Aggregation for general populations without continuity or completeness

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Abstract

We generalize Harsanyi's social aggregation theorem. We allow the population to be infinite, and merely assume that individual and social preferences are given by strongly independent preorders on a convex set of arbitrary dimension. Thus we assume neither completeness nor any form of continuity. Under Pareto indifference, the conclusion of Harsanyi's theorem nevertheless holds almost entirely unchanged when utility values are taken to be vectors in a product of lexicographic function spaces. The addition of weak or strong Pareto has essentially the same implications in the general case as it does in Harsanyi's original setting.

Keywords. Harsanyi's utilitarian theorem; discontinuous preferences; incomplete preferences; infinite populations.

JEL Classification. D60, D63, D81.

Suppose X is the set of lotteries on a finite set of consequences involving a fixed finite population, and suppose also that individual and social preference on X satisfies the expected utility axioms. The social aggregation theorem of Harsanyi (1955) then shows that if Pareto indifference holds, then social preference can be represented by the weighted sum of individual expected utilities. If, in addition, weak [respectively strong] Pareto holds, then the weights can be chosen to be nonnegative [strictly positive].

It is natural to try to generalize this celebrated result in four ways. First, the completeness axiom of expected utility has been heavily criticized, and it is clearly desirable to drop it. Moreover, even if individual preference is assumed complete, it is still far from clear why social preference should be too. Second, the continuity axiom (or axioms) of expected utility is typically regarded as a 'merely technical' axiom, and there are well known cases in which it is far from compelling. Furthermore, it is arguable that once one drops completeness, the case against continuity becomes even stronger. Third, it is of at least theoretical interest to allow the population to be infinite; indeed, it has been argued that one should assign positive probability to the claim that it *is* infinite. Fourth, modelling uncertainty via objective risk on a finite outcome set severely restricts the ways in which uncertainty can be represented.

We present two results that show that provided one accepts well known lexicographic forms of expected utility representation, Harsanyi's result holds almost entirely unchanged when instead of the expected utility axioms we assume only strong independence; when we allow the population to be infinite; and when we merely assume that X is a convex set of possibly infinite dimension. As we elaborate on below, this allows for a vast range of models of uncertainty.

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1 Harsanyi's theorem

We first present Harsanyi's original theorem in a way that facilitates generalization. Recall that a function u on a convex set X is *mixture preserving* if $u(\alpha x + (1-\alpha)x') = \alpha u(x) + (1-\alpha)x'$ for $\alpha \in (0,1)$.

Original Harsanyi Assumptions. Let $X = \Delta(C)$ be the set of lotteries on a finite set C. Let $\mathbb{I}$ be a finite nonempty set, and for each $i \in \mathbb{I} \sqcup \{0\}$,[1] suppose $\succsim_i$ is a complete, continuous, and strongly independent preorder on X. For each $i \in \mathbb{I} \sqcup \{0\}$, let $u_i : X \to \mathbb{R}$ be a mixture preserving function that represents $\succsim_i$. Set $V_{\mathbb{I}} := \prod_{i \in \mathbb{I}} \mathbb{R}$. Define $u : X \to V_{\mathbb{I}}$ by $u(x) = (u_i(x))_{i \in \mathbb{I}}$.

In applications, $\mathbb{I}$ is typically a population of individuals, C a set of social consequences, $\succsim_i$ an individual preference relation for each $i \in \mathbb{I}$, and $\succsim_0$ the social preference relation. The u_i's are standard vNM utility functions representing individual and social preference, and $u(x)$ is then the expected utility profile under the social lottery $x \in X$.

We can now define various Pareto axioms.

Pareto indifference (PI). $x \sim_i x'$ for all $i \in \mathbb{I} \Rightarrow x \sim_0 x'$.

Weak pareto (WP). $x \succsim_i x'$ for all $i \in \mathbb{I} \Rightarrow x \succsim_0 x'$.

Strong pareto (SP). $x \succsim_i x'$ for all $i \in \mathbb{I}$ and $x \succ_j x'$ for some $j \in \mathbb{I} \Rightarrow x \succ_0 x'$.

Clearly

$$\text{SP} \,\&\, \text{PI} \Rightarrow \text{WP} \Rightarrow \text{PI}. \tag{1}$$

Theorem 1 (Harsanyi)**.** *Make the original Harsanyi assumptions. Then PI holds if and only if there exists a linear $L : (V_{\mathbb{I}}, \geq_P) \to \mathbb{R}$ such that $L \circ u$ represents $\succsim_0$. Any such function L is [strictly] positive if and only if WP [SP] holds.*

Here, $L \circ u$ represents $\succsim_0$ if $x \succsim_0 x' \Leftrightarrow (L \circ u)(x) \geq (L \circ u)(x')$ for all $x, x' \in X$; and L is positive if $L(v) \geq 0$ whenever $v_i \geq 0$ for all $i \in \mathbb{I}$, and strictly positive if it is positive and $L(v) > 0$ whenever $v_i \geq 0$ for all i, and $v_j > 0$ for some j.

2 General Harsanyi theorems

2.1 Lexicographic expected utility

When a strongly independent preorder on a convex set violates continuity, it cannot be given a standard expected utility representation. But without continuity, one can still obtain most of expected utility provided one allows utility values in higher dimensional spaces than $\mathbb{R}$. This was shown by Hausner and Wendel (1952) in the case of strongly independent complete preorders on vector spaces. Here we summarize an extension of this approach to possibly incomplete, strongly independent preorders on arbitrary convex sets.

Recall that $(V, \succsim_V)$ is a preordered vector space when V is a vector space, and $\succsim_V$ is a preorder on V satisfying $v \succsim_V w$ and $\alpha > 0 \Rightarrow \alpha v + u \succsim_V \alpha w + u$. It is a partially ordered vector space when in addition $\succsim_V$ is a partial order, and an ordered vector space when $\succsim_V$ is a complete partial order.

Let $(\mathbb{J}, \geq_{\mathbb{J}})$ be an ordered set. The function space $\mathbb{R}^{\mathbb{J}}$ is a vector space, under pointwise function addition and scalar multiplication. Define the *lexicographic order* $\geq_{\mathrm{L}}$ on $\mathbb{R}^{\mathbb{J}}$ by $f \geq_{\mathrm{L}} g$

[1] '$\sqcup$' denotes the disjoint union.

if and only if $f = g$ or there exists a $\geq_{\mathbb{J}}$-least $j \in \mathbb{J}$ such that $f(j) \neq g(j)$, and for that j, $f(j) > g(j)$. Then $(\mathbb{R}^{\mathbb{J}}, \geq_{\mathrm{L}})$ is a *lexicographic function space.* It is a partially ordered vector space. It is not an ordered vector space unless $\geq_{\mathbb{J}}$ is a well-order. But it has a natural ordered vector subspace. Let $\mathbb{R}^{\mathbb{J}}_{\mathrm{W}}$ be the subset of $\mathbb{R}^{\mathbb{J}}$ consisting of functions f such that $\{j \in \mathbb{J} \colon f(j) \neq 0\}$ is well-ordered by $\geq_{\mathbb{J}}$. Then $(\mathbb{R}^{\mathbb{J}}_{\mathrm{W}}, \geq_{\mathrm{L}})$ is an ordered vector space; henceforth $\mathbb{R}^{\mathbb{J}}_{\mathrm{W}}$ will always be such a space.

Given a family of preordered vector spaces $(V_k, \succsim_{V_k})$ indexed by a set $\mathbb{K}$, we define the *product preorder* $\geq_{\mathrm{P}}$ on $\prod_{k\in\mathbb{K}} V_k$ by $v \geq_{\mathrm{P}} w$ if and only if $v_k \succsim_{V_k} w_k$ for all $k \in \mathbb{K}$. The product preorder is a vector preorder. If all the V_k's are partially ordered vector spaces, so is $(\prod_{k\in\mathbb{K}} V_k, \geq_{\mathrm{P}})$. Given preordered sets $(X, \succsim_X)$ and $Y, \succsim_Y)$, we say $f \colon X \to Y$ *represents* $\succsim_X$ if $x \succsim_X x' \Leftrightarrow f(x) \succsim_Y f(y)$ for all $x, x' \in X$.

Definition 2. A *canonical utility space* is any partially ordered vector space V of the form $V = (\prod_{k\in\mathbb{K}} \mathbb{R}^{\mathbb{J}_k}_{\mathrm{W}}, \geq_{\mathrm{P}})$.

This terminology is motivated by the following, which is is McCarthy *et al* (2017b, Thm. 7).

Theorem 3. *Suppose X is a nonempty convex set and $\succsim_X$ is a preorder on X. Then $\succsim_X$ is strongly independent if and only if $\succsim_X$ is represented by a mixture preserving function $u \colon X \to V$, where V is a canonical utility space. Moreover, u can be chosen to be minimal.*

Here, $u \colon X \to (\prod_{k\in\mathbb{K}} \mathbb{R}^{\mathbb{J}_k}_{\mathrm{W}}, \geq_{\mathrm{P}})$ is *minimal* if the $\mathbb{J}_k$'s cannot be made any smaller, and the cardinality of $\mathbb{K}$ cannot be reduced, while still having the corresponding range-restriction of u represent $\succsim_X$; for details, see McCarthy *et al* (2017b).

Theorem 3 says that one obtains a generalized expected utility representation given only that X is convex and $\succsim_X$ strongly independent. The utility values can be thought of as 'matrices' of real numbers.[2] The space $\mathbb{R}^{\mathbb{J}_k}_{\mathrm{W}}$ of k-th row-vectors is lexicographically ordered, and one matrix ranks higher than another if and only if it ranks higher in each row k.

2.2 Main results

General Harsanyi Assumptions. Let X be a nonempty convex set, $\mathbb{I}$ an arbitrary nonempty set, and for each $i \in \mathbb{I} \sqcup \{0\}$, suppose $\succsim_i$ is a strongly independent preorder on X. As in Theorem 3, for each $i \in \mathbb{I} \sqcup \{0\}$, let V_i be a canonical utility space and $u_i \colon X \to V_i$ a minimal mixture preserving function that represents $\succsim_i$. Set $V_{\mathbb{I}} := \prod_{i\in\mathbb{I}} V_i$. Define $u \colon X \to V_{\mathbb{I}}$ by $u(x) = (u_i(x))_{i\in\mathbb{I}}$. Also let $Y := \mathrm{Span}(u(X) - u(X)) \subset V_{\mathbb{I}}$.

Imposing no restrictions on X beyond convexity, thus allowing it to be infinite dimensional,[3] allows for a wide range of models of uncertainty. For example, in the setting of objective risk, X may be the set of Borel probability measures on a compact metric space; in the setting of objective risk and subjective uncertainty, it may be the set of Anscombe-Aumann acts; in the setting of subjective uncertainty, it may be the set of Savage acts when those are equipped with convex structure, as in for example Ghirardoto *et al* (2003); it may be a set of simple lotteries with nonstandard probabilities; or it may be an arbitrary mixture space as in Hausner (1954).

When L is a linear map between preordered vector spaces V and W, we say L is *positive* if $v \succsim_V 0 \Rightarrow L(v) \succsim_W 0$, and *strictly positive* if it is positive and $v \succ_V 0 \Rightarrow L(v) \succ_W 0$.

[2]These 'matrices' need not have rows of the same length. But if one does not require minimality, one can make them of the same length by choosing every $\mathbb{J}_i$ in the family to be the same ordered set $\mathbb{J}$. This is because there is an ordered set $\mathbb{J}$ containing every $\mathbb{J}_i$ as a subset, making each $\mathbb{R}^{\mathbb{J}_i}_{\mathrm{W}}$ a subspace of $\mathbb{R}^{\mathbb{J}}_{\mathrm{W}}$.

[3]The dimension of X is the dimension of the smallest affine space that contains it; equivalently, the dimension of the vector space $\mathrm{Span}(X - X)$.

Theorem 4 (General Harsanyi Theorem I)**.** *Make the general Harsanyi assumptions. Then PI holds if and only if there exists a linear* $L\colon (V_{\mathbb{I}}, \geq_P) \to (V_0, \succsim_{V_0})$ *such that* $L \circ u$ *represents* $\succsim_0$ *and is minimal. Any such function* L *is [strictly] positive on* Y *if and only if WP [SP] holds.*

Theorem 5 (General Harsanyi Theorem II)**.** *Make the general Harsanyi assumptions. Then WP [SP and PI] holds if and only if there exists a canonical utility space* $(V_0', \succsim_{V_0'})$ *and a [strictly] positive linear* $L\colon (V_{\mathbb{I}}, \geq_P) \to (V_0', \succsim_{V_0'})$ *such that* $L \circ u$ *represents* $\succsim_0$*.*

Except for the fact that the general results allow utility values in higher dimensional canonical utility spaces than Harsanyi's original result, the general results are almost identical to the original. Each implies that social preference can be represented by a linear transform of individual utility profiles, with [strict] positivity of the transform then guaranteed by [strong] weak Pareto. Thus, quite remarkably, the essence of Harsanyi's original theorem holds even when we drop completeness and continuity entirely, make room for a very wide range of models of uncertainty, and allow the population to be infinite.

The only real difference is that in Harsanyi's original theorem, $L \circ u$ is minimal *and* L is [strictly] positive on the whole of $V_{\mathbb{I}}$, given [strong] weak Pareto. In the first general Harsanyi theorem, we obtain minimality, but [strict] positivity only on Y; this effectively means all feasible utility profiles, but not in general the whole of $V_{\mathbb{I}}$. In the second general Harsanyi theorem, we obtain [strict] positivity on the whole of $V_{\mathbb{I}}$, but not minimality. One cannot in general guarantee both.

Remark 6. Under the general Harsanyi assumptions, let $v \mapsto v^i$ be the natural embedding of V_i in $V_{\mathbb{I}}$, so that $v^i_j = v$ if $j = i$, 0 otherwise. Given a linear mapping L on $V_{\mathbb{I}}$, define linear L_i on V_i by $L_i(v) := L(v^i)$, and set $u_i' := L_i \circ u_i$. The utilitarian appearance of Harsanyi's original theorem is often emphasized by noting that under SP and PI, its conclusion implies that social preference is represented by the 'utilitarian sum' $\sum_{i \in \mathbb{I}} u_i'$, with u_i' a mixture preserving function representing individual preference for each $i \in \mathbb{I}$. The conclusion of the second general Harsanyi theorem implies exactly the same when the population is finite and individual preference is assumed complete. Thus under this generalization, the utilitarian appearance is maintained, while allowing individual preferences to violate continuity, and social preference to violate both completeness and continuity. For incomplete individual preferences, even continuous ones, this result generally does not hold.

3 Proofs

We begin with two well-known observations about convex sets, cones and vector preorders; see e.g. McCarthy *et al* (2017a) for proofs.

Lemma 7. *Let* X *be a nonempty convex set. Then* $\mathrm{Span}(X - X) = \{\lambda(x - x') : \lambda > 0, x, x' \in X\}$.

Given a vector space V, recall that $C \subset V$ is a *convex cone* if $0 \in C$, $C + C \subset C$ and $\alpha C \subset C$ for all $\alpha > 0$. Clearly C is then a convex cone in any vector space containing V, and $C + C = C$ and $\alpha C = C$ for all $\alpha > 0$.

Lemma 8. *Let* $C \subset V$*, where* V *is a vector space. The relation binary relation* $\succsim$ *on* V *defined by* $v \succsim w \Leftrightarrow v - w \in C$ *is a vector preorder if and only if* C *is a convex cone. Conversely, any vector preorder on* V *is of this form.*

If $\succsim$ is a vector preorder on a vector space V, we set $C_{\succsim} := \{v \in V : v \succsim 0\}$. If $\succsim, \succsim'$ are vector preorders on the same vector space, we call $\succsim'$ a *vector preorder extension of* $\succsim$ if $\succsim \subset \succsim'$ and $\succ \subset \succ'$.

Lemma 9. *Let $\succsim, \succsim'$ be vector preorders on a vector space V.*
(a) The smallest vector preorder $\succsim''$ containing (as subsets) both $\succsim$ and $\succsim'$ satisfies $C_{\succsim''} = C_{\succsim} + C_{\succsim'}$.
(b) Let $Y \subset V$ be a subspace and suppose $C_{\succsim} \cap Y \subset C_{\succsim'} \subset Y$. Let $\succsim''$ be as in (a). Then
(i) $C_{\succsim'} = C_{\succsim''} \cap Y$, so for any $y, z \in Y$ we have $y \succsim' z \Leftrightarrow y \succsim'' z$, and $y \succ' z \Leftrightarrow y \succ'' z$.
(ii) Suppose $y \succ 0 \Rightarrow y \succ' 0$ for all $y \in Y$. Then $v \succ 0 \Rightarrow v \succ'' 0$ for all $v \in V$, implying that $\succsim''$ is a vector preorder extension of $\succsim$.

Proof. (a) If $\succsim, \succsim', \succsim''$ are vector preorders on V with $\succsim, \succsim' \subset \succsim''$, then $v \in C_{\succsim}$, $v' \in C_{\succsim'}$ implies $v + v' \succsim'' 0$, so $C_{\succsim} + C_{\succsim'} \subset C_{\succsim''}$. But $C_{\succsim} + C_{\succsim'}$ is clearly a convex cone, so setting $C_{\succsim''} := C_{\succsim} + C_{\succsim'}$ makes $\succsim''$ the smallest vector preorder containing $\succsim$ and $\succsim'$.

(b)(i) Clearly $C_{\succsim'} \subset C_{\succsim''} \cap Y$. Conversely, suppose $y'' \in C_{\succsim''} \cap Y$. Then $y'' = y + y'$ for some $y \in C_{\succsim}$, $y' \in C_{\succsim'} \subset Y$. But $y = y'' - y' \in Y$, hence $y \in C_{\succsim} \cap Y \subset C_{\succsim'}$, so $y'' \in C_{\succsim'} + C_{\succsim'} = C_{\succsim'}$. Clearly $y \succsim' z \Leftrightarrow y \succsim'' z$, and $y \succ' z \Leftrightarrow y \succ'' z$ for $y, z \in Y$.

(ii) Suppose $y \succ 0 \Rightarrow y \succ' 0$ for all $y \in Y$ $(*)$ and that $v \succ 0$ for some $v \in V$. Clearly $v \succsim'' 0$, so suppose for a contradiction that $0 \succsim'' v$, or equivalently, $-v \in C_{\succsim''}$. Then $-v = w + w'$ for some $w \in C_{\succsim}, w' \in C_{\succsim'}$. This implies $-w' = v + w \in C_{\succsim}$; that is, $-w' \succsim 0$, so $-w' \succsim'' 0$, hence $w' \sim'' 0$. By (i), $w' \sim' 0$. Consequently, $w' \sim 0$ (as $-w' \succ 0 \Rightarrow -w' \succ' 0$ by $(*)$), so $-v = w + w' \in C_{\succsim}$. This implies $0 \succsim v$, contradicting $v \succ 0$, hence $v \succ'' 0$. □

The following is proved in McCarthy *et al* (2017b, Thm. 12).

Theorem 10. *Let X be a nonempty set. Let Y, Z be vector spaces and let $f : X \to Y$, $g : X \to Z$ be such that $(f, g)(X) := \{(f(x), g(x)) : x \in X\}$ is convex. Then we have $g(x) = g(x') \implies f(x) = f(x')$ for all $x, x' \in X$ if and only if $f = L \circ g + y_0$ for some linear $L : Z \to Y$ and some $y_0 \in Y$.*

Proof of Theorem 4. Observe that $(u_0, u)(X)$ is convex. Use Theorem 10 to obtain a linear $L\colon (V_{\mathbb{I}}, \geq_{\mathrm{P}}) \to (V_0, \succsim_{V_0})$ and a constant $v_0 \in V_0$ such that $u_0 = L \circ u + v_0$. Then for $x, y \in X$, $x \succsim_0 y \Leftrightarrow u_0(x) \succsim_{V_0} u_0(y) \Leftrightarrow (L \circ u)(x) + v_0 \succsim_{V_0} (L \circ u)(y) + v_0 \Leftrightarrow (L \circ u)(x) \succsim_{V_0} (L \circ u)(y)$, so $L \circ u$ represents $\succsim_0$. Moreover, $L \circ u$ differs from u_0 by a constant, so $L \circ u$ is minimal.

Now let $L\colon (V_{\mathbb{I}}, \geq_{\mathrm{P}}) \to (V_0, \succsim_{V_0})$ be any linear function such that $L \circ u$ represents $\succsim_0$. Suppose L is [strictly] positive on Y. Then $x \succsim_i y \,\forall i \in \mathbb{I}$ [and $\exists j \in \mathbb{I}\colon x \succ_j y$] $\Rightarrow u(x) - u(y) \geq_{\mathrm{P}} [>_{\mathrm{P}}]\, 0 \Rightarrow L(u(x) - u(y)) \succsim_{V_0} [\succ_{V_0}]\, 0 \Rightarrow (L \circ u)(x) \succsim_{V_0} [\succ_{V_0}]\, (L \circ u)(y) \Rightarrow x \succsim_0 [\succ_0]\, y$, so WP [SP] holds.

Conversely, suppose WP [SP] holds. Suppose that for some $v \in Y$, $v \geq_{\mathrm{P}} [>_{\mathrm{P}}]\, 0$. By Lemma 7, $v = \lambda(u(x) - u(y))$ for some $x, y \in X$ and $\lambda > 0$, implying $x \succsim_i y$ for all $i \in \mathbb{I}$ [and $x \succ_j y$ for some $j \in \mathbb{I}$]. Hence by WP [SP], $x \succsim_0 [\succ_0]\, y$, implying $(L \circ u)(x) \succsim_{V_0} [\succ_{V_0}]\, (L \circ u)(y)$, hence $L(v) \succsim_{V_0} [\succ_{V_0}]\, 0$, so L is [strictly] positive on Y. □

Proof of Theorem 5. Proof of the right to left direction is analogous to the second paragraph in the proof of Theorem 4. For left to right, assume WP [SP & PI]; then PI holds by (1). Let T be the L of Theorem 4, so $T\colon (V_{\mathbb{I}}, \geq_{\mathrm{P}}) \to (V_0, \succsim_{V_0})$ is linear, and $T \circ u\colon X \to (V_0, \succsim_{V_0})$ represents $\succsim_0$.

In Lemma 9, let $V = V_{\mathbb{I}}$, $Y = \mathrm{Span}(u(X) - u(X))$ (as in the general Harsanyi assumptions), and set $\succsim := \geq_{\mathrm{P}}$. Using Lemma 8, define $\succsim'$ by setting $C_{\succsim'} = \{y \in Y : T(y) \succsim_{V_0} 0\}$, which is clearly a convex cone in $V_{\mathbb{I}}$. This defines a vector preorder $\succsim''$ on $V_{\mathbb{I}}$ by Lemma 9(a).

We claim: $C_{\succsim} \cap Y \subset C_{\succsim'} \subset Y$ $(*)$. To prove this, the righthand inclusion is obvious. For the left, suppose $y \in C_{\succsim} \cap Y$. By Lemma 7, this implies $u(x) \geq_{\mathrm{P}} u(x')$ where $y = \lambda(x - x')$ for some $x, x' \in X, \lambda > 0$. WP then implies $x \succsim_0 x'$, hence $(T \circ u)(x) \succsim_{V_0} (T \circ u)(x')$. Linearity of T implies $T(y) \succsim_{V_0} 0$, and since $y \in Y$, we have $y \in C_{\succsim'}$. This establishes $(*)$.

By the definition of $\succsim''$ we have $\geq_{\mathrm{P}} \subset \succsim''$. [By Lemma 9(b)(ii) we also find $>_{\mathrm{P}} \subset \succ''$; this uses SP and an argument similar to proof of $(*)$ to verify the condition that $y >_{\mathrm{P}} 0 \Rightarrow y \succ' 0$ for $y \in Y$.] Therefore, the identity operator $\mathrm{id}\colon (V_{\mathbb{I}}, \geq_{\mathrm{P}}) \to (V_{\mathbb{I}}, \succsim'')$ is [strictly] positive, and clearly linear. By $(*)$ and Lemma 9(b)(i) we find that $\succsim'$ and $\succsim''$ coincide on Y.

One easily shows that a strongly independent preorder on a real vector space is a vector preorder, and that a mixture preserving function between real vector spaces is linear. By Theorem 3, therefore, there is a canonical utility space $(V_0', \succsim_{V_0'})$ and a linear $L'\colon (V_{\mathbb{I}}, \succsim'') \to (V_0', \succsim_{V_0'})$ such that $v \succsim'' v' \Leftrightarrow L'(v) \succsim_{V_0'} L'(v')$ for $v, v' \in V_{\mathbb{I}}$; clearly L' is strictly positive. Setting $L := L' \circ \mathrm{id}$ we find that $L\colon (V_{\mathbb{I}}, \geq_{\mathrm{P}}) \to (V_0', \succsim_{V_0'})$ is [strictly] positive. Moreover, for $x, x' \in X$, we have $x \succsim_0 x' \Leftrightarrow T(u(x) - u(x')) \succsim_{V_0} 0 \Leftrightarrow u(x) - u(x') \succsim' 0 \Leftrightarrow u(x) - u(x') \succsim'' 0 \Leftrightarrow (\mathrm{id} \circ u)(x) \succsim'' (\mathrm{id} \circ u)(x') \Leftrightarrow (L \circ u)(x) \succsim_{V_0'} (L \circ u)(x')$, so $L \circ u$ represents $\succsim_0$. $\square$

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