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HYBRID LOTTERIES FOR FINANCING PUBLIC GOODS[1]

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We propose a new, voluntary mechanism (the "hybrid lottery") as a means for financing the provision of public goods.

We find that, under some conditions, the mechanism can mitigate the free-riding problem and that, for each player, the (weakly) dominant strategy is the one that –in equilibrium– implements the first best.

We also find that the mechanism is quite robust to modifications of the basic model, including heterogeneity in incomes and preferences, different utility functions and incomplete information.

Finally, the mechanism is "self-financed"(i.e., it never runs out of money, neither on- nor off-equilibrium path) and –because of the use of dominant strategies–it is very easy to solve by players. Thus, the mechanism is simple to implement in the real world by charities and other organisations that rely on voluntary contributions.

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1. INTRODUCTION

It is a well known result that the voluntary, private provision of public goods suffers from the free-rider problem and results in a systematic underprovision of the public good when it is socially desirable.

This paper proposes a new mechanism –namely, the "hybrid lottery"–to increase voluntary contributions. It does so by the creation of a subsidy scheme in which the average subsidy rate (ASR) is a non-linear, decreasing function of the total (gross) contributions (TGC) of the players. Thus, if a player decides to increase her contribution it decreases the subsidy enjoyed by everyone else, and for every dollar contributed (not just the marginal one). Thus, if the elasticity of the ASR with respect to the TGC is sufficiently high (at least, higher than 1), then an increase in aggregate contributions leads to a more than proportional decrease in the subsidy rate and –consequently–to a more than proportional increase in the level of public goods provided. This way, the marginal private benefit of contribution increases more than the marginal cost (as compared to the standard voluntary contribution mechanism (VCM)) and the incentives to contribute increase (free-riding decreases).

Further, under some conditions, it can implement the first best allocation. That is, it can totally eliminate the free-riding problem and thus allows for an efficient yet private provision of public goods. This is already a big advantage compared to alternative mechanisms proposed by the literature, as for example the fixed-prize lotteries proposed by Morgan (2000).

On top of the efficiency consideration, the mechanism presents some other desirable features that make it more appealing than competing mechanisms.

For example, it is an horizontally-equitable mechanism. That is, the subsidy received is the same for all those players that contribute the same amount. Further, it can be seen as both a horizontally and vertically equitable mechanism since the subsidy rate is the same for every player. This is a feature shared with Morgan (2000)'s fixed-prize lottery. The latter is, however, regressive (i.e., the fraction of income spent on lottery tickets decreases with the income of the players), while the hybrid lottery does not have to be it.

Another relevant feature is that the mechanism is "self-financed": it never "runs out of money", not even off-the-equilibrium path. This apparently minor point is not minor at all: it is one of the basic constraints present in the standard VCM (the one that demands contributions to be non-negative for each player) and is violated by several mechanisms including Morgan (2000)'s lotteries as well as all-pay auctions. It is really not a surprise that the outcome is then superior to that of the VCM's. On the other hand, the hybrid lottery –though bound by this constraint–also outperforms the standard VCM, does better than the fixed-prize lottery, and does as well as the all-pay auction.

Methodologically, the mechanism can be made "dominance-solvable". That is, the subsidy function can be chosen such that all players have a dominant strategy. This is a relevant result as it means that the cognitive load demanded from players to solve the game is minimal, making it a simple game that anyone can understand and solve. In particular, the mechanism is less cognitive-demanding than alternative mechanisms that rely on the Nash equilibrium concept, such as the provision point mechanisms (eg, Andreoni (1998)) or those that consist of two-stages (eg, Moore and Repullo (1988)). Further, in some cases, the allocation implemented is the first-best one. This means that the efficient outcome can be achieved without people having to make complicated calculations.

Finally, and as a consequence especially of the last two features: "self-sufficiency" and "dominance-solvability", we claim that it would be very easy to implement the hybrid lottery by real-world institutions –such as charities– that rely on voluntary contributions. The self-sufficiency ensures that the mechanism does not need to be called out if not enough money is raised to finance the prizes, while the dominance-solvability implies that people can easily understand the mechanism. If, on top of its simplicity, the mechanism is also efficient and equitable as mentioned above, we claim hybridlotteries can be successfully used as means for financing public goods by charities and other real-world organisations that rely on voluntary contributions.

The paper is organized as follows. A linear version of the public good game with homogeneous agents and complete information is formalized in section 2. Section 3 presents extensions and discusses the mechanism. Finally, section 4 concludes. Proofs are left for the appendix.

2. LINEAR MODEL

2.1. Setup

In an economy there are $N > 1$ homogeneous individuals indexed by $i \in I := \{1, ..., N\}$ and two goods: a private one x and a public one G. Each individual is endowed with wealth $w_i = w > 0 \; \forall i \in I$ that can be used to buy the private good and to contribute to the provision of the public good.[2] The individual budget constraint is thus given by $x_i + g_i \leq w$, where $g_i \geq 0$ is player i's contribution to the public good –measured in units of the private good– and the price of the private good is normalised to 1. The public good is produced out of the private good according to the production function $G := \sum_{i=1}^{N} g_i$.[3] As usual, individuals can consume different quantities of the private good (x_i can be different from x_j, for $i \neq j$) but everyone consumes the same quantity of the public good ($G_i = G_j = G \; \forall i, j \in I$).

[2] The heterogeneous case is analysed in the Extensions section.

[3] Capital letters will be used throughout the paper to indicate aggregate (sociey-wide) variables, while lowercase letters are used for individual variables.

Agents' preferences are given by the linear utility function

$$u(x_i, G) = x_i + \gamma G \tag{1}$$

where

$$\gamma \in \left(\frac{1}{N}, 1\right) \tag{2}$$

is known in the experimental literature as the *marginal per capita return* (MPCR) and is, more generally, the marginal rate of substitution of good G for good x.[4]

2.1.1. First Best (FB) Benchmark

Society's preferences are assumed to be Utilitarian: $\tilde{U}(\mathbf{x}, G) := \sum_{i=1}^{N} u(x_i, G)$, where $\mathbf{x}$ is the private-consumption vector of the economy: $\mathbf{x} := (x_1, ..., x_N)$. Then: $U(X, G) := X + N\gamma G$. A benevolent social planner chooses X and G to maximise social welfare, subject to, $\forall i \in I$:

$$x_i \geq 0 \tag{3}$$

$$g_i \geq 0 \tag{4}$$

$$x_i + g_i \leq w \tag{5}$$

i.e., nonnegative private consumption, nonnegative contributions and budget constraint, respectively.

The efficient, first best solution is to set $(x_i^*, g_i^*) = (0, w)\ \forall i \in I$, so that aggregate consumption is $(X^*, G^*) = (0, W)$. The utility of agent i is thus $u_i^* = x_i^* + \gamma G^* = \gamma W$.

Intuitively, the marginal social cost of contributing (i.e., the loss in utility due to lower private consumption) is $MSC = 1$ and the marginal social benefit of contributing (i.e., the gain in social welfare due to higher public consumption) is $MSB = N\gamma$.

[4] Alternative utility functions are considered in section 3, "Extensions and discussion."

Since (from equation 2) $MSB = N\gamma > 1 = MSC$, then the central planner has incentives to increase contributions as much as possible. Thus, at the optimum each player contributes her entire endowment, consumes 0 units of the private good and W units of the public good.

2.1.2. Nash equilibrium (NE)

Each individual chooses x_i and g_i to maximise her utility (given by equation 1) subject to her budget constraint and the nonnegativity conditions.

The solution is to set $(x_i^{**}, g_i^{**}) = (w, 0)$. This means that $(X^{**}, G^{**}) = (W, 0)$ and that the utility of the agent is $u_i^{**} = x_i^{**} + \gamma G^{**} = w$.

Intuitively, the marginal private cost of contributing is $MPC = 1$ and the marginal private benefit is $MPB = \gamma$. Since (from equation 2) $MPB = \gamma < 1 = MPC$, then the agent has incentives to decrease contributions as much as possible. Thus, in equilibrium each agent contributes 0, consumes w units of the private good and 0 units of the public good. In game-theoretical terms, contributing 0 is a *strictly dominant strategy*: no matter what the other players do, it is always individually optimal to contribute zero.

2.1.3. Comparison: FB v NE

The social optimum is not implemented by the decentralised, voluntary contribution mechanism (VCM). This is so even though the NE is Pareto-dominated by the FB allocation (using equation 2, $u_i^* = \gamma N w > w = u_i^{**}\ \forall i \in I$). The discrepancy is the result of the different incentive schemes (social v individual incentives): $MPB \neq MSB$ because individuals overlook the positive effect (externality) of her contribution on the wellbeing of her fellow agents and thus contributions are suboptimally low.

This is the well-known free-riding problem: the non-excludability of the public good implies that, once provided, everyone will enjoy it, regardless of whether one contributed to its provision or not. Yet, contributions are costly in terms of (foregone) private consumption, so players have incentives to keep their money and spend it on private consumption rather than contributing it, hoping that others will and so benefitting from their contributions. But since every individual is expected to reason in these terms, the end result is that the public good is under-provided as compared to the FB allocation and –in this setup– free-riding is extreme: each player contributes *zero* when it would be optimal that each one of them contributed *their entire endowments*.

Since the utility function is linear, we expect a corner solution. Thus, despite of the fact that the players could –in principle– choose any contribution $g_i \in [0, w]$, for all practical reasons we can restrict our attention to what happens when $g_i = 0$ and when $g_i = w$ and thus transform a continuous game into a discrete one. This is the type of game used by Sanchez Villalba and Martinez Gorricho (2014). In this case, the game can be interpreted as a Prisoners Dilemma in which the dominant strategy is to deviate (contribute zero) and the social optimum is reached when everyone cooperates (contributes everything).

2.2. Hybrid Lottery

2.2.1. Subsidy function

We redefine the problem by introducing a new component: a subsidy or cash rebate. Especifically, we define

$$g_i := \kappa_i - s_i := \kappa_i - s\left(K\right) \cdot \kappa_i = \left[1 - s\left(K\right)\right] \kappa_i \tag{6}$$

where $\kappa_i \in [0, w]$ is player i's *gross* contribution, $s_i := s\left(K\right) \cdot \kappa_i \in [0, \kappa_i]$ is the subsidy (cash rebate) received by player i, $s(K) \in [0, 1]$ is the subsidy rate and $K := \sum_{i=1}^{N} \kappa_i$

is the aggregate (total) gross contribution.

The subsidy function must satisfy three conditions:

$$\begin{aligned} \text{If } \kappa_i &= 0 \text{ then } s_i = 0 && (7)\\ s_i &\geq 0 && (8)\\ s_i &\leq \kappa_i \text{ always, so that } 0 \leq g_i && (9) \end{aligned}$$

The first one simply says that only contributors are entitled to subsidies. The second one that the subsidy is not a tax. And the third one that the mechanism is *self-financing*: the designer always has enough money to pay the subsidies promised, both on- and off-equilibrium.

The third condition, though quite straightforward and apparently self-evident, is however violated by popular mechanisms suggested by the literature like lotteries and auctions, in which the winner ends up paying a "negative contribution" to the public good. Thus, these mechanisms enjoy an unfair advantage over the original VCM model, and so it is not surprising that they fare better than the latter in terms of efficiency: by expanding the choice set (i.e., allowing for $g_i < 0$) things can only improve. We, on the other hand, retain the condition as presented in the original VCM model and, thus, we believe that our comparisons are fair. Actually, our method seems to do better than the VCM (in a fair comparison), but also better than Morgan's fixed-prize and pari-mutual lotteries, and at least as good as auctions in terms of promoting efficiency.

The preferences of the consumers are thus given by

$$u_i(\boldsymbol{\kappa}) = w - [1 - s(K)]\,\kappa_i + \gamma\,[1 - s(K)]\,K \tag{10}$$

2.2.2. *Gross and net contributions*

Thus, g_i can now be interpreted as player i's *net* contribution, such that the level of the public good G is still given by the sum of individual *net* contributions: $G := \sum_{i=1}^{N} g_i$. The choice variable of the optimising player is now the gross contribution κ_i, and her net contribution depends not only on the player's choice κ_i but –through the subsidy $s_i := s(K) \cdot \kappa_i$– on every player's choices K.

2.2.3. *Alternative interpretations*

Lottery with an endogenous prize The subsidy function can be also interpreted as the prize of a lottery similar to the ones presented in Morgan (2000), hence the name of "hybrid lotteries" we gave to our setting. In particular, it is like a lottery in which those who buy tickets (those whose gross contributions are positive) get a prize (the subsidy) that depends on the number of tickets bought (the amount grossly contributed). Also, the "prize per euro spent on tickets" (or "average subsidy rate"), $\frac{s_i}{\kappa_i}$, is the same for every contributor: $\frac{s_i}{\kappa_i} = \frac{s(K)\cdot\kappa_i}{\kappa_i} = s(K)$, thus treating every contributor equally (equality concern: if player i and j grossly contribute the same amount –i.e., $\kappa_i = \kappa_j$– then they receive the same subsidy –i.e., $s_i = s_j$).[5] The total prize/subsidy paid out to ticket-holders/contributors is, however, not a fixed amount like in Morgan, but one that changes with the level of aggregate gross contributions: $S := \sum_{i=1}^{N} s_i = \sum_{i=1}^{N} [s(K) \cdot \kappa_i] = s(K) \cdot K$. It is not, either, a pari-mutual lottery in which the total prize is a constant fraction of the ticket sales, because the fraction of the ticket sales that is allocated to the prize, $\frac{S}{K}$, is a function of the amount of tickets sold: $\frac{S}{K} = s(K)$. Thus, the total prize paid is *endogenous*, as it varies with the level of aggregate gross contributions K and this variation is *nonlinear* in K, so

[5] Note, however, that different players *could* get different subsidies by grossly contributing different quantities, so that the equality here is not ex-post, but ex-ante. In other words, the method ensures equality of opportunity, but allows for ex-post inequality based on the actions taken by the players.

that our lottery shares features of the two lotteries analysed by Morgan: it can be interpreted as a nonlinear subsidy function (just like the fixed-prize lotteries) and the total prize paid changes with the level of aggregate gross contributions (like in pari-mutual lotteries). Thus the "hybrid" lottery of the title.

Third commodity Another interpretation (although related to the previous one) consists in considering the existence of a third commodity in the model, which would correspond to the variable κ_i. This third commodity could be, for example, the "tickets" of the lottery and it has no intrinsic value except for their impact on the size of the prize s_i and on the level of public good provision G. The implicit price of the third commodity would therefore be $p(K) := 1 - s(K)$ and the net contribution of player i would be given by the expression $g_i := p\,(K)\,\kappa_i$. This notation will be very helpful to our analysis and so we will use both throughout the paper. The main advantage of this notation is that the net contribution can be thought as the expenditure on the third commodity and that changes on the net contribution can be decomposed into first-order, quantity effects and second-order, price effects. The preferences of consumers under this interpretation can thus be written as $u_i(\boldsymbol{\kappa}) = w - p\,(K)\,\kappa_i + \gamma \cdot p\,(K) \cdot K$.

2.2.4. Marginal analysis

An agent's problem consists in choosing her gross contribution κ_i in order to maximise her utility (equation 10) subject to her budget constraint (equation 5), the nonnegativity constraints (equations 3 and 4) and the subidy constraints (equations 7, 8 and 9). The first-order condition is thus:

$$\frac{\partial u}{\partial \kappa_i} = -MPC + MPB = -\left[1 - s(K)\left(1 - \frac{\kappa_i}{K}\left|\varepsilon_{sK}\right|\right)\right] + \gamma\left[1 - s(K)\left(1 - \left|\varepsilon_{sK}\right|\right)\right] \tag{11}$$

where $\varepsilon_{sK} := \frac{\partial s(K)}{\partial K}\frac{K}{s(K)}$ is the elasticity of the subsidy rate $s(K)$ with respect to the aggregate level of gross contributions K. Note in particular that both the MPC and the MPB are affected by the introduction of the subsidy.

Comparison to Morgan's lotteries The first-order condition found above is general enough as to encompass Morgan's lotteries as special cases. In our setting, the "prize" is the total amount paid as subsidies to the individuals and is, thus, endogenous: $P(K) := s(K)K$.

Voluntary contributions mechanisms (VCM) Morgan showed that the standard VCM game can be interpreted as a zero-prize lottery: one in which $s(K) = 0$ $\forall K \in [0, W]$ (so that $P(K) = 0\ \forall K$). The FOC then becomes: $\frac{\partial u}{\partial \kappa_i} = -1 + \gamma < 0$ and so the solution is like the one in section 2.1.2. The positive externality that one's contribution has on every other individual leads to suboptimal contributions by all.

Fixed-prize lottery (FPL) Morgan's best lottery is the one in which the prize is fixed: $P(K) = P\ \forall K$, and thus $s(K) := \frac{P}{K}$ and $|\varepsilon_{sK}| = 1$. The FOC then becomes: $\frac{\partial u}{\partial \kappa_i} = -\left[1 - s(K)\left(1 - \frac{\kappa_i}{K}\right)\right] + \gamma$ and we can see that this lottery is superior to the VCM because it decreases the MPC, thus increasing the incentives to contribute. Note, however, that the MPB is not affected. The lottery introduces a negative externality that partially mitigates the positive one (free-riding): when one player buys an extra ticket, she decreases everybody else's chances of winning the prize. Thus, people contribute more than in the VCM.

Pari-mutual lottery (PML) Morgan also explored "pari-mutual" lotteries in which the prize is a fixed proportion of the ticket sales (a very popular format) and found that they are not better than the VCM. In PMLs, $P(K) = c \cdot K$, where $c \in (0, 1)$ is a constant, and so $s(K) := \frac{P(K)}{K} = \frac{cK}{K} = c$. Thus, $|\varepsilon_{sK}| = 0$ and the FOC becomes:

$\frac{\partial u}{\partial \kappa_i} = -(1-c) + \gamma(1-c) = (1-c)(-1+\gamma) < 0$. Thus, the solution is identical to the VCM as Morgan suggested, because the PML does decrease the MPC but it also decreases the MPB in the same proportion, so that the contribution decision is not affected by the lottery. In the terminology of externalities, the PML introduces a negative externality like the FPL, but also a positive externality: when a player buys an extra ticket, the prize is increased, thus benefitting everyone. In this case, these two externalities exactly offset each other and thus the outcome is identical to the VCM one.

Hybrid Lottery The hybrid lottery shares features of the two polar cases analysed by Morgan: it can be reinterpreted as a nonlinear subsidy that is a decreasing function of the contributions (like the FPL) and its prize is not fixed (as in the PML). This flexibility allows us to design hybrid lotteries that are more efficient than the best alternative (Morgan's FPL). Further, if the subsidy rate function is appropriately chosen, $\kappa_i = w$ can become a *weakly dominant strategy*. This is the goal of the following section.

2.3. Mechanics of the mechanism

The goal of this section is to design a subsidy function such that full contribution ($\kappa_i = w \ \forall i \in I$) is a (weakly) *dominant strategy* and the level of net contributions is efficient.

Before we begin with the analysis, however, a couple of concepts need to be defined.

Definition 1. An "Iso-Public Line" (IPL) is the set of all pairs (κ_i, K_{-i}) such that the level of aggregate gross contributions K is constant. Formally, it is the set of all pairs (κ_i, K_{-i}) such that $\kappa_i + K_{-i} = a$, with $a \in [0, W]$.

COROLLARY 1. *The price of contributions* $p(K) := 1 - s(K)$ *is constant along an iso-public line.*

COROLLARY 2. *Since the amount of public good is given by* $G := p(K)K$*, then* G *is constant along an iso-public line.*

Graphically, iso-public lines are straight lines given by the equation $K_{-i} = a - \kappa_i$, $a \in [0, W]$. In figure 1 three iso-public lines are shown: the ones corresponding to the lowest possible value of K ($K = 0$), to an intermediate value ($K = w/\gamma$) and to the highest possible value ($K = W$) (the downward, parallel lines that go through the origin, the intermediate one and the one close to the upper-right corner of the diagram, respectively).[6]

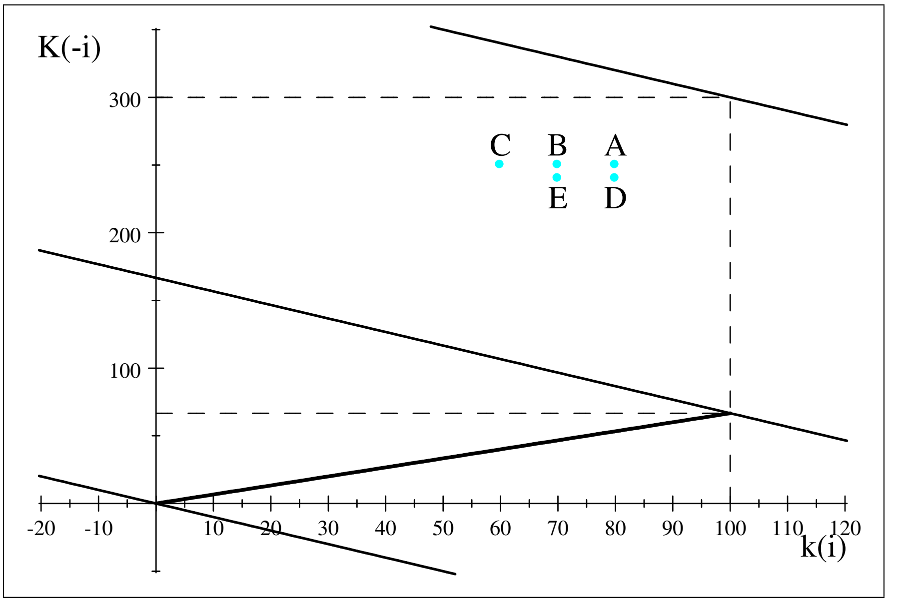

FIG. 1

[6]Variable on x-axis: player i's gross contribution κ_i; variable on y-axis: aggregate contributions of society, excluding player i's gross contributions $K_{-i} := K - \kappa_i$. Paremeters: $\gamma = 0.6$, $N = 4$, $w = 100$, $\kappa_i^A = 80$, $K_{-i}^A = 250$ and $\delta = 10$.

Definition 2. The "Irrelevance Line" is the set of all pairs (κ_i, K_{-i}) such that the utility enjoyed by player i is the same for all possible values of the price of contribution.

This occurs when $\gamma K - \kappa_i = 0$, so that the utility of the player is $u_i = w$ regardless of the value of the price of contributions (note that the utility function can be written as $u_i(\boldsymbol{\kappa}) = w + p(K)(\gamma K - \kappa_i)$, so that the price $p(K)$ is irrelevant when $\gamma K - \kappa_i = 0$). Thus, along this line, the mechanism cannot do anything to modify the situation of the player. Graphically, it is a ray that goes through the origin, has slope $\frac{1-\gamma}{\gamma}$ and takes the value $\frac{1-\gamma}{\gamma}w$ when $\kappa_i = w$ (see figure 1).

Definition 3. The "Lower (Upper) Region" is the area below (above) the irrelevance line. Formally, the set of pairs (κ_i, K_{-i}) such that $\gamma K - \kappa_i < 0$ (> 0).

Note that in the Lower Region, since $p(K) \in [0, 1]$, then the utility of the individual can be, at most, equal to w, which occurs only if contributions are free: $p(K) = 0$. Analogously, in the Upper Region, the utility of the indiviual can be, at least, equal to w, which occurs only if $p(K) = 0$.

2.4. Obtention of the price function

Proposition 1. *If aggregate gross contributions are sufficiently low, then contributions must be free. Formally,* $p(K) = 0 \ \forall K \in \left[0, \frac{w}{\gamma}\right]$.

Proof. On page 28 of the appendix. ∎

Notice that a consequence of proposition 1 is that $u_i = w$ all over the area below the $K = \frac{w}{\gamma}$ iso-public line.

Definition 4. The "Upper-Right Region" is the area above the $K = \frac{w}{\gamma}$ iso-public line. Formally, it is the set of pairs (κ_i, K_{-i}) such that $\kappa_i + K_{-i} \geq \frac{w}{\gamma}$.

Since the "upper-right region" is a subset of the "upper region", it must be true that the utility of the individual must be at least as high as w.

PROPOSITION 2. *Consider* 5 *pairs* (κ_i, K_{-i})*:* $A := \left(\kappa_i^A, K_{-i}^A\right)$*,* $B := \left(\kappa_i^B, K_{-i}^B\right)$*,* $C =: \left(\kappa_i^C, K_{-i}^C\right)$*,* $D := \left(\kappa_i^D, K_{-i}^D\right)$ *and* $E := \left(\kappa_i^E, K_{-i}^E\right)$*, such that all of them belong to the "upper-right region" and* $\kappa_i^B = \kappa_i^A - \delta$*,* $\kappa_i^C = \kappa_i^B - \delta$*,* $\kappa_i^D = \kappa_i^A$*,* $\kappa_i^E = \kappa_i^B$*,* $K_{-i}^B = K_{-i}^A$*,* $K_{-i}^C = K_{-i}^A$*,* $K_{-i}^D = K_{-i}^A - \delta$ *and* $K_{-i}^E = K_{-i}^D$*, with* $\delta > 0$*. Then:*

If (1) $u_i^A > u_i^B$ *and* (2) $u_i^D > u_i^E$*, then* $u_i^A > u_i^C$*.*

Proof On page 29 in the appendix.

Graphically, the 5 pairs are located as shown in figure 1 and are such that B and D belong to the same iso-public line ($K^B = K^D$), as do C and E ($K^C = K^E$). If we consider that the gross contribution κ_i is a discrete variable, then the parameter δ can be interpreted as the difference between two consecutive levels of contribution. The continuous case is obtained by simply assuming that $\delta \to 0$.

The proposition can be interpreted as a sort of "transitivity" feature, so that the following corollary can be stated:

COROLLARY 3. *If* $u(w, K_{-i}) > u(w - \delta, K_{-i})\ \forall K_{-i} \in \left(\frac{1-\gamma}{\gamma}w, W_{-i}\right]$*, then* $u(w, K_{-i}) > u(\kappa_i, K_{-i})\ \forall K_{-i} \in \left(\frac{1-\gamma}{\gamma}w, W_{-i}\right], \kappa_i \in [0, w)$

Proof. Follows directly from the repeated application of proposition 2. ∎

This corollary implies that if full contribution $\kappa_i = w$ is better than the immediately inferior level of contribution $\kappa_i = w - \delta$, then full contribution is a *dominant strategy.* Thus, to achieve our goal of ensuring that full contribution is a weak dominant strategy of the game it is enough to design a price (subsidy rate) function $p(K)$ ($s(K)$) such that the utility of full contribution is weakly greater than the utility of contributing $w - \delta$.

2.4.1. Weak dominance

Remark 1. There are infinite price (subsidy rate) functions that satisfy the conditions of corollary 3.

Table 1 shows an example of one of those price functions that makes full contributions a weakly dominant strategy.[7]

		K_{-i}			
		300	200	100	0
κ_i	100	45 + 132 = 177	60 + 72 = 132	75 + 30 = 105	100 + 0 = 100
	0	100 + 72 = 172	100 + 30 = 130	100 + 0 = 100	100 + 0 = 100

TABLE 1
Dominance-solvable game

Game-theoretically, the price function can transform the Prisoners' Dilemma of the original VCM into a Stag Hunt with Pareto-ranked equilibria and, even, into a game in which full contribution is the (weakly) dominant strategy.

2.4.2. First Best implementation

Furthermore, a subset of those price functions can implement the first best outcome by simply setting $p(W) = 1$. That is, if everyone grossly contributes their entire wealth (so $\kappa_i = w \ \forall i \in I$) then everyone net contributes their entire wealth: $g_i := p(K)\kappa_i = p(W)w = w \ \forall i \in I$, and the level of public good produced is efficient, $G := p(K)K = W$. And since full contribution is a weakly dominant strategy, the first best outcome can be implemented.

PROPOSITION 3. *The first best outcome can be implemented by price functions that ensure that full contribution is a weakly dominant strategy.*

[7] Cell values indicate the utility of the row player, broken down into its two contituent terms, as in equation 1. Parameter values: $w = 100$, $N = 4$, $\gamma = 0.6$, $\delta = 100$.Price function: $p(400) = 0.55$, $p(300) = 0.40$, $p(200) = 0.25$, $p(100) = 0$

Proof. Impose $p(W) = 1$ and, based on this, work out a price function that ensures the weak dominance of full contributions. ∎

Therefore, there are infinite ways to implement the FB and making full contribution the weak dominant strategy. Table 2 shows an example.[8]

		K_{-i}			
		300	200	100	0
κ_i	100	0 + 240 = 240	50 + 90 = 140	75 + 30 = 105	100 + 0 = 100
	0	100 + 90 = 190	100 + 30 = 130	100 + 0 = 100	100 + 0 = 100

TABLE 2
Dominance-solvable, efficient game

It is important to notice that, since dominance of full contributions is only weak and not strict, the implementation of the first best cannot be ensured. However, there is a good case for believing that it will, based on empirical and experimental evidence (Sanchez Villalba and Martinez Gorricho (2014), among others).

2.4.3. "Top 2" (T2) method

One reasonable way of implementing the FB outcome and ensuring that full contribution is a weakly dominant strategy would consist in finding a price function that, on top of satisfying the conditions of corollary 3 and that $p(W) = 1$, maximise the level of utility (public good provision) *for each possible level of* K_{-i} (for each column of a table, as in examples in tables 1 and 2). The designer's problem is thus

$$\max_{\{p(w+K_{-i})\}} u\left(w, K_{-i}\right) \text{ subject to}$$

$$u\left(w-\delta, K_{-i}+\delta\right) \leq u\left(w, K_{-i}+\delta\right) \ \forall K_{-i} \in \left(\frac{1-\gamma}{\gamma}w, W_{-i}\right] \quad (12)$$

[8] Cell values indicate the utility of the row player, broken down into its two contituent terms, as in equation 1. Parameter values: $w = 100$, $N = 4$, $\gamma = 0.6$, $\delta = 100$.Price function: $p(400) = 1$, $p(300) = 0.50$, $p(200) = 0.25$, $p(100) = 0$.

and the process of finding the price function is an *iterative* one: it starts with the highest possible value that K can take, $K = W$ (when $p(W) = 1$), and then considers subsequently lower levels: $K = W - \delta$, $K = W - 2\delta$, *etc.* up to $K = \frac{w}{\gamma}$. Note that pairs (w, K_{-i}) and $(w - \delta, K_{-i} + \delta)$ belong to the same iso-public line ($K = w+K_{-i}$) and so $u(w, K_{-i})$ and $u(w - \delta, K_{-i} + \delta)$ are functions of price $p(w + K_{-i})$, while pair $(w, K_{-i} + \delta)$ belongs to a higher iso-public line ($K = w + K_{-i} + \delta$) and so $u(w, K_{-i} + \delta)$ is the function of price $p(w + K_{-i} + \delta)$. Since the process is iterative, the latter price $p(w + K_{-i} + \delta)$ was determined in a previous iteration, and so is the utility $u(w, K_{-i} + \delta)$. Notice also that $u(w, K_{-i}) = w + p(w + K_{-i})[\gamma(w + K_{-i}) - w]$ and $u(w - \delta, K_{-i} + \delta) = w+p(w + K_{-i})[\gamma(w + K_{-i}) - (w - \delta)]$ are linear, increasing functions of the choice variable $p(w + K_{-i})$,[9] so that higher prices both increase the objective function and makes the constraint more likely to be violated. Thus, the constrained solution can only be either $p(w + K_{-i}) = 1$ if the constraint is not binding or $p(w + K_{-i})$ such that $u(w - \delta, K_{-i} + \delta) = u(w, K_{-i} + \delta)$ if the constraint is binding. Now, the constraint can be re-written as $p(w + K_{-i}) \leq a(K_{-i}, w, \gamma, \delta) \cdot p(w + K_{-i} + \delta)$ where $a(\gamma, w, K_{-i}) := \frac{\gamma(w+K_{-i}+\delta)-w}{\gamma(w+K_{-i})-(w-\delta)}$ is a positive constant that satisfies that $a(K_{-i}, w, \gamma, \delta) < 1 \ \forall K_{-i} \in \left(\frac{1-\gamma}{\gamma}w, W_{-i}\right]$. Thus, the constraint is binding and the solution is

$$p(w + K_{-i}) = a(K_{-i}, w, \gamma, \delta) \cdot p(w + K_{-i} + \delta) \ \ \forall K_{-i} \in \left(\frac{1-\gamma}{\gamma}w, W_{-i}\right] \qquad (13)$$

so that we can obtain the price function in an iterative way, knowing that $p(W) = 1$ and working out the rest using equation 13.

PROPOSITION 4. *The price function $p(K)$ is a strictly increasing function of aggregate gross contribution K, in the range $K \in \left(\frac{1}{\gamma}w, W\right]$.*

[9] This is so because both (w, K_{-i}) and $(w - \delta, K_{-i} + \delta)$ belong to the "upper-right region" and thus the expressions in brackets in the second terms of both utilities are positively signed.

Proof. It follows directly from equation 13. ∎

PROPOSITION 5. *Contributions are free for low values (i.e., $p(K) = 0$ is constant for $K \in \left[0, \frac{1}{\gamma}w\right]$) and they become more expensive as contributions increase (for $K \in \left(\frac{1}{\gamma}w, W\right]$), reaching a maximum when $K = W$: $p(W) = 1$.*

Proof. It follows directly from propositions 1, 4 and 3. ∎

Notice that equation 13 indicates that the relationship between two consecutive values of the price function is an exact one, meaning that we get a unique price function when we use the "Top 2" method. Further, since the constraint is binding it must be the case that $u(w - \delta, K_{-i} + \delta) = u(w, K_{-i} + \delta)\ \forall K_{-i} \in \left(\frac{1-\gamma}{\gamma}w, W_{-i}\right]$, which means that the utility of the "top two" possible levels of individual gross contributions (w and $w - \delta$) must yield the same level of utility (hence the name of the method).

Continuous case The continuous case can be obtained by considering the limiting case when $\delta \to 0$. The constraint thus becomes $\frac{\partial u_i}{\partial \kappa_i}(\kappa_i = w, K_{-i}) = 0$ $\forall K_{-i} \in \left(\frac{1-\gamma}{\gamma}w, W_{-i}\right]$. In this setup we can obtain an explicit solution to the problem, which is given by

$$p(K) = \begin{cases} 0 & \text{if } 0 \leq K \leq \frac{w}{\gamma} \\ \left(\frac{\gamma K - w}{\gamma W - w}\right)^{\frac{1-\gamma}{\gamma}} & \text{if } \frac{w}{\gamma} \leq K \leq W \end{cases} \tag{14}$$

and satisfies the conditions of proposition 5. Graphically:[10]

[10] Parameters: $\gamma = 0.6$, $w = 100$, $W = 400$.

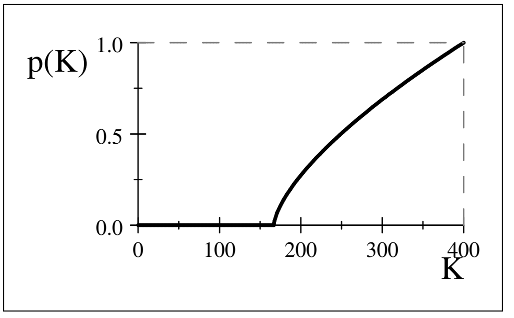

Table 3 shows an example of how the $T2$ method operates.[11]

		K_{-i}												
		300	275	250	225	200	175	150	125	100	75	50	25	0
	100	240	216	194	173	155	139	125	114	105	101	100	100	100
	75	239	215	193	172	154	138	124	112	103	100	100	100	100
κ_i	50	236	212	190	169	150	134	119	106	100	100	100	100	100
	25	231	207	184	163	144	126	109	100	100	100	100	100	100
	0	224	199	176	154	133	111	100	100	100	100	100	100	100

TABLE 3
"Top 2" Method, efficient game

3. EXTENSIONS AND DISCUSSION

3.1. Heterogeneity

3.1.1. Income heterogeneity

Equation 2 becomes $\gamma \in \left(\frac{w_i}{W}, 1\right)$ $\forall i \in I$ (note that the homogeneous case is obtained if $w_i = w$ $\forall i \in I$). This equation ensures that the full contribution allocation (FCA, in which every person gets utility $u_i = \gamma W$) Pareto-dominates the zero contribution allocation (ZCA, in which subjects get utility $u_i = w_i$) or, in other words,

[11] Cell values indicate the utility of the row player. Parameter values: $w = 100$, $N = 4$, $\gamma = 0.6$, $\delta \to 0$.Price function: $p(K) = \begin{cases} 0 & \text{if } 0 \leq K \leq \frac{100}{\frac{3}{5}} \\ \left(\frac{\frac{3}{5}K - 100}{\frac{3}{5}400 - 100}\right)^{\frac{1-\frac{3}{5}}{\frac{3}{5}}} = \left(\frac{3K-500}{700}\right)^{\frac{2}{3}} & \text{if } \frac{100}{\frac{3}{5}} \leq K \leq 400 \end{cases}$

that every player prefers the FCA to the ZCA. Notice that if the richest person (with income w^H) prefers the FCA to the ZCA, then everyone else will also prefer the former to the latter. Thus, we can focus on the richest person and all our results hold if we substitute the highest income w^H for the average income w in our equations.

3.1.2. Heterogeneity in preferences

Equation 2 becomes $\gamma_i \in \left(\frac{1}{N}, 1\right) \; \forall i \in I$ (note that the homogeneous case is obtained if $\gamma_i = \gamma \; \forall i \in I$). As in the previous case, this equation ensures that the full contribution allocation (FCA, in which every person gets utility $u_i = \gamma_i W$) Pareto-dominates the zero contribution allocation (ZCA, in which subjects get utility $u_i = w$). Analogously, if the most "public-averse" person (with the lowest value of gamma, γ^L) prefers the FCA to the ZCA, then everyone else will also prefer the former to the latter. Thus, we can focus on the most "public-averse" person and all our results hold if we substitute the lowest gamma γ^L for the average gamma γ in our equations.

3.1.3. Heterogeneity in income and in preferences

Equation 2 becomes $\gamma_i \in \left(\frac{w_i}{W}, 1\right) \; \forall i \in I$ (note that the homogeneous case is obtained if $\gamma_i = \gamma$ and $w_i = w \; \forall i \in I$). As before, it ensures that the full contribution allocation (FCA, in which every person gets utility $u_i = \gamma_i W$) Pareto-dominates the zero contribution allocation (ZCA, in which subjects get utility $u_i = w_i$). In this setup the "type" of a player can be summarised by the ratio $\tau_i = \frac{\gamma_i}{w_i}$ and so if the person with the lowest type τ^L prefers the FCA to the ZCA, then everyone else will also prefer the former to the latter. Thus, we can focus on the "lowest-type" person and all our results hold if we substitute the "lowest-type" gamma and income for the average gamma and average income in our equations.

3.2. Imperfect information

If the type of the players is not known, and because of the results of the previous three subsections, it can be seen that the designer does not need to know much about the distribution of types (only the “highest” or “lowest” value of the distribution) in order to find the price function. Thus, using conservative estimates of the extreme values of the distribution of types would almost ensure the implementation of the first-best outcome in a (weak) dominance game.

3.3. Replicability of the population

It is straightforward to show that the results of propositions 3 and 5 hold when the population is replicated (e.g., doubled or halved), although the price function can be different from one case to the other.

3.4. Equity considerations

The mechanism is equitative in the sense that each euro contributed is treated equally (i.e., the subsidy rate/price is the same for everyone), although the total subsidy received (the total amount of money spent on the “third commodity”) could differ from one person to the next. Thus, the mechanism is closer to the idea of “equality of opportunity”: each euro contributed is treated equally, but people can obtain different outcomes by choosing different levels of contribution (just like each visit to a doctor might be equally priced for everyone, but different people could pay a different number of visits). Ex-post equality is not expected to arise but in special circumstances (like in the homogeneous case).

Sanchez Villalba and Martinez Gorricho (2014) consider alternative scenarios in which the price/subsidy rate can differ across players by introducing variability in the payments via a lottery. They also explore the effect of risk on the contribution decisions of experimental subjects (the link between risk and inequality being well known since Harsanyi (1955)) and find that the equitable mechanism (the hybrid lottery) yields better results than the inequitable and/or risky ones.

3.5. Non-unanimous societies

When some (one or more) players prefer the Zero Contribution Allocation (ZCA) to the Full Contribution one (FCA) then the hybrid lottery needs to be modified. This occurs when $\gamma_i < \frac{w_i}{W}$ for at least one player, so that the utility of said player if nobody contributes ($u_i^{**} = w_i$) is greater than her utility when everybody contributes ($u_i^* = \gamma_i W$). This means that there is no unanimity regarding the ranking of FCA and ZCA (i.e., FCA and ZCA are not Pareto-ranked): some players ("loners") prefer the ZCA, others ("socials") prefer the FCA. It is therefore impossible for "loners" to be convinced to contribute their entire wealth: even in the best scenario, the most they could get out of it would be a utility equal to $u_i^* = \gamma_i W$, which is lower than the one they would get if they kept all their income for themselves.

It is possible, however, to provide incentives for loners to contribute part of their wealth: intuitively, the price function should be chosen such that the loners' loss due to lower private consumption is (more than) compensated by the increase in public consumption. Table 4 shows an example of how this can be done.[12] The price function becomes a step function that only takes extreme values (0 or 1) but the game is not as simple as in the *"unanimous society"* case (USC): at first sight, it is not straightforward to see that all players have a (weakly) dominant strategy, although they do. It is easy to realise that the rich player has a weakly dominant

[12] Parameter values: $N = 4$, 3 "poor" players with wealth $w_p = 50$ each, and 1 "rich" player with wealth $w_r = 250$. Cell values indicate the utility of the row player: the left/right table (A/B) shows the utility of the/one of the rich/poor player/s. $\delta = 10$. Price function:

$$p(K) = \begin{cases} 0 & if \quad K \in \{0, ..., 360\} \\ 1 & if \quad K \in \{360, ..., 400\} \end{cases} \tag{15}$$

strategy, namely playing $\kappa_r = 220$, but the poor player's case is not self-evident. The poor players can, however, realise that the rich player will play $\kappa_r = 220$, and thus can deduct that K_{-pi} can take values between 220 (if the other poor players contribute nothing) and 320 (if the other poor players contribute all their wealth). The latter upper bound is important because it implies that the poor player can ignore the first four columns of her table and that contributing 50 is a dominant strategy in the "reduced table" resulting from ignoring the first four columns. That is, rich and poor players have a weakly dominant strategy (220 and 50, respectively), but unlike the "unanimous society" case, finding them requires the iterated deletion of weakly dominated strategies ($IDWDS$). This is more demanding for the players than the simple comparison of columns' payoffs required in the USC, and it might generate a suboptimal outcome due to cognitive limitations (Sanchez Villalba (2010) shows that quite a significant number of people do not do even the second iteration). Also, technically, the $IDWDS$ can be criticised by the fact that the actual path of iterations followed can affect the final outcome. But there seems to be evidence that shows that people might treat weakly and strictly dominated strategies as equivalent (Sanchez Villalba and Martinez Gorricho (2014)). Finally, the allocation arrived with this method is very close to the First-Best outcome (in the example in table 4, it corresponds to $\kappa_r^* = 225$ and $\kappa_{pi}^* = 50$) but it might depend crucially on the discreteness of the model (i.e., of δ being finite and positive, the "graininess" of contributions). One of the avenues of research that we plan to follow consists on analysing this case in detail.

A Rich		K_{-r}						
		150	140	130	120	110	...	0
κ_r	250	240	234	228	222	250	250	250
	240	244	238	232	250	250	250	250
	230	248	242	250	250	250	250	250
	220	252	250	250	250	250	250	250
	210	250	250	250	250	250	250	250
	...	250	250	250	250	250	250	250
	0	250	250	250	250	250	250	250

B Poor		K_{-pi}						
		350	340	330	320	310	..	0
κ_{pi}	50	240	234	228	222	50	50	50
	40	244	238	232	50	50	50	50
	30	248	242	50	50	50	50	50
	20	252	50	50	50	50	50	50
	10	50	50	50	50	50	50	50
	0	50	50	50	50	50	50	50

TABLE 4
Non-unanimous society

3.6. Other utility functions

The basic intuitions of the linear model can be extended to quasi-linear and non-linear models, although the technical details are different. This is, at least, the result that we have reached so far in our investigations with alternative utility functions. In particular, the "Top 2" method seems to be robust (with some minor modifications), so we believe that the linear results are likely to be extrapolated to the nonlinear cases. This is the other fundamental avenue of research that we intend to follow in the immediate future. Just as an illustration, table 5 shows an example of the application of the method to a quasi-linear utility function.[13]

[13] Parameter values: $u_i = x_i + v(G)$, where $v(G) = -\frac{1}{3200}G^2 + \frac{3}{8}G$. $N = 4$ players. $w_i = 100$ $\forall i \in I$. $\delta = 10$.

		K_{-i}														
		300	290	280	270	260	250	240	230	220	210	200	190	180	...	0
κ_i	100	113	110	107	105	104	103	102	101	101	101	100	100	100	100	100
	90	113	110	107	105	104	103	102	101	101	100	100	100	100	100	100
	80	112	109	107	105	104	103	102	101	101	100	100	100	100	100	100
	70	111	108	106	105	103	102	102	101	101	100	100	100	100	100	100
	60	110	107	106	104	103	102	101	101	100	100	100	100	100	100	100
	50	108	106	105	103	102	101	101	100	100	100	100	100	100	100	100
	40	107	105	104	103	102	101	100	100	100	100	100	100	100	100	100
	30	106	104	103	102	101	100	100	100	100	100	100	100	100	100	100
	20	105	103	102	101	100	100	100	100	100	100	100	100	100	100	100
	10	104	102	101	100	100	100	100	100	100	100	100	100	100	100	100
	0	102	102	100	100	100	100	100	100	100	100	100	100	100	100	100

TABLE 5
Quasilinear utility function

4. CONCLUSIONS

The private provision of public goods is one of the fundamental topics in Public Economics and has ample application to many real-world situations. Indeed, the presence of public goods is one of the sources of market failure: when left alone, the private market will provide an inefficiently low level of provision of public goods. Examples of this problem can be found in many scenarios, from the low level of charity giving in a society or of foreign aid among states, to the low effort exerted by workers when paid according to the team's output, to the under-investment in private vigilance in a neighborhood, to many others.

The underlying problem behind all of this situations is always the same: the clash between social and individual objectives: while the socially optimal action is to contribute, the individually optimal action is to "free-ride". Several studies have considered different alternative methods designed with the objective of eliminating (or at least mitigating) the inefficiency associated with the private provision of public goods. From pricing strategies (Lindahl (1958)), to truth-telling mechanisms (Groves and Ledyard (1977)), to alternative settings (provision points, money-back guarantees, lotteries à la Morgan (2000)), to a long list of etceteras.

The mechanism we propose in this paper is based on the same idea than Morgan (2000)'s paper, namely, introducing a negative externality among the individuals that (partially) offsets the positive externality present in the voluntary contribution mechanism. We propose that contributors should get a "subsidy" such that the average subsidy rate is a non-linear, decreasing function of aggregate contributions. This decreases the gap between the marginal benefit and the marginal cost of contribution (as compared to the standard VCM), and thus provides incentives for players to contribute more.

Our mechanism is based on voluntary contributions and improves the performance of the best alternative in terms of aggregate contributions and yet it is simple (easy to implement and understand), cheap and self-financed (it is never needed to pour money into it from other sources). Furthermore, it can even generate an equitable outcome by paying the same subsidy rate to everyone.

In summary, we designed a mechanism that is better than the alternative methods suggested in the literature: it produces a larger amount of public good (which is the efficient action to undertake in this setting) than alternative mechanisms, it is simple to understand by the potential contributors and it is easy and cheap to implement by the fundraiser (with special stress on the fact that it is an entirely self-financed mechanism). On top of that, first-best provision in terms of contributions is feasible under some conditions.

Also, the mechanism seems to be robust to the introduction of heterogeneity (say, in terms of income or preferences) and of more general utility functions.

Finally, we consider that –due to its characteristics previously described– the hybrid lottery can be an effective mechanism to finance the provision of public goods, especially for charities and other real-world organisations that rely on voluntary contributions for their operation.

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APPENDIX A: PROOFS

Proposition 1 The proof can be done in two steps:

(1) $p(K) = 0 \ \forall K \in \left[\frac{1-\gamma}{\gamma}w, \frac{1}{\gamma}w\right]$: We know that the utility from pairs (κ_i, K_{-i}) belonging to the "upper region" is greater than or equal to w. In particular, when $K_{-i} = \frac{1-\gamma}{\gamma}w$ then $u\left(\kappa_i, \frac{1-\gamma}{\gamma}w\right) = w + p(K)\left[\gamma\left(\kappa_i + \frac{1-\gamma}{\gamma}w\right) - \kappa_i\right] \geq w$ $\forall \kappa_i \in [0, w]$. Weak dominance of full contribution requires $u\left(w, \frac{1-\gamma}{\gamma}w\right) \geq u\left(\kappa_i, \frac{1-\gamma}{\gamma}w\right) \ \forall \kappa_i \in [0, w]$, which is only possible if $u\left(\kappa_i, \frac{1-\gamma}{\gamma}w\right) = w \ \forall \kappa_i \in [0, w]$, which in turn is only possible if $p\left(\kappa_i + \frac{1-\gamma}{\gamma}w\right) = 0 \ \forall \kappa_i \in [0, w]$. Adding $K_{-i} = \frac{1-\gamma}{\gamma}w$ to the range of κ_i, the latter expression can be re-written as $p(K) = 0 \ \forall K \in \left[\frac{1-\gamma}{\gamma}w, \frac{1}{\gamma}w\right]$.

(2) $p(K) = 0 \ \forall K \in \left[0, \frac{1-\gamma}{\gamma}w\right]$: Consider now the case in which $\kappa_i = 0$ and $K_{-i} \in \left[0, \frac{1-\gamma}{\gamma}w\right]$, which belongs to the "upper region" as well, so $u\left(0, K_{-i}\right) \geq w \ \forall K_{-i} \in \left[0, \frac{1-\gamma}{\gamma}w\right]$. Weak dominance now requires, as a special case, that $u\left(w, K_{-i}\right) \geq u\left(0, K_{-i}\right) \forall K_{-i} \in \left[0, \frac{1-\gamma}{\gamma}w\right]$, which can only occur if $p(0+K_{-i}) = 0 \ \forall K_{-i} \in \left[0, \frac{1-\gamma}{\gamma}w\right]$. Since $K = K_{-i} + \kappa_i = K_{-i}$ in this case, the latter expression becomes $p(K) = 0 \ \forall K \in \left[0, \frac{1-\gamma}{\gamma}w\right]$.

Combining the result of both steps, we obtain the result of proposition 1.

Proposition 2

(1) From the assumption that $u_i^D > u_i^E$ then $w + p(K^D)\left(\gamma K^D - \kappa_i^D\right) > w + p(K^E)\left(\gamma K^E - \kappa_i^E\right)$ and, after some algebraic manipulation, $p\left(K^E\right) < p\left(K^D\right)$.

(2) Notice that u_i^D can be re-written as $u_i^D = u_i^B - p(K^B)\delta$ and u_i^E as $u_i^E = u_i^C - p(K^C)\delta$, so $u_i^D > u_i^E$ implies that $u_i^B - u_i^C > \left[p(K^D) - p(K^E)\right]\delta$.

(3) From **(1)** and **(2)** we get that $u_i^B > u_i^C$.

(4) From the assumption that $u_i^A > u_i^B$ and **(3)** we get that $u_i^A > u_i^C$.