} = (L, L, \cdots, L)$ denote the vector of signal realizations with $t_i = L$ for all i and $\mathbf{t}' = (H, L, \cdots, L)$ the vector with $t_i = L$ for all $i \neq 1$ and $t_1 = H$. Then the ratio can be written as

$$\frac{V_H(0)}{V_L(0)} = \frac{E[v(\theta)|\mathbf{t}']}{E[v(\theta)|\mathbf{t}]} = \frac{\sum_{m=0}^{M-1} p(\theta_m|\mathbf{t}') \, v(\theta_m)}{\sum_{m=0}^{M-1} p(\theta_m|\mathbf{t}) \, v(\theta_m)},$$

where the posterior belief on θ given $\mathbf{t}$ can be calculated with the Bayes rule: for each $\theta = \theta_m$,

$$p\left(\theta_m \,|\, \mathbf{t}'\right) \;=\; \frac{q\left(\theta_m\right) p\left(\mathbf{t}'\,|\,\theta_m\right)}{\sum\limits_{x=0}^{M-1} q\left(\theta_x\right) p\left(\,\mathbf{t}'\,|\,\theta_x\right)} \;=\; \frac{q\left(\theta_m\right) \alpha_m \left(1-\alpha_m\right)^{N-1}}{\sum\limits_{x=0}^{M-1} q\left(\theta_x\right) \alpha_x \left(1-\alpha_x\right)^{N-1}},$$

and

$$p\left(\theta_m \,|\, \mathbf{t}\right) \;=\; \frac{q\left(\theta_m\right) p\left(\mathbf{t}\,|\,\theta_m\right)}{\sum\limits_{x=0}^{M-1} q\left(\theta_x\right) p\left(\mathbf{t}\,|\,\theta_x\right)} \;=\; \frac{q\left(\theta_m\right) \left(1-\alpha_m\right)^{N}}{\sum\limits_{x=0}^{M-1} q\left(\theta_x\right) \left(1-\alpha_x\right)^{N}}.$$

Since we have $\alpha_m < \alpha_{m+1}$ for each m, both posterior beliefs assign a unit mass to $\theta = \theta_0$ as $N \to \infty$. Consequently,

$$\lim_{N\to\infty} \frac{V_H\left(0\right)}{V_L\left(0\right)} \;=\; \frac{v(\theta_0)}{v(\theta_0)} \;=\; 1.$$

We next investigate the right-hand side of (A.4), i.e. likelihood ratio $\frac{p_L(0)}{p_H(0)}$ as $N \to \infty$. We can write this as

$$\frac{p_L\left(0\right)}{p_H\left(0\right)} = \frac{p\left(\theta_0 \,|\, t_i = L\right)\left(1-\alpha_0\right)^{N-1} + \cdots + p\left(\theta_{M-1} \,|\, t_i = L\right)\left(1-\alpha_{M-1}\right)^{N-1}}{p\left(\theta_0 \,|\, t_i = H\right)\left(1-\alpha_0\right)^{N-1} + \cdots + p\left(\theta_{M-1} \,|\, t_i = H\right)\left(1-\alpha_{M-1}\right)^{N-1}}, \quad \text{(A.5)}$$

where

$$p\left(\theta_m \,|\, t_i = L\right) = \frac{q\left(\theta_m\right)\left(1-\alpha_m\right)}{\sum\limits_{x=0}^{M-1} q\left(\theta_x\right)\left(1-\alpha_x\right)} \quad \text{and} \quad p\left(\theta_m \,|\, t_i = H\right) = \frac{q\left(\theta_m\right)\alpha_m}{\sum\limits_{x=0}^{M-1} q\left(\theta_x\right)\alpha_x}$$

are the posteriors of state θ_m after observing signal L and H, respectively, and $\left(1-\alpha_m\right)^{N-1}$ is the probability that all the other $N-1$ players have a low signal, conditional on state. Dividing the top and bottom by $\left(1-\alpha_0\right)^{N-1}$, we have

$$\frac{p_L\left(0\right)}{p_H\left(0\right)} = \frac{p\left(\theta_0 \,|\, t_i = L\right) + p\left(\theta_1 \,|\, t_i = L\right)\left(\frac{1-\alpha_1}{1-\alpha_0}\right)^{N-1} + \cdots + p\left(\theta_{M-1} \,|\, t_i = L\right)\left(\frac{1-\alpha_{M-1}}{1-\alpha_0}\right)^{N-1}}{p\left(\theta_0 \,|\, t_i = H\right) + p\left(\theta_1 \,|\, t_i = H\right)\left(\frac{1-\alpha_1}{1-\alpha_0}\right)^{N-1} + \cdots + p\left(\theta_{M-1} \,|\, t_i = H\right)\left(\frac{1-\alpha_{M-1}}{1-\alpha_0}\right)^{N-1}}.$$

Since $\left(\frac{1-\alpha_m}{1-\alpha_0}\right)^{N-1} \to 0$ for all $m = 1, \cdots, M-1$ as $N \to \infty$, we have

$$\lim_{N\to\infty} \frac{p_L\left(0\right)}{p_H\left(0\right)} \;=\; \frac{p\left(\theta_0 \,|\, t_i = L\right)}{p\left(\theta_0 \,|\, t_i = H\right)} \;>\; 1. \qquad \text{(A.6)}$$

31

The first claim in Proposition 2 is then immediate from Proposition 1.

Part 2: Affiliated Private Value Model

In the private value model, the left-hand side of (A.4) is simply $\frac{v_H}{v_L}$ which is constant over the number of bidders. The likelihood ratio on the other side is as in the mineral rights model, and its limit as $N \to \infty$ is given by (A.6) above. To complete the proof, we prove below that the ratio $\frac{p_L(0)}{p_H(0)}$ is increasing in N. To emphasize its dependence on N, we rewrite (A.5) as

$$\frac{p_L(0;N)}{p_H(0;N)} = \frac{\sum\limits_{m=0}^{M-1} \xi_L(m)}{\sum\limits_{m=0}^{M-1} \xi_H(m)},$$

where

$$\xi_k(m) = p(\theta_m \,|\, t_i = k)(1 - \alpha_m)^{N-1}, \quad t = L, H. \tag{A.7}$$

Note that the ratio

$$\frac{\xi_L(m)}{\xi_H(m)} = \frac{p(\theta_m \,|\, t_i = L)}{p(\theta_m \,|\, t_i = H)} = \frac{1 - \alpha_m}{\alpha_m} \cdot \frac{\sum\limits_{x=0}^{M-1} q(\theta_x)\alpha_x}{\sum\limits_{x=0}^{M-1} q(\theta_x)(1 - \alpha_x)}$$

is decreasing in m by affiliation.

To see how it varies over N, consider next the ratio for $N + 1$:

$$\frac{p_L(0;N+1)}{p_H(0;N+1)} = \frac{p(\theta_0 \,|\, t_i = L)(1 - \alpha_0)^N + \cdots + p(\theta_{M-1} \,|\, t_i = L)(1 - \alpha_{M-1})^N}{p(\theta_0 \,|\, t_i = H)(1 - \alpha_0)^N + \cdots + p(\theta_{M-1} \,|\, t_i = H)(1 - \alpha_{M-1})^N},$$

or with $\xi_k(m)$ defined in (A.7), we can simplify it further into

$$\frac{p_L(0;N+1)}{p_H(0;N+1)} = \frac{\sum\limits_{m=0}^{M-1} \xi_L(m)(1 - \alpha_m)}{\sum\limits_{m=0}^{M-1} \xi_H(m)(1 - \alpha_m)}.$$

The proof is done if we can show that

$$\frac{p_L(0;N+1)}{p_H(0;N+1)} > \frac{p_L(0;N)}{p_H(0;N)},$$

32

that is,

$$\frac{\sum\limits_{m=0}^{M-1} \xi_L(m)(1-\alpha_m)}{\sum\limits_{m=0}^{M-1} \xi_H(m)(1-\alpha_m)} > \frac{\sum\limits_{m=0}^{M-1} \xi_L(m)}{\sum\limits_{m=0}^{M-1} \xi_H(m)}. \tag{A.8}$$

The key here is that both $\frac{\xi_L(m)}{\xi_H(m)}$ and $(1-\alpha_m)$ are decreasing in m. The following lemma establishes (A.8) and hence completes the proof.

Lemma A.6. *Let M be a positive integer and $\{\delta_m\}_{m=0}^{M-1}$, $\{x_m\}_{m=0}^{M-1}$, and $\{y_m\}_{m=0}^{M-1}$ denote sequences with all strictly positive terms (i.e., $\delta_m, x_m, y_m > 0 \; \forall \; m$) such that $\delta_{m-1} > \delta_m$ and $\frac{x_{m-1}}{y_{m-1}} > \frac{x_m}{y_m}$ for all $m = 1, \cdots, M-1$. Then we have*

$$\frac{\sum\limits_{m=0}^{M-1} \delta_m x_m}{\sum\limits_{m=0}^{M-1} \delta_m y_m} > \frac{\sum\limits_{m=0}^{M-1} x_m}{\sum\limits_{m=0}^{M-1} y_m}. \tag{A.9}$$

Proof. In what follows, we will repeatedly use the fact that whenever $A, B, a, b > 0$ and $A/a > B/b$, we have

$$\frac{Aq + B}{aq + b} > \frac{A + B}{a + b} \tag{A.10}$$

for $q > 1$ (this is easy to prove by differentiating the left-hand side with respect to q).

We prove Lemma A.6 using induction. First, (A.9) is clearly true if $M = 2$: If $\delta_0 > \delta_1$ and $\frac{x_0}{y_0} > \frac{x_1}{y_1}$, we have

$$\frac{\delta_0 x_0 + \delta_1 x_0}{\delta_0 y_0 + \delta_1 y_0} = \frac{\frac{\delta_0}{\delta_1} x_0 + x_1}{\frac{\delta_0}{\delta_1} y_0 + y_1} > \frac{x_0 + x_1}{y_0 + y_1},$$

where the inequality uses (A.10).

Fix an integer $M > 2$. As an induction hypothesis, suppose that (A.9) holds when the summation is taken from $m = 1$ to $m = M-1$:

$$\frac{\sum\limits_{m=1}^{M-1} \delta_m x_m}{\sum\limits_{m=1}^{M-1} \delta_m y_m} > \frac{\sum\limits_{m=1}^{M-1} x_m}{\sum\limits_{m=1}^{M-1} y_m},$$

whenever $\delta_{m-1} > \delta_m$ and $\frac{x_{m-1}}{y_{m-1}} > \frac{x_m}{y_m}$ for all $m = 0, \cdots, M-2$. Then, taking the summa-

tion from $m = 0$, we can write

$$\frac{\sum_{m=0}^{M-1} \delta_m x_m}{\sum_{m=0}^{M-1} \delta_m y_m} = \frac{\delta_0 x_0 + \delta_1 \left(x_1 + \frac{\delta_2}{\delta_1} x_2 + \cdots + \frac{\delta_{M-1}}{\delta_1} x_{M-1} \right)}{\delta_0 y_0 + \delta_1 \left(y_1 + \frac{\delta_2}{\delta_1} y_2 + \cdots + \frac{\delta_{M-1}}{\delta_1} y_{M-1} \right)}. \tag{A.11}$$

Let

$$\chi := \frac{x_1 + x_2 + \cdots + x_{M-1}}{x_1 + \frac{\delta_2}{\delta_1} x_2 + \cdots + \frac{\delta_{M-1}}{\delta_1} x_{M-1}}. \tag{A.12}$$

Since $\frac{\delta_k}{\delta_1} < 1$ for all $k = 2, ..., M - 1$, we have $\chi > 1$. Using this defitition, we can write the term in the parantesis in the nominator of (A.11) as:

$$x_1 + \frac{\delta_2}{\delta_1} x_2 + \cdots + \frac{\delta_{M-1}}{\delta_1} x_{M-1} = \frac{1}{\chi} \left(x_1 + x_2 + \cdots + x_{M-1} \right). \tag{A.13}$$

Since $\left(1, \frac{\delta_2}{\delta_1}, \frac{\delta_3}{\delta_1}, \cdots, \frac{\delta_{M-1}}{\delta_1} \right)$ is a decreasing sequence, the induction hypothesis gives:

$$\frac{x_1 + \frac{\delta_2}{\delta_1} x_2 + \cdots + \frac{\delta_{M-1}}{\delta_1} x_{M-1}}{y_1 + \frac{\delta_2}{\delta_1} y_2 + \cdots + \frac{\delta_{M-1}}{\delta_1} y_{M-1}} > \frac{x_1 + x_2 + \cdots + x_{M-1}}{y_1 + y_2 + \cdots + y_{M-1}},$$

which we can rearrange as

$$\begin{aligned}
y_1 + \frac{\delta_2}{\delta_1} y_2 + \cdots + \frac{\delta_{M-1}}{\delta_1} y_{M-1} \;&<\; \frac{x_1 + \frac{\delta_2}{\delta_1} x_2 + \cdots + \frac{\delta_{M-1}}{\delta_1} x_{M-1}}{x_1 + x_2 + \cdots + x_{M-1}} y_1 + y_2 + \cdots + y_{M-1} \\
&=\; \frac{1}{\chi} \left(y_1 + y_2 + \cdots + y_{M-1} \right),
\end{aligned} \tag{A.14}$$

where the last equality uses (A.12). Plugging equality (A.13) and inequality (A.14) in (A.11) gives

$$\begin{aligned}
\frac{\sum_{m=0}^{M-1} \delta_m x_m}{\sum_{m=0}^{M-1} \delta_m y_m} \;&>\; \frac{\delta_0 x_0 + \frac{\delta_1}{\chi} \left(x_1 + x_2 + \cdots + x_{M-1} \right)}{\delta_0 y_0 + \frac{\delta_1}{\chi} \left(y_1 + y_2 + \cdots + y_{M-1} \right)} \\
&=\; \frac{\frac{\delta_0 \chi}{\delta_1} x_0 + x_1 + \cdots + x_{M-1}}{\frac{\delta_0 \chi}{\delta_1} y_0 + y_1 + \cdots + y_{M-1}} \;>\; \frac{\sum_{m=0}^{M-1} x_m}{\sum_{m=0}^{M-1} y_m},
\end{aligned}$$

34

where the last inequality uses (A.10) and the facts that $\frac{\delta_0 \chi}{\delta_1} > 1$ (since $\delta_0 > \delta_1$ and $\chi > 1$) and that $\frac{x_0}{y_0} > \frac{x_1 + \cdots + x_{M-1}}{y_1 + \cdots + y_{M-1}}$ (since $\frac{x_0}{y_0} > \frac{x_m}{y_m}$ for all $m = 1, \cdots, M - 1$). □

A.4. Proof of Proposition 3

For this proof, we introduce some additional notation. Throughout the main body of the proof we assume that $v(\theta, t_i)$ depends non-trivially on θ. The other case, the affiliated private values case, is easier and dealt with at the end. When $v(\theta, t_i)$ depends on θ, it is important to calculate the expected payoff conditional on winning. If there is an atom at $\widehat{b}$ in the bid distribution, a bidder submitting $\widehat{b}$ ties with positive probability for the highest bid, in which case the winner is determined by uniform rationing. Since the probability of winning the rationing depends on the number of bidders that tie, we must take into account the information that winning conveys about θ. Let $T(n, \theta; \widehat{b})$ denote the event that the state is θ, and n (with $0 \leq n \leq N - 1$) bidders amongst $N - 1$ bidders submit bid $\widehat{b}$ and $N - 1 - n$ bidders submit a bid strictly below $\widehat{b}$. Define $P(n, \theta; \widehat{b})$ as the probability of the event $T(n, \theta; \widehat{b})$. The following lemma below determines when a large number of bidders that tie at $\widehat{b}$ is good news and when it is bad news about θ. Denote the probability distribution of Θ conditional on n other bidders tying at $\widehat{b}$ by

$$p_{\widehat{b}}(\theta \,|\, n) := \frac{P(k, \theta; \widehat{b})}{\sum\limits_{\theta \in \Theta} P(k, \theta; \widehat{b})}.$$

The lemma gives a simple criterion whether $p_{\widehat{b}}(\theta \,|\, k)$ is first-order stochastically increasing or decreasing in k. To express this condition, let $F_*^k(\widehat{b}_-) = \lim_{b \uparrow \widehat{b}} F_*^k(b)$ denote the left-hand limit of F_*^k at $\widehat{b}$ for each type $k = L, H$ so that the probability that type k bids at the atom is $\Delta^k(\widehat{b}) = F_*^k(\widehat{b}) - F_*^k(\widehat{b}_-)$.

Lemma A.7. *Probability distribution $p_{\widehat{b}}(\theta \,|\, n)$ is*

- *strictly first-order stochastically increasing in n if*

$$\left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b}) \right) F_*^L(\widehat{b}_-) > \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-) \right) \Delta^L(\widehat{b}).$$

- *strictly first-order stochastically decreasing in n if*

$$\left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b}) \right) F_*^L(\widehat{b}_-) < \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-) \right) \Delta^L(\widehat{b}).$$

35

- *independent of n if*

$$\left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b})\right) F_*^L(\widehat{b}_-) = \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-)\right) \Delta^L(\widehat{b}).$$

Proof. The proof is by noting that when $P(n, \theta; \widehat{b})$ is log-supermodular (log-submodular) in (n, θ), then $p_{\widehat{b}}(\theta | n)$ is first-order stochastically increasing (decreasing) in n. We hence investigate the properties of $P(n, \theta; \widehat{b})$:

$$P(n, \theta; \widehat{b}) = p(\theta) \binom{N-1}{n} \left(\alpha_\theta \Delta^H(\widehat{b}) + (1 - \alpha_\theta) \Delta^L(\widehat{b})\right)^n$$
$$\times \left(\alpha_\theta F_*^H(\widehat{b}_-) + (1 - \alpha_\theta) F_*^L(\widehat{b}_-)\right)^{N-n-1}.$$

Taking logarithms, and collecting into $\eta(n)$ terms that do not depend on θ and into $v(\theta)$ terms that do not depend on n, we have:

$$\ln P(n, \theta; \widehat{b}) = \eta(n) + v(\theta) + n \left[\ln\left(\alpha_\theta \left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b})\right) + \Delta^L(\widehat{b})\right)\right.$$
$$\left. - \ln\left(\alpha_\theta \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-)\right) + F_*^L(\widehat{b}_-)\right)\right]$$
$$= \eta(n) + v(\theta) + n \ln\left[\frac{\alpha_\theta \left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b})\right) + \Delta^L(\widehat{b})}{\alpha_\theta \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-)\right) + F_*^L(\widehat{b}_-)}\right].$$

Since the expression in the bracket on the bottom line above is strictly increasing (decreasing) in α_θ if

$$\left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b})\right) F_*^L(\widehat{b}_-) > (<) \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-)\right) \Delta^L(\widehat{b})$$

and since α_θ is increasing in θ by affiliation, the claim follows. $\qquad\square$

Since winning a rationing event is more likely when n is small, winning is good news on θ whenever $p_{\widehat{b}}(\theta | n)$ is stochastically decreasing in n, and vice versa. Following this reasoning, the next lemma determines whether a small over- or underbidding from an atom increases or decreases the payoff conditional on winning. Let $W_k(b)$ denote the expected value of the object conditional on winning with bid b and with signal $k = L, H$. We have:

Lemma A.8. *Let $\widehat{b}$ be atom of at least one of the bidding distributions. Then:*

- *If*

$$\left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b})\right) F_*^L(\widehat{b}_-) > \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-)\right) \Delta^L(\widehat{b}),$$

we have:

$$\lim_{b \downarrow \widehat{b}} W_k(b) > W_k(\widehat{b}) > \lim_{b \uparrow \widehat{b}} W_k(b).$$

- *If*

$$\left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b})\right) F_*^L(\widehat{b}_-) < \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-)\right) \Delta^L(\widehat{b}),$$

  we have

$$\lim_{b \downarrow \widehat{b}} W_k(b) < W_k(\widehat{b}) < \lim_{b \uparrow \widehat{b}} W_k(b).$$

- *If*

$$\left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b})\right) F_*^L(\widehat{b}_-) = \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-)\right) \Delta^L(\widehat{b}),$$

  we have

$$\lim_{b \downarrow \widehat{b}} W_k(b) = W_k(\widehat{b}) = \lim_{b \uparrow \widehat{b}} W_k(b).$$

Proof. Denote by $V_{\widehat{b}}(n;k)$ the expected value of the object conditional on n other bidders bidding $\widehat{b}$:

$$V_{\widehat{b}}(n;k) = \sum_{\theta \in \Theta} p_{\widehat{b}}(\theta \,|\, n)\, v(\theta, k).$$

By bidding $b = \widehat{b}$, a bidder wins with probability $\frac{1}{n+1}$ if there is a tie with n other bidders. Therefore, conditional on winning, the probability of tying with n other bidders is given by:

$$\frac{\frac{1}{n+1} p_{\widehat{b}}(n)}{\sum\limits_{n=0}^{N-1} \frac{1}{n+1} p_{\widehat{b}}(n)}, \quad n = 0, \cdots, N-1,$$

where $p_{\widehat{b}}(n)$ is the marginal probability of tying with n others at bid $\widehat{b}$. Hence

$$W_k(\widehat{b}) = \sum_{n=0}^{N-1} \frac{\frac{1}{n+1} p_{\widehat{b}}(n)}{\sum\limits_{n=0}^{N-1} \frac{1}{n+1} p_{\widehat{b}}(n)} V_{\widehat{b}}(n;k).$$

By bidding slightly above $\widehat{b}$, a bidder wins against all bidders who pool at $\widehat{b}$, so that winning conveys no additional information on n. Conditional on winning, the probability of n bidders bidding $\widehat{b}$ is hence given by

$$p_{\widehat{b}}(n), \quad n = 0, \cdots, N-1$$

and therefore

$$\lim_{b \downarrow \widehat{b}} W_k(b) = \sum_{n=0}^{N-1} p_{\widehat{b}}(n) \, V_{\widehat{b}}(n; k).$$

By bidding slightly below $\widehat{b}$, a bidder wins only if there is no bidder who bids $\widehat{b}$, and hence

$$\lim_{b \uparrow \widehat{b}} W_k(b) = V_{\widehat{b}}(0; k).$$

Since the probability distribution $\left(p_{\widehat{b}}(0), ..., p_{\widehat{b}}(N-1)\right)$ first-order stochastically dominates (strictly) the distribution $\left(\dfrac{p_{\widehat{b}}(0)}{\sum_{n=0}^{N-1} \frac{1}{n+1} p_{\widehat{b}}(n)}, ..., \dfrac{\frac{1}{N} p_{\widehat{b}}(N-1)}{\sum_{n=0}^{N-1} \frac{1}{n+1} p_{\widehat{b}}(n)}\right)$, which in turn strictly dominates the distribution $(1, 0, ..., 0)$, we have

$$\lim_{b \downarrow \widehat{b}} W_k(b) > (<) \, W_k(\widehat{b}) > (<) \lim_{b \uparrow \widehat{b}} W(b)$$

if $V_{\widehat{b}}(n; k)$ is strictly increasing (decreasing) in n, and

$$\lim_{b \downarrow \widehat{b}} W_k(b) = W_k(\widehat{b}) = \lim_{b \uparrow \widehat{b}} W_k(b)$$

if $V_{\widehat{b}}(n; k)$ does not depend on n. By Lemma A.7, $V_{\widehat{b}}(n; k)$ is strictly increasing (decreasing) in n if

$$\left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b})\right) F_*^L(\widehat{b}_-) > (<) \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-)\right) \Delta^L(\widehat{b})$$

and independent of n if the above holds as equality, and hence the result follows. $\qquad\square$

The next lemma shows that the lowest bid in the support of the bids is made by the low-type bidders only and that it results in a zero payoff.

Lemma A.9. *The lowest symmetric equilibrium bid is $V_L(0)$ and it is in the support of the low-type bidders. High-type bidders do not have an atom at $V_L(0)$. As an implication, equilibrium payoff is zero for the low type.*

Proof. Suppose first that there is no mass point at the lowest bid $\underline{b}$. Then the probability of winning at $\underline{b}$ is zero and hence the expected payoff is also zero. It is not possible that $\underline{b} < V_L(0)$, since a slight overbidding would lead to strictly positive payoffs. It is also not possible that $\underline{b}$ is in the support of H but not L and that $\underline{b} < V_H(N-1)$ since winning at any bid $\underline{b} + \varepsilon$ would imply that all the bidders are of type H and there would be a profitable deviation for H. A bidder of type L never bids above $V_L(N-1) < V_H(N-1)$ in equilibrium. To see that it is not possible that $\underline{b}$ is in both supports, it is enough to

observe that the value of the object conditional on winning is strictly higher to H than to L. Hence they cannot both earn zero expected profit.

The same argument shows that both players cannot have a mass point at $\underline{b}$. The lowest bid $\underline{b}$ cannot have a mass point for low-type bidders with $\underline{b} > V_L(0)$ since that would lead to an expected loss. Hence the claim of the lemma follows. $\qquad \square$

Lemma A.10. *Mass points are possible only at $V_L(0)$.*

Proof. Suppose that there is a mass point at some $\widehat{b} > V_L(0)$. If

$$\left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b}) \right) F_*^L(\widehat{b}_-) > \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-) \right) \Delta^L(\widehat{b}),$$

then by Lemma A.8 the value of the object conditional on winning jumps upwards by bidding slightly above $\widehat{b}$. Since also the probability of winning increases by overbidding, this is a strictly profitable deviation for any bidder bidding $\widehat{b}$.

If

$$\left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b}) \right) F_*^L(\widehat{b}_-) < \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-) \right) \Delta^L(\widehat{b}),$$

then $\Delta^L(\widehat{b}) > 0$ so that a low type must be bidding $\widehat{b}$ with a positive probability. By Lemma A.9, the payoff for the low type is zero, and hence the value of the object conditional on winning at $\widehat{b}$ must be zero for the low type. By Lemma A.8 a slight underbidding would increase the value conditional on winning above zero, which would then be a profitable deviation for the low type bidder.

The only case left is if

$$\left(\Delta^H(\widehat{b}) - \Delta^L(\widehat{b}) \right) F_*^L(\widehat{b}_-) = \left(F_*^H(\widehat{b}_-) - F_*^L(\widehat{b}_-) \right) \Delta^L(\widehat{b}),$$

so that the expected value of the object does not depend on the number of tying bidders. Since the low type has a zero expected profit, the high type makes a strictly positive expected profit at $\widehat{b}$. But overbidding increases discretely the probability of winning without affecting the value conditional on winning, and so bidding $\widehat{b} + \varepsilon$ for ε small enough is a profitable deviation for the high type.

Obviously there cannot be a mass point at some $\widehat{b} < V_L(0)$ since overbidding would be strictly optimal for both types. $\qquad \square$

Lemma A.11. *The support of the low-type bidders cannot have connected components of positive length.*

Proof. Suppose to the contrary that there is such a component and suppose that it is not in the support of the high-type bidder. Then winning at a higher bid implies a lower

expected value and this is not compatible with the zero profit requirement in either a first-price or a second price auction.

Consider next the possibility of overlapping connected components for the two types. In the second-price auction, the bid in a symmetric equilibrium must be the value of the object conditional on tying for the winning bid (otherwise a deviation either up or down would be strictly optimal). This cannot be the same for the two types of bidders.

In the first-price auction, write the payoff of type $k = L, H$ who bids b as

$$U_k(b) = \sum_{n=0}^{N-1} p_k(n) \left(F_*^H(b) \right)^n \left(F_*^L(b) \right)^{N-n-1} (V_k(n) - b).$$

If the bidding supports overlap, then we must have

$$\frac{\partial U_k(b)}{\partial b} = 0$$

for $k = H, L$. We can write the derivative of the payoff function as:

$$
\begin{aligned}
\frac{\partial U_k(b)}{\partial b} &= \sum_{n=0}^{N-1} p_k(n) \left(F_*^H(b) \right)^n \left(F_*^L(b) \right)^{N-n-1} \\
&\quad \times \left[\left(n \frac{f_*^H(b)}{F_*^H(b)} + (N-n-1) \frac{f_*^L(b)}{F_*^L(b)} \right) (V_k(n) - b) - 1 \right]. \quad \text{(A.15)}
\end{aligned}
$$

As a first step towards showing that the supports cannot overlap, we show that there cannot be an interval immediately above $V_L(0)$, where both types have a positive density. Let $\underline{b} := V_L(0)$, and note that by the previous Lemmas we have $F_*^L(\underline{b}) > 0$ and $F_*^H(\underline{b}) = 0$. Then, evaluating (A.15) at $\underline{b}$, we see that all of the terms with $n \geq 2$ vanish, and we are left with

$$
\begin{aligned}
\frac{\partial U_k(\underline{b})}{\partial b} &= p_k(0) \left(F_*^L(\underline{b}) \right)^{N-1} \left[(N-1) \frac{f_*^L(\underline{b})}{F_*^L(\underline{b})} (V_k(0) - \underline{b}) - 1 \right] \\
&\quad + p_k(1) \left(F_*^L(\underline{b}) \right)^{N-2} f_*^H(\underline{b}) (V_k(1) - \underline{b}) \\
&= \left(F_*^L(\underline{b}) \right)^{N-2} p_k(0) \left[(N-1) f_*^L(\underline{b}) (V_k(0) - \underline{b}) - F_*^L(\underline{b}) \right] \\
&\quad + \left(F_*^L(\underline{b}) \right)^{N-2} p_k(1) f_*^H(\underline{b}) (V_k(1) - \underline{b}).
\end{aligned}
$$

Noting that $V_k(1) > V_k(0)$, $V_H(0) > V_L(0)$ and $\frac{p_H(1)}{p_H(0)} > \frac{p_L(1)}{p_L(0)}$, we have

$$\frac{\partial U_L(\underline{b})}{\partial b} = 0 \quad \Longrightarrow \quad \frac{\partial U_H(\underline{b})}{\partial b} > 0,$$

so it is not possible to have a connected component $(V_L(0), V_L(0) + \varepsilon)$ where both types are indifferent.

As a second step, we will rule out overlapping components strictly above $V_L(0)$. By usual arguments, the union of the two supports must be a connected set. Therefore, if the low type is active for $b' > V_L(0)$, there must be a region between $V_L(0)$ and b', where only the high type has a positive density. We will now show that if the high type has a positive density, the value of the low type is strictly decreasing. Since we already know that $U_L(V_L(0)) = 0$, this rules out the possibility that the low type is active for any $b' > V_L(0)$.

Suppose that only the high type has a positive density at b, i.e., $f_*^H(b) > 0$ and $f_*^L(b) = 0$. Then

$$\frac{\partial U_k(b)}{\partial b} = \sum_{n=0}^{N-1} p_k(n) \left(F_*^H(b) \right)^n \left(F_*^L(b) \right)^{N-n-1} \left(\frac{n f_*^H(b)}{F_*^H(b)} (V_k(n) - b) - 1 \right).$$

If the high-type has a positive density, we must have

$$\frac{\partial U_H(b)}{\partial b} = \sum_{n=0}^{N-1} p_H(n) \left(F_*^H(b) \right)^n \left(F_*^L(b) \right)^{N-n-1} \left(\frac{n f_*^H(b)}{F_*^H(b)} (V_H(n) - b) - 1 \right) = 0.$$

Noting that $\frac{n f_*^H(b)}{F_*^H(b)} (V_H(n) - b)$ is increasing in n, we see that

$$p_H(n) \left(F_*^H(b) \right)^n \left(F_*^L(b) \right)^{N-n-1} \left(\frac{n f_*^H(b)}{F_*^H(b)} (V_H(n) - b) - 1 \right)$$

is single crossing in n. Since $\frac{p_L(n)}{p_H(n)}$ is strictly decreasing in n, the single-crossing lemma implies that

$$\sum_{n=0}^{N-1} \frac{p_L(n)}{p_H(n)} \cdot p_H(n) \left(F_*^H(b) \right)^n \left(F_*^L(b) \right)^{N-n-1} \left(\frac{n f_*^H(b)}{F_*^H(b)} (V_H(n) - b) - 1 \right) < 0.$$

Moreover, since $V_L(n) < V_H(n)$ for all n, this implies that

$$\frac{\partial U_H(b)}{\partial b} = \sum_{n=0}^{N-1} p_L(n) \left(F_*^H(b) \right)^n \left(F_*^L(b) \right)^{N-n-1} \left(\frac{n f_*^H(b)}{F_*^H(b)} (V_L(n) - b) - 1 \right) < 0,$$

and hence the value of the low type must be negative for any $b > V_L(0)$. □

Lemma A.12. *In a symmetric equilibrium of the standard second-price auction, low-type bidders all bid $V_L(0)$ and the high-type bidders randomize using an atomless distribution on $[V_H(0), \mathbb{E}[v(\theta, t_i) | t_i = H, Y_i \geq 1]]$. In a symmetric equilibrium of the first-price auction, low-type bidders all bid $V_L(0)$ and the high-type bidders randomize using an atomless distribution on $[V_L(0), \mathbb{E}[v(\theta, H) | t = H] - p_H(0)(V_H(0) - V_L(0))]$.*

Proof. Lemmas A.7 $\sim$ A.11 imply that the low bidders must have a degenerate distribution at the lowest point and that the high-type bidders must play according to an atomless mixed strategy. The support of the high-type bidders distribution is uniquely pinned down by the constant profit condition in both cases. □

Lemma A.12 establishes the uniqueness of a symmetric equilibrium under the assumption, maintained up to this point, that $v(\theta, t)$ depends non-trivially on θ. The case of affiliated private values, where $v(\theta, t) = v(t)$, is easier since no pay-off relevant information can be obtained by the outcome of a rationing event at a mass point. Lemma A.8 does not hold since with private valuations we must have

$$\lim_{b \downarrow \hat{b}} W_k(b) = W_k(\hat{b}) = \lim_{b \uparrow \hat{b}} W_k(b)$$

for any atom $\hat{b}$. This affects the statement of Lemma A.10, according to which no atoms above $V_L(0)$ can exist. It is easy to show that with private valuations, the unique equilibrium in the case of second-price auction involves two atoms: both types bid their own value with probability 1. The nature of the unique equilibrium in the first-price auction is unchanged.

A.5. Probability of misallocation for finite number of bidders

Rewrite equation (6) as:

$$\left[\alpha_1 F_*^H(b) + (1 - \alpha_1) F_*^L(b)\right]^{N-1} = b\gamma_1,$$
$$\left[\alpha_0 F_*^H(b) + (1 - \alpha_0) F_*^L(b)\right]^{N-1} = b\gamma_0,$$

or

$$\alpha_1 F_*^H(b) + (1 - \alpha_1) F_*^L(b) = (b\gamma_1)^{1/(N-1)},$$
$$\alpha_0 F_*^H(b) + (1 - \alpha_0) F_*^L(b) = (b\gamma_0)^{1/(N-1)}.$$

42

Solving for the distribution functions, we have

$$F_*^H(b) = b^{\frac{1}{N-1}}\left[\frac{(1-\alpha_0)(\gamma_1)^{\frac{1}{N-1}}-(1-\alpha_1)(\gamma_0)^{\frac{1}{N-1}}}{\alpha_1-\alpha_0}\right] = \widehat{\Gamma}_H b^{\frac{1}{N-1}}$$

$$F_*^L(b) = b^{\frac{1}{N-1}}\left[\frac{\alpha_1(\gamma_0)^{\frac{1}{N-1}}-\alpha_0(\gamma_1)^{\frac{1}{N-1}}}{\alpha_1-\alpha_0}\right] = \widehat{\Gamma}_L b^{\frac{1}{N-1}},$$

where $\widehat{\Gamma}_H$ and $\widehat{\Gamma}_L$ represent the bracketed terms. By setting $F_*^L\left(\overline{B}_L\right) = 1$, we can find the upper bound of the overlapping support $\overline{B}_L$: $\overline{B}_L = (\widehat{\Gamma}_L)^{-N+1}$.

Let $n = \{0, 1, \cdots, N\}$ denote the realized number of high type bidders. Then for each n, we can derive the distribution function of the highest bid among high types and among low types as follows:

$$\Gamma_H(b;n) = \Pr(\text{highest bid of type } H \text{ below b} \mid n) = \left(F_*^H(b)\right)^n = \left(\widehat{\Gamma}_H\right)^n b^{\frac{n}{N-1}}$$

and

$$\Gamma_L(b;n) = \Pr(\text{highest bid of type } L \text{ below b} \mid n) = \left(F_*^L(b)\right)^{N-n} = \left(\widehat{\Gamma}_L\right)^{N-n} b^{\frac{N-n}{N-1}}.$$

With these two distributions, we can compute the probability of inefficient allocation conditional on n as

$$\begin{aligned}
\Pr(\text{low type wins} \mid n) &= \int_0^{\overline{B}_L} \Gamma_L'(b;n)\,\Gamma_H(b;n)\,db \\
&= \frac{N-n}{N-1}\int_0^{\overline{B}_L} b^{-\frac{n-1}{N-1}}(\widehat{\Gamma}_L)^{N-n}\cdot b^{\frac{n}{N-1}}(\widehat{\Gamma}_H)^n db \\
&= \frac{N-n}{N}\left(\frac{\widehat{\Gamma}_H}{\widehat{\Gamma}_L}\right)^n,
\end{aligned}$$

where the bottom line follows from $\overline{B}_L = (\widehat{\Gamma}_L)^{-N+1}$. The probability of misallocation conditional on state is then

$$\Pr(\text{low type wins} \mid \theta) = \sum_{n=0}^{N}\binom{N}{n}(\alpha_\theta)^n(1-\alpha_\theta)^{N-n}\frac{N-n}{N}\left(\frac{\widehat{\Gamma}_H}{\widehat{\Gamma}_L}\right)^n.$$

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