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Qin, Wei-zhi and Rommeswinkel, Hendrik

National Taiwan University, National Taiwan University

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Conditionally additive utility representations

Wei-zhi Qin, Hendrik Rommeswinkel*

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Abstract

Advances in behavioral economics have made decision theoretic models increasingly complex. Utility models incorporating insights from psychology often lack additive separability, a major obstacle for decision theoretic axiomatizations. We address this challenge by providing a representation theorem for utility functions of the form $u(x, y, z) = f(x, z) + g(y, z)$. We call these representations conditionally additive as they are additively separable only when holding fixed z. We generalize the result to spaces with more than three dimensions. We provide axiomatizations for consumption preferences with reference points, as well as consumption preferences over time with dependence across time periods. Our results also allow us to generalize the theory of additive representations to simplexes.

1 Introduction

In an important contribution to utility theory, Debreu (1959) characterized what is known as additively separable preferences. If preferences are defined on a product space $\prod_{i \in I} X_i$ of goods $x_i \in X_i$, then $\sum_{i \in I} f_i(x_i)$ is an additive utility function. Debreu (1959) showed that certain assumptions on the preferences of a consumer hold if and only if these preferences can be represented by an additive utility function. A wide class of problems can be addressed with such utility functions. In preferences over time, we often assume that the consumption in one time period has no effect on the marginal utility of consumption in another period. Constant elasticity of substitution preferences over goods spaces have an additive representation. In economic policy evaluation, utilitarian policy makers have additively separable preferences across individuals.

However, in the more recent literature, economic models have introduced more nuanced preferences in many of these cases. Consumption preferences for example may depend on reference points. In the case of preferences over time, the marginal utility of consumption in one period may depend on the consumption in the previous period. Policy makers who are not utilitarian may care about inequality, diversity, or the freedom of individuals, which usually lead to preferences which are not additively separable.

In the present paper, we generalize the idea of additively separable preferences to what we call conditionally separable preferences. Consider the example of preferences over consumption x_t in three periods of time t. To make

*National Taiwan University, Department of Economics, No. 1, Sec. 4, Roosevelt Rd., Taipei 106, Taiwan.

the example more salient, let $t = 1$ be breakfast, $t = 2$ lunch, and $t = 3$ dinner. Additively separable preferences yield a utility representation such as $f_1(x_1) + f_2(x_2) + f_3(x_3)$. In this case, the breakfast has no bearing on what one prefers to have for lunch or dinner. However, suppose an individual prefers not to eat the same dish twice in a row or prefers to eat a small dinner if the lunch was large. In this case we instead have a conditionally additive utility representation $f_1(x_1, x_2) + f_2(x_2, x_3)$. In this representation we say that breakfast x_1 and dinner x_3 are additively separable conditionally on lunch x_2. This representation allows for interdependence of preferences between breakfast and lunch and between lunch and dinner, but no interdependence between breakfast and dinner.

We provide an axiomatization for such conditionally additive utility representations. Maintaining the usual continuity and order assumptions, our axiomatization differs from axiomatizations of additive utility representations in two ways.

First, we weaken the independence assumptions such that we require only x_1 and x_3 to be independent of each other for fixed x_2:

$$\begin{array}{rc} & (x_1, x_2, x_3) \succsim (x_1', x_2, x_3) \\ \Leftrightarrow & (x_1, x_2, x_3') \succsim (x_1', x_2, x_3') \\ & \text{and} \\ & (x_1, x_2, x_3) \succsim (x_1, x_2, x_3') \\ \Leftrightarrow & (x_1', x_2, x_3) \succsim (x_1', x_2, x_3'). \end{array} \tag{1}$$

Additive utility functions over all three components would require x_1 to be independent of (x_2, x_3) and x_2 to be independent of (x_1, x_3).

Second, we weaken the Reidemeister condition[1] to a condition we call coseparability. The Reidemeister condition is a necessary condition for additive representations of the kind $f(x_1) + f_2(x_2)$. In two dimensions, it states:

$$\begin{array}{rc} & (x_1, x_2) \sim (\bar{x}_1, \bar{x}_2) \\ \wedge & (x_1', x_2) \sim (\bar{x}_1', \bar{x}_2) \\ \wedge & (x_1, x_2') \sim (\bar{x}_1, \bar{x}_2') \\ \Rightarrow & (x_1', x_2) \sim (\bar{x}_1', \bar{x}_2') \end{array} \tag{2}$$

In additive representation theorems with at least three dimensions the Reidemeister condition is implied by the independence conditions and continuity. However, even though our representation contains three dimensions, we only have two (conditionally) independent dimensions, requiring the use of an additional assumption. If we apply the Reidemeister condition for fixed x_2 only, we would obtain representations of the type $f_1(f_2(x_1, x_2) + f_3(x_2, x_3), x_2)$. Requiring the Reidemeister condition on the entire space would be unnecessarily strong and is not a necessary condition for conditionally additive representabil-

[1]This condition, which first appeared in Reidemeister (1929), is also often called the hexagon condition (e.g., Karni (1998)).

ity. Instead, our coseparability axiom requires:

$$\begin{aligned}
& (x_1, x_2, x_3) \sim (\bar{x}_1, \bar{x}_2, \bar{x}_3) \\
\wedge \quad & (x_1', x_2, x_3) \sim (\bar{x}_1', \bar{x}_2, \bar{x}_3) \\
\wedge \quad & (x_1, x_2, x_3') \sim (\bar{x}_1, \bar{x}_2, \bar{x}_3') \\
\Rightarrow \quad & (x_1', x_2, x_3') \sim (\bar{x}_1', \bar{x}_2, \bar{x}_3')
\end{aligned} \tag{3}$$

Coseparability ensures that the additive utility functions across each value of x_2 are cardinally comparable, yielding a representation $f(x,z) + g(y,z)$.

We extend our results in several ways. Firstly, we extend our results to finitely many dimensions. Unlike additive representations, conditionally additive representations have more than one natural extension to higher dimensions. We provide axiomatizations for the following finite dimensional functional forms:

- A reference dependent representation of the form $\sum_i u_i(x_i, x_1)$ where x_1 can be interpreted as a reference point according to which the other components are evaluated. To show how our results can be used, we axiomatize a generalization of inequity aversion preferences of Fehr and Schmidt (1999).

- A dynamic dependence representation of the form $\sum_i u_i(x_i, x_{i-1})$ where the utility gain of each component x_i depends on x_{i-1}. Such preferences are naturally suited for modeling time preferences. In particular, we axiomatize a generalization of preferences from the macroeconomic literature used in Kydland and Prescott (1982).

- A generalization of rank-dependent utility models of the functional form $\sum_i f_i(x_i, y_i) + g_i(x_i, y_{i-1})$ where the components are ordered by some order $\geqslant$ such that $x_{i+1} \geqslant x_i$ and $y_{i+1} \geqslant y_i$. A classical example of a rank-dependent expected utility model is cumulative prospect theory (Tversky and Kahneman (1992); Wakker and Tversky (1993)). Other potential applications come from the literature on rank-dependent utility models (Abdellaoui (2009)).

Secondly, we use our results to obtain additive representation theorems for spaces previously not covered by the literature, simplices and special types of surfaces. Simplices are often used in economics in division problems, where a finite amount of a resource is shared between n players, and as lottery spaces on finite states. We provide an axiomatization of additive and conditionally additive representation on simplices and a class of hypersurfaces. Previously, the difficulty in axiomatizing additive representations on simplices was their empty interior when viewed as a subset of a product space. The empty interior implies that the usual independence and Reidemeister conditions are either meaningless or only hold when holding fixed the sum of several dimensions. This is where we can use our main result to derive representation theorems despite these difficulties. To our knowledge, this is the first representation theorem of additively separable utility functions on spaces with an *empty* interior in the product topology. Potential applications are utilitarian preferences in cake division problems, von Neumann-Morgenstern preferences on lotteries with arbitrary probability distortions, or measures of diversity and inequality.

Thirdly, we show how our new axiom can also be used for conditionally linear representation theorems on mixture spaces. In real valued vector space where a point can be written as $(x_1, \ldots, x_n, z)$, a conditionally linear representation has the form $\sum_{i=1}^{n} x_i u_i(z)$. We extend the results on measurable utility of Herstein and Milnor (1953) to conditionally measurable utility. As an example application we derive a representation theorem for simultaneous choices under risk with known probabilities and under uncertainty with unknown probabilities. The representation yields a decision maker who behaves like a von Neumann Morgenstern expected utility maximizer on decisions under risk but may have arbitrary preferences when facing uncertainty.

Our results are related to the literature on additive representations (Wakker (1989)). When holding fixed the conditional dimension, we are similarly general as Wakker and Chateauneuf (1993), in fact our main representation result builds on their work. Our result on additive representations on simplices extends their work to a space with a nonempty interior, though with the use of an additional axiom. Other forms of conditional preferences have been explored in Drèze and Rustichini (1999), Wang (2003), Chew and Sagi (2008), and Wakai (2007). Other references specific to particular representations will be given in the main text.

The paper continues as follows. First, we will introduce some basic notation and definitions (Section 2). In Section 3, we state the main representation theorem for the case of subsets of three dimensional product spaces and provide an intuition for the proof. The following Section 4 covers the finite dimensional case. In Section 5 we cover representations on surfaces. Section 6 covers linear representations on mixture spaces. Unless otherwise noted, all proofs are provided in the appendix.

We provide a set of example applications and connections to literatures where conditionally additive representations are being used. In Section 4.1 we provide an extensive example application of our reference dependent representation to inequity aversion preferences. A short example application to preferences used in macroeconomics is provided for our dynamically dependent representations in Section 4.2. In Section 5 we discuss utilitarian preferences in cake division problems. Finally, Section 6 relates our results on mixture spaces to simultaneous decisions under risk and uncertainty.

2 Model and Notation

Let $S \subseteq \prod_{i=0}^{n} X_i$ be a product space where all X_i are connected and separable topological spaces. A generic element of S will be denoted by s and its i'th component by x_i. For notational convenience, we will often gather various dimensions together such that $S \subseteq X \times Y \times Z$ where $X = \prod_{i \in I_X} X_i$, $Y = \prod_{i \in I_Y} X_i$, $Z = \prod_{i \in I_Z} X_i$. The X (and analogously the Y, Z) components of s will be denoted by x (and y, z, respectively). Thus, s can be either written as $(x_0, \ldots, x_n)$ or as (x, y, z).

We will often refer to cylinders of S, i.e. preimages of the canonical projections from S to its components X, Y, Z. If $\hat{Z} \subseteq Z$, then $S_{\hat{Z}} = \{(x, y, z) \in S : z \in \hat{Z}\}$ is the cylinder above $\hat{Z}$ and the corresponding notations for X and Y. For singletons, we denote the cylinder above $x_0 \in X, y_0 \in Y$ by $S_{x_0,y_0} = \{(x, y, z) \in S : x = x_0, y = y_o\}$. Due to the special role of Z in the following we will often call S_z the z-layer for any $z \in Z$. For the images of the canonical projections, we denote $X_{\hat{S}} = \{x \in X : (x, y, z) \in \hat{S}\}$. We can combine these notations, $Z_{S_{x_0,y_0}} = \{z \in Z : (x_0, y_0, z) \in S\}$, etc..

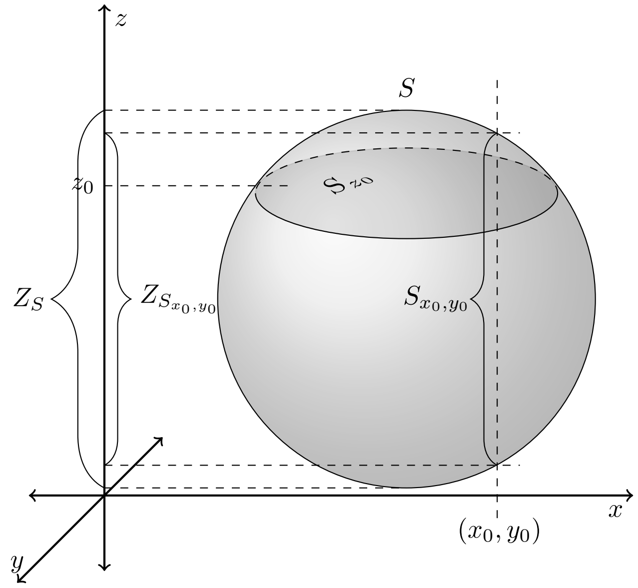

Figure 1: Example of cylinder set notation

As an example (see Figure 1) consider a sphere S as a subset of the product space $\mathbb{R}^3$. S_{z_0} is a disc where all points have the same Z coordinate, z_0, and may also be called the z_0-layer. Similarly, S_{x_0,y_0} is an open segment of a vertical line. $Z_{S_{x_0,y_0}}$ is the projection of S_{x_0,y_0} to the Z dimension. If $\bar{Z} = \{z \in Z : z \geq z_0\}$, then $S_{\bar{Z}}$ is the spherical cap above z_0.

$\succsim$ is a relation on S, i.e., a subset of $S \times S$. We say that s is weakly preferred to s' if $s \succsim s'$. We assume throughout the paper that $\succsim$ is complete and transitive and then call it a preference relation.[2] Let $\succ$ be the strict part of $\succsim$ and $\sim$ the symmetric part. We say that two subsets $S', S'' \subseteq S$ have strictly overlapping indifference curves if there exist points $s, s' \in S'$ and $s'', s''' \in S''$ such that $s \succ s'$, $s'' \succ s'''$ and $s \succsim s'' \succ s'$. A representation is a real valued function $u : S \to \mathbb{R}$ such that $u(s) \geq u(s') \Leftrightarrow s \succsim s'$ for all $s, s' \in S$ and we say that u represents $\succsim$.

All topological concepts used in this paper (product and order topologies, connectedness, separability,...) are standard definitions as can be found in Munkres (2014). To keep track of the different topologies and the continuity

[2]The results can possibly be generalized by dropping the completeness assumption. Vind (1991) gives a representation theorem for additive utility functions without completeness.

properties of various functions, we introduce the following notation.

Definition 1. Topology notation:
i) $t^{\prod_{i\in I^*} X_i}$ denotes the collection of all open sets induced by the product topology on the space $\prod_{i\in I^*} X_i$
ii) For a subset $S \subseteq \prod_{i=0}^n X_i$, $t_S^{\prod_{i\in I^*} X_i}$ denotes the projections of all open sets induced by the subspace topology of $t^{\prod_{i=0}^n X_i}$ to $\prod_{i\in I^*} X_i$
iii) For a subset $S \subseteq \prod_{i=0}^n X_i$ and a relation $\succsim$ on $\prod_{i=0}^n X_i$, τ_S denotes the collection of all open sets induced by the order topology generated by $\succsim$ on S.

Since we assume that S is only a subset of a product space we require further conditions to ensure that this subset is well behaved. The following assumptions are adaptions from the requirements of Wakker and Chateauneuf (1993) to our setting.

Definition 2. The subset S is well behaved given Z if for all $z^* \in Z$
i) S is connected and open in $t^{\prod_{i=1}^n X_i}$
ii,a) for all $i \notin I_Z$ and $x_i^* \in X_i$ $\{(s) \in S : x_i = x_i^* \wedge z = z^*\}$ is connected
ii,b) S_{z^*} is connected
iii) all equivalence classes in $int(S_{z^*})$ are connected.

If the sphere in Figure 1 does not contain its boundary points, it fulfills the well behavedness assumptions subject to $\succsim$ fulfilling iii).

Our topological assumptions on each X_i guarantee that X, Y, Z are connected, separable spaces. If X is a set of breakfast options this excludes finite sets such as $X = \{\text{boiled egg}, \text{sandwich}\}$ but allows for sets which specify the weight or calorie value of the breakfast. For example, each $x \in X$ may be a statement of the kind "100 g of egg, 200 g of sandwich, 400 kcal total" making X a subset of a three-dimensional vector space. However, X may also consist of lotteries over breakfast options. In this case, X is a function space and connectedness and separability can be guaranteed by allowing the consumer to choose any mixture of available lotteries. Later, in Remark 2 and Lemma 8 we will argue that for the Z dimension finite sets such as $Z = \{\text{vegetarian}, \text{beef}, \text{fish}\}$ are also permissible.

The space S is an open subset of $X \times Y \times Z$. This means that choosing a certain breakfast may preclude the consumer to choose a certain dinner (for example due to financial or dietary constraints). The openness of S in the product topology is automatically implied in case $S = X \times Y \times Z$. Thus, a product of closed subsets of $\mathbb{R}$ such as $[0, 400] \times [0, 900] \times [0, 1000]$ is permissible. Spaces excluded by this condition are for example spaces such as $(x, y, z) \in \mathbb{R}^3_{\geq 0}$ with the constraint $x + y + z \leq 100$. The reason for this is the "Eiffel-tower problem" according to which on certain points of the boundary of S we may have infinite utility values. Wakker and Chateauneuf (1993) carefully discuss this problem and state conditions under which additive utility representations can be obtained also on closed subsets of product spaces. Our remaining well-behavedness assumptions imply the well behavedness assumptions of Wakker and Chateauneuf (1993) on every z-layer. In essence, "holes" in the space and cases where indifference curves are disconnected present problems for additive representability, and thus conditionally additive representability.[3]

[3]Segal (1992) showed some of these conditions can be relaxed for subsets of real spaces.

Definition 3. Continuity:
$\succsim$ is continuous if the sets $\overline{S}(s) = \{s' \in S : s' \succ s\}$ and $\underline{S}(s) = \{s' \in S : s \succ s'\}$ belong to $t_S^{\prod_{i=0}^n X_i}$ for all $s \in S$.

The continuity assumption is standard. It requires that for every alternative the sets of strictly better and strictly worse options are open in the subspace topology of the product topology.

Definition 4. Essentiality:
X is essential if for all $x \in X$ there exist $(y, z) \in Y \times Z$ and $(y', z') \in Y \times Z$ such that $(x, y, z) \succ (x, y', z')$.
X is essential given Z if for all $x \in X$ and all $z \in Z$ there exist $y \in Y$ and $y' \in Y$ such that $(x, y, z) \succ (x, y', z)$.
$\succsim$ is essential if X, Y, Z are essential.
$\succsim$ is essential given Z if X and Y are essential given Z.

Essentiality given Z requires the choice of X and Y "to matter" at every point. Having an amazing dinner does not mean that breakfast is irrelevant. However, essentiality given Z allows lunch Z to have no impact on preferences.

Before we discuss the axioms driving our main result, it makes sense to revisit the standard axioms used in additive representation theorems.

Definition 5. Independence:
X is independent (of Y) if for all $x, x' \in X$ and $y, y' \in Y$ such that the following points are in S, we have:

$$\begin{aligned} & (x, y) \succsim (x', y) \\ \Leftrightarrow \quad & (x, y') \succsim (x', y'). \end{aligned} \tag{4}$$

$\succsim$ is independent with respect to X and Y if X and Y are independent.

Definition 6. Reidemeister Condition:
$\succsim$ fulfills the Reidemeister condition with respect to X and Y if for all $x, x', \bar{x}, \bar{x}' \in X$ and all $y, y', \bar{y}, \bar{y}' \in Y$ such that the following points are in S, we have:

$$\begin{aligned} & (x, y) \sim (\bar{x}', \bar{y}') \\ \wedge \quad & (x', y) \sim (\bar{x}, \bar{y}') \\ \wedge \quad & (x, y') \sim (\bar{x}', \bar{y}) \\ \Rightarrow \quad & (x', y') \sim (\bar{x}, \bar{y}). \end{aligned} \tag{5}$$

Together with continuity and essentiality, independence with respect to X and Y, and the Reidemeister condition with respect to X and Y guarantee additive representability of X and Y (Wakker (1989), Wakker and Chateauneuf (1993)). We now present weakenings of the above two axioms which allow for conditionally additive representations.

Definition 7. Conditional independence:
X is independent (of Y) given Z if for all $x, x' \in X$, $y, y' \in Y$, and $z \in Z$ such that the following points are in S, we have:

$$\begin{aligned} & (x, y, z) \succsim (x', y, z) \\ \Leftrightarrow \quad & (x, y', z) \succsim (x', y', z). \end{aligned} \tag{6}$$

$\succsim$ is independent with respect to X and Y given Z if X and Y are independent given Z.

Independence given Z states that if a change in breakfast from x to x' is beneficial, this change in X is beneficial independently of any change in the dinner y. Similarly, a change in dinner may not influence the preferences over breakfast. However, a change in breakfast or dinner may change the preferences over lunch options Z. For example, switching from no breakfast to a large breakfast may make large lunch options worse.

If X is independent of (Y, Z), then X is independent of Y given Z. Therefore, our conditional independence axiom can be seen as a weakening of the independence axiom.

Definition 8. Coseparability:
$\succsim$ fulfills coseparability with respect to X and Y given Z if for all $z, \bar{z} \in Z$ and all $x, x', \bar{x}, \bar{x}' \in X$ and all $y, y', \bar{y}, \bar{y}' \in Y$ such that the following points are in S, we have:

$$\begin{aligned} & (x, y, z) \sim (\bar{x}', \bar{y}', \bar{z}) \\ \wedge \quad & (x', y, z) \sim (\bar{x}, \bar{y}', \bar{z}) \\ \wedge \quad & (x, y', z) \sim (\bar{x}', \bar{y}, \bar{z}) \\ \Rightarrow \quad & (x', y', z) \sim (\bar{x}, \bar{y}, \bar{z}). \end{aligned} \tag{7}$$

Coseparability given Z strengthens the notion of conditional independence. We can interpret coseparability as an independence in improvements and worsenings. Suppose a change from (x, y, z) to (x', y, z) yields an improvement which is as good as an improvement from $(\bar{x}, \bar{y}, \bar{z})$ to $(\bar{x}', \bar{y}, \bar{z})$. Since we only changed x to x' and $\bar{x}$ to $\bar{x}'$, these changes can be seen as improvements in the breakfast of the consumer. The indifferences imply that both improvements are equally beneficial. Similarly, the changes from (x, y, z) to (x', y, z) and $(\bar{x}, \bar{y}, \bar{z})$ to $(\bar{x}, \bar{y}', \bar{z})$ can be seen as equally beneficial dinner improvements. Coseparability given Z holds if combining the breakfast and the dinner improvement for some lunch yields the same improvement as combining equally beneficial breakfast and dinner improvements for another lunch. This is plausible in the case where improvements in breakfast and dinner are comparable. If we instead assume X and Y are the consumption of two different persons (and maybe Z their consumption of a public good), this assumption would imply cardinal comparability of preferences. In welfare analysis, many economists would feel comfortable assuming conditional independence of the consumption of two persons but may not feel comfortable assuming cardinal comparability of their preferences. As a necessary condition for conditionally additive representability, it is important to consider whether one is indeed willing to commit to the coseparability condition before using a conditionally additive utility representation.

If $\succsim$ fulfills the Reidemeister condition with respect to X and $(Y \times Z)$ then it fulfills coseparability of X given Z, but not vice versa. Thus, coseparability is a weakening of the Reidemeister condition. However, assuming that $\succsim$ fulfills the Reidemeister condition with respect to X and Y on every subspace S_z for all $z \in Z$ is weaker than coseparability given Z. This is why assuming an additive representation on each S_z only yields a global representation of the form $h(f(x, z) + g(y, z), z)$.

To have a clean notation, we summarize conditional independence and coseparability in the following way.

Definition 9. To simplify notation, in the following we will write $X \perp Y \mid Z$ if

- $\succsim$ is independent with respect to X and Y given Z and
- $\succsim$ fulfills coseparability with respect to X and Y given Z.

We say that $\succsim$ fulfills restricted solvability given Z if for all $x, x' \in X$, $y, y' \in Y$, $z \in Z$, and $s \in S$: If $(x, y, z) \succsim s \succsim (x', y, z)$ then there exists x'' such that $(x'', y, z) \sim s$. If $(x, y, z) \succsim s \succsim (x, y', z)$ then there exists y'' such that $(x, y'', z) \sim s$.

3 Representation theorem for 3 dimensions

In this section, we will state our representation theorems for three dimensions and prove a lemma from which the main intuition of our result follows. The three dimensional case is the key building block for higher dimensional cases.

Theorem 1. *Let $\succsim$ be a continuous preference relation on a well behaved space $S \subseteq X \times Y \times Z$. Let $\succsim$ fulfill essentiality given Z.*
a) Then the following statements are equivalent:

(i) $\succsim$ fulfills $X \perp Y \mid Z$.

(ii) There exists a representation

$$\begin{aligned} v &: (S, \tau_S) \to \mathbb{R}, \\ f &: (X_S \times Z_S, t_S^{X_S \times Z_S}) \to \mathbb{R}, \\ g &: (Y_S \times Z_S, t_S^{X_S \times Z_S}) \to \mathbb{R}, \text{ and} \\ v(x, y, z) &= f(x, z) + g(y, z). \end{aligned} \tag{8}$$

b) v is continuous and unique up to positive affine transformations. f and g are continuous if Z is normal or if $Z_{S_{x_0, y_0}} = Z_S$. In the latter case, $v(x, y, z) = f(x, z) + g(y, z) + h(z)$ where $f(x_0, z) = 0$, $g(y_0, z) = 0$ and f, g are unique up to linear transformations and h is unique up to affine transformations.

Remark 1. If $S = X \times Y \times Z$, we do not need S to be well behaved. All assumptions summarized by S being well behaved are either trivially fulfilled by product spaces or not needed. In particular if each S_z is a product space we can drop Definition 2 iii,b) which states that all indifference classes of a well behaved space need to be connected on each subset S_z.

Remark 2. According to our assumptions, Z is connected and separable. Instead, we could also assume Z to be countable as long as for any two z_0, z_K there exists a finite sequence $(z_k)_{k=1}^K$ of elements of Z such that for all k the indifference curves of S_{z_k} and $S_{z_{k+1}}$ strictly overlap.

It is important to note that continuity of f and g requires further conditions. For most practical applications, the assumption of normality of Z is not restrictive. The main exception are function spaces from non-compact metric spaces to uncountable spaces. In additive representations, this problem never arises since we can replace the topology of each dimension with the order topology.

Due to its length, we delegate the proof of the representation theorem to the appendix with the exception of a Lemma which provides the main intuition behind the result and the proof of which links well with proofs of additive representations, in particular the proof of Theorem III.4.1. in Wakker (1989).

Lemma 1. *Let $\succsim$ be a continuous preference relation on $S = X \times Y \times Z$ where X, Y, Z are connected and separable topological spaces. Let $\succsim$ be essential given Z and fulfill $X \perp Y \mid Z$. Then:*
For any pair $z', z'' \in Z$ for which $S_{z'}$ and $S_{z''}$ have strictly overlapping indifference curves there exists a utility representation on $S_{\{z',z''\}}$ such that:

$$u(x, y, z) = f(x, z) + g(y, z) + h(z). \tag{9}$$

We included Lemma 1 and its proof into the main text for two reasons. First, it gives an insight into the utility construction process and how this process differs from the procedure for additive representations. Second, Lemma 1 is of independent interest: it proves Remark 2 since given the result for two z-layers, the extension to countably many layers is trivial.

Proof. We start out by constructing a utility function on $S_{z'} = X \times Y \times \{z'\}$. We use the same utility construction process as in Wakker (1989). Using a standard argument, we can ensure $\succsim$ satisfies restricted solvability given Z (see Lemma 2 in the appendix). Essentiality given Z guarantees that there exist x_0,$x_1 \in X$ and y_0 $y_1 \in Y$ such that

$$\begin{aligned}
&(x_0, y_0, z') \prec (x_1, y_0, z') \\
&(x_0, y_0, z') \prec (x_0, y_1, z') \\
&(x_1, y_0, z') \sim (x_0, y_1, z').
\end{aligned} \tag{10}$$

By independence given Z we have $(x_1, y_1, z') \succ (x_1, y_0, z') \succ (x_0, y_0, z')$ and we assign utility values

$$u(x_1, y_1, z') = 2, \quad u(x_1, y_0, z') = 1, \quad u(x_0, y_0, z') = 0. \tag{11}$$

Next, since coseparability with respect to X and Y given Z implies the Reidemeister condition with respect to X and Y on each z-layer $X \times Y \times \{z\}$, we can construct an order grid on the z'-layer such that for any rational numbers n, n', m, m' we have

$$\begin{aligned}
& (x_n, y_m, z') \sim (x_{n'}, y_{m'}, z') \\
\Leftrightarrow \quad & u(x_n, y_m, z') = n + m = n' + m' = u(x_{n'}, y_{m'}, z').
\end{aligned} \tag{12}$$

We provide an intuition for this construction process in Figure 2. For details of how to construct this grid, see Wakker (1989).

We next extend this representation to the z''-layer (Figure 3). Since the indifference curves of $S_{z'}$ and $S_{z''}$ strictly overlap, we can find points

$$(\overline{x'}, \overline{y'}, z') \succsim (\overline{x''}, \overline{y''}, z'') \succ (\underline{x''}, \underline{y''}, z'') \succsim (\underline{x'}, \underline{y'}, z') \tag{13}$$

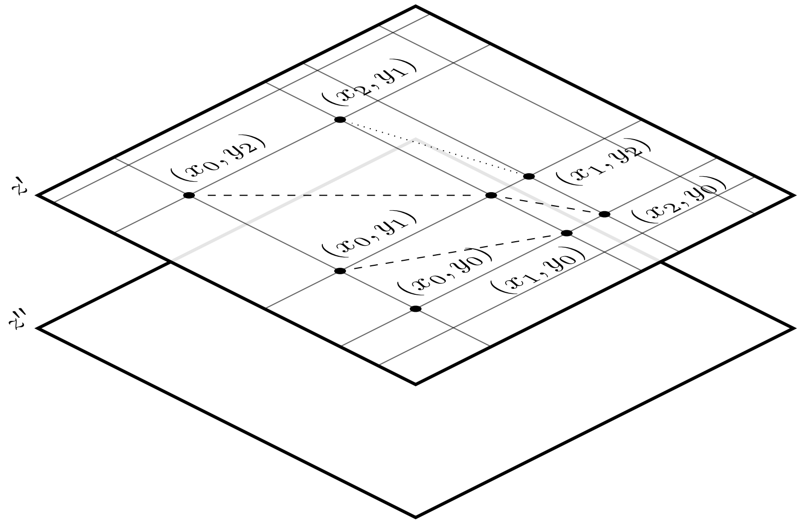

Figure 2: Construction of the utility grid on the z'-layer.

By restricted solvability given Z, the points (x_0, y_1, z'), (x_1, y_0, z') are chosen to be indifferent to each other. Similarly, the points (x_0, y_2, z'), (x_2, y_0, z') are chosen to be indifferent to (x_1, y_1, z') (dashed indifferences). Coseparability given Z then guarantees $(x_2, y_1, z') \sim (x_1, y_2, z')$ (dotted indifference). These indifferences allow us to guarantee $(x_n, y_m, z') \sim (x_{n'}, y_{m'}, z')$ iff $n + m = n' + m'$. In a next step, this grid is made dense in $S_{z'}$.

Since our grid is dense in the z'-layer, we can find a grid point (x_{n_1}, y_{n_2}, z') on the z'-layer such that $(\underline{x'}, \underline{y'}, z') \prec (x_{n_1}, y_{n_2}, z') \prec (\overline{x'}, \overline{y'}, z')$. Therefore, by restricted solvability given Z, we can find a point $(\bar{x}, \bar{y}, z'') \sim (x_{n_1}, y_{n_2}, z')$. Next, we construct the grid on both z-layers in the following way. We use the point $(\bar{x}_0, \bar{y}_0, z'')$ on the z''-layer satisfying $(\bar{x}_0, \bar{y}_0, z'') \sim (x_{n_1}, y_{n_2}, z')$ as the center on the z''-layer and construct the grid with an initial point $\bar{x}_1, \bar{y}_0$ satisfying $(\bar{x}_1, \bar{y}_0, z'') \sim (x_{n_1+1}, y_{n_2}, z')$. These points exist by restricted solvability given Z and by the fact that we can choose our initial points (x_0, y_0, z') and (x_1, y_0, z') to be arbitrarily close to each other.

We now show that the grid points are indeed consistent on both layers (Figure 4). That is, we want to show that

$$
\begin{aligned}
(x_{n+1}, y_m, z') &\sim (x_n, y_{m+1}, z') \\
(\bar{x}_{n+1}, \bar{y}_m, z'') &\sim (\bar{x}_n, \bar{y}_{m+1}, z'') \\
(x_n, y_m, z') &\sim (\bar{x}_{n_1+n}, \bar{y}_{n_2+m}, z'')
\end{aligned} \tag{14}
$$

for all n, m.

Similar to the argument of Wakker (1989), we use induction on our subscripts. For $n + m = 0$, the result directly follows from $(\bar{x}, \bar{y}, z'') \sim (x_{n_1}, y_{n_2}, z')$. For $n+m = 1$ the condition follows from the construction of the grid. For $n+m \geq 2$,

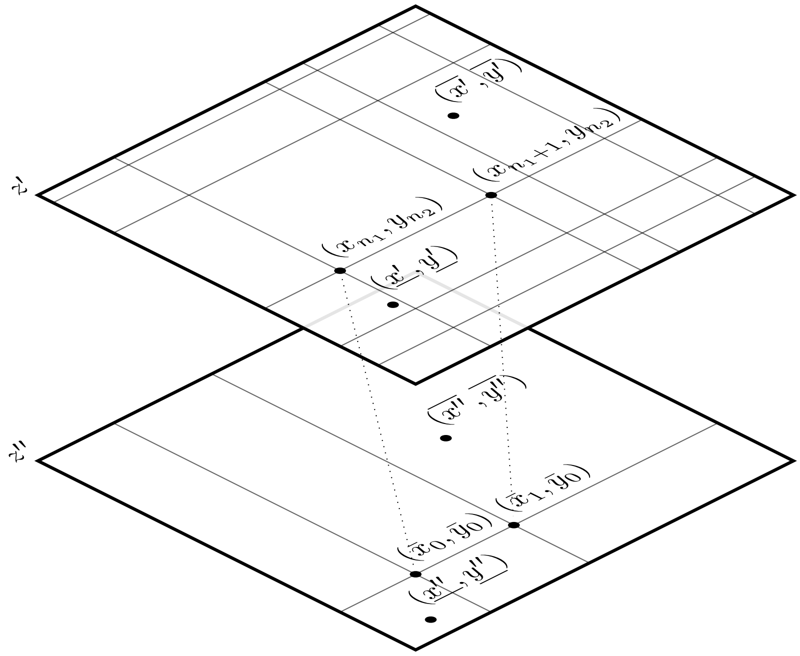

Figure 3: Extension of the utility grid from the z' layer to the z'' layer.
The existence of points (x_{n_1}, y_{n_2}) and (x_{n_1+1}, y_{n_2}) such that both are worse than (x'', y'') and better than $(\underline{x''}, \underline{y''})$ follows from the gridpoints being dense in $S_{z'}$. The existence of the points $(\bar{x}_0, \bar{y}_0)$ and $(\bar{x}_1, \bar{y}_0)$ follows from restricted solvability given Z. indifference $x_{n_1}, y_{n_1} \sim (\bar{x}_0, \bar{y}_0)$

we simply notice that coseparability given Z implies

$$\begin{aligned}
(x_{n_1+n-2}, y_{n_2}, z') &\sim (\bar{x}_{n-2}, \bar{y}_0, z'') \\
(x_{n_1+n-1}, y_{n_2}, z') &\sim (\bar{x}_{n-1}, \bar{y}_0, z'') \\
(x_{n_1+n-2}, y_{n_2+1}, z') &\sim (\bar{x}_{n-2}, \bar{y}_1, z'') \\
&\text{and therefore} \\
(x_{n_1+n-1}, y_{n_2+1}, z') &\sim (\bar{x}_{n-1}, \bar{y}_1, z''). \qquad (15)
\end{aligned}$$

We can extend the integer-valued grid on the z''-layer to a rational-valued grid by the same method as in Wakker (1989). Via transitivity and the fact that for any $x_n, n \in \mathbb{Q}$ we can find y_m such that there exist $x_{n'}, n' \in \mathbb{Z}$ and $y_{m'}, m' \in \mathbb{Z}$ such that $n + m = n' + m'$ and thus $(x_n, y_m, z'') \sim (x_{n'}, y_{m'}, z'')$, the extended grid on the rationals is also consistent.

Next, we define the functions

$$\begin{aligned}
f(x_n, z') &:= n \\
f(\bar{x}_n, z'') &:= n \\
g(y_m, z') &:= m \\
g(\bar{y}_m, z'') &:= m \\
h(z'') &:= n_1 + n_2 \\
h(z') &:= 0 \qquad (16)
\end{aligned}$$

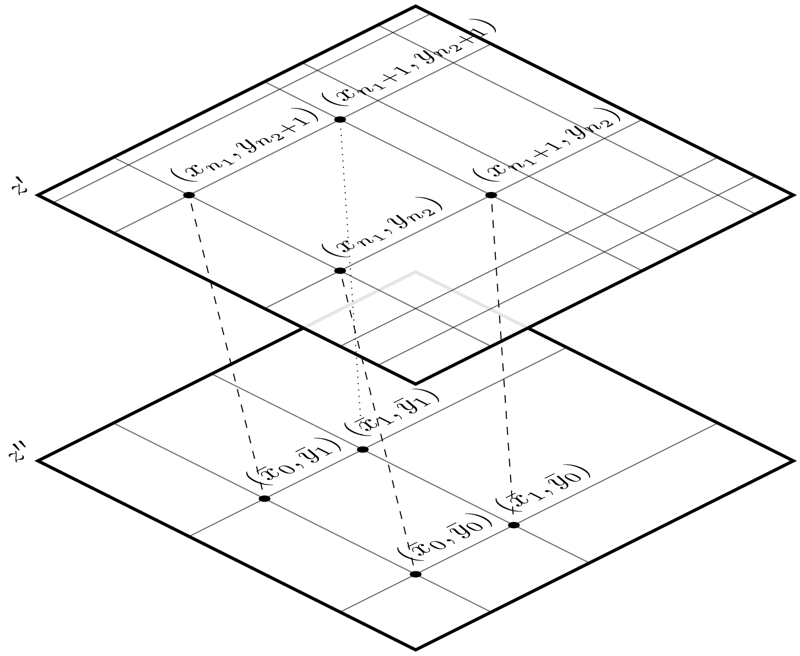

Figure 4: Consistency of the utility grid between the z'-layer and the z''-layer. The dashed indifferences follow from the construction process of the grid. The dashed indifference follows from coseparability given Z.

Since our grid is dense in the z' and z''-layers, due to continuity we can extend the utility functions on the entire z' and z''-layers by taking the limit to obtain a continuous additive utility representation $u(x, y, z) = f(x, z) + g(y, z) + h(z)$ on both layers. □

In summary, the construction of the utility representation on a single layer follows the construction by Wakker (1989). In this step, our coseparability condition fulfills the same role as the Reidemeister condition in Wakker (1989): if a preference relation over a product space $X \times Y$ is continuous, essential, and independent, the Reidemeister condition is necessary and sufficient to ensure that an additive representation exists.

When extending the representation to the second layer, coseparability given Z fulfills an additional role: it makes the additive representations on both layers consistent with each other. Assuming only the Reidemeister condition on each S_z without our generalization, we could obtain an additive representation on each S_z. But notice that for example the preference relation induced by the utility function $(f(x)+g(y))^{h(z)}$ has an additive representation on each z-layer, but does not necessarily have a utility representation of the desired form. The coseparability condition excludes such preferences.

4 Representation theorems for finitely many dimensions

In the following, we will extend our representation result to product spaces of higher dimensions. Notice that as soon as there are more than three dimensions, different extensions are possible. In terms of utility functions, we may for example be interested in the conditions which yield a representation of the kind $f(x_2, x_1) + g(x_3, x_1) + h(x_4, x_1)$ or instead $f(x_1, x_2) + g(x_2, x_3) + h(x_3, x_4)$. In the following, we will consider some cases we found interesting. We hope that our treatment of these cases is instructive for the cases we omit.

4.1 Reference dependent preferences

We start out with the simplest generalization to finite dimensional spaces, the case where we have a conditionally additive representation on $n-1$ dimensions given the first dimension. We call these preferences reference dependent as we can interpret the first dimension as a reference point by which the remaining dimensions are evaluated.

Theorem 2. *Let $\succsim$ be a continuous preference relation on a well behaved space $S \subseteq Z \times \prod_{i=1}^{n} X_i$, $2 \leq i < \infty$ where all X_i and Z are connected and separable topological spaces. Let $\succsim$ be essential given Z. Then the following statements are equivalent:*

(i) $\succsim$ *fulfills for all i:*

- $X_i \perp \prod_{j \neq i} X_j \mid Z$ *and*
- $X_i \times X_{i+1} \perp \prod_{k \neq i, i+1} X_k \mid Z$.

(ii) $\succsim$ *can be represented by a continuous function $v(s) = \sum_{i=2}^{n} f_i(x_i, z)$ unique up to affine transformations.*

Remark 3. Continuity of the additive components f_i can be guaranteed in a similar manner as in Theorem 1 by assuming normality of Z or $Z_{S_{x_0,y_0}} = Z_S$.

In order to extend our results to more than three dimensions we need to impose additional independence conditions $X_i \times X_j \perp \ldots \mid Z$. This is unsurprising given the work of Gorman (1968). Without the additional conditional independence assumptions, additive representations on each z-layer may not exist.

Sugden (2003) axiomatizes reference dependent preferences in a Savage (1972) framework. Due to the differences in the framework, it is hard to compare the two models in their generality. One difference is that our representation is additive given the reference point, while Sugden (2003) axiomatizes a (reference dependent) subjective expected utility representation. Sugden (2003) allows for an uncountable event space as opposed to the finite number of dimensions in our space. The functional form axiomatized by Sugden (2003) is therefore neither a special case nor a generalization of Theorem 2.

We now provide an example application where we derive a generalization of previous axiomatizations of Fehr and Schmidt (1999) using Theorem 2. In a product space $\prod_{i \in I} X_i$ let $x_i \in X_i = \mathbb{R}_{\geq 0}$ be the income of individual i. Fehr and Schmidt (1999) inequity aversion preferences of individual i are represented by a utility function of the form:

$$u_i(s) = x_i - \sum_{j \neq i} \alpha_i \max(0, x_j - x_i) - \beta_i \max(0, x_i - x_j). \qquad (17)$$

Rohde (2010) axiomatized the functional form of Fehr and Schmidt (1999) using the strong linearity assumptions fulfilled by the model. The linearity assumptions prescribe (i) indifference about redistribution of income among individuals with better income, and (ii) redistribution among indiviudals with worse incomes. Neilson (2006) provides a more general representation theorem by dropping linearity of the model but maintaining linearity in the comparisons between individual i's and any other j's income. One avenue for generalizations mentioned in Rohde (2010) are rank-dependent utility models. Here, we try the conditionally additive utility approach.

It is straightforward to obtain the representation $u(x_1, \ldots, x_n) = u_i(x_i) + \sum_{j \in I \setminus \{i\}} u_j(x_j, x_i)$ via Theorem 2. Having obtained such a representation, the remaining axioms for a theory of inequity aversion easily fall into place.

Definition 10. Anonymity:
If $s, s' \in \prod_{i \in I} X_i$, $x_i = x'_i$ and s' is a perturbation of s, then $s \sim s'$.

This axiom requires that the individual cares about others' incomes in an identical manner. The neighbor's income, for example, has exactly the same effect on the individual's income as the income of any other person.

Definition 11. Envy:
If for all $j, k \neq i$ we have $\overline{x}_j > x_j \geq x_i$, then

$$(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) \succ (x_1, \ldots, x_{j-1}, \overline{x}_j, x_{j+1}, \ldots, x_n).$$

The envy axiom implies that the individual cares negatively about anybody's income exceeding their own.

Definition 12. Compassion:
If for all $j, k \neq i$ we have $x_i \geq x_j > \underline{x}_j$ then

$$(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) \succ (x_1, \ldots, x_{j-1}, \underline{x}_j, x_{j+1}, \ldots, x_n).$$

The compassion axiom means that an individual cares positively about anybody's income below their own income. Envy and compassion are slight strengthenings of the inequity aversion axiom in Rohde (2010).

Corollary 1. *Let $S = \prod_{i=1}^n X_i$, and $\succsim$ be a continuous preference relation on S. Let $\succsim$ be essential given X_i. Then the following statements are equivalent:*

(i) $\succsim$ fulfills symmetry, envy, compassion, and for all j, k:

- $X_j \perp \prod_{k \neq i,j} X_k \mid X_i$ *and*
- $X_j \times X_{j+1} \perp_{k \neq i,j,j+1} X_k \mid X_i$.

(ii) $\succsim$ can be represented by:

$$u(x) = v(x_i) + \sum_{j \neq i} u(x_j, x_i) \qquad (18)$$

where u is continuous and strictly increasing (strictly decreasing) in x_j if and only if $x_j \leq (\geq) x_i$.

The usefulness of Theorem 2 is the simplicity with which this result can be proven. We provide the proof directly:

Proof. From Theorem 2, we have that $\succsim$ can be represented by:

$$u(x_1, \ldots, x_n) = v(x_i) + \sum_{j \in I \setminus \{i\}} u_j(x_j, x_i).$$

Using a perturbation of the incomes of three individuals j, k, l, we have

$$\begin{aligned} u_j(x_j, x_i) + u_k(x_k, x_i) &= u_j(x_k, x_i) + u_k(x_j, x_i) \\ u_j(x_j, x_i) + u_l(x_l, x_i) &= u_j(x_l, x_i) + u_l(x_j, x_i) \\ u_k(x_k, x_i) + u_l(x_l, x_i) &= u_k(x_l, x_i) + u_l(x_k, x_i). \end{aligned}$$

If $x_j = x_l = x^*$ it is straightforward to derive $u_j(x^*, x_i) - u_k(x^*, x_i)$ is constant. Since our choice of x^*, x_i, k, l was arbitrary and each u_k is unique up to affine transformations, we have for all k, l: $u_k = u_l = u$. Thus,

$$u(x_1, \ldots, x_n) = v(x_i) + \sum_{j \in I \setminus \{i\}} u(x_j, x_i).$$

The fact that u must be increasing/decreasing in its second argument for $x_j <(>) x_i$ follows directly from the remaining axioms. $\square$

Theorem 2 directly provides us with a functional form on which we can impose axioms descriptive of the behavior we are interested in. The generalization of the functional form of (17) would be of little interest if it could not capture some plausible deviations from the behavior implied by (17). One such deviation is a preference against inequity among the individuals poorer than oneself. Suppose the individuals are ordered by their income such that $x_1 \leq x_2 \leq \ldots \leq x_n$. The linear structure of (17) forces individual i to have preferences $(0, \ldots, 0, x_{i-1}, x_i, \ldots, x_n) \sim (\frac{x_{i-1}}{i-1}, \ldots, \frac{x_{i-1}}{i-1}, x_i, \ldots, x_n)$. Instead, we may want to impose that the individual also cares negatively about inequity of incomes below herself. We can do this by assuming:

$$x \succ (x_1 - \alpha, x_2 + \alpha, x_3, \ldots, x_i, \ldots, x_n) \tag{19}$$

for all x and $\alpha \in \mathbb{R}_{>0}$ such that $x_2 + \alpha \leq x_i$ and $x_2 \leq x_1$. It is then easy to show that $u(x_j, x_i)$ is strictly concave in x_j for $x_j \leq x_i$.

4.2 Dynamic dependence preferences

The previous subsection covers the case where all additive component functions f_i share one argument z. Another interesting representation arises if for all i we assume that the first $i-1$ dimensions are independent of the last $n-i$ dimensions given the i-th dimension.

Theorem 3. *Let $\succsim$ be a continuous preference relation on a well behaved space $S \subseteq \prod_{i=1}^n X_i$, $3 \leq i < \infty$ where all X_i are connected and separable topological spaces. For all i, let $\succsim$ fulfill essentiality given X_i. Then the following statements are equivalent:*

(i) $\succsim$ fulfills for all $i = 2, \ldots, n-1$: $\prod_{j=1}^{i-1} X_j \perp \prod_{k=i+1}^{n} X_k \mid X_i$.

(ii) $\succsim$ can be represented by $v(s) = \sum_{i=2}^n f_i(x_i, x_{i-1})$.

Remark 4. Continuity of each f_i can be guaranteed by each X_i being normal or S being a product space.

A natural application for this representation theorem are preferences over time which exibit dependence across time periods. Preferences over consumption streams need not be additively separable if individuals experience satiation and addiction, or form consumption habits (a growth model with time interdependence is given in Ryder and Heal (1973), for an axiomatization of habit preferences, see Rozen (2010)). In this case, preferences over consumption periods sufficiently distant in time may be additively separable when holding fixed the consumption in between. The above representation theorem captures the case where the marginal utility of consumption depends on the previous period's consumption. The overlapping number of dimensions can of course be increased by a corresponding change in the independence conditions.

More generally, time preferences provide a rich field of applications for our main theorem in deriving axiomatizations in the spirit of Theorem 3. For example, we may consider preferences as in Kydland and Prescott (1982) (for finite time periods):

$$\sum_{i=0}^{n} \beta^i u(x_i, \sum_{j=1}^{i} \alpha_i y_i) \tag{20}$$

where x_i is consumption at time i, y_i is leisure at time i, and α_i is a preference parameter. Using the exact same method of proof as in Theorem 3 we invite the reader to derive:

Corollary 2. *Let $\succsim$ be a continuous preference relation on $S = \prod_{i=1}^{n} X_i \times Y_i$, $3 \leq i < \infty$ where for all i $X_i = \mathbb{R}$ and $Y_i = \mathbb{R}$. For all i, let $\succsim$ fulfill essentiality given $\prod_{l=1}^{i} Y_l$. Then the following statements are equivalent:*

(i) $\succsim$ fulfills for all $i = 1, \ldots, n-1$: $X_i \perp \prod_{j \neq i}^{n} X_j \times \prod_{k=i+1}^{n} Y_k \mid \prod_{l=1}^{i} Y_l$.

(ii) $\succsim$ can be represented by $v(s) = \sum_{i=1}^{n} f_i(x_i, y_1, \ldots, y_i)$ where each f_i is continuous.

The common element in Theorem 3 and Corollary 2 is that both representations originate from combining additive representations in a sequential manner. In Corollary 2 each statement $\ldots \perp \ldots \mid \ldots$ tells us for some dimension i which of the $i-1$ dimensions before and $n-i$ dimensions after are conditionally independent. Theorem 1 gives us conditionally additive representations for each of these statements. The proof of Theorem 3 shows how to combine such representations into a dynamically dependent representation.

4.3 Rank dependent preferences

Lastly, we can obtain a generalization of Rank-dependent expected utility. Rank-dependent expected utility (Quiggin (1982)) has a representation:

$$\sum_{i=1}^{n} (v(y_i) - v(y_{i-1}))u(x_i) \tag{21}$$

where for all $i = 1, \ldots, n$ each x_i is a payoff and y_i is the probability of receiving a payoff worse or equal to x_i, thus $y_0 = 0$.

Theorem 4. *Let $(\mathcal{X}, \gtrdot), (\mathcal{Y}, \gtrdot)$ be ordered, connected, separable spaces endowed with the order topology induced by $\gtrdot$. Assume a well behaved space $S \subseteq \prod_{i=1}^{n} X_i \times Y_i$ such that for all i $X_i = \mathcal{X}$, $Y_i = \mathcal{Y}$ and:*

$$(x_1, y_1, \ldots, x_n, y_n) \in S \Leftrightarrow \forall i \in \{1, \ldots, n-1\} : x_{i+1} \gtrdot x_i, y_{i+1} \gtrdot y_i. \tag{22}$$

Let $\succsim$ be a continuous preference relation on S fulfilling for all i essentiality given X_i. Then the following statements are equivalent:

(i) For all $i = 2, \ldots, n-1$ the relation $\succsim$ fulfills

$$\begin{aligned} X_i \times \prod_{j=1}^{i-1} X_j \times Y_j \perp \prod_{k=i+1}^{n} X_k \times Y_k \mid Y_i, \text{ and} \\ \prod_{j=1}^{i-1} X_j \times Y_j \perp Y_i \times \prod_{k=i+1}^{n} X_k \times Y_k \mid X_i. \end{aligned} \tag{23}$$

(ii) $\succsim$ can be represented by:

$$v(s) = \sum_{i=1}^{n} f_i(x_i, y_i) + g_i(x_i, y_{i-1}) \tag{24}$$

where each f_i and g_i is continuous.

For the connection to Rank-dependent Expected Utility, suppose $\mathcal{X} = \mathbb{R}$, $\mathcal{Y} = [0, 1] \subset \mathbb{R}$, and the subset S is chosen such that for all i we have $x_i < x_{i+1}$, $y_i < y_{i+1}$. x_i can be interpreted as the ith lowest prize and y_i as the probability of receiving a prize lower or equal to x_i. Then setting $f_i(x_i, y_i) = u(x_i)w(y_i)$ and $g_i(x_i, y_{i-1}) = -u(x_i)w(y_{i-1})$ gives the familiar rank-dependent expected utility form. In fact, the representation is more general than Rank-linear utility first axiomatized in Segal (1989) (see also Puppe (1990); Wakker (1993); Segal (1993)). Rank-linear utility can be obtained by removing the subscript from f_i and g_i.

Theorem 4 only generalizes rank-dependent utility models for decisions under risk where the probabilities are known. Thus, the above result holds only for decisions under risk but not uncertainty. However, many results on rank-dependent representations for decisions under uncertainty with unknown probabilities are derived using additive representation theorems.[4] It would be interesting to apply conditionally additive representation theorems to rank dependent models of decisions under uncertainty.

5 Finite dimensional simplices

We can use Theorem 1 to provide an interesting new result on additive representations. So far, additive representations have only been axiomatized for sets with nonempty interiors in the product topology. However, an important class of spaces in economics which do not fulfill this requirement are simplices

[4]For example, see Chapter VI of Wakker (1989).

of the form $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i\theta_i = 1\}$. We encounter simplices for example in lottery spaces, as income shares in fair division problems, or as normalized price vectors. The literature lacks a result for such spaces because in this context the classical independence axiom is not well defined when looking at the independence of a single dimension. Take a statement such as $(x, y) \succsim (x', y) \Rightarrow (x, y') \succsim (x', y')$ where x is the first dimension of the simplex and y the remaining dimensions. Then by the properties of the simplex, $x = x'$ and the axiom is not meaningful. However, even if we consider x to contain more than one dimension, we run into difficulties. To show why, let us focus on a 4 dimensional simplex and let $x = (x_1, x_2)$ contain the first two dimensions and $y = (x_3, x_4)$ contain the other two dimensions. Now a statement such as $(x, y) \succsim (x', y) \Rightarrow (x, y') \succsim (x', y')$ only imposes restrictions on preferences on the subset of alternatives where $x_1 + x_2 = x_1' + x_2'$. Yet, this is exactly the case where conditionally additive representation theorems become useful.

Indeed, Theorem 1 helps us find conditionally additive representations of the form $f(x_1, x_1 + x_2) + g(x_3, x_1 + x_2)$. To see this, consider the following preferences $\succsim^*$ on a subset of a product set $(0, 1)^3$:

$$\begin{aligned} &(x_1, x_2, x_3, x_4) = (x, y) \succsim (x', y') = (x_1', x_2', x_3', x_4') \\ \Leftrightarrow \quad &(x_1, x_3, x_1 + x_2) \succsim^* (x_1', x_3', x_1' + x_2'). \end{aligned} \tag{25}$$

It is straightforward to verify that independence of X and Y in the relation $\succsim$ implies conditional independence of X_1 and X_3 given $X_1 + X_2$ in $\succsim^*$:

$$\begin{aligned} &(x_1, x_3, x_1 + x_2) \succsim^* (x_1', x_3, x_1' + x_2') \\ \Leftrightarrow \quad &(x, y) \succsim (x', y) \\ \Leftrightarrow \quad &(x, y') \succsim (x', y') \\ \Leftrightarrow \quad &(x_1, x_3', x_1 + x_2) \succsim^* (x_1', x_3', x_1' + x_2') \end{aligned} \tag{26}$$

if $x_1 + x_2 = x_1' + x_2'$. Similarly, the Reidemeister condition of X and Y in the relation $\succsim$ implies coseparability given $X_1 + X_2$ of $\succsim^*$. Together with essentiality we obtain the representation $f(x_1, x_1 + x_2) + g(x_3, x_1 + x_2)$ of $\succsim^*$ and thus $\succsim$.

In the same fashion we can obtain a representation $\bar{f}(x_1, x_1 + x_3) + \bar{g}(x_2, x_1 + x_3)$. The existence of the two representations of the same relation gives us the functional equation

$$f(x_1, x_1 + x_2) + g(x_3, x_1 + x_2) = T[\bar{f}(x_1, x_1 + x_3) + \bar{g}(x_2, x_1 + x_3)] \tag{27}$$

for some monotone transformation T. However, this does not yet guarantee additive representability. To make the representation additive among all dimensions, we need to introduce an additional axiom.

Definition 13. Comeasurability:
(X_i, X_n) and (X_j, X_n) are comeasurable if for all x_i, x_j, x_k, x_n, x'_i, x'_j, x'_k, x'_n, x''_i, x''_j, x''_k, x''_n for which the following elements of S are defined:

$$\begin{aligned}
&((x_l)_{l\neq i,j,k,n}, x'_i, x_j, x_k, x'_n)\\
\sim&((x_l)_{l\neq i,j,k,n}, x_i, x'_j, x'_k, x_n)\\
\sim&((x_l)_{l\neq i,j,k,n}, x_i, x''_j, x_k, x''_n)\\
\sim&((x_l)_{l\neq i,j,k,n}, x''_i, x_j, x''_k, x_n)\\
\Rightarrow&\\
&((x_l)_{l\neq i,j,k,n}, x'_i, x'_j, x'_k, x'_n)\\
\sim&((x_l)_{l\neq i,j,k,n}, x''_i, x''_j, x''_k, x''_n).
\end{aligned} \tag{28}$$

Comeasurability is very similar to the Reidemeister condition. In fact, if we set

$$\begin{aligned}
x &= (x_i, x_n) & y &= (x_j, x_k)\\
\bar{x} &= (x_i, x_k) & \bar{y} &= (x_j, x_n)\\
x' &= (x'_i, x'_n) & y' &= (x'_j, x'_k)\\
\bar{x}' &= (x''_i, x''_k) & \bar{y} &= (x''_j, x''_n)
\end{aligned} \tag{29}$$

in (5) then it becomes apparent that for a four-dimensional simplex, comeasurability is simply the Reidemeister condition with a change of dimensions when comparing the LHS to the RHS of (5). Comeasurability guarantees that in the four-dimensional case discussed above, T is affine. In Lemma 12 we solve the functional equation (27) for affine T using a result from Hosszú (1971).

We obtain the following representation theorem.

Theorem 5. *Suppose $S = \{x \in \prod_{i=1}^n X_i : x_n = 1 - \sum_{i=1}^{n-1} x_i\}$ and for all i, $X_i = (0,1)$. Let $\succsim$ be a continuous preference relation fulfilling for all i, j:*

- *(X_i, X_n) and $\prod_{k\neq i,n} X_k$ are independent,*
- *(X_i, X_n) and $\prod_{k\neq i,n} X_k$ are essential,*
- *(X_i, X_n) and $\prod_{k\neq i,n} X_k$ fulfill the Reidemeister condition,*
- *comeasurability of (X_i, X_n) and (X_j, X_n).*

Then $\succsim$ can be represented by a continuous function $u : (S, \tau_S^{\prod_{i=1}^n X_i}) \to \mathbb{R}$:

$$u(x) = \sum_{i=1}^{n} u_i(x_i). \tag{30}$$

Remark 5. We can continuously extend the representation to the closure of the simplex except points where for some i, $x_i = 1$, since it may be that $\lim_{x_i\to 1} u_i(x_i) = \infty$.

This result is very similar to additive representation theorems on product spaces. The only differences are that the independence, essentiality, and Reidemeister conditions have been imposed on pairs of dimensions instead of single dimensions and the additional use of comeasurability between these pairs of dimensions.

Theorem 5 provides an axiomatization for Utilitarianism in cake division problems. In a cake division problem, a fixed resource is divided among n individuals. If we normalize the amount of the resource to 1, we have $\sum_{i=1}^{n} x_i = 1$ as a constraint. Therefore, the space on which preferences are defined is an n-dimensional simplex.[5] If the decision maker's preference is consistent with the axioms in Theorem 5, the decision maker is consistent with Utilitarianism in the following sense. For each individual i, the decision maker can find a utility function u_i, which is cardinally comparable to the utility function of any other j. The decision maker ranks allocations according to the sum of the utility functions. This result can be generalized to cases where agents differ in their productivity with which they convert the resource into some consumption product:

Remark 6. Theorem 5 holds for any space S' where we can find a homeomorphism $h : S' \to S$ such that $h(s') = h(x'_1, \dots, x'_n) = (h_1(x'_1), \dots h_n(x'_n))$ and each h_i is a homeomorphism.

This follows from the fact that our independence, essentiality, coseparability, and comeasurability assumptions do not refer in any way to the linear structure of a simplex. In our cake division problem we could assume that each individual i transforms the resource by multiplying it with a productivity parameter θ_i. Then Theorem 5 holds for $S' = \{x \in \prod_{i=1}^{n} X_i : \theta_n x_n = 1 - \sum_{i=1}^{n-1} \theta_i x_i\}$.

Naturally, it is interesting to consider a reference-dependent version of the Theorem 5. For this, we need to extend the definition of comeasurability to a conditional version:

Definition 14. Conditional Comeasurability:
(X_i, X_n) and (X_j, X_n) are comeasurable given Z if for all x_i, x_j, x_k, x_n, x'_i, x'_j, x'_k, x'_n, x''_i, x''_j, x''_k, x''_n for which the following elements of S are defined:

$$\begin{aligned}
&((x_l)_{l \neq i,j,k,n}, x'_i, x_j, x_k, x'_n, z) \\
\sim &((x_l)_{l \neq i,j,k,n}, x_i, x'_j, x'_k, x_n, z) \\
\sim &((x_l)_{l \neq i,j,k,n}, x_i, x''_j, x_k, x''_n, z) \\
\sim &((x_l)_{l \neq i,j,k,n}, x''_i, x_j, x''_k, x_n, z) \\
\Rightarrow & \\
&((x_l)_{l \neq i,j,k,n}, x'_i, x'_j, x'_k, x'_n, z) \\
\sim &((x_l)_{l \neq i,j,k,n}, x''_i, x''_j, x''_k, x''_n, z).
\end{aligned} \tag{31}$$

[5]If we could force the decision maker to discard some of the resource, we would be back in the standard case of a subset of a product space.

Corollary 3. *Suppose $S = \{x \in \prod_{i=1}^{n} X_i : x_n = 1 - \sum_{i=1}^{n-1} x_i\}$ and for all i: $X_i = (0,1)$. Z is a connected, separable topological space. Let $\succsim$ be a continuous preference relation on $S \times Z$ fulfilling for all i, j:*

- *essentiality given Z,*
- *$X_i \times X_n \perp \prod_{k \neq i,n} X_k \mid Z$,*
- *conditional comeasurability of (X_i, X_n) and (X_j, X_n) given Z.*

Then $\succsim$ can be represented by:

$$u(x,z) = \sum_{i=1}^{n} u_i(x_i, z). \tag{32}$$

This shows that our previous results for conditionally additive representations carry over to simplices under the additional assumption of comeasurability.

6 Mixture Spaces

In some cases, we are interested in even stronger independence conditions which generate not only additive, but linear utility functions. The classical example is expected utility axiomatized by von Neumann and Morgenstern (1944). Herstein and Milnor (1953) generalize their results to mixture spaces, which we will study in the following.

Let Z be a connected, separable set and S an arbitrary set. Let $\xi : S \to Z$ be continuous. For each $z \in Z$ we assume a set $S_z = \{s \in S : \xi(s) = z\}$ with an operator $\oplus$ such that for all $s, s' \in S$ and $\mu, \lambda \in [0,1] \subseteq \mathbb{R}$ we have:

$$\begin{aligned} \mu s \oplus (1-\mu)s' &\in S_z \\ 1s \oplus 0s' &= s \\ \mu s \oplus (1-\mu)s' &= (1-\mu)s' \oplus \mu s \\ \lambda(\mu s \oplus (1-\mu)s') \oplus (1-\lambda)s' &= (\lambda\mu)s \oplus (1-\lambda\mu)s'. \end{aligned} \tag{33}$$

We call $(S, Z, \oplus)$ a conditional mixture space. We call it a continuous conditional mixture space if for all $s, s' \in S$ the map $\alpha \mapsto \alpha s \oplus (1-\alpha)s'$ is a continuous map.

Naturally, conditional independence in this context means:

Definition 15. Conditional independence:
For mixture spaces, $\succsim$ is conditionally independent given Z if for all $z \in Z$ and all $s, s', s'' \in S_z$ and all $\mu \in [0,1]$:

$$s \sim s' \Leftrightarrow \frac{1}{2}s \oplus \frac{1}{2}s'' \sim \frac{1}{2}s' \oplus \frac{1}{2}s''. \tag{34}$$

As usual, we need the technical assumption of essentiality to avoid some pathological cases.

Definition 16. Essentiality:
For mixture spaces, $\succsim$ is essential given Z if for all $z \in Z$ there exist $s, s' \in S_z$ such that $s \succ s'$.

The coseparability condition for product spaces can be translated to this context as:

Definition 17. Coseparability:
For mixture spaces, $\succsim$ fulfills coseparability given Z if for all $z, \bar{z}$ in Z:

$$\begin{aligned} s &\sim \bar{y} \\ \frac{1}{2}s \oplus \frac{1}{2}s' &\sim \frac{1}{2}\bar{s} \oplus \frac{1}{2}\bar{s}' \\ \frac{1}{2}s \oplus \frac{1}{2}s'' &\sim \frac{1}{2}\bar{s} \oplus \frac{1}{2}\bar{s}'' \\ \Rightarrow \frac{1}{2}s' \oplus \frac{1}{2}s'' &\sim \frac{1}{2}\bar{s}' \oplus \frac{1}{2}\bar{s}'' \end{aligned} \tag{35}$$

where $s, s', s'' \in S_z$ and $\bar{s}, \bar{s}', \bar{s}'' \in S_{\bar{z}}$.

Compared to the earlier definition, we have slightly simplified the exposition due to the commutativity of $\oplus$.

Theorem 6. *Suppose $(S, Z, \oplus)$ is a continuous conditional mixture space and $\succsim$ is a continuous preference relation on S. Let $\succsim$ fulfill the following conditions:*

- *conditional independence given Z,*
- *essentiality given Z, and*
- *coseparability given Z.*

Then there exists a representation $u : S \to \mathbb{R}$ and functions $u_z : S_z \to \mathbb{R}$ such that

$$u(s) = u_{\xi(s)}(s) \tag{36}$$

$$u_z(\mu s \oplus (1-\mu)s') = \mu u_z(s) + (1-\mu)u_z(s') \tag{37}$$

for all $s, s' \in S_z$ and all $\mu \in [0, 1]$.

Remark 7. If $S = \prod_{i=1}^{n} X_i \times Z = \mathbb{R}^n \times Z$, $\xi(x_1, \ldots, x_n, z) = z$ and $\oplus$ is defined as vector summation for elements with identical $z \in Z$, then $u(s) = \sum_{i=1}^{n} x_i u_i(z)$.

We apply Theorem 6 to decisions under risk and uncertainty. Suppose Z is a set of acts with uncertain (and possibly unknown) consequences. $x_1, \ldots, x_n$ are known probabilities of lotteries over outcomes $1, \ldots, n$. Many everyday decisions resemble this structure. In a medical context, a decision maker may be informed about the probabilities of the effects of various medical treatments but may be uncertain how choices such as smoking, exercising, or eating interact with the outcomes. For fincancial decisions on insurances and investments, consultants may be able to supply the decision maker with probabilities on the returns of an asset or the payment probabilities of an insurance. However, for a simultaneous decision on whether to switch occupations or employers, the decision maker may not have such data. Nonetheless, the two decisions strongly interact: switching to a lower paid, less stable job may change the risk preferences of the decision maker over other financial decisions.

In the above axiomatization, we impose von Neumann Morgenstern preferences over lotteries when holding the act z fixed. Coseparability ensures comparability of the von Neumann Morgenstern preferences across different acts z, z'. Under these more stringent assumptions on the preferences given Z, the coseparability condition is a powerful, yet highly plausible axiom for rational decision makers. If a decision maker maximizes the expectation of her experienced utility given a decision z, then there is no good reason to believe that this expectation of the experienced utility should not be comparable with that given a decision z'.

In particular, coseparability forbids separate "utility accounts" for z and $x_1, \ldots, x_n$: suppose the decision maker first determines the utility of the lottery and in a second step transforms this utility depending on the act z using a nonlinear function. Then the utility function would instead be $u(s) = v(\sum_i x_i u_i, z)$. We may call this a case of two "utility accounts" as the decision maker first forms an account for the utility of the lottery and then combines this utility with the utility from the act z. An example of an axiomatization which allows for such behavior is given in Karni (2006) where the v function may be nonlinear. Instead, coseparability forces the decision maker to separately consider the utility from each outcome i given act z and take the expectation over these utilities $u_i(z)$.

A special feature of this axiomatization apart is the lack of a specification of how acts relate to outcomes.[6] This is intuitive for applications where there may be unforeseen consequences of acts or where due to bounded rationality it is impossible for a decision maker to relate acts, outcomes, and states of the world. If our axioms are accepted, a decision maker facing a situation where some outcomes are unknown should (when holding fixed z) behave in line with expected utility for simultaneous choices over lotteries and should be able to cardinally compare the expected utility functions across different acts z, z'.

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[6] Giving further structure to the uncertain act component z via a framework like Savage (1972) may be interesting for further research. Chew and Sagi (2008) Theorem 2 axiomatizes risk preferences under uncertainty conditional on "small worlds". This is very much in the spirit of Equation (37) without imposing Equation (36).

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Appendix A Proof of Theorem 1

Necessity of the conditions for the representation and the uniqueness properties are trivial. We therefore focus on proving sufficiency. In the steps up to Lemma 6 we show that for every interior indifference curve we can find a point such that there exists a basic open set $O = O_x \times O_y \times \{z\} \subseteq S_z$ containing the point and points strictly better and worse. This ensures that across different z-layers we can make the conditionally additive representations consistent using 1. In the following steps to Lemma 9 we show sufficiency for the existence of the representation v. The remainder of the proof derives the continuity properties.

A standard result used throughout this paper is:

Lemma 2. *Suppose $\succsim$ is a continuous preference relation on S fulfilling essentiality given Z. Then $\succsim$ satisfies restricted solvability given Z.*

Proof. We can simply apply the proof of Wakker (1989) Lemma III.3.3. to each S_z. □

Definition 18. A point $s = (x, y, z)$ is called locally X-nonsatiated if for every open set $O \in t^{X\times Y\times Z}_{S_{y,z}}$ containing s there exists a point $s' = (x', y, z) \in O$ such that $s' \succ s$.
$s = (x, y, z)$ is called locally X, Y-nonsatiated if it is X-nonsatiated and Y-nonsatiated.

Lemma 3. *Suppose $S \subseteq X \times Y \times Z$ is well behaved given Z. Let $\succsim$ be a continuous preference relation on S with a continuous representation u. Then for every $i^* \in u(S_{x,z})$ such that $i^* \neq \max_{s\in S_{x,z}} u(s)$ there exists a point $s \in S_{x,z}$ such that $u(s) = i^*$ and s is locally Y-nonsatiated.*

Proof. We prove the result by contradiction. By well behavedness of S, all sets $S_{x,z}$ are connected. Moreover, if u is continuous on $S, t^{X\times Y\times Z}_S$, then it is continuous on $S_{x,z}, t^{X\times Y\times Z}_{S_{x,z}}$.

Suppose there exists no point $s \in u^{-1}(i^*) \cap S_{x,z}$ that is locally nonsatiated in Y for $i^* \in int(u(S_{x,z}))$. Then, for every $s \in u^{-1}(i^*)$ there exists an open set $O_s \in t^{X\times Y\times Z}_{S_{x,z}}$ such that s is maximal in O_s. Take the union $O = \bigcup_{s\in u^{-1}(i^*)\cap S_{x,z}} O_s$. As a union of open sets, O is open. Moreover, $O' = \{s \in S_{x,z} : u(s) < i^*\} \in t^{X\times Y\times Z}_{S_{x,z}}$. Thus, $\{s \in S_{x,z} : u(s) \leq i^*\} = O \cup O' \in t^{X\times Y\times Z}_{S_{x,z}}$ is both open and (by continuity) closed in $t^{X\times Y\times Z}_{S_{x,z}}$. Since $i^* \in int(u(S_{x,z}))$, $O \cup O' \neq S_{x,z}$ and $O \cup O' \neq \emptyset$ which means $S_{x,z}$ is disconnected yielding the contradiction. □

Note that the same argument applies to local X-nonsatiation of a nonmaximal indifference curve in any set $S_{y,z}$.

Lemma 4. *Suppose $S \subseteq X \times Y \times Z$ is well behaved given Z. Let $\succsim$ be a continuous preference relation on S with a continuous representation u. Let $\succsim$ fulfill independence given Z. Then if (x, y, z) is locally Y-nonsatiated all $s \in S_{x,z}$ are locally Y-nonsatiated.*

Proof. Suppose $(x^*, y, z) \in S_{x,z}$ is not locally Y-nonsatiated. Note that $Y_{S_{x^*,z}}$ is open in t^Y. Therefore there exists an open set $O_y \in t^Y_{Y_{S_{x^*,z}}}$ such that for all

$y' \in O_y$, $(x, y, z) \succsim (x^*, y', z)$. By independence given Z, for all $y' \in O_y \cap Y_{S_{x,z}}$, we have

$$\begin{aligned} &(x, y, z) \succsim (x, y', z) \\ \Leftrightarrow\ &(x^*, y, z) \succsim (x^*, y', z). \end{aligned} \tag{38}$$

But then, since $O_y \cap Y_{S_{x,z}} \in t^Y_{Y_{S_{x,z}}}$ and $\{x\} \times O_y \times \{z\} \in t^{X \times Y \times Z}_{S_{x,z}}$, the point (x, y, z) is not locally Y-nonsatiated, yielding a contradiction. $\square$

Lemma 5. *Suppose $S \subseteq X \times Y \times Z$ is well behaved given Z. Let $\succsim$ be a continuous preference relation on S with a continuous representation u. Let $\succsim$ fulfill*

- *$X \perp Y \mid Z$ and*
- *essentiality given Z.*

Then for every $i^* \in int(u(S_z))$ there exists a point $s \in S_z$ such that $u(s) = i^*$ and s is locally X, Y-nonsatiated.

Proof. Figure 5 may be useful to follow the steps of the proof. Take a point (x, y, z) such that $u(x, y, z) = i^* \in int(u(S_{x,z}))$ which exists by essentiality. By Lemma 3, there exists a locally Y-nonsatiated point $(x, y^*, z) \sim (x, y, z)$ in $S_{x,z}$.

We distinguish two cases:

- Case $i^* < \max_{s \in S_{y^*,z}} u(s)$: By Lemma 3, on $S_{y^*,z}$ there exists a locally X-nonsatiated point $(x^*, y^*, z) \sim (x, y^*, z)$. By Lemma 4 (x^*, y^*, z) is locally Y-nonsatiated. Thus, (x^*, y^*, z) is locally X, Y-nonsatiated.

- Case $i^* = \max_{s \in S_{y^*,z}} u(s)$: By essentiality given Z and restricted solvability, there exists a point $(x^{**}, y^*, z) \in S_{y^*,z}$ such that $(x^*, y^*, z) \prec (x, y^*, z)$. Since S_z is open in $t^{X \times Y \times \{z\}}$, there exists a basic open set $O = O_x \times O_y \times \{z\}$ of $t^{X \times Y \times \{z\}}$ such that $O \subseteq S_z$ and $(x^*, y^*, z), (x, y^*, z) \in O$. Since (x, y^*, z) is locally Y-nonsatiated, there exists a point $(x, y^{**}, z) \succ (x, y^*, z)$. Due to restricted solvability (Lemma 2), we can assume without loss of generality that $u(x, y^{**}, z) < \max_{s \in S_{x,z}} u(s)$.

  By restricted solvability we can find $(x^{***}, y^{***}, z) \sim (x, y^*, z)$ in O: If $(x^{**}, y^{**}, z) \succ (x, y^*, z) \succ (x^{**}, y^*, z)$ then $x^{***} = x^{**}$. If $(x, y^{**}, z) \succ (x, y^*, z) \succ (x^{**}, y^{**}, z)$ then $y^{***} = y^{**}$.

  Next, take a locally X-nonsatiated point $(\bar{x}^{***}, y^*, z) \sim (x^{***}, y^*, z)$ which exists by Lemma 3. Similarly, let $(x, \bar{y}^{***}, z) \sim (x, y^{***}, z)$ be a locally Y-nonsatiated point. Then by independence given Z and Lemma 4, $(\bar{x}^{***}, \bar{y}^{***}, z)$ is locally X, Y-nonsatiated. By coseparability given Z,

$$\begin{aligned} &(x, y^*, z) \sim (x, y^*, z) \\ \wedge\quad &(x^{***}, y^*, z) \sim (\bar{x}^{***}, y^*, z) \\ \wedge\quad &(x, y^{***}, z) \sim (x, \bar{y}^{***}, z) \\ \Rightarrow\quad &(x^{***}, y^{***}, z) \sim (\bar{x}^{***}, \bar{y}^{***}, z) \end{aligned}$$

  and therefore $u(\bar{x}^{***}, \bar{y}^{***}, z) = i^*$.

$\square$

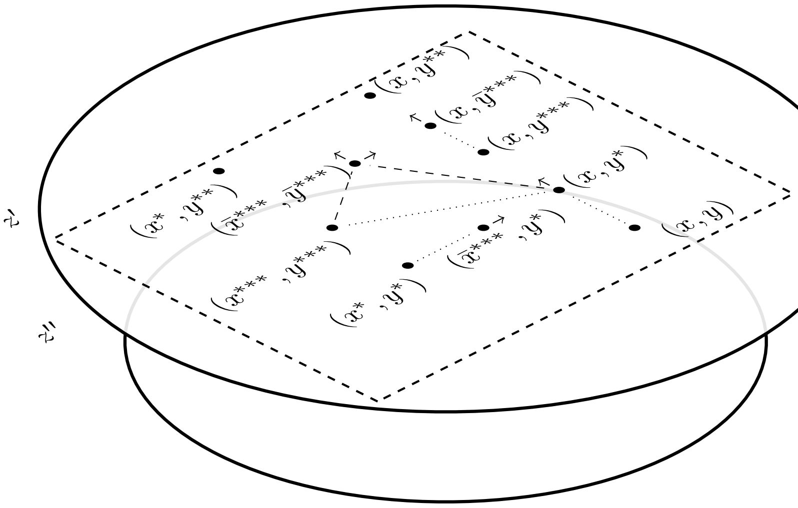

Figure 5: Proof of local X, Y-nonsatiation for all indifference curves.
Case of $i^* = \max_{s \in S_{y^*,z}} u(s)$. The dashed outline marks the open set O. Dotted lines connect indifferent points. The dashed indifferences follow from coseparability given Z. The small arrows mark the direction of local nonsatiation.

Lemma 6. *Suppose $S \subseteq X \times Y \times Z$ is well behaved given Z. Let $\succsim$ be a continuous preference relation on S with a continuous representation u. Let $\succsim$ be conditionally independent given Z and essential given Z. Then for every $z, \bar{z} \in Z$ such that their indifference curves strictly overlap, and every indifference curve $i^* \in int(u(S_z) \cap u(S_{\bar{z}}))$, there exist points in $S_z \cup S_{\bar{z}}$*

$$\begin{aligned} &(x_0, y_0, z) \sim (\bar{x}_0, \bar{y}_0, \bar{z}) \\ \prec\ &(x_1, y_0, z) \sim (\bar{x}_1, \bar{y}_0, \bar{z}) \\ \sim\ &(x_0, y_1, z) \sim (\bar{x}_0, \bar{y}_1, \bar{z}) \end{aligned} \tag{39}$$

such that $u(x_0, y_0, z) = i^*$ and $(x_1, y_1, z) \in S_z$, and $(\bar{x}_1, \bar{y}_1, \bar{z}) \in S_{\bar{z}}$.

Proof. By Lemma 5, we can find locally nonsatiated points $s = (x_0, y_0, z), \bar{s} = (\bar{x}_0, \bar{y}_0, \bar{z})$ in both X and Y such that $u(x_0, y_0, z) = u(\bar{x}_0, \bar{y}_0, \bar{z}) = i^*$. Since $S \in t^{X \times Y \times Z}$, we can find $O = O_x \times O_y \times \{z\}$ and $\bar{O} = \bar{O}_x \times \bar{O}_y \times \{\bar{z}\}$ where $O \subseteq S_z$, $\bar{O} \subseteq S_{\bar{z}}$, $O_x, \bar{O}_x \in t^X$, and $O_y, \bar{O}_y \in t^Y$. Since $s, \bar{s}$ are locally nonsatiated in X and Y, we can find $x_1 \in O_x, \bar{x}' \in \bar{O}_x, y' \in O_y, \bar{y}' \in \bar{O}_y$ such that $(x_1, y, z) \succ s$, $(x, y', z) \succ s$, $(\bar{x}', \bar{y}, \bar{z}) \succ \bar{s}$, $(\bar{x}, \bar{y}', \bar{z}) \succ \bar{s}$. Without loss of generality, let (x_1, y, z) be the worst among the four points. Then by restricted solvability, there exist points

$$\begin{aligned} &(x_1, y_0, z) \sim (\bar{x}_1, \bar{y}_0, \bar{z}) \\ \sim\ &(x_0, y_1, z) \sim (\bar{x}_0, \bar{y}_1, \bar{z}) \end{aligned} \tag{40}$$

in O and $\bar{O}$. Since $O = O_x \times O_y \times \{z\}$ and $\bar{O} = \bar{O}_x \times \bar{O}_y \times \{\bar{z}\}$, we have $(x_1, y_1, z) \in o \subseteq S_z$ and $(\bar{x}_1, \bar{y}_1, \bar{z}) \in \bar{O} \subseteq S_{\bar{z}}$. □

Lemma 7. *Suppose $S \subseteq X \times Y \times Z$ is well behaved given Z. Let $\succsim$ be a continuous preference relation on S with a continuous representation u. Let $\succsim$ be independent and essential given Z. Then for every $i^* \in int(u(S_z))$, there exists a basic open set $b_x \times b_y \times \{z\}$ in $t_{S_Z}^{X \times Y \times Z}$ such that it contain elements with strictly higher and lower utility value than i^*.*

Proof. Using elementary set theory and openness of S, we can show S_z is a union of basic open sets $O = \bigcup_{O_b \in T \subseteq t^{X \times Y \times Z}_{X \times Y \times \{z\}}} O_b$ where for all O_b, we have $O_b = b_x \times b_y \times \{z\}$ and $O_b \in t_{S_Z}^{X \times Y \times Z}$. We will show that if no set $b_x \times b_y \times \{z\}$ in $t_{S_Z}^{X \times Y \times Z}$ contains both points strictly better and worse than i^*, we can derive a contradiction. Suppose for all $O_b \in T$, we have $\forall s \in O_b : u(s) \geq i^*$ or $\forall s \in O_b : u(s) \leq i^*$. Let $\overline{T}, \underline{T}$ partition T into sets fulfilling $\forall s \in O_b : u(s) \geq i^*$ and $\forall s \in O_b : u(s) \leq i^*$, respectively. $\overline{T}$ and $\underline{T}$ are unions of open sets, and thus elements of $t_{S_z}^{X \times Y \times Z}$. As an intersection of open sets, $O' = \overline{T} \cap \underline{T}$ is an element of $t_{S_z}^{X \times Y \times Z}$. By continuity, $O'' = \overline{T} \cap \{s \in S_z : u(s_z) > i^*\}$ and $O''' = \underline{T} \cap \{s \in S_z : u(s_z) < i^*\}$ are elements of $t_{S_z}^{X \times Y \times Z}$. O', O'', O''' partition S_z. Since S_z is connected, it cannot be partitioned into open sets, yielding the desired contradiction. ☐

Lemma 8. *Suppose $S \subseteq X \times Y \times Z$ is well behaved given Z. Let $\succsim$ be a continuous preference relation on S with a continuous representation u. Let $\succsim$ be essential given Z and $X \perp Y \mid Z$. Then for every $z, \bar{z} \in Z$ with strictly overlapping indifference curves, we can find a continuous function $v : (S_z \cup S_{\bar{z}}, \tau_{S_z \cup S_{\bar{z}}}^{X \times Y \times Z}) \to \mathbb{R}$ such that v represents $\succsim$ and $v(x, y, z) = f(x, z) + g(y, z)$.*

Proof. By continuity, we have a utility representation $u : S \to \mathbb{R}$. By Wakker and Chateauneuf (1993), for each $\hat{z} \in \{z, \bar{z}\}$ we can construct a utility function $v_{\hat{z}} : S_{\hat{z}} \to \mathbb{R}$ which is additive in X and Y. Therefore, $v_{\hat{z}}(x, y) = f_{\hat{z}}(x) + g_{\hat{z}}(y)$, where $f_{\hat{z}} : (X_{\hat{z}}, t_{X_{\hat{z}}}^{X \times Y \times Z}) \to \mathbb{R}$ and $g_{\hat{z}} : (Y_{\hat{z}}, t_{Y_{\bar{z}}}^{X \times Y \times Z}) \to \mathbb{R}$.

We may now pick any indifference curve $i^* \in int(u(S_z) \cap u(S_{\bar{z}}))$. By Lemma 6, we can find points

$$\begin{aligned} &(x_0, y_0, z) \sim (\bar{x}_0, \bar{y}_0, \bar{z}) \\ \prec\ &(x_1, y_0, z) \sim (\bar{x}_1, \bar{y}_0, \bar{z}) \\ \sim\ &(x_0, y_1, z) \sim (\bar{x}_0, \bar{y}_1, \bar{z}) \end{aligned} \tag{41}$$

such that $(x_1, y_1, z) \in S_z$ and $(\bar{x}_1, \bar{y}_1, \bar{z}) \in S_{\bar{z}}$. By coseparability given Z, we then have $(x_1, y_1, z) \sim (\bar{x}_1, \bar{y}_1, \bar{z})$.

Without loss of generality, we assume that v_z and $v_{\bar{z}}$ fulfill:

$$\begin{aligned} v_z(x_0, y_0) =& v_{\bar{z}}(\bar{x}_0, \bar{y}_0) = 0 \\ v_z(x_1, y_0) =& v_{\bar{z}}(\bar{x}_1, \bar{y}_0) = 1 \\ v_z(x_0, y_1) =& v_{\bar{z}}(\bar{x}_0, \bar{y}_1) = 1 \\ v_z(x_1, y_1) =& v_{\bar{z}}(\bar{x}_1, \bar{y}_1) = 2, \end{aligned} \tag{42}$$

since otherwise, we can use positive affine transformations of the representations to ensure that this is the case and we can further assume that $v_z = f_z(x) + g_z(y)$ $v_{\bar{z}} = f_{\bar{z}}(x) + g_{\bar{z}}(y)$.

We now show that the two representations are consistent, i.e.,

$$\begin{aligned} & v_z(x,y) \geq (>) v_{\bar{z}}(\bar{x},\bar{y}) \\ \Leftrightarrow \quad & (x,y,z) \succsim (\succ) (\bar{x},\bar{y},\bar{z}). \end{aligned} \tag{43}$$

Since both representations are complete and transitive, we only need to show that

$$\begin{aligned} & (x,y,z) \sim (\bar{x},\bar{y},\bar{z}) \\ \Rightarrow \quad & v_z(x,y,z) = v_{\bar{z}}(\bar{x},\bar{y},\bar{z}). \end{aligned} \tag{44}$$

Without loss of generality, we can assume that $u(x_0,y_0,z) = u(\bar{x}_0,\bar{y}_0,\bar{z}) = 0$.

Therefore, by Lemma 7, for each rational utility level u^* in the interval $[min(0,u(x,y,z)), max(0,u(x,y,z))]$, we can find corresponding open sets $B^z_{u^*}$ and $B^{\bar{z}}_i$ such that $u^* \in int(u(B^z_i) \cap u(B^{\bar{z}}_i))$. Note that we can assume that B^z_1 is a basic open set containing (x_1,y_1,z) and (x_0,y_0,z) and $B^{\bar{z}}_1$ is a basic open set containing $(\bar{x}_0,\bar{y}_0,\bar{z})$ and $(\bar{x}_1,\bar{y}_1,\bar{z})$. Moreover B^z_i and $B^{\bar{z}}_i$ has the utility overlapping with B^z_{i+1} and $B^{\bar{z}}_{i+1}$.

Next, we want to show that v_z and $v_{\bar{z}}$ are consistent in $B^z_i \cup B^{\bar{z}}_i$. We use induction. For $i=1$, it is clear that v_z and $v_{\bar{z}}$ are consistent in $B^z_1 \cup B^{\bar{z}}_1$ by Lemma 8. Suppose consistency holds for $i=k$, then when $i=k+1$, since B^z_{i+1} and $B^{\bar{z}}_{i+1}$ have common indifference curves, by Lemma 8, there exists a consistent additive representation v^* on $B^z_{i+1} \cup B^{\bar{z}}_{i+1}$, unique up to affine transformations. Since B^z_i and $B^{\bar{z}}_i$ have common indifference curves with B^z_{i+1} and $B^{\bar{z}}_{i+1}$, we can find points $s, s' \in B^z_i$, $s'', s''' \in B^z_{i+1}$, $\bar{s}, \bar{s}' \in B^z_i$, and $\bar{s}'', \bar{s}''' \in B^z_{i+1}$ such that

$s \sim s''' \sim \bar{s} \sim \bar{s}''' \succ s' \sim s''' \sim \bar{s}' \sim \bar{s}'''$. We can then define the affine transformations $T(u) = \alpha_z u + \beta_z$ and $\bar{T}(u) = \alpha_{\bar{z}} u + \beta_{\bar{z}}$ where

$$\begin{aligned} \alpha_z = \frac{v_z(s) - v_z(s')}{v^*(s'') - v^*(s''')} &= \frac{v_{\bar{z}}(\bar{s}) - v_{\bar{z}}(\bar{s}')}{v^*(\bar{s}'') - v^*(\bar{s}''')} = \alpha_{\bar{z}} \\ \beta_z = v_z(s) - \frac{v_z(s) - v_z(s')}{v^*(s'') - v^*(s''')} v^*(s'') &= v_{\bar{z}}(\bar{s}) - \frac{v_{\bar{z}}(\bar{s}) - v_{\bar{z}}(\bar{s}')}{v^*(\bar{s}'') - v^*(\bar{s}''')} v^*(\bar{s}'') = \beta_{\bar{z}}. \end{aligned} \tag{45}$$

It follows that for all $s^* \in B^z_{i+1}$ we have $v_z(s^*) = \alpha_z v^*(s^*) + \beta_z$ and for all $\bar{s}^* \in B^{\bar{z}}_{i+1}$ $s^* \in B^{\bar{z}}_{i+1}$ $v_{\bar{z}}(\bar{s}^*) = \alpha_{\bar{z}} v^*(\bar{s}^*) + \beta_{\bar{z}}$.

Since v^* is unique up to affine transformations, $T \circ v^* = \bar{T} \circ v^*$ is consistent on $B^z_{i+1} \cup B^{\bar{z}}_{i+1}$. Thus, v_z and $v_{\bar{z}}$ are consistent on $B^z_{i+1} \cup B^{\bar{z}}_{i+1}$.

Since (x,y,z) and $(\bar{x},\bar{y},\bar{z})$ belong to some $B^z_i \cup B^{\bar{z}}_i$, we have $v_z(x,y,z) = v_{\bar{z}}(x',y',\bar{z})$. Therefore, the following function represents the preference

$$v(\hat{x},\hat{y},\hat{z}) = \begin{cases} v_z(\hat{x},\hat{y}), & \hat{z} = z \\ v_{\bar{z}}(\hat{x},\hat{y}), & \hat{z} = \bar{z}. \end{cases} \tag{46}$$

What is left to show is that v is continuous in the subspace topology of the order topology on S. We show that for every open interval $O_R \subseteq \mathbb{R}$ the preimage $v^{-1}(O_R)$ is open in the subspace topology of the order topology. Since v_z is a continuous representation on S_z, $v_z^{-1}(O_R) = O \cap S_z$ where $O \in \tau_S^{X \times Y \times Z}$ is a connected set. Similarly, $v_{\bar{z}}^{-1}(O_R) = O' \cap S_{\bar{z}}$. Since the two representations are consistent, we may choose $O = O'$. Since $O \cap S_z \cup O \cap S_{\bar{z}} = O \cap (S_z \cup S_{\bar{z}}) = v^{-1}(O_R)$ is open in $S_z \cup S_{\bar{z}}$, v is continuous. $\square$

Lemma 9. *Suppose $S \subseteq X \times Y \times Z$ is well behaved given Z. Let $\succsim$ be a continuous preference relation on S. Let $\succsim$ be essential given Z and $X \perp Y \mid Z$. Then we can find a continuous function $v : (S, \tau_S^{X\times Y\times Z}) \to \mathbb{R}$ such that v represents $\succsim$ and $v(x,y,z) = f(x,z) + g(y,z)$.*

Proof. By continuity, there exists a continuous representation $u : (S, t_S^{X\times Y\times Z}) \to \mathbb{R}$ which we will use to track indifference classes.

We start by making all representations $v_z : (S_z, t_{S_z}^{X\times Y\times Z}) \to \mathbb{R}$ for which $u(S_z)$ intersects with an open interval r consistent with each other. Then we show how we can increase the size of r to $int(u(S))$.

We start with $r = int(u(S_{z_0}))$ for some z_0. r is nonempty and connected by essentiality, connectedness of S_{z_0}, and continuity of u. Let v_{z_0} be the additive representation on S_{z_0}, which we obtain from Lemma 8. By Lemma 8, we can make v_{z_0} consistent with any $v_{z'}$ such that $int(u(S_{z_0}) \cap u(S_{z'})) \neq \emptyset$, i.e., for all $s \in S_{z_0}, s' \in S_{z'}$, $v_{z_0}(s) \geq (>) v_{z'}(s')$ iff $s \succsim (\succ) s'$. We show that any $v_{z'}, v_{z''}$ made consistent with v_{z_0} are also consistent with another: clearly, if $u(S_{z'}) \cap u(S_{z''}) \subseteq u(S_{z_0})$, then $v_{z'}, v_{z''}$ are consistent by transitivity. We therefore consider the case where $u(S_{z'}) \cap u(S_{z''}) \not\subseteq u(S_{z_0})$. By the proof of Lemma 8, if $v_{z'}, v_{z''}$ are additive representations and consistent somewhere, they are consistent everywhere due to coseparability given Z. Therefore, we can obtain a representation $v^r(s) = \begin{cases} v_z(s), & s \in S_z \end{cases}$ on the domain $u^{-1}(r)$.

Next, we show that we can extend the set of indifference classes r (on which all intersecting layers z with representations v_z are consistent) to $int(u(S))$.

Since u is continuous, the sets $A = \{z \in Z : \sup_{s\in S_z} u(s) \leq \sup_{i\in r} i\}$ and $B = \{z \in Z : \inf_{s\in S_z} u(s) \geq \sup_{i\in r} i\}$ are closed. By essentiality, $A \cap B$ is empty. By connectedness, Z cannot be partitioned into two closed, disjoint sets. Thus, there exists z^* such that $\sup_{i\in r} i$ is in the interior of $u(S_{z^*})$. We can then make all layers having indifference classes intersecting $u(S_{z^*})$ consistent with one another, and consistent with all layers intersecting the indifference classes in r. Thus, as long as $\sup_{x\in r} x < \sup_{s\in S} u(s)$, we can always continue making layers consistent. By an analogous argument we can extend our representation until $\inf_{x\in r} x = \inf_{s\in S} u(s)$.

Note that in case $\sup_{i\in u(S)} i = \max_{i\in u(S)} i = u(s^*)$ (or in case $\inf_{i\in u(S)} i = \max_{i\in u(S)} i = u(s^*)$), there must exist some z such that $s^* \in S_z$ and by essentiality $int(r \cap u(S_z))$ is nonempty. Therefore we can extend the representation to all indifference classes.

We claim that v (but not necessarily f and g) is continuous. Since v represents the same relation as u, there must exist a monotone transformation $T : \mathbb{R} \to \mathbb{R}$ such that $T \circ u = v$. Since v is continuous on every S_z and u is continuous when restricted to S_z, T must be continuous. To see this, note that since T is increasing, it can only have jump discontinuities. Let $T(u^*) = v^*$ be such a discontinuity. Then $A = \{s \in S_z : v_z(s) > v^*\}$ is open in $t_{S_z}^{X\times Y\times Z}$ by continuity of v on S_z and $B = \{s \in S_z : u_z(s) \geq u^*\}$ is closed in $t_{S_z}^{X\times Y\times Z}$ by continuity of u on S_z. However, $A = B$ by monotonicity of T and thus A is both closed and open in S_z, contradicting connectedness of S_z. Thus, T is continuous everywhere and thus, v is continuous. $\square$

Lemma 10. *Suppose $S \subseteq X \times Y \times Z$ is well behaved given Z. Let $\succsim$ be a continuous preference relation on S. Let $\succsim$ be essential given Z and $X \perp Y \mid Z$. Then on any set $S_{Z'}$, where $Z' = Z_{S_{x_0,y_0}}$, we can find continuous functions $f_{Z'} : (S_{Z'}, t^{X\times Y\times Z}_{S_{Z'}}) \to \mathbb{R}$ and $g_{Z'} : (S_{Z'}, t^{X\times Y\times Z}_{S_{Z'}}) \to \mathbb{R}$ such that $v(x, y, z) = f_{Z'}(x, z) + g_{Z'}(y, z)$.*

Proof. By Lemma 9 we have a continuous representation $v : (S, \tau^{X\times Y\times Z}_S) \to \mathbb{R}$ with $v(x, y, z) = f(x, z) + g(y, z)$. What remains to be shown is that f, g can be made continuous.

Define $h_{Z'} : (S_{Z'}, \tau_{S_{Z'}}) \to \mathbb{R}$ such that $h_{Z'}(z) = v(x_0, y_0, z)$. We prove that $h_{Z'}$ is continuous. For all open intervals $r \subseteq \mathbb{R}$ we have that $v^{-1}(r) \cap S_{x_0,y_0} \in \tau^{X\times Y\times Z}_{S_{x_0,y_0}} \subseteq \tau_{S_{x_0,y_0}}$ and can be written as $\{x_0\} \times \{y_0\} \times O_z \in \tau_{Z'}$. Since $h^{-1}_{Z'}(O_R) = X \times Y \times O_z \cap S \in t^{X\times Y\times Z}_{S_{Z'}}$ and $h_{Z'}$ is therefore continuous.

Next, let $f_{Z'} : (S_{Z'}, t^{X\times Y\times Z}_{S_{Z'}}) \to \mathbb{R}$ and $g_{Z'} : (S_{Z'}, \tau_{S_{Z'}}) \to \mathbb{R}$ such that $f_{Z'}(x, y, z) = f(x, z) - f(x_0, z)$ and $g_{Z'}(y, z) = g(y, z) - g(y_0, z)$. It then follows, that

$$\begin{aligned} v(x, y, z) &= f(x, z) + g(y, z) \\ &= f(x, z) + g(y, z) + v(x_0, y_0, z) - v(x_0, y_0, z) \\ &= f_{Z'}(x, z) + g_{Z'}(y, z) + h_{Z'}(z). \end{aligned} \tag{47}$$

We claim that $f_{Z'}, g_{Z'}$ are also continuous. By continuity of v and $h_{Z'}$, we only need to show that $f_{Z'}$ is continuous.

$$\begin{aligned} v(x, y_0, z) &= f_{Z'}(x, z) + g_{Z'}(y_0, z) + h_{Z'}(z) \\ &= f_{Z'}(x, z) + h_{Z'}(z) \end{aligned} \tag{48}$$

Let $f^*(x, z) = v(x, y_0, z) - h_{Z'}(z)$ with $f^* : (S_{y_0}, \tau_{S_{y_0}}) \to \mathbb{R}$. Since $t^{X\times Y\times Z}_{S_{Z'}}$ is a subspace topology of a product space, and $f_{Z'}$ is constant in y, continuity of f^* implies continuity of $f_{Z'}$. Since the restrictions of $h_{Z'}$ and v to $(S_{y_0}, t^{X\times Y\times Z}_{S_{y_0}})$ are continuous, f^*, thus $f_{Z'}$, thus $g_{Z'}$ are also continuous. $\square$

Lemma 11. *Suppose $S \subseteq X \times Y \times Z$ is well behaved given Z. Let $\succsim$ be a continuous preference relation on S fulfilling essentiality given Z and $X \perp Y \mid Z$. Further, assume that Z is a normal space. Let v be a representation of $\succsim$. Suppose for connected open sets Z', Z'' with $cl(Z') \cap cl(Z'') \neq \emptyset$ there exist continuous functions*

$$\begin{aligned} f_{Z'} : (X_{S_{Z'}} \times Z', t^{X\times Z}_{X_{S_{Z'}}\times Z'}) \to \mathbb{R}, \quad g_{Z'} : (Y_{S_{Z'}} \times Z', t^{Y\times Z}_{X_{S_{Z'}}\times Z'}) \to \mathbb{R}, \\ f_{Z''} : (X_{S_{Z''}} \times Z'', t^{X\times Z}_{X_{S_{Z''}}\times Z''}) \to \mathbb{R}, \quad g_{Z''} : (Y_{S_{Z''}} \times Z'', t^{Y\times Z}_{X_{S_{Z''}}\times Z''}) \to \mathbb{R} \end{aligned} \tag{49}$$

such that $v = f_{Z'} + g_{Z'}$ and $v = f_{Z''} + g_{Z''}$. Then defining $\bar{Z} = cl(Z') \cup Z''$ we can find continuous functions

$$\begin{aligned} f &:(X_{S_{\bar{Z}}} \times \bar{Z}, t^{X\times Z}_{X_{S_{\bar{Z}}}\times \bar{Z}}) \to \mathbb{R}, \\ g &:(X_{S_{\bar{Z}}} \times \bar{Z}, t^{X\times Z}_{X_{S_{\bar{Z}}}\times \bar{Z}}) \to \mathbb{R} \end{aligned} \tag{50}$$

such that $v = f + g$ and $f_{Z'} = f, g_{Z'} = g$ on the domain of $f_{Z'}, g_{Z'}$, respectively.

Proof. Define $S_0 = S \cap X \times Y \times cl(Z')$ and $S_1 = S \cap X \times Y \times cl(Z'')$. We can continuously extend $f_{Z'}, g_{Z'}$ and $f_{Z''}, g_{Z''}$ to S_0 and S_1, respectively.

Let $s_0 = (x_0, y_0, z_0) \in S_0 \cap S_1$. By the assumption that S is open in the product topology, s_0 is an interior point and therefore there exists an open set $O = O_x \times O_y \times O_z$ containing s_0. By Lemma 10 we can construct on $S^* = S \cap X \times Y \times O_z$ a representation $v = f^* + g^*$ where f^*, g^* are continuous in the subspace topology of the product topology. Note that O intersects with both S_0 and S_1.

On $S^* \cap S_{Z'}$ we have $f^* + g^* = f_{Z'} + g_{Z'}$. Define $h^* : O_z \cap Z' \cap Z_{S_{x_0,y_0}} \to \mathbb{R}$ as $h^*(z) = g^*(y_0, z) - g_{Z'}(y_0, z)$. h^* is continuous in $(O_z \cap Z' \cap Z_{S_{x_0,y_0}}, \tau_{O_z \cap Z' \cap Z_{S_{x_0,y_0}}})$. We can therefore extend h^* to the closure of $O_z \cap Z' \cap Z_{S_{x_0,y_0}}$ in $Z_{S_{x_0,y_0}}$.[7]

Since the topological space (Z, t^Z) is normal, using the Tietze extension theorem we can find a continuous extension of h^* to Z. We now define the function

$$\hat{f}(x,z) = \begin{cases} f_{Z'}(x,z), & z \in cl(Z') \\ f^*(x,z) + h^*(z), & z \in Z_{S_{x_0,y_0}} \backslash Z' \end{cases} \tag{51}$$

$$\hat{g}(y,z) = \begin{cases} g_{Z'}(y,z), & z \in cl(Z') \\ g^*(y,z) - h^*(z), & z \in Z_{S_{x_0,y_0}} \backslash Z'. \end{cases} \tag{52}$$

which is continuous by the pasting lemma.

Next, we extend the representation to S_1 in a similar way. On S_1 we have the representation $v(x,y,z) = f_{Z''}(x,z) + g_{Z''}(y,z)$ where $f_{Z''}, g_{Z''}$ are continuous in the subspace topology of the product topology.

We define $h_{Z''}$ as the continuous extension of $g_{Z''}(y_1, z) - \hat{g}(y_1, z)$ from the closure of $(cl(Z') \cup Z_{S_{x_0,y_0}}) \cap cl(Z'')$ to Z. We obtain the continuous functions

$$f(x,z) = \begin{cases} \hat{f}(x,z), & z \in cl(Z') \cup Z_{S_{x_0,y_0}} \\ f_{Z''}(x,z) + h_{Z''}(z), & z \in Z'' \backslash (cl(Z') \cup Z_{S_{x_0,y_0}}) \end{cases} \tag{53}$$

and

$$g(y,z) = \begin{cases} \hat{g}(y,z), & z \in cl(Z') \cup Z_{S_{x_0,y_0}} \\ g_{Z''}(y,z) - h_{Z''}(z), & z \in Z'' \backslash (cl(Z') \cup Z_{S_{x_0,y_0}}) \end{cases} \tag{54}$$

Since $v(x,y,z) = f(x,z) + g(y,z)$ the conclusion follows. $\square$

We are now ready to state the proof of the main result.

Proof. We use Lemma 9 to construct a continuous utility representation $v(x,y,z) = f(x,z) + g(y,z)$ on S. While v is continuous, Lemma 9 does not guarantee that f and g are continuous. We show continuity of the additive components in two cases.

If $Z_{S_{x_0,y_0}} = Z_S$, we can define $\hat{f}(x,z) = f(x,z) - f(x_0,z)$ and $\hat{g}(y,z) = g(y,z) - g(y_0,z)$ and $h(z) = v(x_0,y_0,z)$. Since v is continuous in the order topology τ_S, h is continuous in $\tau_{S_{x_0,y_0}}$ and by continuity $t_S^{X \times Y \times Z}$. Since on S_{x_0} we have $v(x,y,z) - h(z) = \hat{g}(y,z)$, $\hat{g}$ is continuous in $t_S^{X \times Y \times Z}$ and thus $t_{Y_S,Z_S}^{Y \times Z}$. On S_{y_0}, we have $\hat{f}(x,z) = v(x,y,z) - h(z)$, which is continuous in $t_{X_S,Z_S}^{X \times Z}$ by

[7] By essentiality given Z, the limit points cannot have infinite utility.

a similar argument. Since $v(x,y,z) = \hat{f}(x,z) + \hat{g}(y,z) + h(z)$, we have indeed found a continuous additive representation where all components are continuous as well.

If Z is normal, we use the Tietze extension theorem to prove continuity of f and g:

We first show that we can find a sequence $\{x_k, y_k\}_{k=1}^{\infty}$ such that for all k, $\bigcup_{i=1}^{k} cl(Z_{S_{x_k,y_k}}) \cap cl(Z_{S_{x_{k+1},y_{k+1}}}) \neq \emptyset$ and $\bigcup_{k=1}^{\infty} cl(Z_{S_{x_k,y_k}})$ is dense in Z_S.

Let $\hat{Z}$ be an dense subset of Z which exists by Z being separable. Suppose we have constructed the sequence up to k. Then $A_k = \bigcup_{i=1}^{k} cl(Z_{S_{x_k,y_k}})$ is a closed set as it is a finite union of closed sets. If there exists z' in $\hat{Z} \backslash \{Z_{S_{x_1,y_1}}, \ldots, Z_{S_{x_k,y_k}}\}$ and x', y' such that $(x', y', z') \in S$ and $A_k \cap cl(Z_{S_{x',y'}}) \neq \emptyset$, we can choose $x_{k+1} = x'$ and $y_{k+1} = y'$.

Otherwise, denote $B_k = \bigcup_{i=k+1}^{\infty} cl(Z_{S_{x_k,y_k}})$. Since $cl(A_k \cup cl(B_k)) = A_k \cup cl(B_k) = Z_S$ and Z_S is connected, $A_k \cap cl(B_k) \neq \emptyset$. Take any element $z_{k+1} \in A_k \cap cl(B_k)$ with $(x_{k+1}, y_{k+1}, z_{k+1}) \in S$. Then since $Z_{S_{x_{k+1},y_{k+1}}} \ni z_{k+1}$ is open, it intersects both A_k and B_k. Thus, there exists some x_{k+2}, y_{k+2} such that $Z_{S_{x_{k+1},y_{k+1}}} \cap Z_{S_{x_{k+2},y_{k+2}}} \neq \emptyset$ and we can continue constructing the sequence.

To guarantee continuity, if Z is normal we start by making f and g continuous on $S_{Z_{S_{x_1,y_1}}}$ via Lemma 10. Using Lemma 10, we can ensure the existence of continuous functions

$$\begin{aligned} f_{Z_k} &: (X_{S_{x_k,y_k}} \times Z_{S_{x_k,y_k}}, t^{X \times Z}_{X_{S_{x_k,y_k}} \times Z_{S_{x_k,y_k}}}) \to \mathbb{R}, \\ g_{Z_k} &: (Y_{S_{x_k,y_k}} \times Z_{S_{x_k,y_k}}, t^{Y \times Z}_{X_{S_{x_k,y_k}} \times Z_{S_{x_k,y_k}}}) \to \mathbb{R}, \end{aligned} \tag{55}$$

such that $v = f_{Z_k} + g_{Z_k}$. Since for all k the sets $Z_{S_{x_{k+1},y_{k+1}}}$ and A_k fulfill the assumptions of Lemma 11, we can make the functions f and g continuous on the entire space. $\square$

Appendix B Proof of Theorem 2

Proof. By Theorem 1, we have the case where $n = 3$. Let $k \leq n$ be the largest number such that there exists a representation with the form $v(s) = \sum_{i=1}^{k} f_i(x_i, x_1) + g(x_{k+1}, \ldots, x_n, x_1) + h(x_1)$. Since $X_2 \perp \prod_{j \neq 2,1} X_j \mid X_1$, we have $k \geq 1$. We prove $k = n$ by contradiction. Suppose $k < n$. Since $X_k \times X_{k+1} \perp \prod_{j \neq k,k+1,1} X_j \mid X_1$, we have the representation

$$v_{k,k+1}(s) = F_{k,k+1}(x_1, x_k, x_{k+1}) + g_{k,k+1}(x_1, (x_j)_{j \neq 1,k,k+1}) + h_{k,k+1}(x_1)$$

and by our assumption, we have the representation

$$v(s) = \sum_{i=1}^{k} f_i(x_i, x_1) + g(x_{k+1}, \ldots, x_n, x_1) + h(x_1).$$

Since $v(s)$ and $v_{k,k+1}(s)$ represents the same preference in the space S, $v(s) = T_{k,k+1}(v_{k,k+1}(s))$ for some monotone function $T_{k,k+1}$. Fix any $x^*_{k+1} \in X_{k+1_S}$

and $x_1^* \in X_{1_S}$, we have

$$
\begin{aligned}
&f_k(x_k, x_1^*) + \sum_{i=1}^{k-1} f_i(x_i, x_1^*) + g(x_{k+1}^*, \ldots, x_n, x_1) + h(x_1^*) \\
&= T_{k,k+1}(F_{k,k+1}(x_1^*, x_k, x_{k+1}^*) + g_{k,k+1}(x_1^*, (x_j)_{j\neq 1,k,k+1}) + h_{k,k+1}(x_1^*)). \quad (56)
\end{aligned}
$$

which implies that $v(s)$ and $v_{k,k+1}(s)$ are additive representation which are unique up to an affine transformation with respect to the space $X_k \times \prod_{j\neq 1,k,k+1} X_j \times \{x_1^*\} \times \{x_{k+1}^*\}$. Thus, $T_{k,k+1}$ must be an affine function in the domain $v_{k,k+1}(X_k \times \prod_{j\neq 1,k,k+1} X_j \times \{x_1^*\} \times \{x_{k+1}^*\})$. Since the x_1^* and x_{k+1}^* are arbitrarily chosen,$T_{k,k+1}$ is affine function on the whole domain. Without loss of generality, we can assume $T_{k,k+1}$ is the identity function. We therefore have

$$
\begin{aligned}
&\sum_{i=1}^{k} f_i(x_i, x_1^*) + g(x_{k+1}^*, \ldots, x_n, x_1) + h(x_1^*) \\
&= F_{k,k+1}(x_1^*, x_k, x_{k+1}^*) + g_{k,k+1}(x_1^*, (x_j)_{j\neq 1,k,k+1}) + h_{k,k+1}(x_1^*). \qquad (57)
\end{aligned}
$$

If we can show that $F_{k,k+1}(x_1, x_k, x_{k+1})$ has the form $f_k^*(x_k, x_1)+f_{k+1}^*(x_{k+1}, x_1)$, then the right hand side of above equation is a conditional additive representation in $k + 1$ dimensions, yielding the contradiction. To show this, since $X_{k+1} \perp \prod_{j\neq k+1,1} X_j \mid X_1$, we have the representation

$$v_{k+1}(s) = F_{k+1}(x_1, x_{k+1}) + g_{k+1}(x_1, (x_j)_{j\neq 1,k+1}) + h_{k+1}(x_1).$$

Fix any $x_1^* \in X_{1_S}$ and $x_k^* \in X_{k_S}$, by a similar argument, we can show that $v(s) = T(v_{k+1})$ for an affine function T. Therefore, we can assume that $v(s) = v_{k+1}(s) = v_{k,k+1}(s)$. Therefore,

$$
\begin{aligned}
F_{k,k+1}(x_1, x_k, x_{k+1}) =&F_{k+1}(x_1, x_{k+1}) \\
&+ h_{k+1}(x_1) - h_{k,k+1}(x_1) \\
&+ g_{k+1}(x_1, x_k, (x_j)_{j\neq 1,k+1,k}) - g_{k,k+1}(x_1, (x_j)_{j\neq 1,k,k+1})
\end{aligned} \qquad (58)
$$

and the result follows. □

Appendix C Proof of Theorem 3

Proof. We prove the result by induction on n. For $n = 3$ the result holds in virtue of Theorem 1.

Suppose our result holds for $n = k$. We have $X_1 \times \ldots \times X_{k-1} \perp X_{k+1} \mid X_k$. Thus, we have a representation $u(x) = f(x_1, \ldots, x_k) + g(x_k, x_{k+1})$ by the case of $n = 3$. Since our result holds for $n = k$,

$$
\begin{aligned}
T[f(x_1, \ldots, x_k) + g(x_k, x_{k+1})] =&\sum_{i=2}^{k-1} f_i(x_i, x_{i-1}) \\
&+ f_k((x_k, x_{k+1}), x_{k-1})
\end{aligned} \qquad (59)
$$

for some montone transformation T. Fixing $x_l = x_l^0$ for all $l \neq k+1, k-2$, we have:

$$
\begin{aligned}
T[f((x_l^0)_{l=1}^{k-3}, x_{k-2}, x_{k-1}^0, x_k^0) + g(x_k^0, x_{k+1})] = &\sum_{i=2}^{k-3} f_i(x_i^0, x_{i-1}^0) \\
&+ f_{k-2}(x_{k-2}, x_{k-3}^0) \\
&+ f_{k-1}(x_{k-1}^0, x_{k-2}) \\
&+ f_k((x_k^0, x_{k+1}), x_{k-1}^0). \qquad (60)
\end{aligned}
$$

Noticing that both the term inside $T[\ldots]$ and the RHS are additive representations on the $X_{k-2} \times X_{k+1} \times \prod_{l \neq k-2, k+1} \{x_l^0\}$ space, by the uniqueness of additive representations it follows that T is affine. We may assume without loss of generality that $T[f] = f$. Thus,

$$
\begin{aligned}
f(x_1, \ldots, x_k) + g(x_k, x_{k+1}) = &\sum_{i=2}^{k-1} f_i(x_i, x_{i-1}) \\
&+ f_k((x_k, x_{k+1}), x_{k-1}). \qquad (61)
\end{aligned}
$$

From which follows

$$
\begin{aligned}
f(x_1^0, \ldots, x_{k-2}^0, x_{k-1}, x_k) + g(x_k, x_{k+1}) = &\sum_{i=2}^{k-2} f_i(x_i^0, x_{i-1}^0) \\
&+ f_{k-1}(x_{k-1}, x_{k-2}^0) \\
&+ f_k((x_k, x_{k+1}), x_{k-1}). \qquad (62)
\end{aligned}
$$

Thus, we can write f_k in the form: $f_k((x_k, x_{k+1}), x_{k-1}) = g_k(x_k, x_{k-1}) + g_{k+1}(x_{k+1}, x_k)$ which concludes the proof. □

Appendix D Proof of Theorem 4

Proof. We start out by deriving a representation of the form

$$v(x, y) = \sum_{k=1}^{n} f_k(x_k, y_k, y_{k-1}) \qquad (63)$$

via induction. For the case $n = 3$, the conditions

$$
\begin{aligned}
&X_1 \perp X_2 \times Y_2 \times X_3 \times Y_3 \mid Y_1, \text{ and} \\
&X_1 \times Y_1 \times X_2 \perp X_3 \times Y_3 \mid Y_2 \qquad (64)
\end{aligned}
$$

give us a functional equation:

$$f_1(x_1, y_1) + g(x_2, x_3, y_1, y_2, y_3) = T[\bar{g}(x_1, x_2, y_1, y_2) + f_3(x_3, y_2, y_3)]. \qquad (65)$$

Being additive representations of X_1 and X_3, T is affine via the uniqueness properties of additive representations. Without loss of generality, we assume $T[u] = u$ and obtain:

$$
\begin{aligned}
g(x_2, x_3, y_1, y_2, y_3) = &\bar{g}(x_1^*, x_2, y_1, y_2) + f_3(x_3, y_2, y_3) - f_1(x_1^*, y_1) \\
\equiv &f_2(x_2, y_1, y_2) + f_3(x_3, y_2, y_3). \qquad (66)
\end{aligned}
$$

The representation is therfore:

$$v(x,y) = f_1(x_1,y_1) + f_2(x_2,y_1,y_2) + f_3(x_3,y_2,y_3) \tag{67}$$

where y_0 is constant.

For the induction step we obtain the functional equation:

$$\sum_{k=1}^{n-1} f_k(x_k,y_k,y_{k-1}) + g(x_n,x_{n+1},y_{n-1},y_n,y_{n+1})$$
$$= T[\bar{g}((x_k,y_k)_{k=1}^n) + f_{n+1}(x_{n+1},y_n,y_{n+1})]. \tag{68}$$

Again, we can set $T[u] = u$ due to uniqueness of affine representations on X_1 and X_n. Holding fixed $x_1,\dots,x_{n-1}$ and $y_1,\dots,y_{n-2}$, we get:

$$g(x_n,x_{n+1},y_{n-1},y_n,y_{n+1}) = f_n(x_n,y_{n-1},y_n) + f_{n+1}(x_{n+1},y_n,y_{n+1}) + \alpha \tag{69}$$

for some constant α which we assume without loss of generality to be equal to zero. Thus, we have

$$v(x,y) = \sum_{k=1}^{n} f_k(x_k,y_k,y_{k-1}) \tag{70}$$

for any $n \geq 3$.

It is similarly straightforward to obtain the representation

$$w(x,y) = \sum_{k=1}^{n} g_k(x_k,x_{k+1},y_k) \tag{71}$$

using the conditions of the form $\prod_{j=1}^{i-1} X_j \times Y_j \perp Y_i \times \prod_{k=i+1}^{n} X_k \times Y_k \mid X_i$.

Note that for $n \geq 3$, v and w are affine transformations of each other since both are additive representations on X_1 and X_n when holding fixed the other dimensions. Thus,

$$\sum_{k=1}^{n} f_k(x_k,y_k,y_{k-1}) = \sum_{k=1}^{n} g_k(x_k,x_{k+1},y_k). \tag{72}$$

Holding fixed all dimensions except x_k, y_k, y_{k-1}, we get:

$$\begin{aligned} f_k(x_k,y_k,y_{k-1}) =& g_{k-1}(x_{k-1}^*,x_k,y_{k-1}) + g_k(x_k,x_{k+1}^*,y_k) + \alpha \\ \equiv& \bar{f}_k(x_k,y_{k-1}) + \bar{g}_k(x_k,y_k). \end{aligned} \tag{73}$$

□

Appendix E Proof of Theorem 5

We first provide the solution to a functional equation which will be useful in the remainder of the proof.

Lemma 12. *Let $\mathbb{S},+$ be a cancellative abelian monoid and let $\bar{f}$, $\bar{g}$, f and g be real valued functions defined on $\mathbb{S}^2$ and satisfy the relation*

$$\bar{f}(x_3,x_1+x_2) + \bar{g}(x_1,x_2) = f(x_2,x_1+x_3) + g(x_1,x_3)$$

for all x_1,x_2, x_3 in $\mathbb{S}$. Then $f(x_2,x_1+x_3) + g(x_1,x_3) = v_{123}(x_1+x_2+x_3) + v_1(x_1) + v_2(x_2) + v_3(x_3)$. In particular, $f(a,b) = a_1(a) + a_2(b) + a_3(a+b)$.

Proof. The functional equation to be solved is[8]

$$\bar{g}(x_1, x_2) = f(x_2, x_1 + x_3) + g(x_1, x_3) - \bar{f}(x_3, x_1 + x_2). \tag{74}$$

We set $x_3 = 0$ and define $\bar{u}_1(x_1) = g(x_1, 0)$ and $\bar{u}(x_1) = \bar{f}(0, x_1)$ to obtain:

$$\bar{g}(x_1, x_2) = f(x_2, x_1) + \bar{u}_1(x_1) + \bar{u}_3(x_1 + x_2). \tag{75}$$

By a symmetric argument with $x_2 = 0$, we have

$$g(x_1, x_3) = \bar{f}(x_3, x_1) + u_1(x_1) + u_3(x_1 + x_3). \tag{76}$$

Inserting Equation (76) into Equation (74), we have

$$\begin{aligned} & f(x_2, x_1 + x_3) + \bar{f}(x_3, x_1) + u_1(x_1) + u_3(x_1 + x_3) \\ = & \bar{f}(x_3, x_1 + x_2) + f(x_2, x_1) + \bar{u}_1(x_1) + \bar{u}_3(x_1 + x_2). \end{aligned} \tag{77}$$

Let $x_1 = 0$ in Equation (77). Then we get the following relation between $\bar{f}$ and f

$$\bar{f}(x_3, x_2) = f(x_2, x_3) + A_1(x_2) + A_2(x_3)$$

for some suitably defined functions A_1, A_2. Inserting this result into Equation (77) we get

$$\begin{aligned} & f(x_1 + x_2, x_3) + f(x_2, x_1) + \bar{U}_1(x_1) + \bar{U}_2(x_2) + \bar{U}_3(x_1 + x_2) \\ = & f(x_2, x_1 + x_3) + f(x_1, x_3) + U_1(x_1) + U_2(x_3) + U_3(x_1 + x_3). \end{aligned} \tag{78}$$

We want to characterize the function f, for any $(x_1, x_2) \in \mathbb{S}^2$. Gathering terms, we have

$$\begin{aligned} f(x_1, x_2) = & f(x_1, x_2 + x_3) + f(x_2, x_3) - f(x_1 + x_2, x_3) \\ & + v_1(x_2) + v_2(x_1) + v_3(x_3) + v_{12}(x_1 + x_2) + v_{13}(x_2 + x_3). \end{aligned} \tag{79}$$

Our goal is to prove $f(x, x_2) = a_1(x) + a_2(x_2) + a_3(x + x_2)$. To achieve this, we provide the following lemma:

Lemma 13. *Let $g : \mathbb{S}^2 \to \mathbb{R}$. Then $g(x_1, x_2) = g_1(x_1) + g_2(x_2)$ if and only if $g(x_1', x_2') - g(x_1', 0) - g(0, x_2') + g(0, 0) = 0 \ \forall \ x_1, x_2$.*

Proof. If $g(x_1, x_2) = g_1(x_1) + g_2(x_2)$ then, $g(x_1', x_2') - g(x_1', 0) - g(0, x_2') + g(0, 0) = g_1(x_1') + g_2(x_2') - g_1(x_1') - g_2(0) - g_1(0) - g_2(x_2') + g_1(0) + g_2(0) = 0$.

On the other hand, suppose $g(x_1, x_2)$ satisfies the condition $g(x_1', x_2') - g(x_1', 0) - g(0, x_2') + g(0, 0) = 0$. Then we define the $g_1(x_1) := g(x_1, 0)$ and $g_2(x_2) := g(0, x_2) - g(0, 0)$. Then, by the condition, $g(x_1, x_2) = g(0, x_2) + g(x_1, 0) - g(0, 0) = g_1(x_1) + g_2(x_2)$. □

By Lemma 13, $f(x_1, x_2) = a_1(x_1) + a_2(x_2) + a_3(x_1 + x_2)$ if and only if $f(x_1, x_2) - f(x_1, 0) - f(0, x_2) - f(0, 0) = a_3(x_1 + x_2) - a_3(x_1) - a_3(x_2) + a_3(0)$. Therefore, we define

$$G(x_1, x_2) \equiv f(x_1, x_2) - f(x_1, 0) - f(0, x_2) - f(0, 0). \tag{80}$$

[8]In the remainder of the proof, we will omit stating that equations such as (74) hold for all x_1, x_2, x_3. It will be clear from the context whether a variable is a free variable or not.

Substituting Equation (79) for $f(x_1, x_2)$, we get

$$G(x_1, x_2) = f(x_1, x_2 + x_3) + f(x_2, x_3) - f(x_1 + x_2, x_3) - f(0, x_2 + x_3) \\ +(v_{12}(x_1 + x_2) - v_{12}(x_1) - v_{12}(x_2) + v_{12}(0)). \quad (81)$$

Thus f has the desired functional form if and only if

$$\begin{aligned} N(x_1, x_2) \equiv & f(x_1, x_2 + x_3) + f(x_2, x_3) - f(x_1 + x_2, x_3) - f(0, x_2 + x_3) \\ = & a(x_1 + x_2) - a(x_1) - a(x_2) + a(0) \end{aligned} \quad (82)$$

for some real-valued function a. To show that this is the case, notice that

$$\begin{aligned} N(x_1 + x_2, x_3) =& - [f(x_1 + x_2 + x_3, c) - f(x_3, c)] \\ & + [f(x_1 + x_2, x_3 + c) - f(0, x_3 + c)] \\ N(x_1, x_2) =& - [f(x_1 + x_2, c) - f(x_2, c)] \\ & + [f(x_1, x_2 + c) - f(0, x_2 + c)] \\ N(x_1, x_2 + x_3) =& - [f(x_1 + x_2 + x_3, c) - f(x_2 + x_3, c)] \\ & + [f(x_1, x_2 + x_3 + c) - f(0, x_2 + x_3 + c)] \\ N(x_2, x_3) =& - [f(x_2 + x_3, c) - f(x_3, c)] \\ & + [f(x_2, x_3 + c) - f(0, x_3 + c)]. \end{aligned} \quad (83)$$

We choose $c = 0$ in $N(x_1 + x_2, x_3)$, $N(x_1, x_2 + x_3)$ and $N(x_2, x_3)$, and $c = x_3$ in $N(x_1, x_2)$ to obtain $N(x_1 + x_2, x_3) + N(x_1, x_2) = N(x_1, x_2 + x_3) + N(x_2, x_3)$. By Hosszú (1971), $N(x_1, x_2) = B(x_1, x_2) + a(x_1 + x_2) - a(x_1) - a(x_2)$ where $B(x_1, x_2)$ is a skew-symmetric biadditive function. Since $N(0, 0) = N(x_1, 0) = N(0, x_2) = 0$, $B(x, x_2) = B(0, 0) = a(0) = 0$. Thus, the function f has the functional form

$$f(a, b) = a_1(a) + a_2(b) + a_3(a + b). \quad (84)$$

To show that $f(x_2, x_1 + x_3) + g(x_1, x_3)$ has the desired functional form, we substitute Equation (84) in Equation (78). Then we obtain

$$\begin{aligned} & U_1(x_1) + U_2(x_2) + U_3(x_3) + U_4(x_1 + x_2) \\ = & \bar{U}_1(x_1) + \bar{U}_2(x_2) + \bar{U}_3(x_3) + \bar{U}_4(x_1 + x_3). \end{aligned} \quad (85)$$

Letting $x_3 = 0$, we obtain that $U_4(x_1 + x_2)$ is additively separable in variables x_1 and x_2. Similarly, letting $x_2 = 0$, $\bar{U}_4$ is additively separable in x_1 and x_3. The desired result follows. □

We now prove Theorem 5.

Proof. Define $\hat{S} = \{(x, y_1, \ldots, y_{n-3}, z) \in (0, 1)^{n-1} : 1 - x > z > \sum_{j=1}^{n-3} y_j\} \subset \hat{X} \times \hat{Y} \times \hat{Z}$ where $\hat{X} = (0, 1)$, $\hat{Y} = (0, 1)^{n-3}$, and $\hat{Z} = (0, 1)$. We define a homeomorphism

$$\begin{aligned} & \phi_i : (S, t_S^{\mathbb{R}^n}) \to (\hat{S}, t_{\hat{S}}^{\mathbb{R}^{n-1}}) \text{ such that:} \\ & \phi_i(x_1, \ldots, x_n) = (x_i, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-2}, 1 - x_i - x_n). \end{aligned} \quad (86)$$

Define $\succsim_i$ as $s \succsim_i s'$ if and only if $\phi_i^{-1}(s) \succsim \phi_i^{-1}(s')$. Notice that $\succsim^*$ is continuous if $\succsim$ is continuous since ϕ_i is a homeomorphism. As discussed in the main text for the case of $n = 4$, $\succsim^*$ fulfills $\hat{X} \perp \hat{Y} \mid \hat{Z}$ as can be easily shown by an analogous proof. Similarly, $\succsim^*$ is essential given $\hat{Z}$. Definition 2 i) and ii) are fulfilled as can be easily verified. Definition 2 iii) is not necessary by Remark 1, since S_z is a product set for all $z \in \hat{Z}$. From Theorem 1 we then have the following representation result for all $i < n$:

$$\begin{aligned}
&(x_1, \ldots, x_n) \succsim (x'_1, \ldots, x'_n) \\
&\Leftrightarrow (x, y, z) \succsim_i (x'y'z) \\
&\Leftrightarrow f_i(x, z) + g_i(y, z) \geq f_i(x', z') + g_i(y', z') \\
&\Leftrightarrow f_i(x_i, \sum_{k \neq i,n} x_k) + \hat{g}_i((x_k)_{k \neq i,n-1,n}, \sum_{k \neq i,n} x_k) \geq f_i(x_i, \sum_{k \neq i,n} x_k) + \hat{g}_i((x_k)_{k \neq i,n-1,n}, \sum_{k \neq i,n} x_k) \\
&\Leftrightarrow U_i(x_1, \ldots, x_n) \geq U_i(x'_1, \ldots, x'_n)
\end{aligned} \tag{87}$$

where

$$\begin{aligned}
\phi_i^{-1}(x, y, z) =& (x_1, \ldots, x_n) \\
\phi_i^{-1}(x', y', z') =& (x'_1, \ldots, x'_n) \\
U_i(x) =& f_i(x_i, \sum_{k \neq i,n} x_k) + \hat{g}_i((x_k)_{k \neq i,n-1,n}, \sum_{k \neq i,n} x_k) \\
=& f_i(x_i, \sum_{k \neq i,n} x_k) + g_i((x_k)_{k \neq i,n}).
\end{aligned} \tag{88}$$

Using comeasurability, we can ensure during the utility construction process of U_i, U_j that

$$U_i(x) = U_j(x) = U(x). \tag{89}$$

for all $i, j < n$. Therefore,

$$\begin{aligned}
&f_i(x_i, \sum_{m \neq i,n} x_m) + g_i((x_m)_{m \neq i,n}) \\
=& f_j(x_j, \sum_{m \neq i,n} x_m) + g_j((x_m)_{m \neq i,n}).
\end{aligned} \tag{90}$$

Setting $x_m = \epsilon$ for all $m \neq i, j, k$, we obtain:

$$\begin{aligned}
&f_i(x_i, x_j + x_k) + g_i(\epsilon, \ldots, \epsilon, x_j, x_k) \\
=& f_j(x_j, x_i + x_k) + g_j(\epsilon, \ldots, \epsilon, x_i, x_k).
\end{aligned} \tag{91}$$

By Lemma 12, this functional equation has the solution $f_i(x_i, x_j + x_k) = u_i(x_i) + \bar{u}_i(x_i + x_j + x_k) + \hat{u}_i(x_j + x_k)$ and $f_j(x_j, x_i + x_k) = u_j(x_j) + \bar{u}_j(x_i + x_j + x_k) + \hat{u}_j(x_i + x_k)$. We therefore have for all i

$$\begin{aligned}
U(x) &= u_i(x_i) + \bar{u}_i(\sum_{m \neq n} x_m) + \hat{g}_i((x_m)_{m \neq i,n}) \\
&= u_j(x_j) + \bar{u}_j(\sum_{m \neq n} x_m) + \hat{g}_j((x_m)_{m \neq j,n})
\end{aligned} \tag{92}$$

where $\hat{g}_i((x_m)_{m \neq i,n}) = g_i((x_m)_{m \neq i,n}) - \hat{u}(\sum_{m \neq i,n} x_m)$ and $\hat{g}_j((x_m)_{m \neq j,n}) = g_j((x_m)_{m \neq j,n}) - \hat{u}(\sum_{m \neq j,n} x_m)$. Setting $u_n(x_n) = \bar{u}(1 - x_n)$ we obtain:

$$U(x) = u_i(x_i) + u_n(x_n) + \hat{g}_i((x_m)_{m \neq i,n}). \tag{93}$$

Our initial choice of n was arbitrary. We have thus shown that for any i, n, the utility representation is additively separable. Thus,

$$U(x) = \sum_{i=1}^{n} u_i(x_i). \tag{94}$$

$\square$

Appendix F Proof of Corollary 3

Proof. Using the same homeomorphism $\phi_i : (S, t_S^{\mathbb{R}^n}) \to (\hat{S}, t_{\hat{S}}^{\mathbb{R}^{n-1}})$ in Theorem 5, we can construct $\hat{\phi}_i : (S \times Z, t_{S\times Z}^{\mathbb{R}^n}) \to (\hat{S} \times Z, t_{\hat{S}\times Z}^{\mathbb{R}^{n-1}})$ by $\hat{\phi}_i(s, z) = (\phi_i(s), z)$. As the argument in Theorem 5, $\hat{\phi}_i$ is a homeomorphism and there exist a equivalent relation $\succsim^*$ on $\hat{S} \times Z$ where $\hat{S} \times Z \subset \hat{X} \times \hat{Y} \times \hat{Z}$ where $\hat{X} = (0, 1)$ $\hat{Y} = (0, 1)^{n-3}$ $\hat{Z} = (0, 1) \times Z$. Since comeasurability given Z implies the Reidemeister condition on each z-layer, using the remarks from the main text we can derive $\hat{X} \perp \hat{Y} | \hat{Z}$. Similarly, essentiality given Z gives us essentiality on each z-layer. Therefore, by Theorem 1 there exists a utility representation

$$u(x, z) = f(x_i, \sum_{k\neq i,n} x_k, z) + g((X_k)_{k\neq i,n}, \sum_{k\neq i,n} x_k, z). \tag{95}$$

By Theorem 5, fixing each z we have an additive representation $\sum u_{i,z}(x_i)$ on each layer z. Therefore

$$T_z(\sum u_{i,z}(x_i)) = u(x, z)$$

for some monotone transformation T_z. Next, we claim that T_z is an affine transformation. Indeed, since $\sum u_{i,z}(x_i)$ and $u(x, z)$ are additive representations on the space $S_z = \{(x_i, \prod_{k\neq i,n} X_k, z) | Z = z\}$ via the uniqueness of additive representations T_z is affine. As a result, we can define $u_i(x_i, z) := a_z u_{i,z}(x_i)$ for $i \neq 1$ and $u_1(x_1, z) := a_z u_{i,z}(x_1) + b_z$. Then we have the desired representation

$$u(x, z) = \sum u_i(x_i, z). \tag{96}$$

$\square$

Appendix G Proof of Theorem 6

Proof. (Sketch) Note first that there exists a continuous representation $w : S \to \mathbb{R}$. The representation on each z-layer follows directly from Herstein and Milnor (1953).

Suppose now that we have two layers z, z' which intersect in the indifference classes as characterized by w. Given these two intersecting layers z, z', we can make the representations consistent as follows. There exist $x_1 \sim y_1 \succ x_0 \sim y_0$ such that $x_0, x_1 \in S_z$ and $y_0, y_1 \in S_y$. Next, we have utility representations $v_z, v_{z'}$. Without loss of generality, assume $v_z(x_1) = v_{z'}(y_1) = 1 > 0 = v_z(x_0) =$

$v_{z'}(y_0)$. We need to show that $x \sim y \Rightarrow v_z(x) = v_{z'}(y)$. For this, we construct a sequence $\{x_k\}_{k=2}^{\infty}$ where

$$x_k = \begin{cases} x_{k-1}, & x_{k-1} \sim x \\ \frac{1}{2}x_{k-1} \oplus \frac{1}{2}\underline{x}_k, & x_{k-1} \succ x \\ \frac{1}{2}x_{k-1} \oplus \frac{1}{2}\overline{x}_k, & x \succ x_{k-1} \end{cases} \tag{97}$$

with $\underline{x}_k \in \arg\max_{x' \in \{(x_l)_{l \leq k-1} : x \succ x_l\}} u(x')$ and $\overline{x}_k \in \arg\min_{x' \in \{(x_l)_{l \leq k-1} : x_l \succ x\}} u(x')$.

By continuity, $\{v_z(x_k)\} \to v_z(x)$. Define $\{y_k\}$ analogously, then $v_z(x_k) = v_{z'}(y_k)$ if $x_{k-1} \sim y_{k-1}$ and $v_z(x_{k-1}) = v_{z'}(y_{k-1})$. We now argue that for all k, $x_k \sim y_k$. From coseparability given Z we get:

$$\begin{aligned} x \sim & y \\ \frac{1}{2}x \oplus \frac{1}{2}x' \sim & \frac{1}{2}y \oplus \frac{1}{2}y' \\ \Rightarrow x' \sim & y'. \end{aligned} \tag{98}$$

From this and the linear representations on each respective layer it is straightforward to obtain:

$$\begin{aligned} x \sim & y \\ x' \sim & y' \\ \Rightarrow \frac{1}{2}x \oplus \frac{1}{2}x' \sim & \frac{1}{2}y \oplus \frac{1}{2}y'. \end{aligned} \tag{99}$$

This proves both the case $k = 2$ as well as the induction step, thus for all k we have $x_k \sim y_k$. Therefore, $v_z(x) = \lim_{k\to\infty} v_z(x_k) = \lim_{k\to\infty} v_{z'}(y_k) = v_{z'}(y)$ for all x such that $x_1 \succ x \succ x_0$. Since our choice of points x_0, y_0, x_1, y_1 was arbitrary, $v_z(x) = v_{z'}(y) \Leftrightarrow x \sim y$ in the entire intersection of indifference classes.

The extension of the proof to all z-layers and thus the entire space is similar to the proof of conditionally additive representations. We first show that unless our representation covers all indifference classes, we can extend our representations to an open set of indifference classes around the sup/inf of the indifference classes covered so far. Therefore, we can cover all indifference classes in a countable number of extensions. Having covered all indifference classes, we can extend our representation to a countable dense subset of layers $\hat{Z} \subseteq Z$. From here we can extend the representation to the entire space by taking limits. The only interesting part remaining is to show that $v_{\bar{z}}(\alpha x \oplus (1-\alpha)x') = \alpha v_{\bar{z}}(x) + (1-\alpha)v_{\bar{z}}(x')$ on $\bar{z} \notin \hat{Z}$. To see this, note that using the above procedure we can obtain a consistent, conditionally linear representation $\bar{v}_z, z \in \{\bar{z}, \hat{z}\}$ on $\bar{z}$ and $\hat{z} \in \hat{Z}$. Without loss of generality, assume $\bar{z}$ and $\hat{z}$ intersect in the indifference classes of x, x'. (Otherwise, extend the representation to other z-layers until this is the case.) By an affine transformation, this representation can be made consistent with $v_{\hat{z}}$. But then,

$$\begin{aligned} v_{\bar{z}}(\alpha x \oplus (1-\alpha)x') =& \bar{v}_{\bar{z}}(\alpha x \oplus (1-\alpha)x') \\ =& \alpha \bar{v}_{\bar{z}}(x) + (1-\alpha)\bar{v}_{\bar{z}}(x') \\ =& \alpha v_{\bar{z}}(x) + (1-\alpha)v_{\bar{z}}(x'). \end{aligned} \tag{100}$$

□