# Fully Bayesian Analysis of SVAR Models under Zero and Sign Restrictions

Kocięcki, Andrzej

Narodowy Bank Polski

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# FULLY BAYESIAN ANALYSIS OF SVAR MODELS UNDER ZERO AND SIGN RESTRICTIONS

**Andrzej Kocięcki**
Narodowy Bank Polski
e–mail: andrzej.kociecki@nbp.pl

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**Abstract:** The paper proposes the methodologically sound method to deal with set identified Structural VAR (SVAR) models under zero and sign restrictions. What distinguishes our method from that proposed by Arias, Rubio-Ramírez and Waggoner (2016) is that we isolated many special cases for which we arrive at more efficient algorithms to draw from the posterior. We illustrate our approach with the help of two serious empirical examples. First of all we challenge the output puzzle found by Uhlig (2005). Second, we check the robustness of the results given by Beaudry et al. (2014) concerning impact of optimism shocks on economy.

## I. INTRODUCTION

Following Uhlig (2005), there have been more and more papers that apply sign restrictions in order to decide on most important problems is empirical macroeconomics. It seems that the methodology of sign restrictions is attractive for researchers because it is supposed to be robust with respect to particular identifying scheme imposed on Structural VAR (SVAR) model within the framework of point identification. However some recent papers, notably Baumeister and Hamilton (2015) and Arias, Rubio-Ramírez and Waggoner (2016), point to some pitfalls in appropriate application of set identified SVAR models under zero and sign restrictions.[1]

If only sign restrictions are used, the method proposed by Uhlig (2005) largely survives the passing time test. On the other hand if zero or zero and sign restrictions are used simultaneously in set identified SVAR, the problem with the methodology of Uhlig (2005) and Mountford and Uhlig (2009) was clearly pointed by Arias, Rubio-Ramírez and Waggoner (2016) (to be referred to as ARRW (2016)). This is one of the (not so?) many cases in economics when methodology really matters, and economists who do not pay much attention to the applied methodology could reach economic

[1] By the set identified model we mean a model in which the parameter of interest is not identified in the classical sense (i.e. not point identified).

implications that are intriguing and economically significant yet are based on methodological faults. In particular ARRW (2016) by juxtaposing their results with those of Beaudry et al. (2014), who followed Mountford and Uhlig (2009), show, that contribution of optimism shocks to the Forecast Error Variance Decomposition (FEVD), for whatever horizon and variable, is highly overestimated. To this end, ARRW (2016) developed a new attractive framework which can cope with zero and sign restrictions in set identified SVAR's. The problem is not trivial since we need a method to allow for zero restrictions imposed on various matrices of interest simultaneously (e.g. think of "zeros" put both on instantaneous relations and instantaneous impulse response matrices in SVAR). Unfortunately due to extensive use of abstract differential calculus on manifolds many of results in ARRW (2016) may be less than transparent for researchers who want to apply their methodology. On the contrary, using classical calculus we show that 1) the density underlying their algorithms may by analytically given and 2) in some special but important cases their algorithms may be substantially simplified. All in all, our main contribution are algorithms that are in many cases more efficient and in all cases easier to implement than those proposed by ARRW (2016).

Having developed appropriate tool, we applied it to challenge output puzzle found in Uhlig (2005). Further we try to obtain reliable estimates of Impulse Responses Functions (IRF's) and FEVD's due to optimism shocks in a model considered by Beaudry et al. (2014).

## II. THE MODEL AND NOTATION

Our model framework is the standard SVAR model

$$A_0 y_t = A_1 y_{t-1} + A_2 y_{t-2} + \cdots + A_p y_{t-p} + c + \varepsilon_t \tag{1}$$

where $A_0$ is an $(n \times n)$ nonsingular matrix measuring contemporaneous relations between $n \times 1$ observables $y_t$, $c \in \mathbb{R}^{n \times 1}$ is vector of constants, and $A_1, \ldots, A_p$ are $(n \times n)$ matrices of coefficients on lagged data. We assume that structural shocks i.e. $\varepsilon_t$, are independently, identically and normally distributed with identity covariance matrix i.e. $\varepsilon_t \sim i.i.d.\ N(0, \mathrm{I}_n)$. Let $B = [A_1\ A_2\ \ldots\ A_p\ c] \in \mathbb{R}^{n \times (np+1)}$, $y = [y_1\ y_2\ \ldots\ y_T] \in \mathbb{R}^{n \times T}$, $T$ denotes the sample size and $\underset{(np+1)\times T}{X'} = \begin{bmatrix} y_0 & y_1 & \cdots & y_{T-1} \\ y_{-1} & y_0 & \cdots & y_{T-2} \\ \vdots & \vdots & \cdots & \vdots \\ y_{-p+1} & y_{-p+2} & \ddots & y_{T-p} \\ 1 & 1 & \cdots & 1 \end{bmatrix}$.

Reduced form coefficients induced by (1) will be denoted as $\Pi = A_0^{-1}B$ and the reduced form covariance as $\Sigma = A_0^{-1}A_0'^{-1}$. Let us denote by $\Psi_h$ an $n \times n$ matrix of impulse responses after $h$ periods of time, $\Psi_0 = A_0^{-1}$ being the instantaneous response. Prominent role in our paper play orthogonal matrices. The space of those matrices is denoted as $O(n) = \{Q \in \mathbb{R}^{n\times n} \mid Q'Q = \mathrm{I}_n\}$. Sometimes the following partitioning of $Q \in O(n)$ will be used: $Q = [q_1 \ldots q_n]$ and $Q_i = [q_1 \ldots q_i]$, where $q_i$ is the $i-$th column of $Q$. We will frequently use the $QR$ decomposition of the instantaneous impulse response matrix i.e. $A_0^{-1} = LQ$, where $L$ is an $(n \times n)$ lower triangular matrix with positive diagonal elements and $Q \in O(n)$. Note that $\Sigma = LL'$. Lastly let $e_j$ denote the $j-$th column of $\mathrm{I}_n$.

Following ARRW (2016) let us define selection matrices for zero restrictions as $Z_i$ ($i = 1,\ldots,k \leq n$) and those for sign restrictions as $S_i$ ($i = 1,\ldots,k \leq n$). The rationale behind introduction of $Z_i$ and $S_i$ was given by Rubio–Ramírez et al. (2010). Suffice it to say the notation is instrumental to write down all interesting restrictions appearing in SVAR literature. We assume that each $Z_i$ and $S_i$ has full row rank, in particular $\mathrm{rank}(Z_i) = z_i$ and $\mathrm{rank}(S_i) = s_i$, and $Z_i$ captures all zero restrictions implicitly imposed on the $i-$th column of $Q$ i.e. $q_i$, and $S_i$ those sign restrictions implicitly imposed on $q_i$. W.l.o.g. we assume $z_1 \geq z_2 \geq \ldots \geq z_k$. For example zero restriction imposed on $(i,j)$ element of $A_0^{-1}$ may be written as follows $0 = e_i'A_0^{-1}e_j = e_i'LQe_j = e_i'Lq_j$. Thus all $z_j$ zero restrictions imposed on the $j-$th column of $A_0^{-1}$ may be written as $Z_jLq_j = 0$, where $Z_j$ is the selection matrix. Further zero restriction imposed on $(i,j)$ element of $A_0$ reads as $0 = e_i'A_0e_j = e_i'Q'L^{-1}e_j = e_j'L'^{-1}Qe_i = e_j'L'^{-1}q_i$, hence all $z_i$ restrictions imposed on the $i-$th row of $A_0$ may be written as $Z_iL'^{-1}q_i = 0$. Similar reasoning yields restrictions put on lagged coefficients $B$ and $\Psi_h$ for $h = 0,1,2,\ldots$, including $h = \infty$. The latter relates to the long–run impulse response, provided the data are in differences. In general, this allows us to write all linear restrictions imposed on either $A_0$, $B$ or $\Psi_h$ as $Z_jf(\Pi,L)q_j = 0$ for $j = 1,\ldots,k \leq n$, where $f(\Pi,L)$ is a matrix whose all entries are functions of the reduced form parameters $\Pi, L$ only, see Giacomini and Kitagawa (2015).[2] Repeating the above reasoning in the context of sign restrictions one may write all these restrictions as $S_jf(\Pi,L)q_j \geq 0$, where $j = 1,\ldots,k \leq n$.

[2] In what follows we refer interchangeably to both $\Pi, L$ and $\Pi, \Sigma$ as the reduced form parameters.

## III. UNRESTRICTED POSTERIOR

An unrestricted posterior is just the posterior without introducing any (sign or exact) restrictions. In our notation it will always be identified with the subscript "*ur*". In order to derive it one should take a stand on a prior distribution. Since there is no universal uninformative prior it is a good idea to state ignorance with respect to aspects of phenomenon that your model is intended to cope with. In our case these aspects are Impulse Response Functions (IRF's). That is why we think that unrestricted posterior should be derived under ignorance prior explicitly stated in the context of IRF's.[3] It seems that this methodological stance addresses some points raised by Baumeister and Hamilton (2015), at least those that can be operationally solved.

Being consistent with the above insight let us start with the assumption of the flat prior for the first $p+1$ IRF's i.e. $p(\Psi_0, \Psi_1, \ldots, \Psi_p) \propto 1$. Needless to say such a prior is agnostic in the terminology of ARRW (2016). Given that the model is completely unidentified this induces the prior for structural parameters (see e.g. Kocięcki (2010))

$$p(A_0, B) \propto |\det(A_0)|^{-2n(p+1)} \tag{2}$$

which leads to the following unrestricted posterior of SVAR model

$$p_{ur}(A_0, B \mid y) \propto |\det(A_0)|^{T-2n(p+1)} \operatorname{etr}\{-\tfrac{1}{2} A_0 M A_0' - \tfrac{1}{2}(B - \hat{B}) X'X (B - \hat{B})'\} \tag{3}$$

where $M = y[\mathrm{I}_T - X(X'X)^{-1}X']y'$; $\hat{B} = A_0\hat{\Pi}$; $\hat{\Pi} = yX(X'X)^{-1}$, $\operatorname{etr}\{\cdot\} := \exp\{trace\{\cdot\}\}$ and subscript "*ur*" signifies that the posterior under consideration is unrestricted. Following e.g. ARRW (2016), Moon et al. (2013) or Giacomini and Kitagawa (2015), let us decompose the impact response as $A_0^{-1} = LQ$, where $L$ is lower triangular with positive diagonal elements and $Q \in O(n)$. However in contrast to Moon et al. (2013) or Giacomini and Kitagawa (2015) but following ARRW (2016), to proceed

[3] In general the other option would be to state ignorance prior over the space of structural parameters of the model (1). As emphasized in ARRW (2016), the point is that when zero restrictions are imposed, one should take an a priori stand whether to be ignorant over IRF's or structural parameters space. This will influence the posterior conclusions. ARRW (2016) overcome this conundrum introducing the notion of base parameterization. We prefer being explicit about the base parameterization at the outset since it results in more transparent results. However the analogous results when being ignorant over the structural parameters space may be easily obtained since they require only changing $T - 2n(p+1)$ to $T$ in the exponent of the determinant in (3) and using probability rules to derive corresponding versions of (4) and/or (5).

further, we follow the logic of our (fully Bayesian) approach and in order to work with $L,Q$ instead of $A_0$, we must take into account the Jacobian $J(A_0 \to L,Q) = J(A_0 \to A_0^{-1})J(A_0^{-1} \to L,Q)$. Moreover changing variables from $B$ to the reduced form coefficients $\Pi = A_0^{-1}B$, it is easy to show that the joint posterior may be decomposed as follows[4]

$$p_{ur}(\Pi, L, Q \mid y) = p_{ur}(\Pi \mid L, y) p_{ur}(L \mid y) p_{ur}(Q) \tag{4}$$

where

$p_{ur}(\Pi \mid L, y) \propto \det(LL')^{-\frac{1}{2}(np+1)} \operatorname{etr}\{-\frac{1}{2}(LL')^{-1}(\Pi - \hat{\Pi}) X'X(\Pi - \hat{\Pi})'\}$

$p_{ur}(L \mid y) \propto \prod_{i=1}^{n} l_{ii}^{-(T-2np)+n-i} \operatorname{etr}\{-\frac{1}{2}(LL')^{-1} M\}$

$p_{ur}(Q) \propto (Q'dQ)$

where $l_{ii}$ denotes $(i,i)$ element of $L$. In the last expression, $dQ$ denotes elementwise differential of all elements in $Q$ and $(Q'dQ)$ denotes the product of elements of $Q'dQ$ below the diagonal and is known as the Haar measure on $O(n)$. The differential form $Q'dQ$ appears as a result of changing measures using exterior algebra. It means that $p_{ur}(Q)$ is the flat pdf with respect to the Haar measure, see James (1954) or Muirhead (1982) for details. It is well known that the normalizing constant connected with $p_{ur}(Q)$ is given by $C_{ur}^{-1} = [\int_{O(n)} (Q'dQ)]^{-1} = 2^{-n}\pi^{-\frac{1}{2}n^2}\Gamma_n(\frac{n}{2})$, where $\Gamma_n(\frac{n}{2}) = \pi^{\frac{1}{4}n(n-1)}\prod_{i=1}^{n}\Gamma(\frac{n-i+1}{2})$. We note in passing that it is not a coincidence that $C_{ur}$ is just the surface area along $Q'Q = \mathrm{I}_n$ i.e. $C_{ur} = \int_{O(n)} (Q'dQ) = \int_{O(n)} dQ$, hence in the sequel we will write $p_{ur}(Q) = \frac{1}{\int_{O(n)} dQ} dQ \propto dQ$, which is in line with ARRW (2016).

Equivalently, using Choleski decomposition of the reduced form covariance matrix $\Sigma = LL'$, one may rewrite (4) as

$$p_{ur}(\Pi, \Sigma, Q \mid y) = p_{ur}(\Pi \mid \Sigma, y) p_{ur}(\Sigma \mid y) p_{ur}(Q) \tag{5}$$

where

[4] Note that lack of conditioning on $y$ in the last density on the right of (4) means that it does not depend on the data $y$. Moreover there is no $Q$ among conditioning set in the first density on the right of (4), which means that $\Pi$ is independent of $Q$ given $L$. This notation will be generic to our paper, so please be careful.

$$p_{ur}(\Pi \mid \Sigma, y) \propto \det(\Sigma)^{-\frac{1}{2}(np+1)} \operatorname{etr}\{-\tfrac{1}{2}\Sigma^{-1}(\Pi - \hat{\Pi})X'X(\Pi - \hat{\Pi})'\}$$
$$p_{ur}(\Sigma \mid y) \propto \det(\Sigma)^{-\frac{1}{2}(T-2np+1)} \operatorname{etr}\{-\tfrac{1}{2}\Sigma^{-1}M\}$$

Hence we arrive at $p_{ur}(\Pi \mid \Sigma, y)$, being matricvariate Normal, and $p_{ur}(\Sigma \mid y)$, being inverted Wishart i.e. the framework adopted e.g. by Uhlig (2005), ARRW (2016) or Giacomini and Kitagawa (2015) and many others. Overall, advantage of our approach is that the prior $p_{ur}(Q)$ is not imposed like e.g. in Uhlig (2005), but retrieved from our basic postulates (being ignorant about IRF's) as e.g. in ARRW (2016). It is worth noting that since we started with agnostic prior the resultant posterior (4) or (5) is also agnostic using the terminology of ARRW (2016).

## IV. THE PROBLEM STATEMENT AND OUR SOLUTION

As demonstrated by ARRW (2016), if there are only sign restrictions (and no zero restrictions) then dealing with set identified SVAR is relatively easy. In the context of our model setup it amounts to independent drawing from the Normal–Inverted Wishart distribution for $\Pi, \Sigma$, uniform distribution for $Q$ and keeping the draw only if the underlying sign restrictions are fulfilled. The complications arise when one imposes zero restrictions.

To understand the nature of the problem suppose we want to introduce the restriction that $(3,1)$ element of $A_0^{-1}$ is zero. Working with the model (4), this imposes the restriction $e_3' A_0^{-1} e_1 = e_3' L Q e_1 = l_3 q_1 = 0$, where $l_3$ is the third row of $L$. Note that the restriction involves both $L$ (i.e. reduced form parameter) and $Q$. Hence although this restriction does not "touch" permissible $\Pi$'s, the underlying spaces of $L$'s, say $\Theta$, and $Q$'s i.e. $O(n)$, are no longer variation free i.e. it is not a product space $\Theta \times O(n)$. It follows that traditional factorization into marginal and conditional posterior densities becomes less obvious. That is apart from the conditional posterior of $\Pi$ given $L, Q$ and the restriction (which is $p_{ur}(\Pi \mid L, y)$ from (4)), the decomposition of the joint density of $L, Q$ subject to the restriction is not readily seen. In fact the relevant questions are 1) What is the marginal posterior of the reduced form parameter $L$ subject to $l_3 q_1 = 0$? and 2) What is the conditional posterior of $Q$ given $L$ and the restriction $l_3 q_1 = 0$? Finding a method to derive these two posterior densities would be instrumental in solving the general inference problem in the set identified SVAR model under zero (and sign) restrictions i.e. the one in which the restrictions involve all parameters $\Pi, L, Q$ through the general restriction $Z_j f(\Pi, L) q_j = 0$. Since we are seeking the posterior decomposition subject to zero restrictions as being consistent with probability rules, it explains “fully Bayesian” in the paper title.

In fact the whole problem boils down to evaluation of the integral $\int_{O(n),Z_i f(\Pi,L)q_i=0;i=1,\ldots,k\le n} dQ$. Absent zero restrictions this is just $C_{ur}$. With zero restrictions this is not the case. If we manage to do that then, as a byproduct, we obtain conditional posterior of $Q$ given $\Pi, L$ and zero restrictions and the marginal posterior of $\Pi, L$ given zero restrictions. Perhaps surprisingly, evaluation of this integral is cumbersome in general. In a nutshell, our solution to evaluate the integral hinges on the following trick. Consider our simple example. Treating the restriction as a “new” variable i.e. $r = l_3 q_1$, we express the underlying measure of $Q$ in terms of $r, Q^*$, where $Q^*$ comprises “part” of functionally independent elements of $Q$, so that the number of functionally independent elements in $Q$ is equal to that in $r, Q^*$. Having the posterior $p(L, r, Q^* \mid y)$, the conditional posterior $p(L, Q^* \mid r = 0, y)$ is just proportional to $p(L, r = 0, Q^* \mid y)$. Since there is a 1–1 correspondence between $Q$ subject to the restriction and $(r = 0, Q^*)$, we are done. The conditional posterior $p(L, Q^* \mid r = 0, y)$ will be the counterpart of the underlying density in algorithm 4 in ARRW (2016). The merit of our approach is that we arrive at the analytical form of this distribution whereas those who follow ARRW (2016) must spend much computing time to get it numerically. Moreover, exploiting our approach we can go one step further. We will show that in some special but important cases we can obtain the marginal posterior of the reduced form parameters (given restrictions), which makes drawing even more efficient. In particular in our simple case we do find $p(L \mid r = 0, y)$. These insights are missing in the approach of ARRW (2016).

## V. THE SET IDENTIFIED SVAR UNDER ZERO RESTRICTIONS

Bearing in mind our proposition from appendix 3 (see also lemma 5.1 in Giacomini and Kitagawa (2015)), from now on we will confine to the case when $z_i \le n - i$, hence also $k < n$. Let us denote symbolically all zero restrictions $Z_i f(\Pi, L) q_i = 0; i = 1,\ldots,k < n$ as $R$, and first $k$ columns of $Q$ subject to zero restrictions as $\Lambda_k$ i.e. $\Lambda_k \in \{Q_k \in \mathbb{R}^{n\times k} \mid Q_k' Q_k = \mathrm{I}_k, R\}$. Let us choose any $n \times (n-k)$ matrix $W$ (being a function of $\Lambda_k$), such that $[\Lambda_k \vdots W] \in O(n)$. It follows that all matrices orthogonal to $\Lambda_k$ and having orthogonal columns can be obtained as $W\tilde{Q}$, where $\tilde{Q} \in O(n-k)$. Then using lemma 9.5.3 in Muirhead (1982) we can decompose $\int_{O(n),R} dQ$ as $\int_{Q_k'Q_k=\mathrm{I}_k,R} \int_{\tilde{Q}\in O(n-k)} d\tilde{Q} dQ_k$. To be sure, the measure decomposition is interpreted as: first integrate over $\tilde{Q}$ for fixed $Q_k$ (subject to zero restrictions) and then integrate over $Q_k$ subject to zero restrictions i.e. $\Lambda_k$. The immediate consequence is that conditional prior of the last $n-k$ columns of $Q$ given $\Lambda_k$ is not influenced by the zero restrictions, which will be exploited in our algorithms. So in further theoretical development our starting point will be $p_{ur}(\Pi, L, Q \mid y)$ integrated out with respect to the last $n-k$ columns of $Q$ i.e. $p_{ur}(\Pi, L, Q_k \mid y) = p_{ur}(\Pi \mid L, y) p_{ur}(L \mid y) p_{ur}(Q_k)$, where $p_{ur}(Q_k) \propto dQ_k$, and we focus on the evaluation of the integral $\int_{Q_k'Q_k = \mathrm{I}_k, R} dQ_k$.

The following key proposition will be instrumental in the decomposition of the posterior under zero restrictions and designing algorithms to sample from

**Proposition 1:** *Assume that* $\begin{bmatrix} Z_i f(\Pi, L) \\ Q_{i-1}' \end{bmatrix}$ *has full row rank for each* $i = 1, \ldots, k$. *Then*

$$\int_{Q_k'Q_k = \mathrm{I}_k, R} dQ_k = \int_{x_i'x_i = 1; i = 1, \ldots, k} J \, dx_1 \ldots dx_k$$

*where* $J = \prod_{i=1}^{k} | Z_i f(\Pi, L)(\mathrm{I}_n - \Lambda_{i-1}\Lambda_{i-1}') f'(\Pi, L) Z_i' |^{-\frac{1}{2}}$, *each* $x_i$ *has dimension* $(n - z_i - i + 1) \times 1$, $\Lambda_{i-1} = [G_1 x_1 \vdots G_2 x_2 \vdots \ldots \vdots G_{i-1} x_{i-1}] \in \{Q_{i-1} \in \mathbb{R}^{n \times (i-1)} \mid Q_{i-1}'Q_{i-1} = \mathrm{I}_{i-1}, R\}$ *(with the convention that* $\Lambda_0$ *is empty) and* $G_i$ *is any fixed* $n \times (n - z_i - i + 1)$ *full column rank matrix such that* $G_i'G_i = \mathrm{I}_{n - z_i - i + 1}$ *and* $\begin{bmatrix} Z_i f(\Pi, L) \\ Q_{i-1}' \end{bmatrix} G_i = 0$ *(with the convention that* $Q_0$ *is empty).*

We think that a few remarks how to read the content of proposition 1 will be useful. First of all, the proposition is obtained assuming $(f'(\Pi, L) Z_i' \vdots Q_{i-1})'$ has full row rank. As usual with rank conditions, this is a generic property of a model so when it holds for one (e.g. randomly selected) $\Pi, L, Q_{i-1}$, it holds for almost all $\Pi, L, Q_{i-1}$'s. Note that when $k = 1$ and $f(\Pi, L)$ has full row rank, the rank condition always holds. For example this is the case when one imposes zeros only on instantaneous impulse responses to one shock (since then $f(\Pi, L) = L$). Importantly in appendix 3 we show that sufficient condition for existence of orthogonal matrix subject to zero and sign restrictions implies $(f'(\Pi, L) Z_i' \vdots Q_{i-1})'$ has full row rank. This gives additional rationale for making our rank assumption. Further, $x_1, \ldots, x_k$ comprise all functionally independent elements of $\Lambda_k$ (i.e. $Q_k$ subject to zero restrictions).[5] In fact the proof of

[5] As a useful crosscheck one may consider the case when there are no zero restrictions i.e. each $Z_i$ is empty and $z_i = 0$. Then modifying the proof of proposition 1 we get $J = 1$ and integrating the

the proposition reveals that there is a 1–1 correspondence between $\Lambda_k$ and $x_1,\dots,x_k$. Expressing the underlying measure for $Q_k$ subject to zero restrictions in terms of $x_1,\dots,x_k$, we directly obtain the posterior

$$p(\Pi, L, x_1,\dots,x_k \mid y, R) \propto J \cdot p_{ur}(\Pi \mid L, y) p_{ur}(L \mid y)\, dx_1 \dots dx_k \tag{6}$$

where $J = \prod_{i=1}^{k} | Z_i f(\Pi, L)(\mathrm{I}_n - \Lambda_{i-1}\Lambda_{i-1}') f'(\Pi, L) Z_i' |^{-\frac{1}{2}}$. As a result we explicitly obtained the posterior density, which is the counterpart of the implicit density in algorithms 3 and 4 in ARRW (2016).

From (6) it easily follows that $p(x_1,\dots,x_k \mid \Pi, L, y, R) = p(x_1,\dots,x_k \mid \Pi, L, R)$ $\propto J\, dx_1 \dots dx_k$. However to obtain the marginal posterior $p(\Pi, L \mid y, R)$ we have to integrate (6) with respect to $x_1,\dots,x_k$, where each $x_i$ is constrained as $x_i' x_i = 1$. Unfortunately this is surprisingly difficult to do since $J$ also involves those $x_i$'s. Though when the dimension of a model does not exceed 3 and/or only two columns of $Q$ are restricted this is feasible, the resultant formula is quite complicated and not easy to work with.[6] Dealing with restrictions embracing more than two columns of $Q$ seems to be condemned to failure. However in some special but important cases integration becomes trivial. Two cases will be distinguished. The first one is when only one column of $Q$ is subject to the restrictions i.e. $k = 1$, and the second one, in which restrictions follow some pattern. Considering the former, since $\Lambda_0$ is empty in proposition 1 we get

**Corollary:** *Assume that* $k = 1$, *then*

$$\int_{q_1' q_1 = 1, R} dq_1 = | Z_1 f(\Pi, L) f'(\Pi, L) Z_1' |^{-\frac{1}{2}} \int_{x_1' x_1 = 1} dx_1 = \tfrac{2\pi^{\frac{1}{2}(n - z_1)}}{\Gamma(\frac{n - z_1}{2})} | Z_1 f(\Pi, L) f'(\Pi, L) Z_1' |^{-\frac{1}{2}}$$

On the other hand, recalling that we ordered restrictions so as $z_1 \geq z_2 \geq \dots \geq z_k$, suppose that $Z_i$ contains all rows that appear among those in $Z_{i+1}, Z_{i+2},\dots, Z_k$, for each $i = 1,\dots,k-1$. Equivalently, $Z_i$ is a submatrix of all $Z_1,\dots,Z_{i-1}$. Although (partially) recursive identifying schemes conform to this pattern, the pattern proper is more general. This pattern of the underlying restrictions will be denoted as $\overline{R}$ and SVAR

formula on the right in proposition 1 gives surface area along $Q_k' Q_k = \mathrm{I}_k$, as expected, see James (1954).

[6] The formula involves infinite series.

subject to these kind of restrictions will be termed as $\bar{R}-$restricted SVAR. Then we have useful

**Lemma:** *Suppose that SVAR is* $\bar{R}-$*restricted, then*

$$\int_{Q_k'Q_k=I_k,R,\bar{R}} dQ_k = \prod_{i=1}^{k} | Z_i f(\Pi,L) f'(\Pi,L) Z_i' |^{-\frac{1}{2}} \int_{x_i'x_i=1;i=1,\dots,k} dx_1 \dots dx_k$$
$$= \prod_{i=1}^{k} \frac{2\pi^{\frac{1}{2}(n-z_i-i+1)}}{\Gamma(\frac{n-z_i-i+1}{2})} | Z_i f(\Pi,L) f'(\Pi,L) Z_i' |^{-\frac{1}{2}}$$

It turns out that depending on $k$, whether the restrictions conform to $\bar{R}$ or not or the restrictions involve only $A_0$ and/or $A_0^{-1}$, the joint posterior $p(\Pi, L, x_1, \dots, x_k \mid y, R)$ may be decomposed in various ways. To save the space, and for readers' convenience, we only summarize in Table 1 all possible variants stemming from our considerations. Since in each case the proof is rather trivial we only point to key results justifying each claim.

**Table 1:** Decompositions of the posterior (6)

| | **Restrictions involve only $A_0$ and/or $A_0^{-1}$ i.e. $f(\Pi,L) \equiv f(L)$** | **General restrictions $f(\Pi,L)$** |
|---|---|---|
| $k=1$ | $p(\Pi,L,x_1 \mid y,R) = p(\Pi \mid L,y,R)p(L \mid y,R)p(x_1 \mid L,R)$<br>where<br>$p(\Pi \mid L,y,R) = p_{ur}(\Pi \mid L,y)$<br>$p(L \mid y,R) \propto \mid Z_1 f(L) f'(L) Z_1' \mid^{-\frac{1}{2}} p_{ur}(L \mid y)$<br>$p(x_1 \mid L,R) \propto dx_1$<br>Proof: see Corollary | $p(\Pi,L,x_1 \mid y,R) = p(\Pi,L \mid y,R)p(x_1 \mid \Pi,L,R)$<br>where<br>$p(\Pi,L \mid y,R) \propto \mid Z_1 f(\Pi,L) f'(\Pi,L) Z_1' \mid^{-\frac{1}{2}} p_{ur}(\Pi \mid L,y) p_{ur}(L \mid y)$<br>$p(x_1 \mid \Pi,L,R) \propto dx_1$<br>Proof: see Corollary |
| $k>1$ **and** $\bar{R}$ | $p(\Pi,L,x_1,\dots,x_k \mid y,R) = p(\Pi \mid L,y,R)p(L \mid y,R)p(x_1,\dots,x_k \mid L,R)$<br>where<br>$p(\Pi \mid L,y,R) = p_{ur}(\Pi \mid L,y)$<br>$p(L \mid y,R) \propto \prod_{i=1}^{k} \mid Z_i f(L) f'(L) Z_i' \mid^{-\frac{1}{2}} p_{ur}(L \mid y)$<br>$p(x_1,\dots,x_k \mid L,R) \propto dx_1 \dots dx_k$<br>Proof: see Lemma | $p(\Pi,L,x_1,\dots,x_k \mid y,R) = p(\Pi,L \mid y,R)p(x_1,\dots,x_k \mid \Pi,L,R)$<br>where<br>$p(\Pi,L \mid y,R) \propto \prod_{i=1}^{k} \mid Z_i f(\Pi,L) f'(\Pi,L) Z_i' \mid^{-\frac{1}{2}} p_{ur}(\Pi \mid L,y) p_{ur}(L \mid y)$<br>$p(x_1,\dots,x_k \mid \Pi,L,R) \propto dx_1 \dots dx_k$<br>Proof: see Lemma |
| $k>1$ | $p(\Pi,L,x_1,\dots,x_k \mid y,R) = p(\Pi \mid L,y,R)p(L,x_1,\dots,x_k \mid y,R)$<br>where<br>$p(\Pi \mid L,y,R) = p_{ur}(\Pi \mid L,y)$<br>$p(L,x_1,\dots,x_k \mid y,R) \propto J \cdot p_{ur}(L \mid y) dx_1 \dots dx_k$<br>Proof: The Jacobian $J$ does not involve $\Pi$ | $p(\Pi,L,x_1,\dots,x_k \mid y,R) \propto J \cdot p_{ur}(\Pi \mid L,y) p_{ur}(L \mid y) dx_1 \dots dx_k$<br>(the posterior does not conform to any useful decomposition in general) |

Note that when $k=1$ the restrictions trivially conform to $\bar{R}$ hence the first row in the table is just the second one putting $k=1$. We distinguish these two cases only for ease of reference for applied researchers. Hence when designing the algorithm to draw from the posterior we just need the one for the case "$k>1$ and $\bar{R}$". In addition notice that when zero restrictions are confined only to $A_0$ and/or $A_0^{-1}$, we always have $p(\Pi \mid L, y, R) = p_{ur}(\Pi \mid L, y)$, so that the conditional posterior of $\Pi$ is standard and not affected by the zero restrictions at all. This will speed up the algorithm considerably (a point which is absent from the considerations leading to algorithms in ARRW (2016)). Interestingly, in the context of $\bar{R}-$restricted SVAR you may think of $\prod_{i=1}^{k} | Z_i f(\Pi, L) f'(\Pi, L) Z_i' |^{-\frac{1}{2}}$ as the "additional" prior. Since the restriction $Z_i f(\Pi, L) q_i = 0$ must hold, the prior (quite rationally) favors reduced form parameter space that result in small values for $Z_i f(\Pi, L)$. Hence it works towards shrinking of those functions of reduced form parameters that are directly connected with restrictions.

## VI. THE ALGORITHM FOR $\bar{R}-$ RESTRICTED SVAR

We are in a position to state the (fully Bayesian) algorithm to sample from $\bar{R}-$ restricted SVAR. Needless to say the below algorithm is necessarily valid for the case when only one column of $Q$ is subject to linear restrictions (just setting $k = 1$). In what follows recall that $Q_{i-1} = [q_1 \ldots q_{i-1}]$, with the convention that $Q_0$ is empty. The unit sphere in $\mathbb{R}$ (or uniform distribution on $O(1)$) consists of two points i.e. $\{-1,1\}$, to which we attach equal probability $\frac{1}{2}$. We continue to assume $z_i \leq n - i$.

**Algorithm 1:**

1) *Draw from* $p(\Pi, L \mid y, R) \propto p_{ur}(\Pi \mid L, y) p_{ur}(L \mid y) \cdot \prod_{i=1}^{k} | Z_i f(\Pi, L) f'(\Pi, L) Z_i' |^{-\frac{1}{2}}$
2) *Set* $i = 1$
3) *Draw an* $(n - z_i + 1 - i) \times 1$ *vector* $x_i$ *from the uniform distribution on the unit sphere in* $\mathbb{R}^{n - z_i + 1 - i}$
4) *Find any* $n \times (n - z_i + 1 - i)$ *matrix* $G_i$ *(of full column rank) such that* $G_i' G_i = \mathrm{I}_{n - z_i + 1 - i}$ *and* $\begin{bmatrix} Z_i f(\Pi, L) \\ Q_{i-1}' \end{bmatrix} G_i = 0$
5) *Set* $q_i = G_i x_i$
6) *Set* $i = i + 1$*, go to 3), and repeat until* $i = k$ *to get* $Q_k = [q_1 \ldots q_k]$
7) *Find any* $n \times (n - k)$ *matrix* $W$ *such that* $W'W = \mathrm{I}_{n-k}$ *and* $Q_k' W = 0$
8) *Draw* $\tilde{Q}$ *from the uniform distribution on* $O(n - k)$ *and set* $[q_{k+1} q_{k+2} \ldots q_n] = W\tilde{Q}$
9) *Stack* $Q = [Q_k \vdots q_{k+1} q_{k+2} \ldots q_n]$
10) *Go to 1), and repeat* $N$ *times*

Justification of the algorithm 1 follows from appropriate posterior decomposition given in Table 1, proposition 1 and considerations on pp. 7–8 (i.e. conditional prior of the last $n-k$ columns of $Q$ given $Q_k$ and zero restrictions is not influenced by zero restrictions). The sampling in steps 3) and 8) could be made as explained e.g. in ARRW (2016). On the other hand sampling in the step 1) could be accomplished using the Independence Metropolis Hastings (IMH) algorithm:

**Algorithm 2:**

0) *Take the starting values by setting* $\Pi^{(0)} = \hat{\Pi}$, $\Sigma^{(0)} = \frac{1}{T-2np-2n-1}M$ *and applying the Choleski decomposition* $\Sigma^{(0)} = L^{(0)}(L^{(0)})'$

*Update* $(\Pi^{(j)}, L^{(j)})$ *to* $(\Pi^{(j+1)}, L^{(j+1)})$ *as follows:*

1) *Draw* $(\Pi^{(*)}, \Sigma^{(*)})$ *from the Normal–Inverted Wishart posterior* $p_{ur}(\Pi \mid \Sigma, y)p_{ur}(\Sigma \mid y)$
2) *Obtain the Choleski decomposition* $\Sigma^{(*)} = L^{(*)}(L^{(*)})'$ *and compute*

$$\alpha = \min\left\{1, \prod_{i=1}^{k} \mid Z_i f(\Pi^{(*)}, L^{(*)}) f'(\Pi^{(*)}, L^{(*)}) Z_i' \mid^{-\frac{1}{2}} \mid Z_i f(\Pi^{(j)}, L^{(j)}) f'(\Pi^{(j)}, L^{(j)}) Z_i' \mid^{\frac{1}{2}}\right\}$$

3) *Take* $(\Pi^{(j+1)}, L^{(j+1)}) = \begin{cases} (\Pi^{(*)}, L^{(*)}), \ \textit{with probability}\ \alpha \\ (\Pi^{(j)}, L^{(j)}), \ \textit{with probability}\ 1-\alpha \end{cases}$

A few comments are in order. First of all, the algorithm 1 in its part to draw from the conditional posterior of $Q$ given the reduced form and zero restrictions, essentially appears in the older version of ARRW (2016) (dated 2014). We think it is useful to know when this algorithm is still correct (i.e. when the SVAR is $\bar{R}-$ restricted), since it is a) the exact sampling and b) much easier to implement than the sampling from the latest version of ARRW (2016). When $z_i = n-i;\ \forall i$, the algorithm collapses to finding a unique orthogonal matrix (up to the sign of each column or row) that is consistent with the restrictions. This is just algorithm 1 in Rubio–Ramírez et al. (2010). Lastly it should be clear that if restrictions concern only $A_0$ and/or $A_0^{-1}$, the step 1) in algorithm 1 should be modified in the interest of the efficiency. As evident from Table 1, if restrictions concern only $A_0$ and/or $A_0^{-1}$, the algorithm 2 could be made more efficient since we can draw exactly from $p(\Pi \mid L, y, R) = p_{ur}(\Pi \mid L, y)$ and the IMH algorithm is confined only to drawing from the marginal posterior $p(L \mid y, R)$.

On the other hand, the IMH algorithm is instructive. In general, if functions of the candidate reduced form parameters involved in the restrictions i.e. $Z_i f(\Pi^{(*)}, L^{(*)})$, are closer to zero than those in the previous draw $Z_i f(\Pi^{(j)}, L^{(j)})$, then we always accept a candidate draw. This is consistent with our discussion on rationale of the "additional" prior. Hence during the sampling process we penalize reduced form parameters that are probably inconsistent with the restrictions.

The tough part to draw from the set identified SVAR under zero and sign restrictions relates to taking into account the "zeros" in the algorithm. Sign restrictions present no problems for we have

**Algorithm 3:**

1) *Draw* $\Pi, L, Q$ *from the joint posterior subject to zero restrictions using algorithm 1*
2) *Keep the draw if all sign restrictions are satisfied*
3) *Go to 1) and repeat* $N$ *times*

## VII. THE ALGORITHM IN THE GENERAL CASE

What about the case when SVAR is not $\bar{R}-$restricted? The present section proposes a numerical method to deal with this case, which is the counterpart of the algorithms 3 and 4 in ARRW (2016). It should not be surprising that working with SVAR which is not $\bar{R}-$restricted will be a little more computationally demanding. In particular, unlike in the $\bar{R}-$restricted case, the joint posterior does not conform to any useful decomposition in general (except the case $f(\Pi, L) \equiv f(L)$). Hence we can only sample from the joint posterior of $\Pi, L, Q$ subject to zero restrictions

**Algorithm 4:**

0) *Take the starting values for reduced form parameters by setting* $\Pi^{(0)} = \hat{\Pi}$, $\Sigma^{(0)} = \frac{1}{T-2np-2n-1} M$ *and apply the Choleski decomposition* $\Sigma^{(0)} = L^{(0)}(L^{(0)})'$. *As a starting value for* $Q^{(0)}$ *take any draw, which is made applying steps 2) to 9) in algorithm 1.*
   *Update* $(\Pi^{(j)}, L^{(j)}, Q^{(j)})$ *to* $(\Pi^{(j+1)}, L^{(j+1)}, Q^{(j+1)})$ *as follows:*

1) *Draw* $(\Pi^{(*)}, \Sigma^{(*)})$ *from the Normal–Inverted Wishart posterior* $p_{ur}(\Pi \mid \Sigma, y) p_{ur}(\Sigma \mid y)$ *and obtain the Choleski decomposition* $\Sigma^{(*)} = L^{(*)}(L^{(*)})'$
2) *Set* $i = 1$
3) *Draw an* $(n - z_i + 1 - i) \times 1$ *vector* $x_i^{(*)}$ *from the uniform distribution on the unit sphere in* $\mathbb{R}^{n-z_i+1-i}$
4) *Find any* $n \times (n - z_i + 1 - i)$ *matrix* $G_i$ *(of full column rank) such that* $G_i' G_i = \mathrm{I}_{n-z_i+1-i}$ *and* $\begin{bmatrix} Z_i f(\Pi^{(*)}, L^{(*)}) \\ Q_{i-1}' \end{bmatrix} G_i = 0$
5) *Set* $q_i^{(*)} = G_i x_i^{(*)}$
6) *Set* $i = i + 1$, *go to 3), and repeat until* $i = k$ *to get* $Q_k^{(*)} = [q_1^{(*)} q_2^{(*)} \ldots q_k^{(*)}]$

7) *Find any* $n \times (n-k)$ *matrix* $W$ *such that* $W'W = \mathrm{I}_{n-k}$ *and* $(Q_k^{(*)})'W = 0$

8) *Draw* $\tilde{Q}^{(*)}$ *from the uniform distribution on* $O(n-k)$ *and set* $[q_{k+1}^{(*)} q_{k+2}^{(*)} ... q_n^{(*)}] = W\tilde{Q}^{(*)}$

9) *Stack* $Q^{(*)} = [Q_k^{(*)} \vdots q_{k+1}^{(*)} q_{k+2}^{(*)} \dots q_n^{(*)}]$

10) *Defining* $Q_{i-1}^{(*)} = [q_1^{(*)} q_2^{(*)} \dots q_{i-1}^{(*)}]$ *and* $Q_{i-1}^{(j)} = [q_1^{(j)} q_2^{(j)} \dots q_{i-1}^{(j)}]$ *compute*

$$\alpha = \min\left\{1, \prod_{i=1}^{k} \frac{| Z_i f(\Pi^{(*)}, L^{(*)})(\mathrm{I}_n - Q_{i-1}^{(*)}(Q_{i-1}^{(*)})') f'(\Pi^{(*)}, L^{(*)}) Z_i' |^{-\frac{1}{2}}}{| Z_i f(\Pi^{(j)}, L^{(j)})(\mathrm{I}_n - Q_{i-1}^{(j)}(Q_{i-1}^{(j)})') f'(\Pi^{(j)}, L^{(j)}) Z_i' |^{-\frac{1}{2}}}\right\}$$

*and take* $(\Pi^{(j+1)}, L^{(j+1)}, Q^{(j+1)}) = \begin{cases} (\Pi^{(*)}, L^{(*)}, Q^{(*)}), \ \text{with probability } \alpha \\ (\Pi^{(j)}, L^{(j)}, Q^{(j)}), \ \text{with probability } 1-\alpha \end{cases}$

11) *Go to 1), and repeat* $N$ *times*

Again the justification of algorithm 4 follows from the posterior decomposition in Table 1, proposition 1 and considerations on pp. 7–8. Suffice it to say that the step 10) stems from the IMH algorithm, when the candidate generating distribution is a product of Normal-Inverted Wishart marginal posterior for reduced form parameters and some $g(Q \mid \Pi, L)$ subject to zero restrictions. The (exact) drawing from the latter distribution is made using steps 2) to 9) in algorithm 1. Of course when sign restrictions are present (in addition to zero restrictions) then we use the algorithm 3 except that the first step in this algorithm should be made using algorithm 4. Lastly when $f(\Pi, L) \equiv f(L)$, further gain in efficiency is possible. That is we should apply IMH algorithm only to $L, Q$. Drawing $\Pi$ should be made from the (exact) conditional posterior $p(\Pi \mid L, y, R) = p_{ur}(\Pi \mid L, y)$, which is easily realized looking at Table 1.

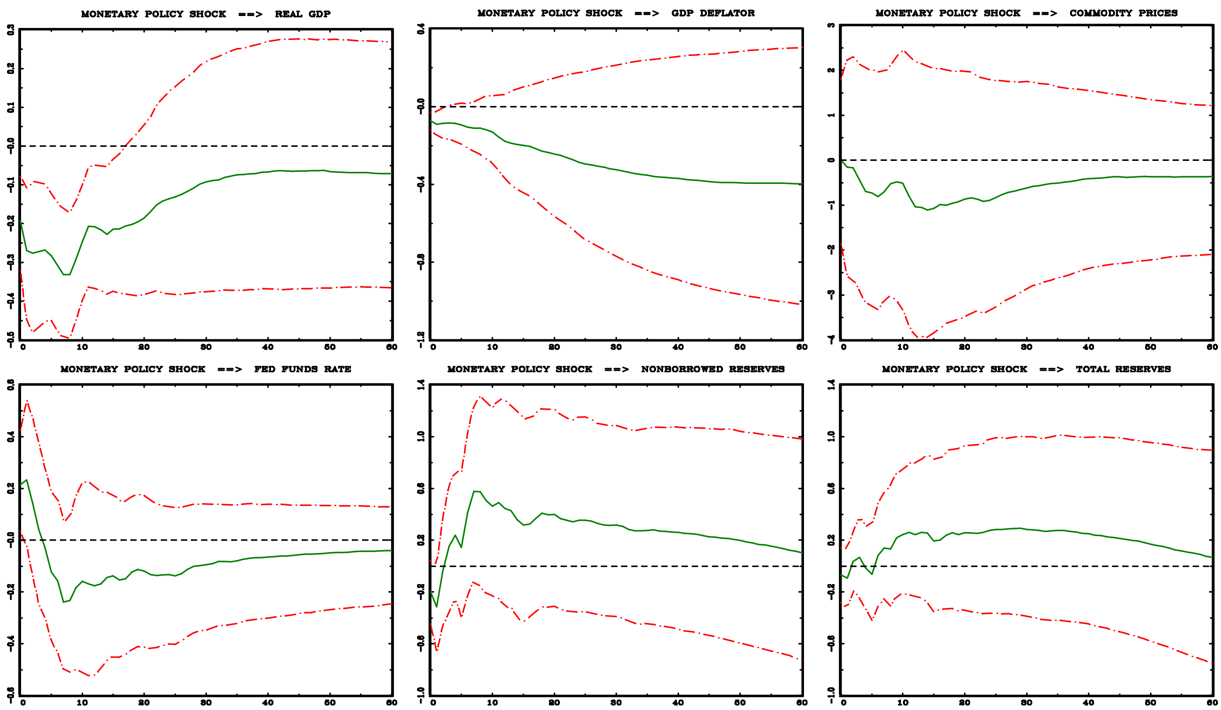
## VIII. EFFECTS OF MONETARY POLICY ON OUTPUT

In the well-known article, Uhlig (2005) argues that the effects of monetary policy shock on real output are uncertain. Hence he challenged the whole literature on identified SVAR that reached the consensus that a contractionary monetary policy shock should lower the real output in a significant way. The conclusion was made on the basis of pure sign restricted SVAR. In a recent paper, Arias, Caldara and Rubio-Ramírez (2016) (henceforth ACRR (2016)) conclude that the pitfall in the approach of Uhlig (2005) was the fact, that his sign restrictions imposed on a model to identify monetary policy shock accommodated unreasonable systematic monetary policy behavior. That is that the induced probability that interest rates rise in response to an increase in output was not close to 1 at all (as one may conjecture), but only 0.34. According to ACRR (2016) this fact discredits the approach of Uhlig (2005) i.e. the shock identified by Uhlig (2005) is not the monetary policy shock. The suggestion of ACRR (2016) was to impose explicit zero and sign restrictions on "monetary policy" or "feedback rule" equation in SVAR. Quite naturally, since they challenged results of Uhlig (2005), they worked with the same 6-variable SVAR. W.l.o.g suppose the first equation in this SVAR is labeled as "monetary policy" or "feedback rule" equation. Since what matters are the coefficients of the contemporaneous relations matrix $A_0$, denoting by $lags$ all remaining terms in the first equation we get

$$a_{0,14} r_t = -a_{0,11} y_t - a_{0,12} p_t - a_{0,13} p_{c,t} - a_{0,15} nbr_t - a_{0,16} tr_t + lags + \varepsilon_{t,1} \tag{7}$$

where $a_{0,ij}$ is the $(i,j)$ element of $A_0$, $r_t$ is the U.S federal funds rate, $y_t$ is the real GDP, $p_t$ is the GDP deflator, $p_{c,t}$ is the commodity price index, $nbr_t$ denotes nonborrowed reserves and $tr_t$ total reserves (particular ordering of variables follows that in Uhlig (2005)). In particular in their baseline specification, ACRR (2016) imposed the following restrictions: $a_{0,14} > 0$, $a_{0,11} \leq 0$, $a_{0,12} \leq 0$, $a_{0,15} = 0$ and $a_{0,16} = 0$, so that the fed funds rate only reacts contemporaneously to output, prices, and commodity prices and the reaction to output and prices is positive.

**Figure 1:** The baseline specification in ACRR (2016). The data span is 1965:01–2003:12. Green line denotes the median response, and two dot-dashed lines restrict the area of 68% posterior error bands (pointwise).



As a result they identified the structural shock connected with this equation as the monetary policy shock. Figure 1 presents IRF's to this shock.[7] Although our approaches slightly differ along many dimensions we roughly got the same picture as in ACRR (2016).[8] The most important difference is related to IRF of real GDP. In our case the negative response of real GDP to the contractionary monetary policy shock is obtained with more confidence. In addition, according to our median results, the maximum response suggests lowering real GDP by about 0.35 percent, whereas ACRR (2016) estimated this effect as 0.2 percent. Moreover the liquidity effect in nonborrowed reserves is better manifested than in ACRR (2016). Lastly it seems that all our 68% error bands are a little bit wider than those given in ACRR (2016).

In the development of their arguments ACRR (2016) put much emphasize on probabilities of negative coefficients of real GDP and GDP deflator in monetary policy equation. As demonstrated empirically by ACRR (2016), what drives their result concerning the IRF shape of real GDP is the restriction $a_{0,11} \leq 0$ (coefficient of real GDP is nonnegative). Below we explain theoretically the underlying statistics. In the baseline specification of ACRR (2016) we have

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{Z_1} L^{-1\prime} q_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8}$$

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{S_1} L^{-1\prime} q_1 \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

Note that by our proposition from appendix 3, the set of $q_1$'s that fulfills (8)-(9) is nonempty. Let us denote the $i-$th element of $q_1$ as $q_{i1}$. The instantaneous response of real GDP to monetary policy shock is given by the $(1,1)$ element of $A_0^{-1} = LQ$ i.e. $l_{11}q_{11}$. Since $l_{11}$ is strictly positive, this response will be negative iff $q_{11} < 0$. So the whole problem amounts to introducing sign restriction that induces $q_{11} < 0$ but avoids the explicit assumption that instantaneous response of real GDP to monetary policy shock is negative (which would violate the Uhlig's imperative to be agnostic

[7] We used Uhlig's monthly data set available at https://estima.com/procs_perl/uhligjme2005.zip. All variables in logs but the federal funds rate. The data we used to produce Figure 1 span 1965:01–2003:12. All computations in the paper are on the basis of 10.000 draws from the posterior. For monthly data we always set the number of lags in SVAR to 12.

[8] We used slightly different prior (we are agnostic over the IRF's, whereas ACRR (2016) chose to be agnostic over structural parameter space), we included a constant in SVAR, the dataset is shorter by 4 years.

along this dimension). Now, zero restrictions (8) imply that the last two elements in $q_1$ are zeros i.e. $q_{51} = q_{61} = 0$. Let $l^{ij}$ denote the $(i,j)$ element of $L^{-1}$. Clearly $l^{ii} = l_{ii}^{-1} > 0$. Taking all considerations into account the first inequality in (9) reads as $-l_{11}^{-1}q_{11} - l^{21}q_{21} - l^{31}q_{31} - l^{41}q_{41} \geq 0$ and the second one as $-l_{22}^{-1}q_{21} - l^{32}q_{31} - l^{42}q_{41} \geq 0$.[9] The very reason why baseline identification in ACRR (2016) works is that diagonal elements $l^{ii}$ dominate the remaining ones in the corresponding column of $L^{-1}$. The median estimates (together with 68% credible interval) of the first two sign restrictions in (9) are

$$-\underset{(2.25,2.44)}{2.35}\,q_{11} + \underset{(0.49,0.76)}{0.62}\,q_{21} + \underset{(-0.06,0.22)}{0.08}\,q_{31} + \underset{(0.08,0.36)}{0.22}\,q_{41} \geq 0$$
$$-\underset{(6.31,6.84)}{6.58}\,q_{21} + \underset{(0.68,1.44)}{1.06}\,q_{31} + \underset{(-0.11,0.65)}{0.27}\,q_{41} \geq 0$$

Clearly the second inequality highly favors negative or very small positive $q_{21}$'s. This reinforces the requirement that $q_{11}$ should be negative to fulfill the first inequality. In particular 90% of $q_{21}$'s consistent with sign restrictions are contained in the interval $(-0.8, 0.04)$. This in turn implies that 90% of $q_{11}$'s consistent with the first sign restriction belong to the interval $(-0.89, -0.05)$. As a result $q_{11} < 0$ holds with high probability.

Anyhow, we asked ourselves the question whether the negative response of GDP to the contractionary monetary policy shock could be obtained 1) without explicit sign restrictions for this IRF (as in Uhlig (2005)), 2) avoiding clever restrictions on monetary policy equation proposed by ACRR (2016), 3) without standard (and commonly criticized) restriction that monetary policy shock could not influence GDP and prices on impact. Using the whole data set from Uhlig (2005) (up to and including the year 2003) we could not produce such a response. But with data ending in the mid 90's, this was quite easy. In fact when Uhlig (2005) applied standard Choleski decomposition to the data up to year 2003 he obtained "price puzzle", which he commented as "*It may well be that the additional decade of data since 1992 has made this route* [ i.e. introducing commodity prices] *to resolving the price puzzle more difficult*". Since the data set was prepared for the problem from the perspective of mid 90's we think that it is fair to play with the data constrained by that time. The clue how it may be accomplished was given by Uhlig (2005), since he wrote that "*the identification of additional shocks can help in principle, as orthogonality between the shocks provides an additional restriction for identifying the monetary policy shock*". Hence in

[9] Although the last sign restriction does not play any role in our reasoning it is very easy to see that it leads to $q_{41} \geq 0$.

addition to monetary policy shock we thought seriously about minimal zero restrictions for additional three shocks. Two of them may be identified as production shocks and the last one the information shock. Table 2 presents zero restrictions used by us.[10] The entries in the Table 2 may be read as elements of the contemporaneous matrix $A_0$.

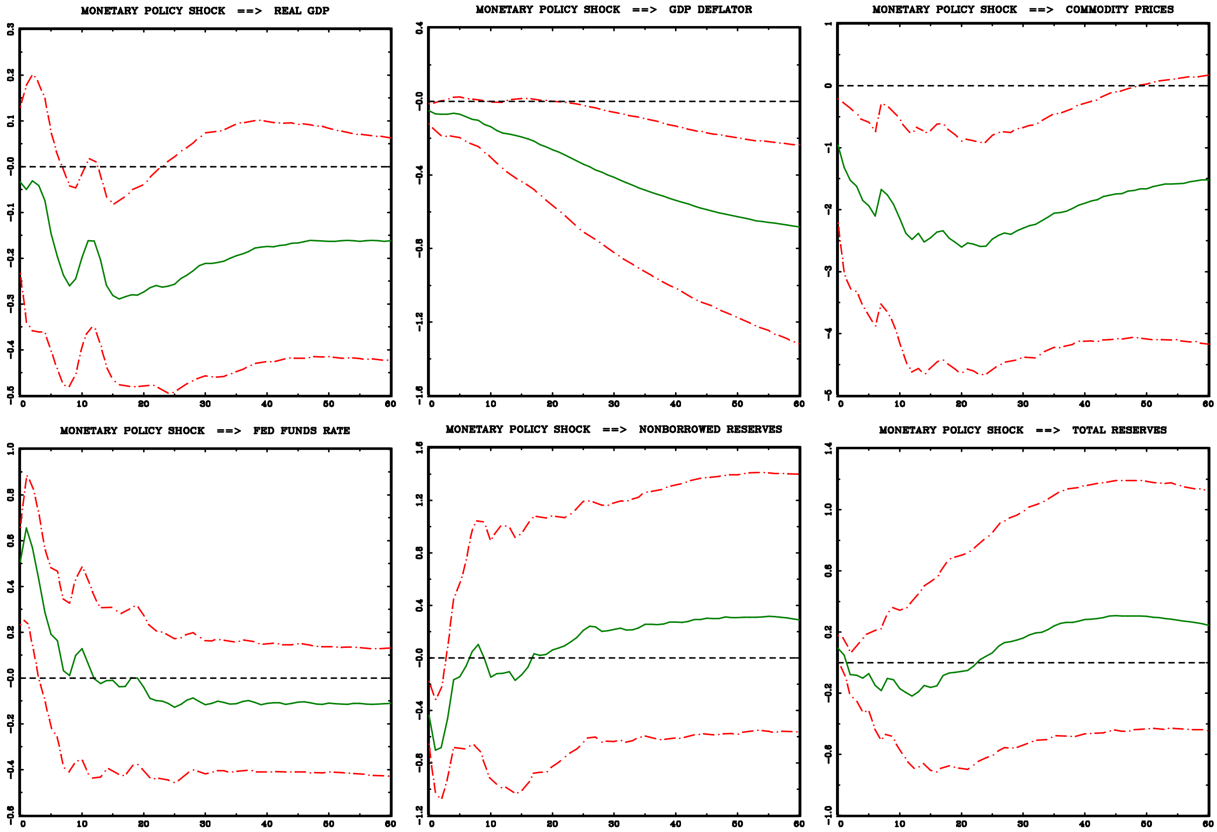
**Table 2**: Zero restrictions imposed on $A_0$

| | $y_t$ | $p_t$ | $p_{c,t}$ | $r_t$ | $nbr_t$ | $tr_t$ |
|---|---|---|---|---|---|---|
| Prod | X | X | X | 0 | 0 | 0 |
| Prod | X | X | X | 0 | 0 | 0 |
| Inf | X | X | X | X | 0 | 0 |
| MP | X | X | X | X | 0 | 0 |
| Fin | X | X | X | X | X | X |
| Fin | X | X | X | X | X | X |

"X" denotes unrestricted elements in $A_0$ and "0" those restricted to zero. The first column contains the reference names for the equation in SVAR. "Prod" refers to production sector, "Inf" refers to informational equation and "MP" is the monetary policy equation ("Fin" refers to financial sector but these equations play no role in our analysis). The main assumption underlying Table 2 is that nonpolicy variables do not respond contemporaneously to the policy variables (though instantaneous responses of these variables to the monetary policy shock are not restricted to zero). Clearly what makes the difference is the $(3,4)$ element in $A_0$ i.e. unrestricted coefficient in the informational equation of the federal funds rate. The rationale for this is that commodity price world market should respond immediately to the main indicators of monetary policy in a very large economy like the US. In addition to zeros induced by Table 2, we imposed Uhlig's sign restrictions but confined only to the instantaneous response $\Psi_0 = A_0^{-1}$. Specifically, responses of GDP deflator, commodity price index, nonborrowed reserves are nonpositive, and those of federal funds rate nonnegative on impact. Hence we substantially weakened sign restrictions used by Uhlig (2005), who imposed them for horizons from 0 to 5 months. Using dataset spanning 1965:01–1995:12 we obtained IRF's to the monetary policy shock identified by zero restrictions summarized in Table 2 and the sign restrictions confined to the impact responses, that are shown in Figure 2.

[10] It is easy to realize that SVAR under consideration is $\overline{R}-$restricted.

**Figure 2:** Monetary policy shock identified by zero restrictions summarized in Table 2 and the sign restrictions confined to the impact responses. The data span is 1965:01–1995:12. Green line denotes the median response, and two dot-dashed lines restrict the area of 68% posterior error bands (pointwise).




Now we do not observe the output puzzle. The one standard deviation monetary policy shock leads to the significant decrease in the real GDP by about 0,3% after 7 months (and up to two years). Overall all the remaining plots look more reasonable in comparison with those presented in Figure 1. Both GDP deflator and commodity price index respond negatively in a significant way in all horizons. Responses of federal funds rate are sharper. As a consequence, the liquidity effect in responses of nonborrowed reserves is more revealed.

Following ACRR (2016), to gain some intuition we present the median estimates of the monetary policy equation (7) normalized on $r_t$ and using our zero and sign restrictions (68% credible interval in parentheses)

$$r_t = \underset{(-0.41,0.65)}{0.14} y_t + \underset{(0.008,2.72)}{0.81} p_t + \underset{(0.01,0.15)}{0.05} p_{c,t} + lags + \varepsilon_{t,1} \tag{10}$$

Note that although we did not impose any sign restrictions on monetary policy equations, coefficients of prices and commodity prices are sharply constrained, i.e. the zero is outside the 68% error bands, and both coefficients admit the expected signs. The only problem is with coefficient of real GDP. Our interpretation of this is that the assumed restrictions favor models that have moderate positive coefficients of the real GDP. If this is the case then the non-negligible negative support of the underlying marginal posterior for the coefficient of the real GDP may be accepted.

On the other hand the monetary policy rule estimated using baseline identification in ACRR (2016) (based on dataset truncated at 1995:12) looks as follows

$$r_t = \underset{(0.32,4.62)}{1.20} y_t + \underset{(0.71,10.75)}{2.92} p_t - \underset{(-0.45,0.23)}{0.06} p_{c,t} + lags + \varepsilon_{t,1} \tag{11}$$

Since we explicitly imposed sign restrictions on coefficients of $y_t$ and $p_t$, the corresponding credible intervals cover only positive values. It is clear that baseline identification in ACRR (2016) implies that the set of possible monetary policy equations includes those highly responsive to the present state of the economy, see in particular the large upper bound in 68% credible interval for coefficient of the GDP deflator. Although ACRR (2016) addressed this point carefully, it still may be the case that the set of monetary policy rules consistent with the baseline restrictions in ACRR (2016) contain some implausible models of monetary policy behavior. In addition, the median estimate of the coefficient of commodity prices is negative and 68% credible interval contains both negative and positive values. Since the equation was estimated using the sample 1965:01–1995:12, to some extent this undermines the well thought and successful route to include commodity prices in the monetary policy function in order to avoid the price puzzle as advocated by Sims (1992) (see also Christiano et al. (1999)). Hence even if inclusion of commodity prices was found essential in the literature from 90's (using the same dataset), the sign restrictions proposed by ACRR (2016) suggest something different. Finally we note that although our IRF's presented in Figure 2 are quite different to those presented in Figure 1 (so as the estimated feedback rules), the FEVD's in two models are remarkably similar.[11]

## IX. OPTIMISM SHOCKS

Beaudry et al. (2014) used zero and sign restrictions to identify so-called optimism shocks. They applied Mountford and Uhlig's (2009) penalty function approach (PFA) and obtain quite sharp results in terms of IRF's and FEVD's, which were criticized by ARRW (2016). For example, the latter authors, using their methodology, claim that contribution of optimism shocks to FEVD's of many variables for whatever horizon was highly overestimated by Beaudry et al. (2014). We decided to check the robustness of these results with respect to our methodology.

[11] These are available from the author upon request.

The benchmark model of Beaudry et al. (2014) contains seven variables: Total Factor Productivity (TFP), stock price index, consumption, the real interest rate, hours worked, investment and output. The variables were logged (but the real interest rate) and taken in levels. For obvious reasons we use the same dataset and adopt the same model specification (i.e. four lags) as in Beaudry et al. (2014).[12] They considered three basic identification schemes. All of them amounted to putting one zero and several sign restrictions, but were confined to the first column of $\Psi_0 = A_0^{-1}$ only. They called them identification I, II and III. In all these identifications the optimism shock was assumed to have zero impact on TFP (in horizon "0"). In addition, in identification I, stock prices rise in response to optimism shock on impact, in identification II: stock prices and consumption rise in response to optimism shock on impact, and in identification III: stock prices, consumption and real interest rate increase in response to optimism shock on impact.

Main findings in Beaudry et al. (2014), who applied the PFA, were that optimism shock is crucial for business fluctuations since it leads to significant rise in consumption, hours worked, investment and output and results in the large corresponding FEVD's. In fact, looking at Figure 1 in Beaudry et al. (2014) one may have impression that even identification I does a good job. Hence using only one zero and one sign restriction on impact, one may find that optimism shock is essential driver of the business fluctuations.

Figures 3, 4, 5 present IRF's to optimism shock adopting identification I, II and III, respectively, using our algorithms. They are strikingly different in terms of IRF's uncertainty to those showed in Beaudry et al. (2014). Hence we confirm the conclusion in ARRW (2016), who claimed that IRF's bands presented in Beaudry et al. (2014) were artificially narrow and blamed the PFA for this. In particular, in contrast to Beaudry et al. (2014), identification I does not prove to be successful in obtaining sharp (i.e. statistically significant) results.

[12] The author is extremely grateful to Jian Wang for making available the dataset used in Beaudry et al. (2014). The quarterly data span is 1955:1–2012:4. For detailed information about the sources and construction of this dataset we refer to Beaudry et al. (2014).

**Figure 3:** Identification I. Green line denotes the median response, and two dot-dashed lines restrict the area of 68% posterior error bands (pointwise).

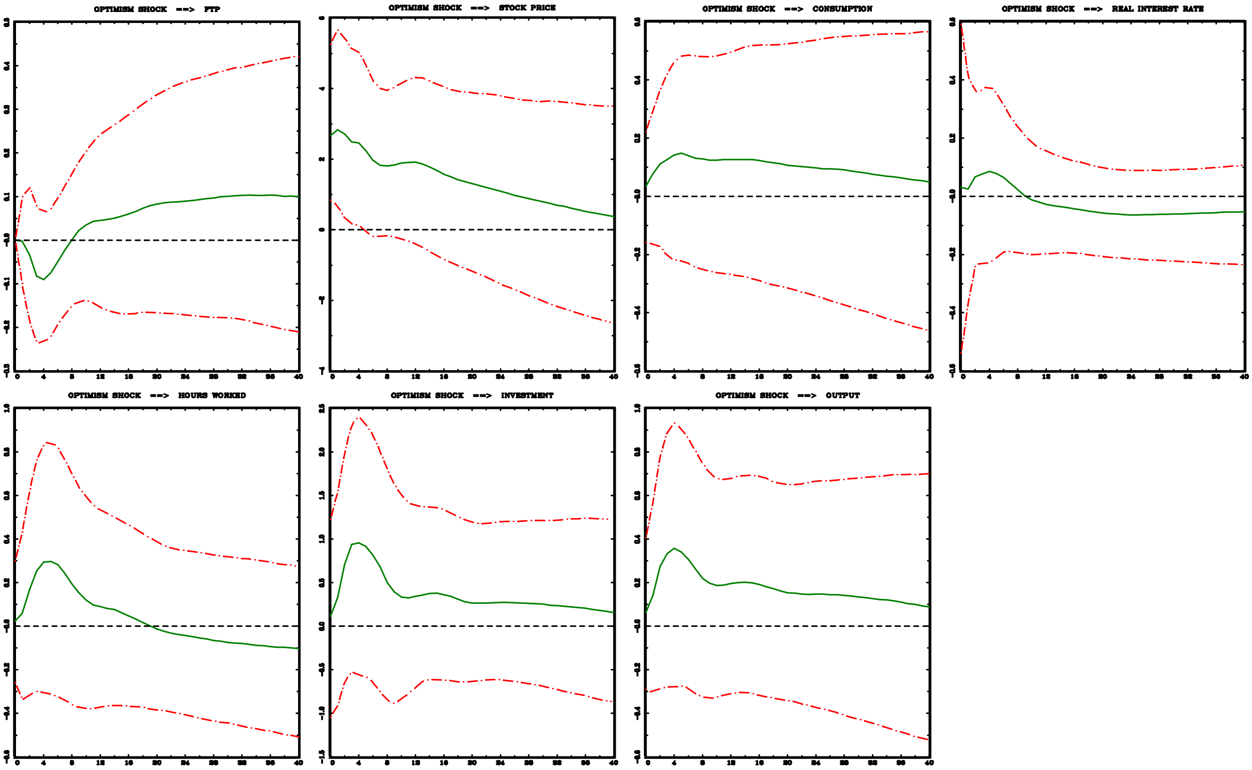



**Figure 4**: Identification II. Green line denotes the median response, and two dot-dashed lines restrict the area of 68% posterior error bands (pointwise).

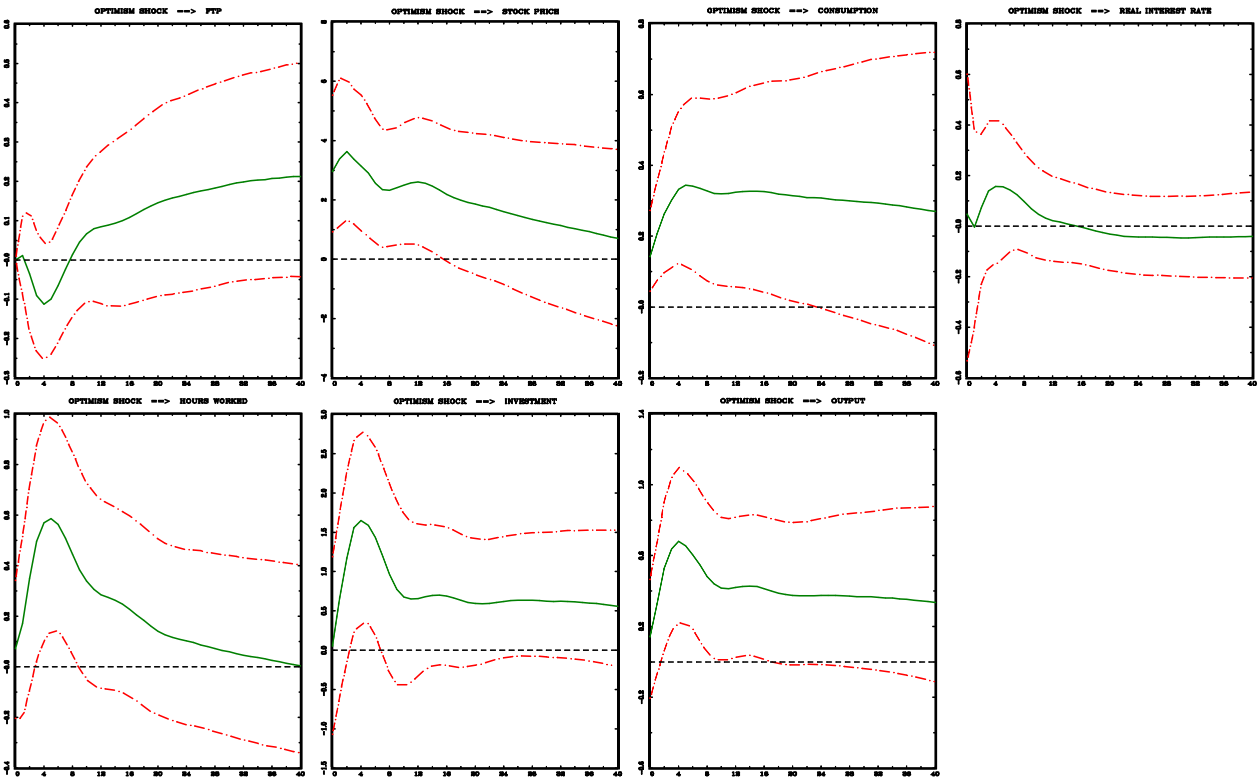


**Figure 5:** Identification III. Green line denotes the median response, and two dot-dashed lines restrict the area of 68% posterior error bands (pointwise).

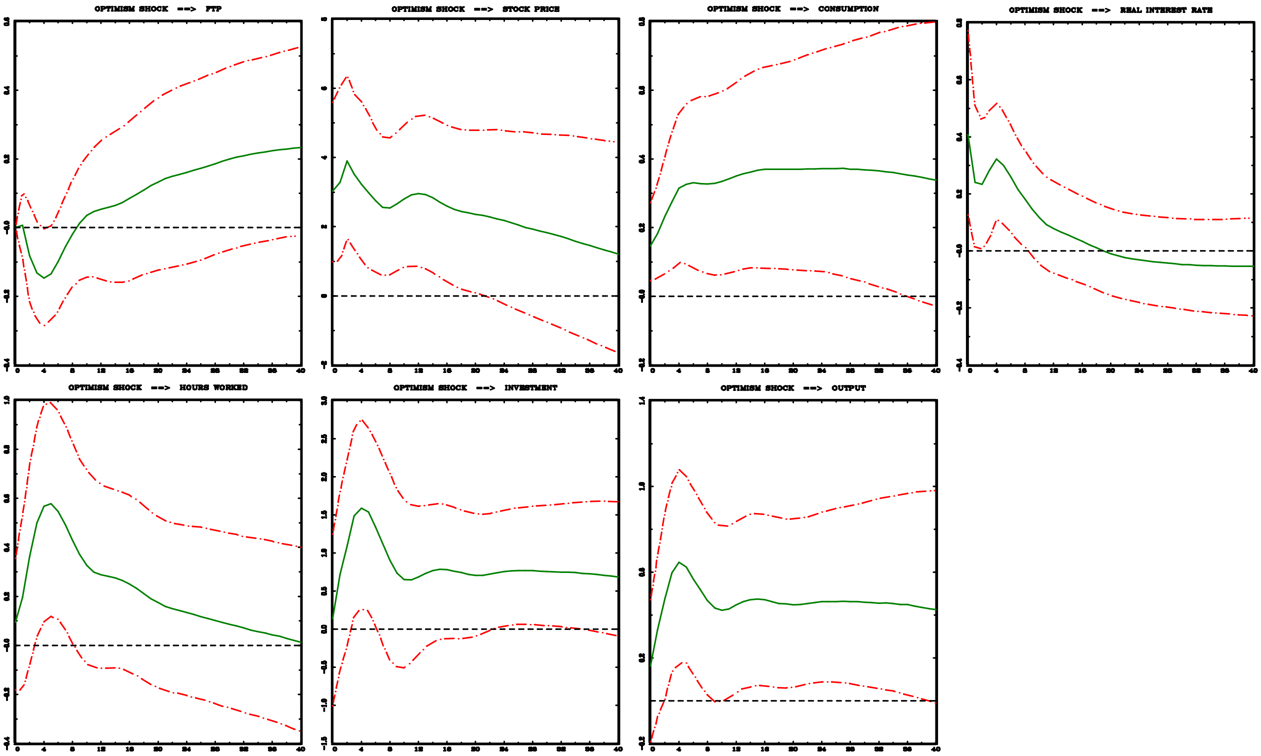



Table 3 contains median estimates of FEVD due to optimism shock (together with 68% credible intervals) using our algorithms.

**Table 3:** FEVD's due to optimism shocks for 4 and 40 quarters. Our methodology. Values in brackets show 68% posterior credible interval.

| | Identification I | | Identification II | | Identification III | |
|---|---|---|---|---|---|---|
| | $h = 4$ qtrs. | $h = 40$ qtrs. | $h = 4$ qtrs. | $h = 40$ qtrs. | $h = 4$ qtrs. | $h = 40$ qtrs. |
| TFP | 0.02<br>[0.004, 0.05] | 0.07<br>[0.02, 0.2] | 0.02<br>[0.005, 0.05] | 0.08<br>[0.02, 0.23] | 0.02<br>[0.005, 0.06] | 0.08<br>[0.02, 0.23] |
| stock prices | 0.1<br>[0.02, 0.34] | 0.09<br>[0.03, 0.25] | 0.14<br>[0.03, 0.4] | 0.12<br>[0.04, 0.28] | 0.15<br>[0.03, 041] | 0.13<br>[0.04, 0.31] |
| consumption | 0.09<br>[0.01, 0.34] | 0.1<br>[0.02, 032] | 0.15<br>[0.03, 0.42] | 0.12<br>[0.03, 0.36] | 0.12<br>[0.02, 0.38] | 0.15<br>[0.3, 0.39] |
| interest rate | 0.1<br>[0.03, 0.3] | 0.13<br>[0.05, 0.26] | 0.1<br>[0.03, 0.3] | 0.12<br>[0.05, 0.25] | 0.12<br>[0.03, 0.33] | 0.14<br>[0.06, 0.28] |
| hours worked | 0.1<br>[0.02, 0.33] | 0.11<br>[0.04, 0.26] | 0.13<br>[0.03, 0.4] | 0.13<br>[0.04, 0.29] | 0.13<br>[0.03, 0.41] | 0.12<br>[0.04, 0.29] |
| investment | 0.1<br>[0.03, 0.29] | 0.12<br>[0.04, 0.27] | 0.13<br>[0.04, 0.35] | 0.15<br>[0.06, 0.33] | 0.12<br>[0.04, 0.35] | 0.16<br>[0.06, 0.33] |
| output | 0.09<br>[0.02, 0.31] | 0.11<br>[0.03, 0.28] | 0.15<br>[0.03, 0.39] | 0.15<br>[0.05, 0.35] | 0.13<br>[0.03, 0.37] | 0.16<br>[0.05, 0.37] |

Again, they are orthogonal to those presented in Beaudry et al. (2014). In fact we get quite striking uniformity of median estimates of FEVD's for all variables (except

TFP). One may say that each FEVD is about $1/7 \approx 0.14$. We interpret this as a complete lack of identification. Further identifying (zero and/or sign) restrictions are probably needed to obtain economically significant results. On the other hand, the FEVD's presented in Table 2 in Beaudry et al. (2014) point to substantially different results. There are many drastic discrepancies between our estimates. For example, using identification II, median estimates of FEVD of consumption and hours worked (for 4 quarters) are 0.74 and 0.46, respectively. Further, using identification III, Beaudry et al. (2014) presented median estimates of consumption and output (for 40 quarters), which were both equal to 0.52. Needless to say, in all these cases (and many other ones not mentioned) FEVD's median estimates in Beaudry et al. (2014) are outside the 68% credible intervals given in our Table 3.

## X. CONCLUSIONS

The paper presents new algorithms to deal with set identified SVAR models under zero and/or sign restrictions. Our methodology is similar to that presented in ARRW (2016), however we differ in some details. Paying special attention to many popular patterns of zero restrictions, we managed to simplify algorithms to draw from the posterior. We applied our methodology to challenge the output puzzle found in Uhlig (2005). Staying a priori agnostic about responses of output to monetary policy shock, we showed that it is not necessary to adopt sign restrictions on the systematic monetary policy equation proposed by ACRR (2016) to obtain significant real output drop as a result of contractionary monetary policy shock. In the second exercise, we largely confirm conclusions from ARRW (2016) concerning the results in Beaudry et al. (2014). Specifically the uncertainty in IRF's given by Beaudry et al. (2014) is highly underestimated, and the estimates of FEVD's presented in Beaudry et al. (2014) should be divided by two, three or even four to be consistent with ours.

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**APPENDICES**

**Appendix 1 (proof of Proposition 1):**

Our goal is to evaluate $\int_{Q_k'Q_k=\mathrm{I}_k, Z_i f(\Pi,L)q_i=0; i=1,\ldots,k<n} dQ_k$ . Consider the transformation

$$\begin{bmatrix} Z_i f(\Pi,L) \\ Q_{i-1}' \\ G_i' \end{bmatrix} q_i = \begin{bmatrix} \alpha_i \\ \beta_i \\ x_i \end{bmatrix}, \text{ for each } i = 1,\ldots,k \quad \text{(A1)}$$

where $\alpha_i : (z_i \times 1)$, $\beta_i : (i-1)\times 1$, $x_i : (n - z_i - i + 1)\times 1$ are "new" variables. Note that in case $i = 1$, $\beta_1$ is empty. Further, $Q_{i-1} = [q_1 \ldots q_{i-1}]$, with the convention that $Q_0$ is empty and $G_i : n\times(n - z_i - i + 1)$ is any fixed (full column rank) matrix such that $G_i'G_i = \mathrm{I}_{n-z_i-i+1}$ and $\begin{bmatrix} Z_i f(\Pi,L) \\ Q_{i-1}' \end{bmatrix} G_i = 0$. The transformation (A1) serves its purpose as long as $(f'(\Pi,L)Z_i' \vdots Q_{i-1} \vdots G_i)'$ is nonsingular for each $i = 1,\ldots,k$, which holds iff $(f'(\Pi,L)Z_i' \vdots Q_{i-1})'$ has full row rank.

We need the Jacobian underlying the transformation (A1). Due to recursiveness of the transformation one has

$J(Q_k \to \alpha_i,\beta_i,x_i; i = 1,\ldots,k) = \prod_{i=1}^{k} J(q_i \to \alpha_i,\beta_i,x_i)$, so that

$$J(Q_k \to \alpha_i,\beta_i,x_i; i = 1,\ldots,k) = \prod_{i=1}^{k} \left| \begin{pmatrix} Z_i f(\Pi,L) \\ Q_{i-1}' \end{pmatrix} \begin{pmatrix} Z_i f(\Pi,L) \\ Q_{i-1}' \end{pmatrix}' \right|^{-\frac{1}{2}} =$$

$$= \prod_{i=1}^{k} | Z_i f(\Pi,L)(\mathrm{I}_n - Q_{i-1}Q_{i-1}')f'(\Pi,L)Z_i' |^{-\frac{1}{2}} \quad \text{(A2)}$$

Implicit assumption in Jacobian derivation is that we always choose $G_i$ such that the determinant of $(f'(\Pi,L)Z_i' \vdots Q_{i-1} \vdots G_i)'$ is positive.

Of course $Q_k$ will be the function of the "new" variables. To be specific

$$q_i = \begin{bmatrix} Z_i f(\Pi,L) \\ Q_{i-1}' \\ G_i' \end{bmatrix}^{-1} \begin{bmatrix} \alpha_i \\ \beta_i \\ x_i \end{bmatrix} = [* \vdots * \vdots G_i] \begin{bmatrix} \alpha_i \\ \beta_i \\ x_i \end{bmatrix} \quad \text{(A3)}$$

What the above formula states is that the last $n - z_i - i + 1$ columns in the inverse of $\begin{bmatrix} Z_i f(\Pi,L) \\ Q_{i-1}' \\ G_i' \end{bmatrix}$ are necessarily equal to $G_i$, which will be of great importance for us (the proof of this assertion is trivial hence omitted). Note that the correspondence between $q_i$ and $\alpha_i,\beta_i,x_i$ will be 1–1 since $G_i$ is arbitrary but fixed. In particular setting $\alpha_i = 0, \beta_i = 0, x_i$ for each $i = 1,\dots,k$, we get a 1–1 correspondence between $Q_k$ subject to zero restrictions and $x_i$.

The purpose of using the transformation (A1) should be evident. We use (A1) because the restrictions explicitly show up among the "new" variables. So $Q_k'Q_k = \mathrm{I}_k, Z_i f(\Pi, L)q_i = 0 \Leftrightarrow \alpha_i = 0$, $\beta_i = 0$, $x_i'x_i = 1; i = 1,\dots,k$. We have

$$\int_{Q_k'Q_k=\mathrm{I}_k, Z_i f(\Pi,L)q_i=0; i=1,\dots,k} dQ_k = \int_{x_i'x_i=1; i=1,\dots,k} J(Q_k \to \alpha_i = 0, \beta_i = 0, x_i; i = 1,\dots,k)\, dx_1 \dots dx_k$$

where $J(Q_k \to \alpha_i = 0, \beta_i = 0, x_i; i = 1,\dots,k)$ denotes (A2) evaluated at $\alpha_i = 0, \beta_i = 0$:

$$J(Q_k \to \alpha_i = 0, \beta_i = 0, x_i; i = 1,\dots,k) = \prod_{i=1}^{k} | Z_i f(\Pi, L)(\mathrm{I}_n - \Lambda_{i-1}\Lambda_{i-1}')f'(\Pi, L)Z_i' |^{-\frac{1}{2}}$$

where $\Lambda_{i-1} = [G_1x_1 \vdots G_2x_2 \vdots \dots \vdots G_{i-1}x_{i-1}]$ using (A3).

**Appendix 2 (proof of Lemma):**

All we have to do is to show that $J = \prod_{i=1}^{k} | Z_i f(\Pi, L)(\mathrm{I}_n - \Lambda_{i-1}\Lambda_{i-1}')f'(\Pi, L)Z_i' |^{-\frac{1}{2}} = \prod_{i=1}^{k} | Z_i f(\Pi, L)f'(\Pi, L)Z_i' |^{-\frac{1}{2}}$. Since $\Lambda_0$ is empty, $Z_1 f(\Pi, L)(\mathrm{I}_n - \Lambda_0\Lambda_0')f'(\Pi, L)Z_1' = Z_1 f(\Pi, L)f'(\Pi, L)Z_1'$ holds trivially. We will show that $Z_i f(\Pi, L)\Lambda_{i-1} = 0$ for $i > 1$. Since $\Lambda_{i-1} = [G_1x_1 \vdots G_2x_2 \vdots \dots \vdots G_{i-1}x_{i-1}]$, we must show that $Z_i f(\Pi, L)[G_1 \vdots G_2 \vdots \dots \vdots G_{i-1}] = 0$. By construction $Z_i f(\Pi, L)G_i = 0$. Since $Z_i$ contains all rows that appear in $Z_{i+1}, Z_{i+2}, \dots, Z_k$, $Z_i f(\Pi, L)G_i = 0$ implies $Z_{i+1} f(\Pi, L)G_i = 0$, …, $Z_k f(\Pi, L)G_i = 0$. In particular $Z_i f(\Pi, L)G_1 = 0$ for $i > 1$, $Z_i f(\Pi, L)G_2 = 0$ for $i > 2$ and so on. This proves the lemma.

**Appendix 3 (existence of orthogonal matrix subject to zero and sign restrictions):**

Unfortunately our algorithms implicitly assume the existence of orthogonal matrices that are consistent with zero/sign restrictions. In this appendix we address this point. In this appendix, $x \in \mathbb{R}^n$ denotes a column vector. Let us denote the unit sphere in $\mathbb{R}^n$ as $\mathcal{S}^{n-1} = \{x \in \mathbb{R}^n \mid x'x = 1\}$, and define our object of interest as

$$\Gamma_i = \begin{bmatrix} Z_i f(\Pi, L) \\ Q_{i-1}' \\ S_i f(\Pi, L) \end{bmatrix} \qquad i = 1,\dots,k \le n \tag{A4}$$

Note that $\Gamma_i$ is a $(z_i + s_i + i - 1) \times n$ matrix. Let us denote $\mathcal{C}_i = \{x \in \mathbb{R}^n \mid \Gamma_i x \ge 0\}$, where $\Gamma_i x \ge 0$ signifies $Z_i f(\Pi, L)x = 0$, $Q_{i-1}'x = 0$, $S_i f(\Pi, L)x \ge 0$. Since $\mathcal{C}_i$ is the solution set of a finite number of homogenous equalities and inequalities it is the polyhedral convex cone, to be called the cone, see e.g. Schrijver (1986) p. 87. Although properties of $\mathcal{C}_i$ are fundamental for our considerations, the ultimate object of interest is the set being an intersection of the cone with a unit sphere in $\mathbb{R}^n$ i.e. $\mathcal{C}_i \cap \mathcal{S}^{n-1}$. This is just the set of unit length vectors $q_i$ that are consistent with zero and sign restrictions and are orthogonal to all previous $q_{i-1},\ldots,q_1$. Immediate thing to notice is that although $\mathcal{C}_i$ is never empty (since it is a cone hence must contain the origin 0), $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ may be empty since intersection of the origin with the unit sphere $\mathcal{S}^{n-1}$ is empty. However as long as $\mathcal{C}_i$ contains $x \neq 0$, $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ will be non-empty, since $\mathcal{C}_i$ is the cone. That is if $x \neq 0 \in \mathcal{C}_i$ then $\lambda x \in \mathcal{C}_i$ for all $\lambda \geq 0$ so that $(x'x)^{-\frac{1}{2}}x \in \mathcal{C}_i \cap \mathcal{S}^{n-1}$. Hence the problem of existence of orthogonal matrix consistent with zero/sign restrictions amounts to checking whether each $\mathcal{C}_i$ contains at least one nonzero point. The following proposition gives sufficient conditions

**Proposition:** *Let the zero and/or sign restrictions be imposed on the first $k$ columns of $Q$. Then for each $i = 1,\ldots,k$, in case $s_i = 0$, assume $z_i \leq n - i$ and in case $s_i \geq 1$ assume the matrix $\Gamma_i$ is of full row rank i.e. $1 \leq \operatorname{rank}(\Gamma_i) = z_i + s_i + i - 1 \leq n$. Then $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ is not empty for each $i = 1,\ldots,k$.*

**Proof:** Recall that $\mathcal{C}_i \cap \mathcal{S}^{n-1} \neq \varnothing$ iff $\mathcal{C}_i \neq \{0\}$. Hence we must show that $\mathcal{C}_i$ contains at least one nonzero point. Suppose $s_i \geq 1$. Under the hypothesis that $\Gamma_i$ is of full row rank, by Motzkin's theorem (see e.g. Mangasarian (1994), p. 29), $Z_i f(\Pi, L)x = 0$, $Q'_{i-1}x = 0$, $S_i f(\Pi, L)x > 0$, possesses a solution, which is nonzero solution (because $S_i f(\Pi, L)x > 0$). This solution must be also the solution of $\Gamma_i x \geq 0$. Hence $\mathcal{C}_i \neq \{0\}$, i.e. $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ is nonempty. On the other hand suppose $s_i = 0$. Then $\mathcal{C}_i = \{x \in \mathbb{R}^n \mid \Gamma_i x = 0\}$. By assumption, $z_i \leq n - i$, so $z_i + i - 1 \leq n - i + i - 1 < n$. Thus $\operatorname{rank}(\Gamma_i) < n$. It follows that $\mathcal{C}_i$ contains at least one nonzero point i.e. $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ is nonempty.

First part of assumptions i.e. for the case $s_i = 0$, is quite standard in the literature e.g. Giacomini and Kitagawa (2015), Gafarov and Montiel Olea (2015). When $z_i = n - i$, for each $i = 1,\ldots,k$, we get exact identification for the first $k$ equations, see Rubio–Ramírez et al. (2010).

On the other hand in case $s_i \geq 1$ the assumption that $\Gamma_i$ is of full row rank (to be called the assumption) appears to be new to the literature. It may be treated as restrictive, since it imposes the maximal number of equality and sign restrictions. For example the sum of equality and sign restrictions imposed on the first shock cannot be greater than $n$. In fact much of applied work violates this assumption including the seminal work by Uhlig (2005). But the point is that when the assumption does not hold, one can construct simple examples as those in Moon et al. (2013), Giacomini and Kitagawa (2015) (and many more), such that $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ is empty for a subset of $\Pi, L$ that has positive posterior probability. In other words if the assumption is not satisfied it may impose implicit sign (or even zero) restrictions on the reduced form itself i.e. $\Pi, L$, which may be awkward when working in partially identified environment. On the other hand if the assumption is satisfied for one (e.g. randomly selected) $\Pi, L, Q$, then it holds for almost all $\Pi, L$ and $Q$. The merit of the assumption is obvious: you can estimate your set identified SVAR in a sound way without bothering about existence of unit length vectors consistent with the restrictions. Of course dealing with SVARs that violate the assumption is possible. We should only verify for given $\Pi, L$, whether each $\mathcal{C}_i$ contains nonzero element. As noted by Gafarov and Montiel Olea (2015), the main challenge is not to state the conditions *per se* but making them computationally efficient so as it could be easily verified in an algorithmic way.[13]

Lastly it is easy to show that if $\Gamma_i$ has full row rank then $(f'(\Pi, L)Z_i' \vdots Q_{i-1})'$ is also full row matrix. Hence under our sufficient condition for existence of orthogonal matrix subject to zero and sign restrictions, the content of proposition 1 is also valid.

[13] Giacomini and Kitagawa (2015) proposed very crude method to do that. If, for given reduced form parameter values, you cannot find $Q$ satisfying sign restrictions among quite large number of draws, say 10.000, from the posterior, you consider at least one $\mathcal{C}_i$ contains only the origin.

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