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VON NEUMANN–MORGENSTERN UTILITIES AND CARDINAL PREFERENCES*†

GRACIELA CHICHILNISKY

Columbia University

We study the aggregation of preferences when intensities are taken into account: the aggregation of cardinal preferences, and also of von Neumann–Morgenstern utilities for choices under uncertainty. We show that with a finite number of choices there exist no continuous anonymous aggregation rules that respect unanimity, for such preferences or utilities. With infinitely many (discrete sets of) choices, such rules do exist and they are constructed here. However, their existence is not robust: each is a limit of rules that do not respect unanimity. Both results are for a finite number of individuals.

The results are obtained by studying the global topological structure of spaces of cardinal preferences and of von Neumann–Morgenstern utilities. With a finite number of choices, these spaces are proven to be noncontractible. With infinitely many choices, on the other hand, they are proven to be contractible.

1. Introduction. The methods of preference aggregation studied in social choice theory typically describe an individual preference as a ranking among choices, i.e., in *ordinal* terms. In most of the literature following Black [3] and Arrow [1], intensities of preferences are not recorded; in particular, given three choices x, y, z, individuals are not able to say whether x is preferred to y by more than y is preferred to z. Since most of the results in the aggregation of ordinal preferences are negative, it seems natural to inquire whether more positive results can be obtained when this property is relaxed and intensities of preferences are recorded.

A significant step in allowing the consideration of intensities is introduced with *cardinal preferences*. These preferences express precisely the notion that x is preferred to y by more than y is preferred to z. The space of cardinal preferences can be shown to have the same mathematical structure as the space of *von Neumann–Morgenstern utilities*, the numerical representations of preferences over lotteries for the case of choice under uncertainty. These utilities are denoted NM utilities, and have frequently been used in the operations research literature and in decision theory ever since their introduction by von Neumann and Morgenstern [19]. NM utilities are also widely used to represent individual behavior in game theory, see Fishburn [13].

Both cardinal preferences and NM utilities can be represented by 'utilities', i.e. by real valued functions over the choice space. However, a whole (equivalence) class of such functions will define *one* preference or NM utility; the representation assigns therefore a family of real valued functions to each preference or NM utility. Ordinal preferences are also represented by classes of real valued functions.

The main distinction between ordinal and cardinal preferences lies in the size of the equivalence classes of their utility representations. Cardinal preferences have smaller equivalence classes than ordinal preferences: the representation of a *cardinal preference* by a utility function is unique up to (and only up to) positive *linear* transformations of

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the real line. For *ordinal preferences*, however, the representation is unique up to *any* positive transformation.

The weaker the invariance, the 'closer' are preferences to utilities, and utilities pose no problem of aggregation, since they define a linear space.[1] Therefore one may expect that the task of aggregation is easier with cardinal rather than ordinal preferences. However, this is not the case. It was shown in Chichilnisky [4] and in Chichilnisky and Heal [10] that the crucial element in our ability to aggregate preferences is the global topological structure of the space of preferences considered. In order to admit appropriate aggregation rules, these spaces must be contractible, i.e. topologically trivial. However, the topological structure of spaces of preferences may be complex even when less invariance is required. For instance, NM utilities with finite prizes are shown here to define a *noncontractible space*, i.e. a space with a nontrivial topological structure (see §4).

By investigating the global topology of spaces of cardinal preferences and of NM utilities, we prove here that, with *finitely many choices*, there exists no continuous anonymous social aggregation rule that respects unanimity.[2] Such forms of aggregation are impossible for cardinal preferences and for NM utilities.

With *infinitely many choices* we show instead that such aggregation rules do exist. However, their existence is not robust in the sense that they are the limit of rules which are defined on subsets of finitely many choices and which do not respect unanimity. The same results apply to von Neumann–Morgenstern utilities defined over infinitely many lotteries. A finite number of individuals is considered through the paper.

The rest of the paper is organized as follows: §2 gives notation and definitions; §3 discusses previous literature; and §4 gives the results.

2. Notation and definitions. In the case of finite choices the *choice space* X is a finite set of points in Euclidean space $X = \{x_i\}$, $i = 1, \ldots, n$, $n \geqslant 3$.

A *preference with intensity* or *cardinal preference* p is identified with a nonnegative real valued function on X, i.e. with a nonnegative vector in R^n, $p \in R^{n+}$, $p = (p_1, \ldots, p_n)$; p_i denotes the utility value attached to the choice x_i. The *total indifference* preference is thus the vector with all coordinates equal. Our space of preferences contains this total indifference preference as well.

The following step is to normalize utility vectors in order to obtain a unique representation of each cardinal preference by a vector in Euclidean space. This normalization is a standard one; see, e.g., Kalai and Schmeidler [14]. Formally, if $p = (p_1, \ldots, p_n)$ is a utility vector in R^n, p is normalized by subtracting from its coordinates the vector with all components identical to the minimum utility value $p \to (p_1 - m, \ldots, p_n - m)$ where $m = \min_i\{p_i\}$, and then dividing the outcome by its

[1] The problem of aggregation of ordinal preferences is significantly different from that of aggregating utility functions because the aggregation of ordinal preferences must be independent of the choice of their utility representation. For instance, if u is a utility funciton and F is a strictly increasing numerical function, the ordinal preference associated with the utility u must be the same as that associated with the function $F \circ u$. Therefore, the rule for aggregating utilities will only induce a rule for aggregating ordinal preferences if it is invariant under any such increasing transformation of utilities. This is indeed a rather strong condition, and several possible relaxations have been studied, for instance, by Sen [16], [17], Kalai and Schmeidler [14] and more recently by d'Aspremont and Gevers [11]. Sen [17] concentrates on the relaxation of the assumption of no interpersonal comparisons. d'Aspremont and Gevers discuss the characterize a wide combination of assumptions that relax both the interpersonal comparison and the ordinality assumptions. Our framework here is most closely related to axiom (CN) of d'Aspremont and Gevers, which assumes that individual utility functions are cardinal and noncomparable, and to the cardinality assumptions of Sen and of Kalai and Schmeidler.

[2] Respect of unanimity requires that if all individuals agree unanimously *over all choices*, so does the aggregate. This condition does not imply that if one choice x is preferred to another y by all individuals, then the aggregate prefers x to y.

maximum component M if $M \neq 0$, i.e.,

$$p \to \frac{p_1 - m}{M}, \ldots, \frac{p_n - m}{M},$$

where $M = \max_i \{p_i - m\}$. The total indifference preference is identified therefore with the vector $(0, \ldots, 0)$. The fact that all normalized preferences have the same minimum and maximum utility values can be considered a weak form of interpersonal comparison.[3]

It follows therefore that with finitely many choices the space of cardinal preferences is $P = Q \cup \{0\}$, where Q is the subspace of nonzero cardinal preferences,

$$Q = \{p \in R^{n+} : p_i \leqslant 1, p_j = 0 \text{ and } p_k = 1 \text{ for some } k, j \in \{1, \ldots, n\}\},$$

and $\{0\}$ denotes the total indifference preference. In order to define continuity of the social choice rule, P is given the natural topology it inherits from R^n. The space P has two connected components, Q and $\{0\}$.[4] An equivalent way of defining the space P of cardinal preferences over n choices is as the space of equivalence classes of positive vectors in R^n, where the equivalence relation is $p^1 \sim p^2$ iff $p^1 = a + bp^2$, where b is a nonnegative number and a is a nonnegative vector with all its coordinates equal.

We shall now define the space of cardinal preferences P^∞ for the case of infinitely many choices.

Assume now that the choice space is N, the set of positive integers. A preference p is assumed to be a nonnegative real valued function on N, i.e. a sequence of nonnegative numbers. Since we are concerned with bounded sequences, without loss of generality, we may assume that $\sum p(n)\mu(n) < \infty$, for some *finite* measure μ on N given by a density function $\mu(n)$.

The space of preferences P^∞ is therefore strictly contained in the space of all bounded sequences, and this is in turn a subset (the positive cone) of weighted l_1 space.[5]

As in the case of finitely many choices, we normalize the vector p in order to obtain a unique representation of cardinal preferences. An equivalence relation $\sim$ is defined by $p^1 \sim p^2$ if and only if $p^1 = a + bp^2$, b a nonnegative number and a a nonnegative element in l_1 with all coordinates equal.

A preference over N is thus an equivalence class of positive vectors under the relation $\sim$. A space which is in a one-to-one correspondence with the space of

[3] This normalization has in particular the effect that the sum total of intensities over choices is uniformly bounded over agents; this was a suggestion of L. Gevers.

[4] A *connected component* of a topological space Y is a maximum connected subspace of Y. A space X is *connected* if it cannot be decomposed as a union $X = X_1 \cup X_2$, where $X_1 \neq \emptyset$, $X_2 \neq \emptyset$, and X_1 and X_2 are both open and closed sets. This extends the notion that any point in X can be joined to another in X by a path contained in X.

[5] (l_1, μ) is the Banach space of infinite sequences of real numbers $\{x_n\}$ $n = 1, 2, \ldots$ such that

$$\|x\|_1 = \sum_{n=1}^{\infty} |x_n| \mu(n) < \infty,$$

see [12]. The measure μ is *finite* if $\sum_N \mu(n) < \infty$.

Note that we could instead embed P^∞ into l_∞, the space of all bounded sequences with the sup norm, $\|x\|_\infty = \sup_{n=1,2,\ldots} |x_n|$. However, the space l_1 with the (finite) weight $\mu(n)$ contains l_∞ as a subspace. Therefore, if one defines an aggregation map ϕ for l_1 one has automatically defined an aggregation map for l_∞, given by the restriction of ϕ on l_∞ considered as a subspace of l_1. The topology induced by l_1 is different than the sup norm on l_∞; but since our aim is to prove an existence theorem for *some* adequate topology, this procedure seems appropriate. In any case, P^∞ is a *strict* subset of l_∞ as well as of l_1, and is significantly smaller than either l_∞ or l_1. Therefore, neither l_∞ nor l_1 coincides with P^∞ and thus the choice of topology is best made on the basis of mathematical adequacy. Theorem 2 shows that l_1 is an adequate space; the spaces l_1, or more generally l_p (with $1 \leqslant p \leqslant \infty$) have been used previously in the economic literature; see e.g. Chichilnisky [6].

preferences is obtained by considering all vectors with coordinates smaller than 1, with at least one coordinate zero, and with one of its nonzero coordinates (if it exists) equal to 1. Therefore with infinitely many choices the space of cardinal preferences is $P^{\infty} = Q^{\infty} \cup \{0\}$, where $Q^{\infty} = \{f \in l_1^+;$ for all $i, f_i \leq 1; f_j = 0$ and $f_k = 1$ for some $j, k\}$. P^{∞} inherits the topology of l_1 and is a closed subset of a Banach space. As P, P^{∞} consists of exactly two connected components.

We consider now the case of choice under uncertainty. There are finitely many prizes $x_1, \ldots, x_n$. Von Neumann–Morgenstern axioms [19] characterize choice under uncertainty by the maximization of *expected utility* over lotteries. A von Neumann–Morgenstern utility, denoted NM, assigns the expected value $\sum_{i=1}^{n} q_i(U(x_i))$ to the lottery x which offers the prize (or outcome) x_i with probability q_i, for $i = 1, \ldots, n$. The vector of real numbers $U = (U_1, \ldots, U_n)$ where $U_i = U(x_i)$, characterizes therefore the NM utility. However, any positive linear transformation of this vector of the form $V = a + bU$, where b is a positive real number and a is a nonnegative vector with all coordinates equal, defines also the same NM utility. This is because the expected value of a lottery x is higher than the expected value of another lottery y according to U, if and only if it is higher according to V. The space of von Neumann utilities on n prizes is therefore the space of equivalence classes of nonnegative vectors in R^n with the equivalence relation $U \sim V$ iff $V = aU + b$, where a is a nonnegative number and b is a nonnegative vector with all its coordinates equal. The definitions of the spaces of von Neumann–Morgenstern utilities and of cardinal preferences are identical. These two spaces are therefore the same mathematical object.

Assume now there are k agents, $k \geq 2$. With finite choices a *profile of cardinal preferences* is a vector $(p^1, \ldots, p^k) \in (P^{\infty})^k$. Similar definitions apply to profiles of NM utilities.

An aggregation rule for k individuals is a map from profiles into preferences $\phi: P^k \to P$. ϕ is said to *respect unanimity* when $\phi(p, \ldots, p) = p$ $\forall p \in P$, i.e. if all individuals have identical preferences over all choices, so does the social preference.[6]

A rule ϕ is *anonymous* when the outcome is independent of the order of the individuals, i.e.,

$$\phi(p^1, \ldots, p^k) = \phi(p^{\eta^1}, \ldots, p^{\eta^k}) \qquad \text{where} \quad \eta: \{1, \ldots, k\} \to \{\eta_1, \ldots, \eta_k\}$$

is any permutation of the set $\{1, \ldots, k\}$.

Continuity of a rule is defined with respect to the usual product topologies of the spaces of preferences as subsets of R^{nk} or $(l_1)^k$ in the finite and infinite cases, respectively.

We now discuss a basic topological concept used in the following.

A topological space A is *contractible* if there exists a continuous map $f: A \times [0, 1] \to A$ such that $f(x, 0) = x$ $\forall x$ in A, and $f(x, 1) = x_0$, for some $x_0 \in A$. Intuitively, A is contractible if it can be deformed continuously through itself, into one of its points, x_0. Linear spaces and convex sets are contractible. Spaces of real valued utility functions are contractible, since they are linear. Topologically speaking, these are all trivial spaces, since they are topologically equivalent to (i.e., continuously deformable into) points. A hollow sphere in R^n is not contractible. As we shall prove below, both the nontrivial connected component of the space of cardinal preferences Q, and that the von Neumann–Morgenstern utilities, are *not* contractible. This proves to be important for the aggregation results of this paper.

[6] Note that this condition is binding, only when all voters have identical preferences. It is therefore a strictly weaker condition than the Pareto condition, since a rule ϕ satisfies the Pareto condition if, whenever a choice $x \in X$ is preferred to another $y \in X$ for all preferences $p^1, \ldots, p^k$, then $\phi(p^1, \ldots, p^k)$ also prefers x to y.

3. Previous work. Before proving the results, it may be useful to discuss the relationship of the spaces of preferences studied here with earlier concepts of cardinal preferences used in the literature, and also earlier results in this area.

The space of non-zero cardinal preferences Q was defined for instance, by Kalai and Schmeidler [14]: two vectors p^1 and p^2 define the same cardinal preference when there exist a nonnegative number b and a nonnegative vector a such that $p^1 = a + bp^2$, where a has all its coordinates equal. As pointed out in §2, the space of von Neumann–Morgenstern utilities, i.e., numerical representation of preferences over lotteries, corresponds precisely to this definition of the space of cardinal preferences. For further discussion see, e.g., [14] and [19].

The specification of Q given here is also related to one of the forms of relaxation of usual ordinality and comparability assumptions discussed in d'Aspremont and Gevers [11]. Their condition CN of *cardinality* and *noncomparability* requires that if u_1 and u_2 are two utilities, then they define the same preference whenever

$$u_1(x) = \alpha_v + \beta_v(u_2(x))$$

where $v = 1, \ldots, l$ is the index for the individual, x denotes a choice, and where $\{\alpha_v\}$ and $\{\beta_v\}$ are nonnegative real numbers. In our framework CN means that for each voter the vector p^1 represents the same preference as another p^2 when there exist a nonnegative vector α with all its coordinates equal and a nonnegative number β such that $p^1 = \alpha + \beta p^2$, which is precisely the cardinality condition discussed above:

Let p^1 and p^2 satisfy $p^1 = \alpha + \beta p^2$. Then they yield the same element in P, since for any $p = (p_1, \ldots, p_n)$, and any $j = 1, \ldots, n$

$$\frac{\alpha + \beta p_j - \min_i(\alpha + \beta p_i)}{\mathrm{Max}_i[\alpha + \beta(p_i) - \min_i(\alpha + \beta p_i)]} = \frac{[p_j - \min_i(p_i)]}{[\mathrm{Max}_i(p_i - \min_i(p_i))]} .$$

Conversely, if two utility vectors p^1 and p^2 in R^n yield the same element in the space of preferences Q, then the kth coordinates of p^1 and p^2, p_k^1 and p_i^2, satisfy:

$$\frac{p_k^1 - \min_i(p_i^1)}{\mathrm{Max}_i(p_i^1 - \min_i(p_i^1))} = \frac{p_k^2 - \min_i(p_i^2)}{\mathrm{Max}_i(p_i^2 - \min_i(p_i^2))}$$

which implies that $p_k^1 = \beta p_k^2$ for

$$\alpha = \min(p_i^1) - \frac{(\min_i(p_i^2))\mathrm{Max}_i(p_i^1 - \min_j(p_j^1))}{\mathrm{Max}_i(p_i^2 - \min_j(p_j^2))} \quad \text{and} \quad \beta = \frac{\mathrm{Max}_i(p_i^1 - \min_j(p_j^1))}{\mathrm{Max}_i(p_i^2 - \min_j(p_j^2))} .$$

Therefore social choice rules which are well defined on our space of cardinal preferences Q correspond to those satisfying condition CN in d'Aspremont and Gevers.

Several authors have studied the problems involved in aggregating cardinal preferences, e.g., Sen [16] and Kalai and Schmeidler [14]. It has been shown [14] that Arrow-like paradoxes may exist even with cardinal preferences, provided Arrow-like conditions are required of the aggregation procedure. These are the conditions of independence from irrelevant alternatives, Pareto and nondictatorship. Such conditions may be too strong: it is of interest to examine weaker conditions than Pareto and independence. Also, while making the problem amenable to a combinatorial analysis, such conditions tend to leave out its intrinsic geometry. We study here different conditions of the aggregation rule: continuity, anonymity and respect of unanimity. These conditions admit a ready geometrical interpretation, and help to exhibit the topological nature of the problem.

4. The results. The following result establishes the impossibility of aggregation for cardinal preferences on finitely many choices. The space of preferences is therefore $P = Q \cup \{0\}$, as defined above. There are n choices.

THEOREM 1. *There exists no continuous aggregation rule for cardinal preferences $\phi : P^k \to P$ which respects unanimity and is anonymous, $k \geq 2$. This includes cases where individual and social preferences may be indifferent among all choices.*

PROOF. An aggregation rule for cardinal preferences is a map $\phi : P^k \to P$. Now, as discussed in §2, the space P has exactly two connected components, Q and $\{0\}$ (Figure 1 illustrates the case of three choices). Therefore the product space P^k has exactly 2^k connected components.

The map ϕ is a continuous function from a topological space with 2^k components into another with 2 components. It follows from continuity of ϕ that each of the connected components of P^k must be mapped by ϕ into one connected component of P. Consider in particular Q^k, which is the connected component of P^k consisting of all nonzero cardinal preferences. Then either $\phi(Q^k) \subset Q$, or else $\phi(Q^k) \subset \{0\}$. However, by the condition of respect of unanimity $\phi(p, \ldots, p) = p$ for all p in Q, implying that $\phi(Q^k) \not\subset \{0\}$: it follows that $\phi(Q^k) \subset Q$.

Therefore, the axioms of continuity and of respect of unanimity taken together rule out the possibility that a profile with all preferences different from the zero vector may be mapped into the zero vector. A continuous rule for cardinal preferences which respects unanimity will only assign the total indifference to a set of voters if at least one of them is totally indifferent among all choices.[7]

The map ϕ induces therefore a continuous map $\bar{\phi} : Q^k \to Q$, which is also continuous, anonymous and respects unanimity. Since Q^k and Q are both connected spaces, we can now use the results in [4], which establish that the existence of such a map ϕ depends on certain topological invariants of the space Q. The next step of the proof

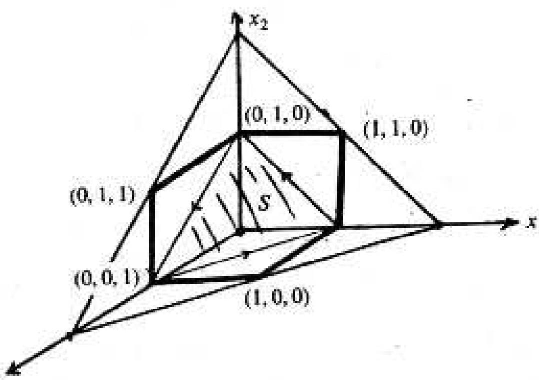

FIGURE 1. The space of preferences P with three choices is indicated as the union of the point $\{0\}$ with the set drawn with a heavy line. The nonzero connected component of P, Q (indicated with the heavy line) is on a one-to-one continuous correspondence with the boundary ∂S of the simplex S in R^3. In particular, it is not contractible.

[7]In addition to violating our axioms, rules that assign zero outcomes to nonzero vectors are clearly undesirable for other reasons: It is easy to check that if ϕ is a continuous map from $(R^n)^3$ into R^n that maps a profile of three voters with nonzero vectors into the trivial (zero) social preference, it will necessarily map the equivalent of *some* Condorcet triple into the trivial outcome $(0, \ldots, 0)$. The 'Condorcet triple' we are referring to is obtained by choosing three points (xyz) in R^n, so that the three vectors giving the voters' preferences rank these choices in the orders (xyz) (zxy) and (yzx) respectively. Such aggregation map would give a *trivial* (total indifference) solution to the Condorcet triple, which is clearly not an acceptable solution; for instance Arrow's impossibility theorem is based *inter alia* on the fact that Condorcet triples cannot be aggregated. If the total indifference was an acceptable aggregate of a Condorcet triple, Arrow's impossibility theorem ceases to be valid.

consists therefore of investigating the topology of the space of nonzero cardinal preferences, for an finite number of choices $n \geqslant 3$.

Consider now a family of sets $\{V_i\}_{i=1,\ldots,n} \subset Q$, $V_i = \{p \in Q : p_i = 0\}$. The union of the V_i's is Q. Each V_i is a union of n sets V_{ij}, $j = 1, \ldots, n$, where $V_{ij} = \{p \in Q : p_i = 0 \text{ and } p_j = 1\}$, i.e. $V_i = \bigcup_{j=1}^n V_{ij}$. It is easy to check that V_i is a contractible set: it can be continuously deformed through itself into the point $(1, \ldots, 1, 0, 1, \ldots, 1)$. Consider any subfamily W_k of the family $\{V_i\}_{i=1,\ldots,n}$. Then $\bigcap_k W_k$, if nonempty, is a contractible set. It follows that $\{V_i\}$ is an acyclic family in R^n; it can easily be made into an open family by replacing the number 0 by the interval $(-\epsilon, \epsilon)$ in the definition of the sets V_i. Since $\{V_i\}$ is an acyclic family, the Čech homology of its nerve is isomorphic to the singular homology of the union $\bigcup_{i=1}^n V_i$: this is a Leray's acyclic cover theorem.

It suffices now to compute the Čech homology of the nerve of $\{V_i\}$. Note that all subfamilies of $\{V_i\}$ with at most $n-1$ elements have a nonempty intersection, while the intersection of the whole family is empty. The nerve of $\{V_i\}$ is therefore an $n-2$ sphere. Therefore Q is a homology-$(n-2)$ sphere. Because the sets V_i are piecewise linear, Q is also homeomorphic to an $(n-2)$ sphere.

In particular, Q is not contractible. We can now apply the results of Chichilnisky [4], [9], to deduce that there exists no continuous anonymous rule $\bar{\phi} : Q^k \to Q$ which respects unanimity, since Q is homeomorphic to a sphere of dimension at least one. This completes the proof. ∎

Since as discussed above the space of NM utilities can be identified with P if there are finitely many prizes, we have therefore obtained

COROLLARY 1. *The space of von Neumann–Morgenstern utilities with finitely many prizes is not contractible. With n prizes, this space is a sphere of dimension $n-2$, union the origin* $\{0\}$.

PROOF. This follows from the proof of Theorem 1. ∎

COROLLARY 2. *With finitely many lotteries there exists no continuous anonymous aggregation rule for von Neumann–Morgenstern utilities which respects unanimity. This includes cases where individual and social utilities are indifferent among all lotteries.*

PROOF. This follows from Chichilnisky [4]. ∎

REMARK. Even though our framework and conditions on the aggregation rule are rather different from those of Kalai and Schmeidler, our impossibility result is consistent with theirs in cases of finitely many choices.

However, this is not the case for infinitely many choices. Instead, we obtain in the following a positive aggregation result for his latter case. This contrasts with Kalai and Schmeidler, who obtain an impossibility result with infinitely many choices. The difference emerges because they require the axiom of independence of irrelevant alternatives, which effectively reduces the problem of aggregation with infinitely many choices to one of aggregation with finitely many choices. Different sets of axioms are utilized: we do not require independence of irrelevant alternatives, but require continuity; we do not require the Pareto condition, but rather a (weaker) condition of respect of unanimity.

We now turn to the case of infinitely many choices. Our space of cardinal preferences is therefore P^∞. As before, there are a finite number of individuals, $k \geqslant 2$.

THEOREM 2. *With infinitely many choices, there exists a continuous aggregation rule $\phi : (P^\infty)^k \to P^\infty$ for cardinal preferences respecting unanimity and anonymity, given by a continuous deformation of a Bergsonian rule i.e., a convex addition rule. However, any* *such rule is a limit of rules defined on arbitrarily large finite sets of choices and which do not respect unanimity; in particular they are not Pareto.*

PROOF. As in Theorem 1, we may consider a continuous aggregation rule $\phi:(P^\infty)^k \to P^\infty$ assigning to each profile of k (nontrivial) preferences in P^∞, an element of P^∞, where $P^\infty = Q^\infty \cup \{0\}$, and $Q^\infty = \{p \in l_1^+ : p_i \leqslant 1$ for all i, $p_j = 0$ and $p_k = 1$ for some $j, k\}$.

As in the finite dimensional case one can show that Q^∞ is in a one-to-one continuous correspondence with the boundary of a disk in l_1, i.e., with an infinite dimensional sphere in l_1.

Now, by Corollary 5.1, p. 109 of Bessaga and Pelczynski [2] and Kuiper [15] the space Q^∞ is homeomorphic to l_1, and, in particular, is contractible. This contrasts with the finite dimensional case, where spheres are not contractible and indeed not homeomorphic to euclidean space. Let H be the homeomorphism, $H: Q^\infty \to l_1$. Since the convex addition C in l_1 exists and it satisfies anonymity, continuity and respect of unanimity, the composition map $\phi = H^{-1} \circ C \circ H^k$ defined by

$$(Q^\infty)^k \xrightarrow{H^k} (l_1)^k \xrightarrow{C} l_1 \xrightarrow{H^{-1}} Q^\infty$$

is a continuous function $\phi:(Q^\infty)^k \to Q^\infty$ satisfying anonymity and respect of unanimity. Since we can repeat this procedure for each connected component of $(P^\infty)^k$, this proves existence. Clearly the map ϕ is a deformation of the convex addition rule C, i.e., a deformation of a Bergsonian rule.

Consider now the space of truncated sequences $T \subset Q$,

$$T = \{\{p\} \in Q \text{ with } p_M = 0 \text{ for } M \geqslant M_0\}.$$

This space is dense in l_1 with a (finite) measure. Consider the pointwise convergence topology of the space F of continuous functions $F = \{f:(l_1)^k \to l_1\}$.[8] The sequence of maps $\{\phi_d\}$, defined by the restrictions of ϕ to finite dimensional linear subspaces L_d whose dimensions define an unbounded sequence of integers $\{d\}$, converges to ϕ. Note that when restricted to any finite subspace of choices (i.e., when restricted to vectors of finite length) each map ϕ_d is anonymous. It follows by Theorem 1 that ϕ_d cannot respect unanimity on such subspace; in particular, it is not Pareto. Since the sequence of maps $\{\phi_d\}$ converges to the map ϕ, this completes the proof. ∎

From Theorem 2 we obtain immediately the analogue of Corollaries 1 and 2 for von Neumann–Morgenstern utilities:

COROLLARY 3. *With infinitely many prizes, the space of von Neumann–Morgenstern utilities is contractible.* ∎

COROLLARY 4. *With infinitely many prizes there exists a continuous anonymous aggregation rule for von Neumann–Morgenstern utility functions which respects unanimity. However, this rule is not robust since it is the limit of non-Pareto rules on arbitrarily large sets of lotteries.* ∎

[8] The *pointwise* convergence topology τ on F is defined by the convergence rule

$$\{f^i\} \xrightarrow{\tau} f_0 \quad \text{iff} \quad f^i(p) \to f_0(p) \quad \forall p \text{ in } l_1,$$

see [12].

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DEPARTMENT OF ECONOMICS, INTERNATIONAL AFFAIRS BUILDING, COLUMBIA UNIVERSITY, NEW YORK, NEW YORK 10027