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Chatelain, Jean-Bernard and Ralf, Kirsten

Paris School of Economics, Université Paris 1 Panthéon Sorbonne, ESCE

26 August 2017

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Jean-Bernard Chatelain*and Kirsten Ralf†

August 26, 2017

Abstract

This algorithm extends Ljungqvist and Sargent (2012) algorithm of Stackelberg dynamic game to the case of dynamic stochastic general equilibrium models including exogenous forcing variables. It is based Anderson, Hansen, McGrattan, Sargent (1996) discounted augmented linear quadratic regulator. It adds an intermediate step in solving a Sylvester equation. Forward-looking variables are also optimally anchored on forcing variables. This simple algorithm calls for already programmed routines for Ricatti, Sylvester and Inverse matrix in Matlab and Scilab. A final step using a change of basis vector computes a vector auto regressive representation including Ramsey optimal policy rule function of lagged observable variables, when the exogenous forcing variables are not observable.

JEL classification numbers: C61, C62, E47, E52, E58.

Keywords: Ramsey optimal policy, algorithm, forcing variables, augmented linear quadratic regulator.

1 Introduction

Ljungqvist and Sargent (2012, chapter 19) offer an elegant algorithm of Stackelberg dynamic game used for Ramsey optimal policy. All dynamic stochastic general equilibrium (DSGE) models include exogenous auto-regressive forcing variables, which are not included in their algorithm. This algorithm extends Ljungqvist and Sargent (2012, chapter 19) algorithm of dynamic Stackelberg game to the case of DSGE models including exogenous forcing variables.

We use Anderson, Hansen, McGrattan, Sargent (1996) discounted augmented linear quadratic regulator. After the usual algorithm for solving the Riccati equation of the linear quadratic regulator (Amman (1996)), *this algorithm adds another step in solving a Sylvester equation for completing the policy rule. It also adds a term for the optimal initial anchor of forward-looking variables on the predetermined forcing variables.*

This algorithm is easy to code and check. It is simple because it only calls already optimized routines solving Ricatti and Sylvester equations and inverse matrix in Matlab

*Paris School of Economics, Université Paris I Pantheon Sorbonne, PjSE, 48 Boulevard Jourdan, 75014 Paris. Email: jean-bernard.chatelain@univ-paris1.fr

†ESCE International Business School, 10 rue Sextius Michel, 75015 Paris, Email: Kirsten.Ralf@esce.fr.

and Scilab. A final step using a change of basis vector computes a vector auto regressive representation of Ramsey optimal policy. In this representation of the Ramsey optimal policy rule, policy instruments respond to lagged observable variables if all the exogenous forcing variables are not observable.

2 A Simple Algorithm

2.1 The Stackelberg problem

We refer to Ljungqvist and Sargent (2012), chapter 19, step by step. The Stackelberg leader is the government and the Stackelberg follower is the private sector.

Let $\mathbf{k}_t$ be an $n_k \times 1$ vector of controllable predetermined state variables with initial conditions $\mathbf{k}_0$ given, $\mathbf{x}_t$ an $n_x \times 1$ vector of endogenous variables free to jump at t without a given initial condition for $\mathbf{x}_0$, and $\mathbf{u}_t$ a vector of government policy instruments. Let $\mathbf{y}_t = (\mathbf{k}_t^T, \mathbf{x}_t^T)^T$ be an $(n_k + n_x) \times 1$ vector.

Our only addition to Sargent and Ljungvist (2012) Stackelberg problem is to include $\mathbf{z}_t$, which an $n_z \times 1$ vector of non-controllable, exogenous forcing state variables such as auto-regressive shocks. All variables are expressed as absolute or proportional deviations about a steady state.

Subject to an initial condition for $\mathbf{k}_0$ and $\mathbf{z}_0$, but not for $\mathbf{x}_0$, a government wants to maximize:

$$-\frac{1}{2}\sum_{t=0}^{+\infty}\beta^t \left(\mathbf{y}_t^T\mathbf{Q}_{yy}\mathbf{y}_t + 2\mathbf{y}_t^T\mathbf{Q}_{yz}\mathbf{z}_t + \mathbf{u}_t^T\mathbf{R}\mathbf{u}_t\right) \tag{1}$$

where β is the policy maker's discount factor and her policy preference are the relative weights included matrices $\mathbf{Q}, \mathbf{R}$. $\mathbf{Q}_{yy} \geq \mathbf{0}$ is a $(n_k + n_x) \times (n_k + n_x)$ positive symmetric semi-definite matrix, $\mathbf{R} > \mathbf{0}$ is a $p \times p$ *strictly* positive symmetric definite matrix so that policy maker's has at least a very small concern for the volatility of policy instruments. The cross-product of controllable policy targets with non-controllable forcing variables $\mathbf{y}_t^T\mathbf{Q}_{yz}\mathbf{z}_t$ is introduced by Anderson, Hansen, McGrattan and Sargent (1996). To our knowledge, it has always been set to zero $\mathbf{Q}_{yz} = \mathbf{0}$ so far in models of Ramsey optimal policy. This simplifies the Sylvester equation in step 3.

The policy transmission mechanism of the private sector's behavior is summarized by this system of equations written in a Kalman controllable staircase form:

$$\begin{pmatrix} E_t\mathbf{y}_{t+1} \\ \mathbf{z}_{t+1} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{yy} & \mathbf{A}_{yz} \\ \mathbf{0}_{zy} & \mathbf{A}_{zz} \end{pmatrix} \begin{pmatrix} \mathbf{y}_t \\ \mathbf{z}_t \end{pmatrix} + \begin{pmatrix} \mathbf{B}_y \\ \mathbf{0}_z \end{pmatrix} \mathbf{u}_t \tag{2}$$

$\mathbf{A}$ is $(n_k + n_x + n_z) \times (n_k + n_x + n_z)$ matrix. $\mathbf{B}$ is the $(n_k + n_x + n_z) \times p$ matrix of the marginal effects of policy instruments $\mathbf{u}_t$ on next period policy targets $\mathbf{y}_{t+1}$.

The government minimizes his discounted objective function by choosing sequences $\{u_t, x_t, k_{t+1}, z_{t+1}\}_{t=0}^{+\infty}$ subject to the policy transmission mechanism (2) and subject to $2(n_x + n_k + n_z)$ boundary conditions detailed below.

The certainty equivalence principle of the linear quadratic regulator (Simon (1956)) allows us to work with a non stochastic model. "*We would attain the same decision rule if we were to replace x_{t+1} with the forecast $E_t x_{t+1}$ and to add a shock process $C\varepsilon_{t+1}$ to the right hand side of the private sector policy transmission mechanism, where ε_{t+1} is an* *i.i.d. random vector with mean of zero and identity covariance matrix.*" (Ljungqvist and Sargent, 2012 p.767).

The policy maker's choice can be solve with Lagrange multipliers using Bellman's method (Ljungqvist and Sargent (2012)). It is practical (but not necessary) to solve the policy maker's choice by attaching a sequence of Lagrange multipliers $2\beta^{t+1}\mu_{t+1}$ to the sequence of private sector's policy transmission mechanism constraints and then forming the Lagrangian:

$$-\frac{1}{2}\sum_{t=0}^{+\infty}\beta^t \left[\begin{array}{c} \mathbf{y}_t^T\mathbf{Q}_{yy}\mathbf{y}_t + 2\mathbf{y}_t^T\mathbf{Q}_{yz}\mathbf{z}_t + \mathbf{u}_t^T\mathbf{R}\mathbf{u}_t + \\ 2\beta^{t+1}\mu_{t+1}\left(\mathbf{A}_{yy}\mathbf{y}_t + \mathbf{B}_y u_t - \mathbf{y}_{t+1}\right) \end{array} \right] \tag{3}$$

The non-controllable variables dynamics can be excluded from the Lagrangian (Anderson, Hansen, McGrattan and Sargent (1996)). It is important to partition the Lagrange multipliers μ_t conformable with our partition of $\mathbf{y}_t = \begin{bmatrix} \mathbf{k}_t \\ \mathbf{x}_t \end{bmatrix}$, so that $\mu_t = \begin{bmatrix} \mu_{k,t} \\ \mu_{x,t} \end{bmatrix}$, where $\mu_{x,t}$ is an $n_x \times 1$ vector of Lagrange multipliers of forward-looking variables.

The first order conditions with the policy transmission mechanism leads to the linear Hamiltonian system of the discrete time linear quadratic regulator (Anderson, Hansen, McGrattan and Sargent (1996)).

$2(n_x+n_k+n_z)$ boundary conditions determining the policy maker's Lagrangian system with $2(n_x+n_k+n_z)$ variables $(\mathbf{y}_t, \mu_t, \mathbf{z}_t)$ with μ_t the policy maker's Lagrange multipliers related to each of the controllable variables $\mathbf{y}_t$ (table 1).

Table 1: $2(n_x+n_k+n_z)$ boundary conditions

Number	Boundary conditions
n_z	$\lim_{t\to+\infty}\beta^t\mathbf{z}_t = \mathbf{z}^* = \mathbf{0}$, $\mathbf{z}_t$ bounded
$+n_k+n_x$	$\lim_{t\to+\infty}\beta^t\mathbf{y}_t = \mathbf{y}^* = \mathbf{0} \Leftrightarrow \lim_{t\to+\infty}\frac{\partial L}{\partial \mathbf{y}_t} = \mathbf{0} = \lim_{t\to+\infty}\beta^t\mu_t$, μ_t bounded
$+n_k+n_z$	$\mathbf{k}_0$ and $\mathbf{z}_0$ predetermined (given)
$+n_x$	$\mathbf{x}_0 = \mathbf{x}_0^* \Leftrightarrow \frac{\partial L}{\partial \mathbf{x}_0} = 0 = \mu^*_{\mathbf{x},t=0}$ predetermined

Essential boundary conditions are the initial conditions of predetermined variables $\mathbf{k}_0$ and $\mathbf{z}_0$ which are given.

Natural boundary conditions are such that the policy maker's anchors unique optimal initial values of private sectors forward-looking variables. The policy maker's Lagrange multipliers of private sector's forward (Lagrange multipliers) variables are *predetermined at the value zero:* $\mu_{\mathbf{x},t=0} = 0$ in order to determine the unique optimal initial value $\mathbf{x}_0 = \mathbf{x}_0^*$ of private sector's forward variables.

Bryson and Ho ((1975), p.55) explains natural boundary conditions as follows. "*If x_t is not prescribed at $t = t_0$, it does not follow that $\delta x(t_0) = 0$. In fact, there will be an optimum value for $x(t_0)$ and it will be such that $\delta L = 0$ for arbitrary small variations of $x(t_0)$ around this value. For this to be the case, we choose $\frac{\partial L}{\partial x(t_0)} = \mu_{x,t_0} = 0$ (1) which simply says that small changes of the optimal initial value of the forward variables $x(t_0)$ on the loss function is zero. We have simply traded one boundary condition: $x(t_0)$ given, for another, (1). Boundary conditions such as (1) are sometimes called "natural boundary conditions" or transversality conditions associated with the extremum problem.*"

Anderson, Hansen, McGrattan and Sargent (1996) assume a bounded discounted quadratic loss function:

$$E\left(\sum_{t=0}^{+\infty}\beta^t\left(\mathbf{y}_t^T\mathbf{y}_t+\mathbf{z}_t^T\mathbf{z}_t+\mathbf{u}_t^T\mathbf{u}_t\right)\right)<+\infty \tag{4}$$

This implies a stability criterion for eigenvalues of the dynamic system such that $\left|(\beta\lambda_i^2)^t\right| < |\beta\lambda_i^2| < 1$, so that stable eigenvalues are such that $|\lambda_i| < 1/\sqrt{\beta} < 1/\beta$. A preliminary step is to multiply matrices by $\sqrt{\beta}$as follows $\sqrt{\beta}\mathbf{A}_{yy}$ $\sqrt{\beta}\mathbf{B}_y$ in order to apply formulas of Riccati and Sylvester equations for the non-discounted augmented linear quadratic regulator (Anderson, Hansen, McGrattan and Sargent (1996)).

2.2 Preliminary step: Check if the system is stabilizable

Assumption 1: The matrix pair ($\sqrt{\beta}\mathbf{A}_{yy}$ $\sqrt{\beta}\mathbf{B}_y$) is controllable (all forward-looking variables are controllable).

The matrix pair ($\sqrt{\beta}\mathbf{A}_{yy}$ $\sqrt{\beta}\mathbf{B}_y$) is controllable if the Kalman (1960) controllability matrix has full rank:

$$\text{rank}\left(\sqrt{\beta}\mathbf{B}_y \quad \beta\mathbf{A}_{yy}\mathbf{B}_y \quad \beta^{\frac{3}{2}}\mathbf{A}_{yy}^2\mathbf{B}_y \quad ... \quad \beta^{\frac{n_k+n_x}{2}}\mathbf{A}_{yy}^{n_k+n_x-1}\mathbf{B}_y\right)=n_k+n_x \tag{5}$$

Assumption 2: The system is stabilizable when the transition matrix $\mathbf{A}_{zz}$ for the non-controllable variables has stable eigenvalues, such that $|\lambda_i| < 1/\sqrt{\beta}$.

2.3 Step 1: Stabilizing solution of a linear quadratic regulator

"*Step 1 and 2 seems to disregard the forward-looking aspect of the problem (step 3 will take account of that). If we temporarily ignore the fact that the x_0 component of the state y_0 is not actually a state vector, then superficially the Stackelberg problem has the form of an optimal linear regulator.*" (Ljungqvist and Sargent (2012, p.769)).

When the forcing variables are set to zero $\mathbf{z}_t=\mathbf{0}$, a stabilizing solution of the linear quadratic regulator satisfies:

$$\mu_t=\mathbf{P}_y\mathbf{y}_t \tag{6}$$

where $\mathbf{P}_y$ solves the matrix Riccati equation (Anderson, Hansen, McGrattan and Sargent (1996)):

$$\mathbf{P}_y{=}\mathbf{Q}_y+\beta\mathbf{A}_{yy}^{'}\mathbf{P}_y\mathbf{A}_{yy}-\beta^{'}\mathbf{A}_{yy}^{'}\mathbf{P}_y\mathbf{B}_y\left(\mathbf{R}+\beta\mathbf{B}_y^{'}\mathbf{P}_y\mathbf{B}_y\right)^{-1}\beta\mathbf{B}_y^{'}\mathbf{P}_y\mathbf{A}_{yy} \tag{7}$$

The optimal rule of the linear quadratic regulator is:

$$\mathbf{u}_t=\mathbf{F}_y\mathbf{y}_t \tag{8}$$

where $\mathbf{F}_y$ is computed knowing $\mathbf{P}_y$ (Anderson, Hansen, McGrattan and Sargent (1996)):

$$\mathbf{F}_y=\left(\mathbf{R}{+}\beta\mathbf{B}_y^{'}\mathbf{P}_y\mathbf{B}_y\right)^{-1}\beta\mathbf{B}_y^{'}\mathbf{P}_y\mathbf{A}_{yy} \tag{9}$$

As demonstrated by Simon (1956) certainty equivalence principle and by Kalman (1960) solution, the optimal rule parameters $\mathbf{F}_y$ and $\mathbf{P}_y$ of the linear quadratic regulator are independent of additive random shocks and of initial conditions. This confirms that it is correct to temporarily ignore the fact that $\mathbf{x}_0$ is not a state vector.

2.4 Step 2: Stabilizing solution of an augmented linear quadratic regulator

This is the additional step missing in Ljungqvist and Sargent (2012) algorithm. A stabilizing solution of the augmented linear quadratic regulator satisfies (Anderson, Hansen, McGrattan and Sargent (1996)):

$$\mu_t = \mathbf{P}_y \mathbf{y}_t + \mathbf{P}_z \mathbf{z}_t \tag{10}$$

where $\mathbf{P}_z$ solves the matrix Sylvester equation:

$$\mathbf{P}_z = \mathbf{Q}_{yz} + \beta \left(\mathbf{A}_{yy} + \mathbf{B}_y \mathbf{F}_y\right)' \mathbf{P}_y \mathbf{A}_{yz} + \beta \left(\mathbf{A}_{yy} + \mathbf{B}_y \mathbf{F}_y\right)' \mathbf{P}_z \mathbf{A}_{zz} \tag{11}$$

The optimal rule of the augmented linear quadratic regulator is:

$$\mathbf{u}_t = \mathbf{F}_y \mathbf{y}_t + \mathbf{F}_z \mathbf{z}_t \tag{12}$$

where $\mathbf{F}_z$ is computed knowing $\mathbf{P}_z$:

$$\mathbf{F}_z = \left(\mathbf{R} + \beta \mathbf{B}_y' \mathbf{P}_y \mathbf{B}_y\right)^{-1} \beta \mathbf{B}_y' \left(\mathbf{P}_y \mathbf{A}_{yz} + \mathbf{P}_z \mathbf{A}_{zz}\right) \tag{13}$$

As demonstrated by Simon (1956) certainty equivalence principle and by Anderson, Hansen, McGrattan and Sargent (1996) solution, the optimal rule parameters $\mathbf{F}_z$ and $\mathbf{P}_z$ of the augmented linear quadratic regulator are independent of additive random shocks and of initial conditions. This confirms that it is correct to temporarily ignore the fact that $\mathbf{x}_0$ is not a state vector, until step 3.

2.5 Step 3: Solve for x_0, the optimal initial anchor of forward-looking variables

The policy maker's Lagrange multipliers on private sector forward-looking variables are such that $\mu_{0,x} = \mathbf{0}$, at the initial date. The optimal stabilizing condition is:

$$\begin{pmatrix} \mu_{0,k} \\ \mu_{0,x} \end{pmatrix} = \begin{pmatrix} \mathbf{P}_{y,k} & \mathbf{P}_{y,kx} \\ \mathbf{P}_{y,kx} & \mathbf{P}_{y,x} \end{pmatrix} \begin{pmatrix} \mathbf{k}_0 \\ \mathbf{x}_0 \end{pmatrix} + \begin{pmatrix} \mathbf{P}_{z,k} \\ \mathbf{P}_{z,x} \end{pmatrix} \mathbf{z}_0 = \begin{pmatrix} \mu_{0,k} \\ \mathbf{0} \end{pmatrix} \tag{14}$$

This implies

$$\mathbf{P}_{y,kx} \mathbf{k}_0 + \mathbf{P}_{y,x} \mathbf{x}_0 + \mathbf{P}_{z,x} \mathbf{z}_0 = \mathbf{0} \tag{15}$$

Which provides the optimal initial anchor:

$$\mathbf{x}_0 = \mathbf{P}_{y,x}^{-1} \mathbf{P}_{y,kx} \mathbf{k}_0 + \mathbf{P}_{y,x}^{-1} \mathbf{P}_{z,x} \mathbf{z}_0 \tag{16}$$

The exogenous forcing variables adds the term $\mathbf{P}_{y,x}^{-1} \mathbf{P}_{z,x} \mathbf{z}_0$ with respect to Ljungqvist and Sargent (2012) algorithm.

2.6 Step 4: Compute impulse response functions and optimal loss function

The transmission mechanism is given. Computing $\mathbf{F}_y$ and $\mathbf{F}_z$ provides a reduced form of the optimal policy rule. Computing $\mathbf{P}_y$ and $\mathbf{P}_z$ provides the missing initial conditions.

$$\begin{aligned} \begin{pmatrix} E_t\mathbf{y}_{t+1} \\ \mathbf{z}_{t+1} \end{pmatrix} &= \begin{pmatrix} \mathbf{A}_{yy} & \mathbf{A}_{yz} \\ \mathbf{0}_{zy} & \mathbf{A}_{zz} \end{pmatrix} \begin{pmatrix} \mathbf{y}_t \\ \mathbf{z}_t \end{pmatrix} + \begin{pmatrix} \mathbf{B}_y \\ \mathbf{0}_z \end{pmatrix} \mathbf{u}_t \\ \mathbf{u}_t &= \mathbf{F}_y\mathbf{y}_t + \mathbf{F}_z\mathbf{z}_t \\ \mathbf{x}_0 &= \mathbf{P}_{y,x}^{-1}\mathbf{P}_{y,kx}\mathbf{k}_0 + \mathbf{P}_{y,x}^{-1}\mathbf{P}_{z,x}\mathbf{z}_0,\ \mathbf{k}_0 \text{ and } \mathbf{z}_0 \text{ given} \end{aligned}$$

This information is sufficient to compute impulse response functions (the optimal path of the expected values of variables $\mathbf{y}_t$ $\mathbf{z}_t$ and $\mathbf{u}_t$) and to sum up over time their value in the the discounted loss function.

By contrast to other algorithms based on Miller and Salmon (1985) solution, *it is not necessary* to compute all the values over time of all policy-makers Lagrange multipliers μ_t. These algorithms then add a step which is a change of vector basis for eliminating Lagrange multipliers. Knowing the optimal path of variables ($\mathbf{y}_t$ $\mathbf{z}_t$), one can compute the Lagrange multipliers at the end of this algorithm:

$$\mu_t = \mathbf{P}_y\mathbf{y}_t + \mathbf{P}_z\mathbf{z}_t \tag{17}$$

2.7 Step 5 (optional): An implementable representation of Ramsey optimal policy

Policymakers cannot implement a Ramsey optimal policy rule where policy instruments responds to non-observable variables, such as the shocks $\mathbf{u}_t$ or the Lagrange multipliers μ_t. They can implement an observationally equivalent representation of the Ramsey optimal policy rule where policy instruments responds to lagged observable variables, including the lags of the policy instruments. This is also a useful representation for testing Ramsey optimal policy using vector auto-regressive system of equation.

$$(H) \left\{ \begin{aligned} \begin{pmatrix} E_t\mathbf{y}_{t+1} \\ \mathbf{z}_{t+1} \end{pmatrix} &= \begin{pmatrix} \mathbf{A}_{yy} + \mathbf{B}_y\mathbf{F}_y & \mathbf{A}_{yz} + \mathbf{B}_y\mathbf{F}_z \\ \mathbf{0}_{zy} & \mathbf{A}_{zz} \end{pmatrix} \begin{pmatrix} \mathbf{y}_t \\ \mathbf{z}_t \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varepsilon_t \\ \mathbf{u}_t &= \mathbf{F}_y\mathbf{y}_t + \mathbf{F}_z\mathbf{z}_t \\ \mathbf{x}_0 &= \mathbf{P}_{y,x}^{-1}\mathbf{P}_{y,kx}\mathbf{k}_0 + \mathbf{P}_{y,x}^{-1}\mathbf{P}_{z,x}\mathbf{z}_0,\ \mathbf{k}_0 \text{ and } \mathbf{z}_0 \text{ given} \end{aligned} \right.$$

$$\Leftrightarrow \left\{ \begin{aligned} \begin{pmatrix} E_t\mathbf{y}_{t+1} \\ \mathbf{u}_{t+1} \end{pmatrix} &= \mathbf{M}^{-1}\left(\mathbf{A} + \mathbf{B}\mathbf{F}\right)\mathbf{M} \begin{pmatrix} \mathbf{y}_t \\ \mathbf{u}_t \end{pmatrix} + \mathbf{M}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varepsilon_t \\ \mathbf{z}_t &= \mathbf{F}_z^{-1}\mathbf{u}_t - \mathbf{F}_z^{-1}\mathbf{F}_y\mathbf{y}_t \\ \mathbf{x}_0 &= \mathbf{P}_{y,x}^{-1}\mathbf{P}_{y,kx}\mathbf{k}_0 + \mathbf{P}_{y,x}^{-1}\mathbf{P}_{z,x}\mathbf{z}_0,\ \mathbf{k}_0 \text{ and } \mathbf{z}_0 \text{ given} \end{aligned} \right.$$

where

$$\mathbf{A}+\mathbf{BF}=\begin{pmatrix} \mathbf{A}_{yy}+\mathbf{B}_y\mathbf{F}_y & \mathbf{A}_{yz}+\mathbf{B}_y\mathbf{F}_z \\ \mathbf{0}_{zy} & \mathbf{A}_{zz} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{y}_t \\ \mathbf{u}_t \end{pmatrix}=\mathbf{M}^{-1}\begin{pmatrix} \mathbf{y}_t \\ \mathbf{z}_t \end{pmatrix} \text{ with } \mathbf{M}^{-1}=\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{F}_y & \mathbf{F}_z \end{pmatrix}$$

In the estimation of dynamic stochastic general equilibrium model, the controllable predetermined variables are usually set to zero at all periods. They are as many auto-regressive forcing variables than controllable forward-looking variables. If the number of policy instrument is equal to the number of controllable forward-looking policy targets, $\mathbf{F}_z$ is a square matrix which can be invertible. One eliminates forcing variables $\mathbf{z}_t$ and replace them by policy instruments $\mathbf{u}_t$ in the recursive equation, doing a change of vector basis. There is then of a representation of forward-looking variables and policy instruments rule optimal policy dynamics in a vector auto-regressive model. This representation of Ramsey optimal policy rule is such that policy instruments $\mathbf{u}_t$ responds to lags of policy instruments $\mathbf{u}_{t-1}$ and of lags of the observable policy targets $\mathbf{y}_{t-1}$. This representation can be implemented by policy makers. It can be estimated by econometricians (Chatelain and Ralf (2017a)). .

2.8 Examples

Chatelain and Ralf (2017a) use this algorithm for the new-Keynesian Phillips curve as a monetary policy transmission mechanism. They check that it is equivalent to Gali (2015) solution who used the method of undetermined coefficients. They use the implementable representation of step 5 to estimate structural parameters.

Chatelain and Ralf (2017b) use this algorithm for the new-Keynesian Phillips curve and the consumption Euler equation as a monetary policy transmission mechanism. They check the determinacy property of step 2 reduced form of the Ramsey optimal policy rule.

Chatelain and Ralf (2016) use this algorithm for Taylor (1999) monetary policy transmission mechanism. They check whether Taylor principle applies to Ramsey optimal policy.

3 Conclusion

This algorithm complements Ljungqvist and Sargent (2012) algorithm taking into account forcing variables. It is easy to code, check and implement.

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