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Bibi, Abdelouahab and Ghezal, Ahmed

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Asymptotic properties of $QMLE$ for periodic asymmetric strong and semi-strong $GARCH$ models.

ABDELOUAHAB BIBI AND AHMED GHEZAL
Department of Mathematics, UMC (1) Constantine, Algeria
E-mails addresses: a.bibi@umc.edu.dz and a.ghezal@centre-univ-mila.dz

Abstract

In this paper, we propose a natural extension of time-invariant coefficients threshold $GARCH$ ($TGARCH$) processes to periodically time-varying coefficients ($PTGARCH$) one. So some theoretical probabilistic properties of such models are discussed, in particular, we establish firstly necessary and sufficient conditions which ensure the strict stationarity and ergodicity (in periodic sense) solution of $PTGARCH$. Secondary, we extend the standard results for the limit theory of the popular quasi-maximum likelihood estimator ($QMLE$) for estimating the unknown parameters of the model. More precisely, the strong consistency and the asymptotic normality of $QMLE$ are studied in cases when the innovation process is an $i.i.d$ ($Strong\ case$) and/or is not ($Semi-strong\ case$). The finite-sample properties of $QMLE$ are illustrated by a Monte Carlo study. Our proposed model is applied to model the exchange rates of the Algerian Dinar against the U.S-dollar and the single European currency ($Euro$).

MR(2010) subject classification: 62G20, 62M10.

Keywords: Periodic asymmetric $GARCH$ model, Stationarity, Strong consistency, Asymptotic normality.

1 Motivation

Autoregressive conditionally heteroskedastic ($ARCH$) processes were introduced firstly by Engle [10] and their generalized $GARCH$ version by Bollerslev [6], are certainly the great deal of research on modelling volatility dynamics (denoted by $(h_n)_{n\in\mathbb{Z}}$ throughout) clustering in financial and econometric time-series $(\varepsilon_n)_{n\in\mathbb{Z}}$. These models belong to symmetric models (in the sense that h_n is formulated as a linear function of the past values of $\varepsilon^2_{n-i}, i \geq 1$) and hence past positive and negative values of observed process have the same effect on the current volatility which is in contradiction with many empirical evidences of volatilities arising mainly from the series of stocks. Indeed, it is well known that if h_t were symmetric, a negative correlation between the squared current innovation and the past one would be equal to zero and hence the asymmetry property is violated. However, and to remedy this fact, some issue were proposed in the literature, citing, among the asymmetric $GARCH$ models, threshold $GARCH$ ($TGARCH$) models, already pioneered by Zakoïan [29], is now the most popular model in asymmetric volatility (see also Rabemananjara and Zakoïan [26] for a comprehensive review). It become increasingly important in modelling and forecasting financial time series and continues to gain a growing interest of researchers. The main purpose of $TGARCH$ processes is to allow the parameters in volatility to depend on the sign of observed process $(\varepsilon_n)_{n\in\mathbb{Z}}$ in order to capture asymmetric and leverage effects on the volatility dynamics. In other words the volatility may be regarded as a switched process between two regimes often specified by $\{n : \varepsilon_n < 0\}$ and $\{n : \varepsilon_n \geq 0\}$. This structural changes, we allows to assume that the parameters of each regime are different or more generally varying according with time. This assumption can cause however unstable (integrated or explosive) volatility process which plays an important interest in macroeconomic and in financial datasets (see for instance Francq and Zakoian [14] and the references therein). This interest is due to the fact that the unstable volatility present a persistent property, contrary to the stable case. So, this paper is mainly concerned with stable (but non-stationary) volatility in $TGARCH$ models in which the parameters may be depending on a known periodic sequence $(s_n)_n$ which refers to the stage of the periodic cycle at time n. This specification is inherent in many economic time series. Seasonal fluctuations have been found to significantly account for most of the variation in many macroeconomic time series (see Bibi and Aknouche [3] for further discussions). Periodicity is often removed either by using seasonally adjusted data or by including seasonal intercept dummies in the models. In this paper, periodicity is treated as one of the features to be explained within the $TGARCH$ model.

The mains purposes of the present paper are twofold, the first one is related to the probabilistic properties of $PTGARCH$ specification. In particular, after a general presentation of the threshold processes and its Markovian representation, in next section, in Sec. 3, our attention is focussed on traditional and alternative formulations of the $PTGARCH$ model, emphasizing the strict relation between its structure and the so-called periodic random coefficients autoregressive ($PRCA$) models. Starting from this relation, we study the necessary and sufficient conditions ensuring the strict (in periodic sense) of the $PTGARCH$ model. The second aim of the paper is purely statistical, i.e., we apply the standard quasi-maximum likelihood (QML) for estimating the parameters of model. So, in Sec. 4, we give explicit formulae for QML estimator of the parameters in $PTGARCH$ model in strong and/or in semi-strong cases, then the proofs of main theorems are relegated in Sec. 5. Numerical illustrations are given in Section 6 and an empirical application to the daily series of exchange rates from January $3, 2000$ to September $29, 2011$ of the Algerian Dinar against the U.S. Dollar and the single European currency is provided in Section 7. Section 8 concludes the article.
Before we proceed, let us introduce some symbolism and definitions.

1.1 Algebraic notation

Throughout, the following notations are used

- $I_{(n)}$ is the $n \times n$ identity matrix and $\mathbb{I}_\Delta$ denotes the indicator function of the set Δ.
- $O_{(n,m)}$ denotes the matrix of order $n \times m$ whose entries are zeros, for simplicity we set $O_{(n)} := O_{(n,n)}$ and $\underline{O}_{(n)} := O_{(n,1)}$.
- The spectral radius of squared matrix M is noted $\rho(M)$. Moreover, for any sequence of squared matrices (M_i) we set sometimes $M_i^l = M_i M_{i+1} ... M_l$ if $i \leq l$ and $M_i M_{i-1} ... M_l$ otherwise.
- $\|.\|$ refers to the standard norm in $\mathbb{R}^n$ or the uniform induced norm in the space $\mathcal{M}(n)$ of $n \times n$ matrices, for instance, the norm of matrix $M = (m_{ij})$ is defined by $\|M\| = \sum |m_{ij}|$.

2 The model and its Markovian representation

A process $(\varepsilon_n)_{n\in\mathbb{Z}}$ defined on some probability space $(\Omega, \Im, P)$ is called a periodic $TGARCH(p, q)$ process with period $s > 0$ abbreviated by $PTGARCH_s(p, q)$, if it is solution to the following stochastic difference equation $\varepsilon_n = h_n e_n$ and conditionally on the $\sigma-$field $\Im_n = \sigma(\varepsilon_{n-i}, i \geq 0)$, h_n satisfy

$$h_n = \alpha_0(s_n) + \sum_{i=1}^{q} \left(\alpha_i(s_n)\varepsilon_{n-i}^+ + \beta_i(s_n)\varepsilon_{n-i}^-\right) + \sum_{j=1}^{p} \gamma_j(s_n) h_{n-j} \tag{2.1}$$

where $\varepsilon_n^+ = \varepsilon_n \mathbb{I}_{\{\varepsilon_n \geq 0\}}$, $\varepsilon_n^- = -\varepsilon_n \mathbb{I}_{\{\varepsilon_n < 0\}}$ so, $\varepsilon_n = \varepsilon_n^+ - \varepsilon_n^-$ and $|\varepsilon_n| = \varepsilon_n^+ + \varepsilon_n^-$. In (2.1), $(s_n)_n$ is a periodic sequence of positive integers with finite state space $\mathbb{S} = \{1, ..., s\}$ defined by $s_n := \sum_{k=1}^{s} k\mathbb{I}_{\Delta(k)}(n)$ with $\Delta(k) := \{sn + k, n \in \mathbb{Z}\}$ that refers to the stage or "season" of the periodic cycle at time n, the innovation sequence $(e_n)_{n\in\mathbb{Z}}$ is subject to the following assumption:

Assumption 1 *$(e_n)_{n\in\mathbb{Z}}$ is a sequence of independent identically distributed (i.i.d.) random variables defined on the same probability space $(\mathbf{\Omega}, \mathcal{A}, P)$ with zero mean and unit variance and e_k is independent of ε_n for $k > n$.*

The $PTGARCH_s(p,q)$ models with an $i.i.d.$ innovations are often called periodic strong $TGARCH(p,q)$ models. Now, setting $n = st + v$, $\varepsilon_{st+v} = \varepsilon_t(v)$, $h_{st+v} = h_t(v)$ and $e_{st+v} = e_t(v)$, Model (2.1) may be equivalently written as

$$\varepsilon_t(v) = h_t(v) e_t(v) \text{ and } h_t(v) = \alpha_0(v) + \sum_{i=1}^{q} \left(\alpha_i(v)\varepsilon_t^+(v-i) + \beta_i(v)\varepsilon_t^-(v-i)\right) + \sum_{j=1}^{p} \gamma_j(v) h_t(v-j), \tag{2.2}$$

which we will make heavy use of (2.2), in wherein, $\alpha_0(v)$, $\alpha_i(v)$, $\beta_i(v)$ and $\gamma_j(v)$ with $i \in \{1, ..., q\}$ and $j \in \{1, ..., p\}$ are positive coefficients with $\alpha_0(v) > 0$ for any $v \in \mathbb{S}$, and $\varepsilon_t(v)$ refers to ε_t during the $v - th$ "season" or regime $v \in \mathbb{S}$ of cycle t, so the process $(h_n)_{n\in\mathbb{Z}}$ may be interpreted as the conditional standard deviation of $(\varepsilon_n)_{n\in\mathbb{Z}}$. For the convenience, $\varepsilon_t(v) = \varepsilon_{t-1}(v+s)$, $h_t(v) = h_{t-1}(v+s)$ and $e_t(v) = e_{t-1}(v+s)$ if $v < 0$. The non-periodic notations (ε_t), (h_t), (e_t) etc.,... will be used interchangeably with the periodic one $(\varepsilon_t(v))$, $(h_t(v))$, $(e_t(v))$ etc.,.... The process $(\varepsilon_n)_{n\in\mathbb{Z}}$ is globally non stationary, but is stationary within each period, it becoming an appealing tool for investigating both asymmetric volatility and distinct "seasonal" patterns for modelling financial time series and monetary economics.

A large lot of models may be defined from (2.1) including among others are for instance

i. The standard asymmetric $TGARCH(p,q)$ models and many extended $TGARCH(p,q)$ to periodic one

ii. Periodic version of Glosten et al.[18] models (denoted by $GJR - PGARCH_s$) obtained from (2.2) as

$$h_t(v) = \alpha_0(v) + \sum_{i=1}^{q} \left(\alpha_i(v) + \beta_i(v)\mathbb{I}_{\{\varepsilon_t(v-i)>0\}}\right) \varepsilon_t(v-i) + \sum_{j=1}^{p} \gamma_j(v) h_t(v-j), \ t \in \mathbb{Z} \tag{2.3}$$

iii. Periodic absolute value $GARCH$ models ($PAGARCH_s$): This class of models are obtained by assuming that $\alpha_i(v) - \beta_i(v) = 0$, $v \in \mathbb{S}$ and the volatility may be rewritten as

$$h_t(v) = \alpha_0(v) + \sum_{i=1}^{q} \alpha_i(v) |\varepsilon_t(v-i)| + \sum_{j=1}^{p} \gamma_j(v) h_t(v-j), \ t \in \mathbb{Z} \tag{2.4}$$

(see Bollerslev [7] for further discussion and recent inference on the area).

2.1 Markovian representation

Now, define $p-$vector $\underline{\gamma}_{1:p}(v) := (\gamma_1(v), ..., \gamma_p(v))'$, $2q-$vector $\underline{\zeta}_{1:q}(v) := (\alpha_1(v), \beta_1(v), ..., \alpha_q(v), \beta_q(v))'$, $r = (2q+p)-$vectors $\underline{H} = (1, -1, 0, ..., 0)'$, $r-$random vectors, $\underline{e}_t(v) := \alpha_0(v)\left(e_t^+(v), e_t^-(v), \underline{O}'_{(2(q-1))}, 1, 0...0\right)'$, $\underline{\varepsilon}_t(v) := (\varepsilon_t^+(v), \varepsilon_t^-(v), ..., \varepsilon_t^+(v-q+1), \varepsilon_t^-(v-q+1), h_t(v), ..., h_t(v-p+1))'$ and $r \times r-$ random matrix

$$\Gamma_v(e_t(v)) = \begin{pmatrix} \underline{\zeta}_{1:q-1}(v)\, e_t^+(v) & \alpha_q(v)\, e_t^+(v) & \beta_q(v)\, e_t^+(v) & \underline{\gamma}_{1:p-1}(v)\, e_t^+(v) & \gamma_p(v)\, e_t^+(v) \\ \underline{\zeta}_{1:q-1}(v)\, e_t^-(v) & \alpha_q(v)\, e_t^-(v) & \beta_q(v)\, e_t^-(v) & \underline{\gamma}_{1:p-1}(v)\, e_t^-(v) & \gamma_p(v)\, e_t^-(v) \\ I_{(2(q-1))} & \underline{O}_{(2(q-1))} & \underline{O}_{(2(q-1))} & O_{(2(q-1),p-1)} & \underline{O}_{(2(q-1))} \\ \underline{\zeta}_{1:q-1}(v) & \alpha_q(v) & \beta_q(v) & \underline{\gamma}_{1:p-1}(v) & \gamma_p(v) \\ O_{(p-1,2(q-1))} & \underline{O}_{(p-1)} & \underline{O}_{(p-1)} & I_{(p-1)} & \underline{O}_{(p-1)} \end{pmatrix}_{r\times r}. \tag{2.5}$$

With this notation, Equation (2.2) may be rewritten in state-space form $\varepsilon_t(v) = \underline{H}'\underline{\varepsilon}_t(v)$ and

$$\underline{\varepsilon}_t(v) = \Gamma_v(e_t(v))\underline{\varepsilon}_t(v-1) + \underline{e}_t(v). \tag{2.6}$$

Equation (2.6) is the same as the defining equation for independent periodic distribution ($i.p.d$) random coefficient autoregressive models introduced recently by Aknouche and Guerbyenne [2]. In this paper, we are interested in causal solution of equation (2.6), i.e., solution such that $\underline{\varepsilon}_t$ is independent of e_k for $t < k$. Hence, it is useful to write (2.6) in some equivalent Markovian representation in order to facilitate its study. For this purpose, iterating Equation (2.6) $s-$time to get

$$\underline{\varepsilon}_t(s) = \left\{\prod_{v=0}^{s-1}\Gamma_{s-v}(e_t(s-v))\right\}\underline{\varepsilon}_{t-1}(s) + \sum_{k=1}^{s}\left\{\prod_{v=0}^{s-k-1}\Gamma_{s-v}(e_t(s-v))\right\}\underline{e}_t(k)$$

and by setting $\underline{\varepsilon}(t) = \underline{\varepsilon}_t(s)$, then the above equation can be rewritten as

$$\underline{\varepsilon}(t) = \Lambda(\underline{e}_t)\underline{\varepsilon}(t-1) + \underline{\eta}(\underline{e}_t). \tag{2.7}$$

wherein $\underline{e}_t = (e_t(s), e_t(s-1), ..., e_t(1))'$, $\Lambda(\underline{e}_t) = \left\{\prod_{v=0}^{s-1}\Gamma_{s-v}(e_t(s-v))\right\}$ and $\underline{\eta}(\underline{e}_t) = \sum_{k=1}^{s}\left\{\prod_{v=0}^{s-k-1}\Gamma_{s-v}(e_t(s-v))\right\}\underline{e}_t(k)$.
Notice here that our formulation in Equation (2.7), the random matrix $\Lambda(\underline{e}_t)$ is independent of $\underline{\varepsilon}(t')$ for all $t' < t$ and $(\Lambda(\underline{e}_t))_{t\in\mathbb{Z}}$ (resp. $\left(\underline{\eta}(\underline{e}_t)\right)_{t\in\mathbb{Z}}$) is a sequence of $i.i.d.$ of random matrices (resp. $i.i.d.$ vectors). So the process $(\underline{\varepsilon}(t))_{t\in\mathbb{Z}}$ is Markov chain with state-space $\mathbb{R}^r$ and one-step transition probability $P(\underline{\varepsilon}, C) = P\left(\Lambda(\underline{e}_0)\underline{\varepsilon} + \underline{\eta}(\underline{e}_0) \in C\right)$ for any Borel $C \in \mathbb{B}_{\mathbb{R}^r}$.

3 Strict periodic stationarity

The existence of causal solution of (2.1) is now equivalent to the existence of the one of (2.7). Indeed, it is obvious that any causal solution of (2.1) leads via (2.6) to one of (2.7) and vice versa, that any components of a stationary solution of the dual process $\left((\underline{\varepsilon}'_t(1), ..., \underline{\varepsilon}'_t(s))'\right)_{t\in\mathbb{Z}}$ (see Gladyshev [17] for more details) are one of (2.1). So, in what follows, we examine the necessary and sufficient conditions ensuring the strict stationarity of the models (2.7) and hence the corresponding solution of equation (2.6) is called strictly periodic stationary (SPS). Note here that Equations similar to (2.7) were studied successfully in literature (e.g., Bougerol and Picard [8] and the reference therein). The key tool in studying the strict stationarity of (2.7) is however the top-Lyapunov exponent associated with the sequence of *i.i.d* random matrices $(\Lambda_t)_t$ and defined by

$$\gamma_L^{(s)}(\Lambda) := \inf_{t>0}\left\{\frac{1}{t}E\left\{\log\left\|\prod_{j=0}^{t-1}\Lambda(\underline{e}_{t-j})\right\|\right\}\right\} \overset{\text{a.s.}}{=} \lim_{t\longrightarrow\infty}\left\{\frac{1}{t}\log\left\|\prod_{j=0}^{t-1}\Lambda(\underline{e}_{t-j})\right\|\right\} \tag{3.1}$$

in which the second equality can be justified using Kingman's [23] subadditive ergodic theorem and the existence of $\gamma_L^{(s)}(A)$ is guaranteed however by the fact that $E\left\{\log^+\|\Lambda(\underline{e}_t)\|\right\} \leq E\left\{\|\Lambda(\underline{e}_t)\|\right\} < +\infty$, where $\log^+(x) = \max(\log x, 0)$ for any $x > 0$. Moreover, since $(e_t)_{t\in\mathbb{Z}}$ is a stationary and ergodic process, then $\left(\Lambda(\underline{e}_t), \underline{\eta}(\underline{e}_t)\right)_{t\in\mathbb{Z}}$ is also a stationary and ergodic process and since $E\left\{\log^+\|\Lambda(\underline{e}_0)\|\right\} < \infty$ and $E\left\{\log^+\left\|\underline{\eta}(\underline{e}_0)\right\|\right\} < \infty$, then we have

Theorem 3.1 *Equation* (2.7) *has a causal strictly stationary solution given by the series*

$$\underline{\epsilon}(t) = \sum_{k\geq 0}\left\{\prod_{j=0}^{k-1}\Lambda(\underline{e}_{t-j})\right\}\underline{\eta}(\underline{e}_{t-k}) \tag{3.2}$$

if and only if $\gamma_L^{(s)}(\Lambda) < 0$*. Moreover, the series* (3.2) *converges absolutely almost surely and constitute the unique ergodic solution process to* (2.7) *and hence Equation* (2.6) *is SPS process and admits a causal solution given by the series*

$$\underline{\epsilon}_t(v) = \sum_{k=0}^{\infty}\left\{\prod_{i=0}^{k-1}\Gamma_{v-i}(e_t(v-i))\right\}\underline{e}_t(v-k) \tag{3.3}$$

which converges absolutely almost surely and the process $\left(\underline{H}'\underline{\epsilon}_t(v)\right)_{t\in\mathbb{Z}}$ *constitute the unique, causal, SPS and periodically ergodic solution of equation* (2.1).

Corollary 3.1 *If* $PTGARCH_s(p,q)$ *model* (2.2) *has an SPS solution, then*

$$\rho(\mathbf{\Omega}_1^s) < 1 \text{ where } \mathbf{\Omega}_1^s = \prod_{v=1}^{s}\mathbf{\Omega}_v \text{ with } \mathbf{\Omega}_v = \begin{pmatrix} \underline{\gamma}_{1:p-1}(v) & \gamma_p(v) \\ I_{(p-1)} & \underline{O}_{(p-1)} \end{pmatrix}$$

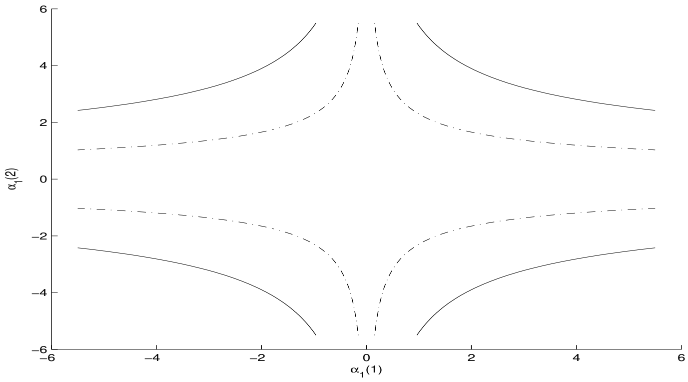
Proof. See Aknouche and Bibi [1].■

Example 3.1 *In the following table, we summarize the condition* $\gamma_L^{(s)}(\Lambda) < 0$ *for some particular cases*

Specifications	*Condition* $\gamma_L^{(s)}(\Lambda) < 0$
$TGARCH_1(1,1)$	$E\left\{\log\left\{\alpha_1 e_0^+ + \beta_1 e_0^- + \gamma_1\right\}\right\} < 0$
$PTGARCH_s(1,1)$	$\sum_{v=1}^{s} E\left\{\log\left\{\alpha_1(v)e_t^+(v-1) + \beta_1(v)e_t^-(v-1) + \gamma_1(v)\right\}\right\} < 0$
$PAGARCH_s(1,1)$	$\sum_{v=1}^{s} E\left\{\log\left\{\alpha_1(v)\lvert e_0\rvert + \gamma_1(v)\right\}\right\} < 0.$
$GJR-PGARCH_s(1,1)$	$\sum_{v=1}^{s} E\left\{\log\left\lvert\left(\alpha_1(v) + \beta_1(v)\mathbb{I}_{\{e_0>0\}}\right)e_0 + \gamma_1(v)\right\rvert\right\} < 0$

Table(1): Condition $\gamma_L^{(s)}(\Lambda) < 0$ *for some specifications*

Noting that the existence of "explosive regimes" does not preclude the existence of SPS solution. In particular, for $PTARCH_2(1)$ *with* $\alpha_1(2) = 0.5\alpha_1(1)$, $\beta_1(2) = 0.25\beta_1(1)$ *and* $e_t \rightsquigarrow t(5)$*, the stationarity zone is showed in Fig* (1) *below*



Fig(1). The stationary areas of $TARCH(1)$ (discontinuous line) and $PTARCH(1)$ (continuous line)

Corollary 3.2 *IIf $\gamma_L^{(s)}(\Lambda) < 0$ then there is $\delta > 0$ such that $E\left(h_t^{\delta}\right) < \infty$ and $E\left(|\varepsilon_t|^{\delta}\right) < \infty$ for all t.*

Remark 3.1 *If in distribution $\underline{\varepsilon}(0) = \sum\limits_{k\geq 0}\left\{\prod\limits_{j=0}^{k-1}\Lambda(\underline{e}_j)\right\}\underline{\eta}(\underline{e}_k)$, then $(\underline{\varepsilon}(t))_{t\in\mathbb{Z}}$ is strictly stationary and the above series converges absolutely with probability one.*

Remark 3.2 *Though, the condition $\gamma_L^{(s)}(\Lambda) < 0$ could be used as a necessary and sufficient condition for the strict stationarity of equation similar to (2.7), it is of little use for practical checking of stationarity since this condition involve the limit of products of infinitely many random matrices. Hence, some simple sufficient conditions ensuring the negativity of $\gamma_L^{(s)}(\Lambda)$ can be given.*

1. *If $E\left\{\log\left\|\prod\limits_{v=0}^{s-1}\Gamma_{s-v}(e_t(s-v))\right\|\right\} < 0$ or $E\left\|\prod\limits_{v=0}^{s-1}\Gamma_{s-v}(e_t(s-v))\right\| < 1$ then $\gamma_L^{(s)}(\Lambda) < 0$.*
2. *If $\rho\left(E\left\{\prod\limits_{v=0}^{s-1}\Gamma_{s-v}(e_t(s-v))\right\}\right) < 1$, then $\gamma_L^{(s)}(\Lambda) < 0$.*

Remark 3.3 *It is worth noting that the condition $\gamma_L^{(s)}(\Lambda) < 0$ provide a certain global stability of model (2.2). However when $\gamma_L^{(s)}(\Lambda) \geq 0$, the model (2.2) is said to be unstable and hence does not admit a SPS solution. As an example, consider $PTARCH_s(1)$ define by $\varepsilon_t(v) = h_t(v)e_t(v)$ and*

$$h_t(v) = \alpha_0(v) + \alpha_1(v)|e_t(v-1)|h_t(v-1), \tag{3.4}$$

then it is not difficult to verify that $\gamma_L^{(s)}(\Lambda) = \log\left\{\prod\limits_{v=0}^{s-1}\alpha_1(v)\right\} + sE\{\log|e_0|\} \geq 0$ if and only if $\exp(-sE\{\log|e_0|\}) \leq \prod\limits_{v=0}^{s-1}\alpha_1(v)$. Moreover, if $e_t \rightsquigarrow \mathcal{N}(0,1)$, $E\{\log|e_0|\} = \frac{1}{2}(\log(2) + \dfrac{\Gamma'(0.5)}{\Gamma(0.5)})$ where $\Gamma(.)$ and $\Gamma'(.)$ are the Gamma function and its first derivative respectively, so, $\exp(-sE\{\log|e_0|\}) \approx \exp(0.1048s)$. Hence the existence of some (not all) "stable regimes" (i.e., $E\{\log\alpha_1(v)\} < 0$) does not guarantees the existence of SPS solution. More generally we have the following convergence of the volatility to infinity for $PTARCH_s(1)$ process encompassing (2.2).

Proposition 3.1 *For $PTARCH_s(1)$, the following assertions hold*

1. *When $\gamma_L^{(s)}(\Lambda) > 0$, almost surely $h_t \to +\infty$ at an exponential rate, i.e., $\rho^t h_t \to +\infty$ and $\rho^t \varepsilon_t^2 \to +\infty$ as $t \to +\infty$ for any $\rho > e^{-\gamma_L^{(s)}(\Lambda)}$*

2. *When $\gamma_L^{(s)}(\Lambda) = 0$, in distribution $h_t \to +\infty$, and $\varepsilon_t^2 \to +\infty$ as $t \to +\infty$.*

4 QML estimator

In this section we consider the quasi-maximum likelihood estimator ($QMLE$) for the $PTGARCH_s$ parameter gathered in vector $\underline{\theta}' := (\underline{\alpha}', \underline{\beta}', \underline{\gamma}') := (\underline{\theta}'(1), ..., \underline{\theta}'(s)) \in \Theta \subset]0, +\infty]^s \times [0, +\infty[^{s(2q+p)}$ where $\underline{\alpha}' := (\underline{\alpha}'_0, \underline{\alpha}'_1, ..., \underline{\alpha}'_q)$, $\underline{\beta}' := \left(\underline{\beta}'_1, ..., \underline{\beta}'_q\right)$, $\underline{\gamma}' := \left(\underline{\gamma}'_1, ..., \underline{\gamma}'_p\right)$ and $\underline{\theta}'(v) := (\alpha_0(v), \alpha_1(v), ..., \alpha_q(v), \beta_1(v), ..., \beta_q(v), \gamma_1(v), ..., \gamma_p(v))$, $v \in \mathbb{S}$ with $\underline{\alpha}'_i := (\alpha_i(1), ..., \alpha_i(s))$, $\underline{\beta}'_k := (\beta_k(1), ..., \beta_k(s))$ and $\underline{\gamma}'_j := (\gamma_j(1), ..., \gamma_j(s))$ for all $0 \leq i, k \leq q$ and $1 \leq j \leq p$. The true parameter value denoted by $\underline{\theta}'_0 := \left(\underline{\alpha}'_0, \underline{\beta}'_0, \underline{\gamma}'_0\right) \in \Theta \subset]0, +\infty[^s \times [0, +\infty[^{s(2q+p)}$, is unknown and therefore it must be estimated. For this purpose, consider a realization $\{\varepsilon_1, ..., \varepsilon_n; n = sN\}$ from the unique, causal and SPS solution of (2.2), and let $h_t^2(\underline{\theta})$ be the conditional variance of ε_t given $\mathcal{F}_{t-1}$. The Gaussian likelihood function of $\underline{\theta} \in \Theta$ conditional on initial values $\varepsilon_0, ..., \varepsilon_{1-q}, h_0, ..., h_{1-p}$, which may be chosen as

$$\varepsilon_0^+ = \varepsilon_0^- = h_0 = \varepsilon_{-1}^+ = \varepsilon_{-1}^- = h_{-1} = ... = \varepsilon_{1-\max(p,q)}^+ = \varepsilon_{1-\max(p,q)}^- = h_{1-\max(p,q)} = 0 \tag{4.1}$$

is given by

$$\widetilde{L}_n(\underline{\theta}) = \left\{\prod_{t=1}^{n} \frac{1}{\left(2\pi \widetilde{h}_t^2(\underline{\theta})\right)^{\frac{1}{2}}}\right\} \exp\left\{-\sum_{t=1}^{n} \frac{\varepsilon_t^2}{2\widetilde{h}_t^2(\underline{\theta})}\right\} \tag{4.2}$$

in which $\widetilde{h}_t^2(\underline{\theta})$ are constructed under the initial values (4.1) and defined recursively by

$$\widetilde{h}_t(\underline{\theta}) = \alpha_0(t) + \sum_{i=1}^{q} \left(\alpha_i(t)\varepsilon_{t-i}^+ + \beta_i(t)\varepsilon_{t-i}^-\right) + \sum_{j=1}^{p} \gamma_j(t) \widetilde{h}_{t-j}(\underline{\theta}).$$

A $QMLE$ of $\underline{\theta}$ is defined as any measurable solution $\widehat{\underline{\theta}}_n$ of

$$\widehat{\underline{\theta}}_n = Arg\max_{\underline{\theta} \in \Theta} \widetilde{L}_n(\underline{\theta}) = Arg\min_{\underline{\theta} \in \Theta} \left(\widetilde{I}_n(\underline{\theta})\right)$$

where (ignoring the constants) $\widetilde{I}_n(\underline{\theta}) = (sN)^{-1} \sum_{t=1}^{N} \sum_{v=0}^{s-1} \widetilde{l}_{st+v}(\underline{\theta})$ with $\widetilde{l}_t(\underline{\theta}) = \dfrac{\varepsilon_t^2}{\widetilde{h}_t^2(\underline{\theta})} + \log \widetilde{h}_t^2(\underline{\theta})$. In view of the strong dependency of $\widetilde{h}_t(\underline{\theta})$ on initial values (4.1), $\left(\widetilde{l}_t(\underline{\theta})\right)_{t \geq 1}$ is not a SPS nor a periodically ergodic (PE) process, and therefore, it will however convening to work with a SPS and PE approximate version $I_n(\underline{\theta})$ of the likelihood (4.2) i.e., $I_n(\underline{\theta}) = (sN)^{-1} \sum_{t=1}^{N} \sum_{v=0}^{s-1} l_{st+v}(\underline{\theta})$ with $l_t(\underline{\theta}) = \dfrac{\varepsilon_t^2}{h_t^2(\underline{\theta})} + \log h_t^2(\underline{\theta})$. In what follows, we will give conditions ensuring the strong consistency of $\widehat{\underline{\theta}}_n$ and its asymptotic normality. Our approach is principally benefitted from the paper by Aknouche and Bibi [1].

4.1 Asymptotic properties for $QMLE$ of strong $PTGARCH_s$ models

To study the strong consistency of $\widehat{\underline{\theta}}_n$, we first define the polynomials $\mathbf{a}_{0,v}(z) = \sum_{i=1}^{q} \alpha_{0,i}(v) z^i$, $\mathbf{b}_{0,v}(z) = \sum_{i=1}^{q} \beta_{0,i}(v) z^i$ and $\mathbf{c}_{0,v}(z) = 1 - \sum_{i=1}^{p} \gamma_{0,i}(v) z^i$, by convention $\mathbf{a}_{0,v}(z) = 0$ and $\mathbf{b}_{0,v}(z) = 0$ if $q = 0$ and $\mathbf{c}_{0,v}(z) = 1$ if $p = 0$, for all $v \in \{1, ..., s\}$. Now, consider the following regularities conditions

A.0 $\underline{\theta}_0 \in \Theta$ and Θ is a compact subset of $\mathbb{R}^{s(1+2q+p)}$.

A.1 If $p > 0$, $\mathbf{a}_{0,v}(z)$ and $\mathbf{b}_{0,v}(z)$ have no common roots with $\mathbf{c}_{0,v}(z)$ for all v. Moreover, $\mathbf{a}_{0,v}(1) + \mathbf{b}_{0,v}(1) \neq 0$ and $\alpha_{0,q}(v) + \beta_{0,q}(v) + \gamma_{0,p}(v) \neq 0$ for all $v \in \mathbb{S}$.

A.2 $\gamma_L^{(s)}(\Lambda_0) < 0$ and $\rho(\mathbf{\Omega}_1^s) < 1$ where $\gamma_L(\Lambda_0)$ is the Lyapunov exponent associated with the random matrix $\Lambda(\underline{e}(t))$ evaluate under the true value $\underline{\theta}_0$.

A.3 $(e_t)_{t\in\mathbb{Z}}$ is non-degenerate and $P(e_t > 0) \in (0, 1)$.

We are now in a position to state the following result.

Theorem 4.1 *Under Assumption 1 and the conditions A.0−A.3, almost surely* $\widehat{\underline{\theta}}_{sN} \to \underline{\theta}_0$ *as* $N \to \infty$.

To show the asymptotic normality of $\widehat{\underline{\theta}}_{sN}$, the following additional assumptions are made.

A.4 $\underline{\theta}_0 \in \mathring{\Theta}$, with $\mathring{\Theta}$ denotes the interior of Θ.

A.5 $\kappa = E\left\{e_t^4\right\} < \infty$.

The second main result of this section is the following

Theorem 4.2 *Under the Assumption 1 and the condition A.0−A.5,* $\sqrt{sN}\left(\widehat{\underline{\theta}}_{sN} - \underline{\theta}_0\right) \rightsquigarrow \mathcal{N}\left(\underline{O}, (\kappa - 1) J^{-1}\right)$ *as* $N \to \infty$ *where the matrix* J *given by*

$$J := \sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{\frac{\partial^2 l_{st+v}}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\theta}_0)\right\} = 4\sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{\frac{1}{h_{st+v}^2(\underline{\theta}_0)}\frac{\partial h_{st+v}}{\partial\underline{\theta}}(\underline{\theta}_0)\frac{\partial h_{st+v}}{\partial\underline{\theta}'}(\underline{\theta}_0)\right\},$$

is block-diagonal. In particular, for $PTARCH_s(1)$ *we have* $J = diag\{J_v, v \in \mathbb{S}\}$ *with*

$$J_v = E_{\underline{\theta}_0}\begin{pmatrix} \frac{1}{h_{st+v}^2(\underline{\theta}_0)} & \frac{\varepsilon_t^+(v-1)}{h_{st+v}^2(\underline{\theta}_0)} & \frac{\varepsilon_t^-(v-1)}{h_{st+v}^2(\underline{\theta}_0)} \\ \frac{\varepsilon_t^+(v-1)}{h_{st+v}^2(\underline{\theta}_0)} & \frac{\varepsilon_t^{+2}(v-1)}{h_{st+v}^2(\underline{\theta}_0)} & 0 \\ \frac{\varepsilon_t^-(v-1)}{h_{st+v}^2(\underline{\theta}_0)} & 0 & \frac{\varepsilon_t^{-2}(v-1)}{h_{st+v}^2(\underline{\theta}_0)} \end{pmatrix}.$$

Now, a few comments can be made, the compactness of Θ is assumed in order that several results from real analysis may be used. Condition $A.1$, is a standard identifiability assumption. Condition $A.2$, implies that for the true value $\underline{\theta}_0$, the model (2.2) admits a SPS, PE solution and ensures the existence of a finite moment (see, Corollary 3.2). The second part of Condition $A.2$ ensure that $h_t(\underline{\theta})$ has a causal solution of $(e_t, e_{t-1}, ...)$, i.e., $h_t(v) = \phi_{0,v} + \sum_{j\geq 0}\phi_{j,v}\left(e_t^+(v-j), e_t^-(v-j)\right)$ for

$v \in \mathbb{S}$, where $\max_{1\leq v\leq s} E\left\{\phi_{j,v}\left(e_t^+(v-j), e_t^-(v-j)\right)\right\} = O\left(\lambda^j\right)$, with $0<\lambda<1$. Condition $A.3$, is made for identifiability purpose, it ensures also that the process (ε_t) takes positive and negative values with a positive probability. Condition $A.4$ is standard and allow to validate the first-order condition on the maximizer of the log-likelihood function while Condition $A.5$ is necessary for the existence of the limiting covariance matrix of the $QMLE$.

Remarks

1. Regarding to the asymptotic inference of stationary asymmetric $GARCH$ models allowing a signed volatility, the consistency and the asymptotic normality of the $QMLE$ have been established under different conditions see for instance Pen et al [25], Wang and Pen [28] and Hamadeh and Zakoïan [20]. However, Gonzalez-Rivera and Drosi [19] have established the loss of asymptotic efficiency of $QMLE$ relative gain in its robustness.

2. Non stationarity in the volatility process has been well documented for financial time series data. Indeed, Jensen and Rehbek [21], [22] and recently Chan [9] established asymptotic properties of $QMLE$ for non stationary time-invariant $ARCH/GARCH$ models, where non stationarity stems from the fact that the strict stationarity condition is not met, i.e., $\gamma_L^{(1)}(\Lambda)>0$. Hence, it is fruitful the study the asymptotic properties of $QMLE$ for non-stationary (i.e., $\gamma_L^{(s)}(\Lambda)>0$) $PGARCH_s$ (resp. $PTGARCH_s$) models generalizing thus the time-invariant cases.

3. Based on a general quasi-likelihood distribution, Francq and Zakoian [14] proposed a class of $QMLE$ for time-invariant non-stationary asymmetric $ARCH$ models and established the efficiency test for symmetry and stationarity assumptions.

4. It is worth noting that the asymptotic properties of $QMLE$ are also valid for the particular periodic integrated $TGARCH$ model obtained from the $PTGARCH_s$ model when the parameters are subject to be on the boundary of the second-order periodic stationarity domain. This is due to the strict inclusion of the latter domain into the strict stationarity one.

5. Noting here that the asymptotic properties for $TGARCH$ case can be acquired when the period is assumed to be equal to one and hence supports a parametric estimate method for $TGARCH$ model.

4.2 QMLE of semi-strong $PTGARCH_s$ models

Now, we extend the above results to the so-called semi-strong $PTGARCH_s$ models, i.e., when the *i.i.d.* assumption in innovation sequence is violated. In this case the Assumption 1 is replaced by the following

Assumption 2 $(e_n)_{n\in\mathbb{Z}}$ *is strictly stationary and ergodic sequence satisfying*

$$E\left\{e_t^2|\Im_{t-1}\right\}=1, E\left\{e_t^+|\Im_{t-1}\right\}=\mu_+ \text{ and } E\left\{e_t^-|\Im_{t-1}\right\}=\mu_- \text{ a.s.}$$

for some constants μ_+ *and* μ_-.

Remark 4.1 *It is worth noting that under the Assumption 2, the condition* $\gamma_L^{(s)}(\Lambda)<0$ *is not however necessary in theorem 3.1. Moreover, Corollary 3.2 is no longer under the Assumption 2, and hence we shall assume that*

A.6 there exists some positive τ such that $E\left\{|\varepsilon_n|^\tau\right\}<+\infty$.

The following theorem extends Theorem 4.1

Theorem 4.3 *Under Assumption 2 and the conditions A.0−A.3, A.6, almost surely* $\widehat{\underline{\theta}}_{sN} \rightarrow \underline{\theta}_0$ *as* $N \rightarrow \infty$.

For the asymptotic normality of semi-strong $PTGARCH_s$ models, we need to assume that

A.7 $E\left\{e_t^{4(1+\tau)}\right\} < +\infty$ for some $\tau > 0$

Theorem 4.4 *Under the Assumption 2 and the conditions A.0−A.7,* $\sqrt{sN}\left(\widehat{\underline{\theta}}_{sN} - \underline{\theta}_0\right) \rightsquigarrow \mathcal{N}\left(\underline{O}, \Sigma\left(\underline{\theta}_0\right)\right)$ *as* $N \rightarrow \infty$ *where* $\Sigma\left(\underline{\theta}\right) = J^{-1}\left(\underline{\theta}\right) I\left(\underline{\theta}\right) J^{-1}\left(\underline{\theta}\right)$ *where the matrix* $I\left(\underline{\theta}\right)$ *given by*

$$I\left(\underline{\theta}_0\right) := \sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{\left(E\left\{e_t^4 | \Im_{t-1}\right\} - 1\right) \frac{1}{h_{st+v}^2\left(\underline{\theta}_0\right)} \frac{\partial h_{st+v}}{\partial \underline{\theta}}\left(\underline{\theta}_0\right) \frac{\partial h_{st+v}}{\partial \underline{\theta}'}\left(\underline{\theta}_0\right)\right\},$$

is block-diagonal.

Remark 4.2 *Escanciano [11] and Lee and Hansen [24] established asymptotic results for a standard semi-strong* $GARCH$ *models when* $\left(e_n\right)_{n\in\mathbb{Z}}$ *is martingale difference sequence. Hamadeh and Zakoian [20] studied in general context the asymptotic behavior of* $QMLE$ *for a class of power-transformed threshold* $GARCH$ *models. In this paper, we extend the above results for a periodic version of* $TGARCH$.

5 Proofs

Sketch of Proof of Theorem 3.1. Following Bougerol and Picard [8], it is obviously that if (3.1) holds, then the solution must be given by (3.2). By subadditive ergodic theorem (see Kingman [23]), the Series (3.2) exists *a.s.*, whenever $\gamma_L^{(s)}\left(\Lambda\right) < 0$. The stationarity and ergodicity are immediate consequence of Theorem 3.5.8 in Stout [27].■

Proof of Corollary 3.2. In this proof, we have to show that if $\gamma_L^{(s)}\left(\Lambda\right) < 0$ *then there is* $\delta > 0$ *and* m_0 *such that*

$$E\left\{\left\|\prod_{k=0}^{sm_0-1} \Gamma_{sm_0-k}(e_{sm_0-k})\right\|\right\} < 1. \tag{5.1}$$

Since $\gamma_L^{(s)}\left(\Lambda\right) < 0$, there is a positive integer m_0 such that $E\left\{\log\left\|\prod_{k=0}^{sm_0-1} \Gamma_{sm_0-k}(e_{sm_0-k})\right\|\right\} < 0$. On the other hand, working with a multiplicative norm and by the *i.p.d.* property of the sequence $\left(\Gamma_t\left(e_t\right), t \in \mathbb{Z}\right)$ we have

$$\begin{aligned} & E\left\{\left\|\prod_{k=0}^{sm_0-1} \Gamma_{sm_0-k}(e_{sm_0-k})\right\|\right\} \\ & = \left\|E\left\{\prod_{k=0}^{sm_0-1} \Gamma_{sm_0-k}\left(e_{sm_0-k}\right)\right\}\right\| = \left\|E\left\{\left(\prod_{v=0}^{s-1} \Gamma_{s-v}\left(e_{s-v}\right)\right)^{m_0}\right\}\right\| \leq \left\|E\left\{\prod_{v=0}^{s-1} \Gamma_{s-v}\left(e_{s-v}\right)\right\}\right\|^{m_0} < \infty. \end{aligned}$$

Let $g\left(t\right) = E\left\{\left(\prod_{k=0}^{sm_0-1} \Gamma_{sm_0-k}\left(e_{sm_0-k}\right)\right)^t\right\}$. Since $g'\left(0\right) = E\left\{\log\left\|\prod_{k=0}^{sm_0-1} \Gamma_{sm_0-k}(e_{sm_0-k})\right\|\right\} < 0$, $g\left(t\right)$ decrease in a neighborhood of 0 and since $g(0) = 1$, it follows that there exists $0 < \delta < 1$ such that Eq (5.1) holds. Now for all $v \in \mathbb{S}$

$$E\left\{\left\|\underline{\epsilon}_t(v)\right\|^\delta\right\} \leq \sum_{k=0}^{\infty}\left\{E\left\{\left\|\prod_{i=0}^{k-1} \Gamma_{v-i}(e_t\left(v-i\right))\right\|^\delta\right\}\right\} E\left\{\left\|\underline{e}_t(v-k)\right\|^\delta\right\} \leq \sigma\left(\delta\right) \sum_{k=0}^{\infty}\left\{E\left\{\left\|\prod_{i=0}^{k-1} \Gamma_{v-i}(e_t\left(v-i\right))\right\|^\delta\right\}\right\},$$

where $\sigma(\delta) = \max_{v\in\mathbb{S}} E\left\{\|\underline{e}_t(v-k)\|^{\delta}\right\}$. Using Eq (5.1) there exist $a_v > 0$ and $0 < b_v < 1$ such that

$$E\left\{\left\|\prod_{i=0}^{k-1}\Gamma_{v-i}(e_t(v-i))\right\|^{\delta}\right\} \leq a_v b_v^k \leq ab^k := \max_{v\in\mathbb{S}} a_v b_v^k.$$

showing that $E\left(|\varepsilon_t|^{\delta}\right) < \infty$.■

Proof of Propositio 3.1. First, iterate (3.4), $s-$time to get the following equality

$$h_t(s) = \sum_{k=0}^{s-1}\left\{\prod_{i=0}^{k-1}\alpha_1(s-i)\,|e_t(s-i-1)|\right\}\alpha_0(s-k) + \left\{\prod_{i=0}^{s-1}\alpha_1(s-i)\,|e_t(s-i-1)|\right\}h_t(0). \tag{5.2}$$

Now, set

$$\omega(\underline{e}_t(1)) = \sum_{k=0}^{s-1}\left\{\prod_{i=0}^{k-1}\alpha_1(s-i)\,|e_t(s-i-1)|\right\}\alpha_0(s-k),\ \alpha(\underline{e}_t(0)) = \left\{\prod_{i=0}^{s-1}\alpha_1(s-i)\,|e_t(s-i-1)|\right\}, h(t+1) = h_t(s)$$

and rewriting (5.2) as $h(t+1) = \alpha(\underline{e}_t(0))\,h(t) + \omega(\underline{e}_t(1))$ with $\underline{e}_t(l) = (e_{st+l}, ..., e_{st+s-1})$. Note that $\alpha(\underline{e}_t(0))$ is a sequence of *i.i.d.* non negative random variables and independent of $h(k)$ for any $k < t$. With this notation, the proof follows essentially the same arguments as in Francq and Zakoian [13].■

Proof of Theorem 4.1

Rewrite (2.1) in vector form as

$$\underline{h}_t = \mathbf{\Omega}_t \underline{h}_{t-1} + \underline{\epsilon}_t \tag{5.3}$$

where $\underline{h}_t := (h_t, ..., h_{t-p+1})'$ and $\underline{\epsilon}_t := (\alpha_0(s_t) + \sum_{i=1}^{q}\left(\alpha_i(s_t)\varepsilon_{t-i}^{+} + \beta_i(s_t)\varepsilon_{t-i}^{-}\right),\ \underline{O}'_{(p-1)})'$. We will establish the following assertions gathered in the following lemma

Lemma 5.1 *Under Assumptions A.0−A.3, we have*

- i $\lim_{N\to\infty}\sup_{\underline{\theta}\in\Theta}\left|\widetilde{L}_{sN}(\underline{\theta}) - L_{sN}(\underline{\theta})\right| = 0$ a.s.
- ii There is $t \in \mathbb{Z}$ such that $h_t(\underline{\theta}) = h_t(\underline{\theta}_0)$ a.s. $\Rightarrow \underline{\theta} = \underline{\theta}_0$.
- iii $\sum_{v=1}^{s} E_{\underline{\theta}_0}\{l_{st+v}(\underline{\theta}_0)\} < \infty$ and if $\underline{\theta} \neq \underline{\theta}_0$, then $\sum_{v=1}^{s} E_{\underline{\theta}_0}\{l_{st+v}(\underline{\theta})\} > \sum_{v=1}^{s} E_{\underline{\theta}_0}\{l_{st+v}(\underline{\theta}_0)\}$.
- iv Any $\underline{\theta} \neq \underline{\theta}_0$ there is a neighborhood $\mathcal{V}(\underline{\theta})$ such that $\liminf_{N\to\infty}\inf_{\underline{\theta}^*\in\Theta}\left(\widetilde{L}_{sN}(\underline{\theta}^*)\right) > \sum_{v=1}^{s} E_{\underline{\theta}_0}\{l_{st+v}(\underline{\theta}_0)\}$ a.s.

Proof. To prove i, we note first that by corollary 3.1 and the compactness of Θ, we have $\sup_{\underline{\theta}\in\Theta}\rho(\mathbf{\Omega}_1^s) < 1$. Hence, iterating (5.3), we get

$$\underline{h}_t = \sum_{k=0}^{\infty}\mathbf{\Omega}_t^{t-k+1}\underline{\epsilon}_{t-k}. \tag{5.4}$$

where, as usual, empty products are set equal to $I_{(.)}$. Now, let $\widetilde{\underline{h}}_t$, $\widetilde{\underline{\epsilon}}_t$ be the vectors obtained from $\underline{h}_t$, $\underline{\epsilon}_t$, respectively, by replacing $\varepsilon_0^{+}, \varepsilon_0^{-}, ..., \varepsilon_{1-q}^{+}, \varepsilon_{1-q}^{-}$ by their initial values (4.1), so from (5.3), we obtain

$$\widetilde{\underline{h}}_t = \mathbf{\Omega}_t^0\widetilde{\underline{h}}_0 + \sum_{k=0}^{t-q-1}\mathbf{\Omega}_t^{t-k+1}\underline{\epsilon}_{t-k} + \sum_{k=t-q}^{t-1}\mathbf{\Omega}_t^{t-k+1}\widetilde{\underline{\epsilon}}_{t-k}$$

and hence, similarly to equation $(A.7)$ in Aknouche and Bibi [1] (hereafter AB), almost surely for all $t \geq 0$

$$\sup_{\underline{\theta}\in\Theta}\left\|\underline{\widetilde{h}}_t - \underline{h}_t\right\| = \sup_{\underline{\theta}\in\Theta}\left\|\mathbf{\Omega}_t^0\left(\underline{\widetilde{h}}_0 - \underline{h}_0\right) + \sum_{k=t-q}^{t-1}\mathbf{\Omega}_t^{t-k+1}\left(\underline{\widetilde{\epsilon}}_{t-k} - \underline{\epsilon}_{t-k}\right)\right\| \leq K\tau^t.$$

Moreover, since $\min(\widetilde{h}_t(\underline{\theta}), h_t(\underline{\theta})) \geq \max_{v\in\mathbb{S}}\{\alpha_0(v)\} = \overline{\alpha}_0$, then by, the mean value theorem we obtain for all t

$$\sup_{\underline{\theta}\in\Theta}\left|\widetilde{h}_t^2(\underline{\theta}) - h_t^2(\underline{\theta})\right| \leq 2\sup_{\underline{\theta}\in\Theta}\max\left(\widetilde{h}_t(\underline{\theta}), h_t(\underline{\theta})\right)\sup_{\underline{\theta}\in\Theta}\left|\widetilde{h}_t(\underline{\theta}) - h_t(\underline{\theta})\right| \leq K\sup_{\underline{\theta}\in\Theta}\max\left(\widetilde{h}_t^2(\underline{\theta}), h_t^2(\underline{\theta})\right)\tau^t.$$

Using the inequality $\log x \leq x - 1$ for $x > 0$, we deduce that

$$\begin{aligned}\sup_{\underline{\theta}\in\Theta}\left|\widetilde{L}_n(\underline{\theta}) - L_n(\underline{\theta})\right| &\leq n^{-1}\sum_{t=1}^{n}\sup_{\underline{\theta}\in\Theta}\left\{\frac{\left|\widetilde{h}_t^2(\underline{\theta}) - h_t^2(\underline{\theta})\right|}{\widetilde{h}_t^2(\underline{\theta})\,h_t^2(\underline{\theta})}\varepsilon_t^2 + \left|\log\left(\frac{\widetilde{h}_t^2(\underline{\theta})}{h_t^2(\underline{\theta})}\right)\right|\right\}\\ &\leq n^{-1}K\sum_{t=1}^{n}\tau^t\varepsilon_t^2 + n^{-1}K\sum_{t=1}^{n}\sup_{\underline{\theta}\in\Theta}\left(\widetilde{h}_t(\underline{\theta}) + h_t(\underline{\theta})\right)\tau^t.\end{aligned}$$

By Assumption $A.2$**,** and corollary 3.2, we have

$$\begin{aligned}&E\left\{\sup_{\underline{\theta}\in\Theta}h_t^\delta(\underline{\theta})\right\} \leq E\left\{\sup_{\underline{\theta}\in\Theta}\|\underline{h}_t\|^\delta\right\} \leq \sum_{k\geq 0}E\left\{\sup_{\underline{\theta}\in\Theta}\left\|\mathbf{\Omega}_t^{t-k+1}\right\|^\delta\left\|\underline{\widetilde{\epsilon}}_{t-k}\right\|^\delta\right\}\\ &\leq \sum_{k\geq 0}\tau^{\delta k}\max_{1\leq v\leq s}\left\{\sup_{\underline{\theta}\in\Theta}\left\{\alpha_0^\delta(v)\right\} + \sum_{i=1}^{q}\left(\sup_{\underline{\theta}\in\Theta}\left\{\alpha_i^\delta(v)E\left\{\left(\varepsilon_{st+v-i}^+\right)^\delta\right\}\right\} + \sup_{\underline{\theta}\in\Theta}\left\{\beta_i^\delta(v)E\left\{\left(\varepsilon_{st+v-i}^-\right)^\delta\right\}\right\}\right)\right\} < \infty\end{aligned}$$

and hence

$$E\left\{\sup_{\underline{\theta}\in\Theta}\left\|\underline{\widetilde{h}}_t\right\|^\delta\right\} \leq E\left\{\sup_{\underline{\theta}\in\Theta}\|\underline{h}_t\|^\delta\right\} + \tau^{\delta t}\sum_{k=1}^{q}\left\{\tau^{-\delta k}E\left\{\sup_{\underline{\theta}\in\Theta}\|\underline{\widetilde{\epsilon}}_k - \underline{\epsilon}_k\|^\delta\right\} + E\left\{\sup_{\underline{\theta}\in\Theta}\left\|\underline{\widetilde{h}}_0 - \underline{h}_0\right\|^\delta\right\}\right\} < K$$

The Borel–Cantelli lemma shows that almost surely $\tau^t\varepsilon_t^2 \to 0$, and to deduce i it suffices to use the Cesàro lemma. Turning to ii, assume that $h_t(\underline{\theta}) = h_t(\underline{\theta}_0)$, $a.s.$, and by Condition $A.2.$, the polynomial $(\mathbf{c}_{0,v}(z))_{v\in\mathbb{S}}$ is invertible. By second Equation in (2.2), we have $a.s.$

$$\left(\frac{\mathbf{a}_v(L)}{\mathbf{c}_v(L)} - \frac{\mathbf{a}_{0,v}(L)}{\mathbf{c}_{0,v}(L)}\right)\varepsilon_{st+v}^+ + \left(\frac{\mathbf{b}_v(L)}{\mathbf{c}_v(L)} - \frac{\mathbf{b}_{0,v}(L)}{\mathbf{c}_{0,v}(L)}\right)\varepsilon_{st+v}^- = \left(\frac{\alpha_{0,0}(v)}{\mathbf{c}_{0,v}(1)} - \frac{\alpha_0(v)}{\mathbf{c}_v(1)}\right) \text{ for all } 1 \leq v \leq s$$

where L is the lag operator. If $\frac{\mathbf{a}_v(L)}{\mathbf{c}_v(L)} \neq \frac{\mathbf{a}_{0,v}(L)}{\mathbf{c}_{0,v}(L)}$ or $\frac{\mathbf{b}_v(L)}{\mathbf{c}_v(L)} \neq \frac{\mathbf{b}_{0,v}(L)}{\mathbf{c}_{0,v}(L)}$ for some $v \in \mathbb{S}$, then there exist $k > 0$ and $(a(v), b(v))' \in \mathbb{R}^2\backslash\{(0,0)\}$ such that $(a(v), b(v))\left(\varepsilon_{st+v-k}^+, \varepsilon_{st+v-k}^-\right)'$ is a measurable function of the e_{st+v-l}, $l > k$. Then, we have a.s.

$$\begin{aligned}&(a(v), b(v))\left(\left(\varepsilon_{st+v-k}^+, \varepsilon_{st+v-k}^-\right)' - E_{\underline{\theta}_0}\left\{\left(\varepsilon_{st+v-k}^+, \varepsilon_{st+v-k}^-\right)'\middle|\mathcal{F}_{st+v-k-1}\right\}\right)\\ &= h_{st+v-k}(\underline{\theta}_0)(a(v), b(v))\left(e_{st+v-k}^+ - E\left\{e_{st+v-k}^+\right\}, e_{st+v-k}^- - E\left\{e_{st+v-k}^-\right\}\right)' = 0.\end{aligned}$$

Since $h_{st+v-k}(\underline{\theta}_0) > 0$, we deduce that $a(v)e_{st+v-k}^+ + b(v)e_{st+v-k}^- = c(v)$, $a.s.$, for some constant $c(v)$. If $a(v) = 0$ and $b(v) \neq 0$ then $e_{st+v-k}^- = 0$, $a.s$, which is in contradiction with $A.3$. If $a(v).b(v) \neq 0$, e_{st+v-k} takes at most two different values, which is contradiction with $A.3$. Thus we deduce that $a(v) = b(v) = 0$ and hence $\mathbf{a}_v(z) = \mathbf{a}_{0,v}(z)$, $\mathbf{b}_v(z) = \mathbf{b}_{0,v}(z)$, $\mathbf{c}_v(z) = \mathbf{c}_{0,v}(z)$, for any $z \in \mathbb{C} : |z| \leq 1$ and $\alpha_0(v) = \alpha_{0,0}(v)$ for all $v \in \mathbb{S}$, proving ii. To show iii, we have by Corollary 3.2

$$\sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{\log h_{st+v}^2(\underline{\theta}_0)\right\} = \frac{2}{\delta}\sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{\log h_{st+v}^{\delta}(\underline{\theta}_0)\right\} \leq \frac{2}{\delta}\sum_{v=1}^{s}\log E_{\underline{\theta}_0}\left\{h_{st+v}^{\delta}(\underline{\theta}_0)\right\} < \infty,$$

from which it follows that

$$\sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{l_{st+v}(\underline{\theta}_0)\right\} = \sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{\frac{h_{st+v}^2(\underline{\theta}_0)e_{st+v}^2}{h_{st+v}^2(\underline{\theta}_0)} + \log h_{st+v}^2(\underline{\theta}_0)\right\} = s + \sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{\log h_{st+v}^2(\underline{\theta}_0)\right\} < \infty,$$

and since $\log x \leq x - 1$ for all $x > 0$ with equality if and only if $x = 1$, we obtain

$$\begin{aligned}&\sum_{v=1}^{s}\left(E_{\underline{\theta}_0}\left\{l_{st+v}(\underline{\theta})\right\} - E_{\underline{\theta}_0}\left\{l_{st+v}(\underline{\theta}_0)\right\}\right)\\ &= \sum_{v=1}^{s}\left(\log\frac{h_{st+v}^2(\underline{\theta})}{h_{st+v}^2(\underline{\theta}_0)} + \frac{h_{st+v}^2(\underline{\theta}_0)}{h_{st+v}^2(\underline{\theta})} - 1\right) \geq \sum_{v=1}^{s}\left(\log\frac{h_{st+v}^2(\underline{\theta})}{h_{st+v}^2(\underline{\theta}_0)} + \log\frac{h_{st+v}^2(\underline{\theta}_0)}{h_{st+v}^2(\underline{\theta})}\right) = 0,\end{aligned}$$

which shows that the limit criterion is minimized at $\underline{\theta}_0$. It remains to show iv. For all $\underline{\theta} \in \Theta$ and all integer k, let $\mathcal{V}_k(\underline{\theta})$ be an open sphere of centre $\underline{\theta}$ and radius $\dfrac{1}{k}$. Using (i) we have

$$\begin{aligned}\liminf_{N\to\infty}\inf_{\underline{\theta}^*\in\Theta\cap\mathcal{V}_k(\underline{\theta})}\left(\widetilde{L}_{sN}(\underline{\theta}^*)\right) &\geq \liminf_{N\to\infty}\inf_{\underline{\theta}^*\in\Theta\cap\mathcal{V}_k(\underline{\theta})}\left(L_{sN}(\underline{\theta}^*)\right) - \limsup_{N\to\infty}\sup_{\underline{\theta}\in\Theta}\left(L_{sN}(\underline{\theta}) - \widetilde{L}_{sN}(\underline{\theta})\right)\\ &\geq \liminf_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\sum_{v=1}^{s}\inf_{\underline{\theta}^*\in\Theta\cap\mathcal{V}_k(\underline{\theta})} l_{st+v}(\underline{\theta}^*).\end{aligned}$$

Applying the ergodic theorem for the sequence $\left(\sum_{v=1}^{s} l_{st+v}(\underline{\theta})\right)_t$ with $E\left\{\sum_{v=1}^{s} l_{st+v}(\underline{\theta})\right\} \in \mathbb{R}\cup\{\infty\}$ (cf. Billingsley [5], p. 284, 495) it follows that

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\sum_{v=1}^{s}\inf_{\underline{\theta}^*\in\Theta\cap\mathcal{V}_k(\underline{\theta})} l_{st+v}(\underline{\theta}^*) = \sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{\inf_{\underline{\theta}^*\in\Theta\cap\mathcal{V}_k(\underline{\theta})} l_{st+v}(\underline{\theta}^*)\right\}$$

and by the Beppo-Levi theorem (e.g. Billingsley [5], p. 219), we have

$$\sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{\inf_{\underline{\theta}^*\in\Theta\cap\mathcal{V}_k(\underline{\theta})} l_{st+v}(\underline{\theta}^*)\right\} \longrightarrow \sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{l_{st+v}(\underline{\theta})\right\} \text{ as } k\to\infty,$$

which complete the proof of the lemma.■

The proof of the theorem 4.1 is completed upon the observation that for any neighborhood $\mathcal{V}(\underline{\theta}_0)$ of $\underline{\theta}_0$ we have

$$\limsup_{N\to\infty}\inf_{\underline{\theta}^*\in\mathcal{V}(\underline{\theta}_0)}\left(\widetilde{L}_{sN}(\underline{\theta}^*)\right) \leq \lim_{N\to\infty}\left(\widetilde{L}_{sN}(\underline{\theta}_0)\right) = \lim_{N\to\infty}\left(L_{sN}(\underline{\theta}_0)\right) = \sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{l_{st+v}(\underline{\theta}_0)\right\}.$$

The compact Θ is recovered by a union of a neighborhood $\mathcal{V}(\underline{\theta}_0)$ and the set of neighborhoods $\mathcal{V}(\underline{\theta})$, $\underline{\theta} \in \Theta\setminus\mathcal{V}(\underline{\theta}_0)$. Therefore, there exists a finite sub-covering of Θ by $\mathcal{V}(\underline{\theta}_0), \mathcal{V}(\underline{\theta}_1), ..., \mathcal{V}(\underline{\theta}_k)$ such that

$$\inf_{\underline{\theta}^*\in\mathcal{V}(\underline{\theta}_0)}\left(\widetilde{L}_{sN}(\underline{\theta}^*)\right) = \min_{j\in\{1,...,k\}}\inf_{\underline{\theta}^*\in\Theta\cap\mathcal{V}(\underline{\theta}_j)}\left(\widetilde{L}_{sN}(\underline{\theta}^*)\right).$$

The latter relation shows that $\widehat{\underline{\theta}}_{sN} \in \mathcal{V}(\underline{\theta}_0)$ for N sufficiently large, which complete the proof of the theorem.■

Proof of Theorem 4.2

This proof rests classically on a Taylor series expansion of $\dfrac{\partial L_{sN}}{\partial \underline{\theta}}(\underline{\theta})$ around $\underline{\theta}_0$ i.e.,

$$\underline{0} = (sN)^{-\frac{1}{2}} \sum_{t=1}^{sN} \frac{\partial l_t}{\partial \underline{\theta}} \left(\widehat{\underline{\theta}}_{sN}\right) = (sN)^{-\frac{1}{2}} \sum_{t=1}^{sN} \frac{\partial l_t}{\partial \underline{\theta}} (\underline{\theta}_0) + \left((sN)^{-1} \sum_{t=1}^{sN} \frac{\partial^2 l_t}{\partial \underline{\theta} \partial \underline{\theta}'} (\underline{\theta}^*) \right) (sN)^{\frac{1}{2}} \left(\widehat{\underline{\theta}}_{sN} - \underline{\theta}_0 \right)$$

where the coordinates of $\underline{\theta}^*$ are between the corresponding entries of $\widehat{\underline{\theta}}_{sN}$ and those of $\underline{\theta}_0$. We will thus show that

$$(sN)^{-\frac{1}{2}} \sum_{t=1}^{sN} \frac{\partial l_t}{\partial \underline{\theta}} (\underline{\theta}_0) \rightsquigarrow \mathcal{N}(\underline{0}, (\kappa - 1) J) \text{ as } N \to \infty \text{ and } \lim_{n \to \infty} \left((sN)^{-1} \sum_{t=1}^{sN} \frac{\partial^2 l_t}{\partial \underline{\theta} \partial \underline{\theta}'} (\underline{\theta}^*) \right) \overset{a.s.}{=} J.$$

The partial derivatives of $l_t(\underline{\theta})$ are given by

$$\begin{aligned} \frac{\partial l_t}{\partial \underline{\theta}} (\underline{\theta}) &= \frac{2}{h_t} \left(1 - \frac{\varepsilon_t^2}{h_t^2} \right) \frac{\partial h_t}{\partial \underline{\theta}}, \\ \frac{\partial^2 l_t}{\partial \underline{\theta} \partial \underline{\theta}'} (\underline{\theta}) &= \frac{2}{h_t} \left(1 - \frac{\varepsilon_t^2}{h_t^2} \right) \frac{\partial^2 h_t}{\partial \underline{\theta} \partial \underline{\theta}'} + \frac{2}{h_t^2} \left(3 \frac{\varepsilon_t^2}{h_t^2} - 1 \right) \frac{\partial h_t}{\partial \underline{\theta}} \frac{\partial h_t}{\partial \underline{\theta}'}. \end{aligned}$$

in which, in periodic notations

$$\left. \begin{aligned} \frac{\partial \underline{h}_{st+v}}{\partial \alpha_0(n)} &= \sum_{k \geq 0} \boldsymbol{\Omega}_v^{v-k+1} \underline{1} \mathbb{I}_{\{v-k \equiv n[s]\}} \\ \frac{\partial \underline{h}_{st+v}}{\partial \alpha_i(n)} &= \sum_{k \geq 0} \boldsymbol{\Omega}_v^{v-k+1} \varepsilon_{st+v-k-i}^{+} \underline{1} \mathbb{I}_{\{v-k \equiv n[s]\}} \\ \frac{\partial \underline{h}_{st+v}}{\partial \beta_i(n)} &= \sum_{k \geq 0} \boldsymbol{\Omega}_v^{v-k+1} \varepsilon_{st+v-k-i}^{-} \underline{1} \mathbb{I}_{\{v-k \equiv n[s]\}} \\ \frac{\partial \underline{h}_{st+v}}{\partial \gamma_j(n)} &= \sum_{k \geq 0} \left(\sum_{m=0}^{k-1} \boldsymbol{\Omega}_v^{v-m+1} \boldsymbol{\Omega}^{(j)} \mathbb{I}_{\{v-m \equiv n[s]\}} \boldsymbol{\Omega}_{v-m-1}^{v-k+1} \right) \underline{\varepsilon}_{st+v-k} \end{aligned} \right\} \tag{5.5}$$

where $\underline{1}$ denotes an $s \times 1$ unit vector whose entries are all zero except for a one in the $v^{th}-$row and $\boldsymbol{\Omega}^{(j)}$ is a $p \times p$ matrix with $(1, j)^{th}$ $-$element equal 1, and all other elements are equal to zero. Moreover,

$$\frac{\partial^2 h_{st+v}}{\partial \underline{\alpha}_0 \partial \underline{\alpha}_0'}, \frac{\partial^2 h_{st+v}}{\partial \underline{\alpha}_0 \partial \underline{\alpha}_i'}, \frac{\partial^2 h_{st+v}}{\partial \underline{\alpha}_0 \partial \underline{\beta}_i'}, \frac{\partial^2 h_{st+v}}{\partial \underline{\alpha}_i \partial \underline{\alpha}_j'}, \frac{\partial^2 h_{st+v}}{\partial \underline{\alpha}_i \partial \underline{\beta}_j'}, \frac{\partial^2 h_{st+v}}{\partial \underline{\beta}_i \partial \underline{\beta}_j'}$$

are null matrices and

$$\left. \begin{aligned} \frac{\partial^2 \underline{h}_{st+v}}{\partial \alpha_0(n) \partial \gamma_j(n_1)} &= \sum_{k \geq 0} \sum_{m=0}^{k-1} \boldsymbol{\Omega}_v^{v-m+1} \boldsymbol{\Omega}^{(j)} \boldsymbol{\Omega}_{v-m-1}^{v-k+1} \underline{1} \mathbb{I}_{\{v-k \equiv n[s]\}} \mathbb{I}_{\{v-m \equiv n_1[s]\}} \\ \frac{\partial^2 \underline{h}_{st+v}}{\partial \alpha_i(n) \partial \gamma_j(n_1)} &= \sum_{k \geq 0} \sum_{m=0}^{k-1} \boldsymbol{\Omega}_v^{v-m+1} \boldsymbol{\Omega}^{(j)} \boldsymbol{\Omega}_{v-m-1}^{v-k+1} \varepsilon_{st+v-k-i}^{+} \underline{1} \mathbb{I}_{\{v-k \equiv n[s]\}} \mathbb{I}_{\{v-m \equiv n_1[s]\}} \\ \frac{\partial^2 \underline{h}_{st+v}}{\partial \beta_i(n) \partial \gamma_j(n_1)} &= \sum_{k \geq 0} \sum_{m=0}^{k-1} \boldsymbol{\Omega}_v^{v-m+1} \boldsymbol{\Omega}^{(j)} \boldsymbol{\Omega}_{v-m-1}^{v-k+1} \varepsilon_{st+v-k-i}^{-} \underline{1} \mathbb{I}_{\{v-k \equiv n[s]\}} \mathbb{I}_{\{v-m \equiv n_1[s]\}}, \\ \frac{\partial^2 \underline{h}_{st+v}}{\partial \gamma_j(n) \partial \gamma_{j_1}(n_1)} &= \sum_{k \geq 0} \sum_{m=0}^{k-1} \sum_{\tau=0}^{m-1} \boldsymbol{\Omega}_v^{v-\tau+1} \boldsymbol{\Omega}^{(j_1)} \boldsymbol{\Omega}_{v-\tau-1}^{v-m+1} \boldsymbol{\Omega}^{(j)} \boldsymbol{\Omega}_{v-m-1}^{v-k+1} \underline{\varepsilon}_{st+v-k} \mathbb{I}_{\{v-m \equiv n[s]\}} \mathbb{I}_{\{v-\tau \equiv n_1[s]\}} \\ &+ \sum_{k \geq 0} \sum_{m=0}^{k-1} \sum_{\tau=m+1}^{k-1} \boldsymbol{\Omega}_v^{v-m+1} \boldsymbol{\Omega}^{(j)} \boldsymbol{\Omega}_{v-m-1}^{v-\tau+1} \boldsymbol{\Omega}^{(j_1)} \boldsymbol{\Omega}_{v-\tau-1}^{v-m+1} \underline{\varepsilon}_{st+v-k} \mathbb{I}_{\{v-m \equiv n[s]\}} \mathbb{I}_{\{v-\tau \equiv n_1[s]\}} \end{aligned} \right\} \tag{5.6}$$

Again, we will split the proof of theorem 4.2 into several intermediate results.

Lemma 5.2 *Under Conditions* $A0-A5$, *we have*

(a) $\sum\limits_{v=1}^{s} E_{\underline{\theta}_0}\left\{\sup\limits_{\underline{\theta}\in\Theta}\left\|\dfrac{\partial l_{st+v}}{\partial\underline{\theta}}(\underline{\theta}_0)\dfrac{\partial l_{st+v}}{\partial\underline{\theta}'}(\underline{\theta}_0)\right\|\right\}<\infty$ and $\sum\limits_{v=1}^{s} E_{\underline{\theta}_0}\left\{\sup\limits_{\underline{\theta}\in\Theta}\left\|\dfrac{\partial^2 l_{st+v}}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\theta}_0)\right\|\right\}<\infty.$

(b) J is invertible and $\sum\limits_{v=1}^{s} Var_{\underline{\theta}_0}\left\{\dfrac{\partial l_{st+v}}{\partial\underline{\theta}}(\underline{\theta}_0)\right\}=(\kappa-1)J$

(c) There is a neighborhood $\mathcal{V}(\underline{\theta}_0)$ of $\underline{\theta}_0$ such that $\sum\limits_{v=1}^{s} E_{\underline{\theta}_0}\left\{\sup\limits_{\underline{\theta}\in\mathcal{V}(\underline{\theta}_0)}\left\|\dfrac{\partial^3 l_{st+v}}{\partial\theta_i\partial\theta_j\partial\theta_k}(\underline{\theta}_0)\right\|\right\}<\infty$, for all $i,j,k\in\{1,...,$ $s(1+2q+p)\}$

(d) The following limit hold true

$$p\lim\left\|N^{-\frac{1}{2}}\sum_{t=1}^{N}\sum_{v=1}^{s}\left(\frac{\partial\widetilde{l}_{st+v}}{\partial\underline{\theta}}(\underline{\theta}_0)-\frac{\partial l_{st+v}}{\partial\underline{\theta}}(\underline{\theta}_0)\right)\right\|=0,$$
$$p\lim\sup_{\underline{\theta}\in\mathcal{V}(\underline{\theta}_0)}\left\|N^{-1}\sum_{t=1}^{N}\sum_{v=1}^{s}\left(\frac{\partial^2\widetilde{l}_{st+v}}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\theta}_0)-\frac{\partial^2 l_{st+v}}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\theta}_0)\right)\right\|=0,$$

(e) $(sN)^{-\frac{1}{2}}\sum\limits_{t=1}^{sN}\frac{\partial l_t}{\partial\underline{\theta}}(\underline{\theta}_0)\rightsquigarrow\mathcal{N}(\underline{O},(\kappa-1)J)$ as $N\to\infty$ and almost surely $\lim\limits_{n\to\infty}\left((sN)^{-1}\sum\limits_{t=1}^{sN}\frac{\partial^2 l_t}{\partial\underline{\theta}\partial\underline{\theta}'}\left(\widetilde{\underline{\theta}}\right)\right)=J.$

Proof. To prove (a), it is sufficient to show that

$$\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{\left\|\frac{1}{h_{st+v}}\frac{\partial\underline{h}_{st+v}}{\partial\underline{\theta}}(\underline{\theta}_0)\right\|\right\}<\infty,$$
$$\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{\left\|\frac{1}{h_{st+v}}\frac{\partial^2\underline{h}_{st+v}}{\partial\underline{\theta}\partial\underline{\theta}'}(\underline{\theta}_0)\right\|\right\}<\infty,$$
$$\sum_{v=1}^{s}E_{\underline{\theta}_0}\left\{\left\|\frac{1}{h_{st+v}^2}\frac{\partial\underline{h}_{st+v}}{\partial\underline{\theta}}(\underline{\theta}_0)\frac{\partial\underline{h}_{st+v}}{\partial\underline{\theta}'}(\underline{\theta}_0)\right\|\right\}<\infty.$$

Now, by the positivity of the coefficients in (5.5) and in (5.6), the derivatives of h_t are non-negatives. It's clear that $\dfrac{\partial h_{st+v}}{\partial\alpha_0(n)}$ is bounded. Since $h_t\geq\underline{a}_0=\inf\limits_{v\in\mathbb{S}}a_0(v)$, then $\sum\limits_{v=1}^{s}E_{\underline{\theta}_0}\left\{\left\|\frac{1}{h_{st+v}}\frac{\partial h_{st+v}}{\partial\alpha_0(n)}(\underline{\theta}_0)\right\|\right\}<\infty$ and

$$\max\left(\alpha_i(n)\frac{\partial\underline{h}_{st+v}}{\partial\alpha_i(n)},\beta_i(n)\frac{\partial\underline{h}_{st+v}}{\partial\beta_i(n)}\right)\leq\underline{h}_{st+v}\text{ and }\gamma_j(n)\frac{\partial\underline{h}_{st+v}}{\partial\gamma_j(n)}\leq\sum_{k\geq1}k\mathbf{\Omega}_v^{v-k+1}\underline{\varepsilon}_{st+v-k}\text{ for all }i,j,v,$$

from which we deduce $\sum\limits_{v=1}^{s}E_{\underline{\theta}_0}\left\{\left\|\frac{1}{h_{st+v}}\frac{\partial h_{st+v}}{\partial\alpha_i(n)}(\underline{\theta}_0)\right\|\right\}<\infty$ and $\sum\limits_{v=1}^{s}E_{\underline{\theta}_0}\left\{\left\|\frac{1}{h_{st+v}}\frac{\partial h_{st+v}}{\partial\beta_i(n)}(\underline{\theta}_0)\right\|\right\}<\infty$. Using $(A.2)$, we have $\left\|(\mathbf{\Omega}_1^s)^k\right\|\leq K\mu^k$ for all k with $K>0$ and $\mu\in]0;1[$. Using the inequality $(a+b)^\delta\leq a^\delta+b^\delta$ for all $a,b\geq0$, we deduce that $\underline{1}'\underline{\varepsilon}_{st+v}$ has a moment of order δ, for some $\delta \in]0,1[$. By the inequality $x \leq (1+x)\, x^{\delta}$ for all $x \geq 0$, we obtain

$$\begin{aligned} E_{\underline{\theta}_0}\left\{\left\|\frac{1}{h_{st+v}}\frac{\partial h_{st+v}}{\partial \gamma_j(n)}\left(\underline{\theta}_0\right)\right\|\right\} &\leq \frac{1}{\gamma_j(n)}E_{\underline{\theta}_0}\left\{\sum_{k\geq 1} k\left\|\frac{\left(\mathbf{\Omega}_v^{v-k+1}\underline{\varepsilon}_{st+v-k}\right)(1)}{\alpha_0(v)+\left(\mathbf{\Omega}_v^{v-k+1}\underline{\varepsilon}_{st+v-k}\right)(1)}\right\|\right\} \\ &\leq \frac{1}{\gamma_j(n)\,\alpha_0^{\delta}(v)}\sum_{k\geq 1} kE_{\underline{\theta}_0}\left\{\left\|\left(\mathbf{\Omega}_v^{v-k+1}\underline{\varepsilon}_{st+v-k}\right)^{\delta}(1)\right\|\right\} \\ &\leq \frac{K^{\delta}}{\gamma_j(n)\,\alpha_0^{\delta}(v)}\sum_{k\geq 1} k\mu^{\delta k}E_{\underline{\theta}_0}\left\{\left\|\left(\underline{1}'\underline{\varepsilon}_{st+v-k}\right)^{\delta}\right\|\right\} \leq K. \end{aligned} \tag{5.7}$$

Let us now turn to the second derivatives of h_t. It follows that

$$\left.\begin{aligned} \gamma_j(n_1)\,\tfrac{\partial^2 \underline{h}_{st+v}}{\partial\alpha_0(n)\partial\gamma_j(n_1)} &\leq \sum_{k\geq 1} k\mathbf{\Omega}_v^{v-k+1}\underline{1} \text{ for all } j,v \\ \max\left(\alpha_i(n)\,\gamma_j(n_1)\,\tfrac{\partial^2 \underline{h}_{st+v}}{\partial\alpha_i(n)\partial\gamma_j(n_1)},\beta_i(n)\,\gamma_j(n_1)\,\tfrac{\partial^2 \underline{h}_{st+v}}{\partial\beta_i(n)\partial\gamma_j(n_1)}\right) &\leq \sum_{k\geq 1} k\mathbf{\Omega}_v^{v-k+1}\underline{\varepsilon}_{st+v-k} \text{ for all } i,j,v \\ \gamma_j(n)\,\gamma_{j_1}(n_1)\,\tfrac{\partial^2 \underline{h}_{st+v}}{\partial\gamma_j(n)\partial\gamma_{j_1}(n_1)} &\leq \sum_{k\geq 2} k(k-1)\,\mathbf{\Omega}_v^{v-k+1}\underline{\varepsilon}_{st+v-k} \text{ for all } j,j_1,v \end{aligned}\right\}$$

Using the same arguments as for (5.7), we can conclude that $\dfrac{1}{h_{st+v}}\dfrac{\partial^2 \underline{h}_{st+v}}{\partial\underline{\theta}\partial\underline{\theta}'}\left(\underline{\theta}_0\right)$ is integrable. The proof of assertion **(b)** follows essentially the same arguments as in Francq and Zakoian [12]. To proof **(c)**, we have $\frac{1}{2}\dfrac{\partial^3 l_t(\underline{\theta})}{\partial\theta_i\partial\theta_j\partial\theta_k} = I_1 + I_2 + I_3$ where

$$\begin{aligned} I_1 &:= \left(1-\frac{\varepsilon_t^2}{h_t^2}\right)\left(\frac{1}{h_t}\frac{\partial^3 h_t}{\partial\theta_i\partial\theta_j\partial\theta_k}\right), \\ I_2 &:= \left(3\frac{\varepsilon_t^2}{h_t^2}-1\right)\left(\frac{1}{h_t}\frac{\partial h_t}{\partial\theta_i}\right)\left(\frac{1}{h_t}\frac{\partial^2 h_t}{\partial\theta_j\partial\theta_k}\right)+\left(3\frac{\varepsilon_t^2}{h_t^2}-1\right)\left(\frac{1}{h_t}\frac{\partial h_t}{\partial\theta_j}\right)\left(\frac{1}{h_t}\frac{\partial^2 h_t}{\partial\theta_i\partial\theta_k}\right)+\left(3\frac{\varepsilon_t^2}{h_t^2}-1\right)\left(\frac{1}{h_t}\frac{\partial h_t}{\partial\theta_k}\right)\left(\frac{1}{h_t}\frac{\partial^2 h_t}{\partial\theta_i\partial\theta_j}\right), \\ I_3 &:= 2\left(1-6\frac{\varepsilon_t^2}{h_t^2}\right)\left(\frac{1}{h_t}\frac{\partial h_t}{\partial\theta_i}\right)\left(\frac{1}{h_t}\frac{\partial h_t}{\partial\theta_j}\right)\left(\frac{1}{h_t}\frac{\partial h_t}{\partial\theta_k}\right). \end{aligned}$$

First, we will show that $\dfrac{\varepsilon_t^2}{h_t^2}$ is uniformly integrable in a neighborhood of $\underline{\theta}_0$. Let Θ^* be a compact included in $\dot{\Theta}$ such that $\underline{\theta}_0 \in \Theta^*$. Denote by $\mathbf{\Omega}_t$ the matrix $\mathbf{\Omega}_t$ evaluated at $\underline{\theta} = \underline{\theta}_0$. For all $r > 0$, there exists a neighborhood $\mathcal{V}\left(\underline{\theta}_0\right)$ of $\underline{\theta}_0$, with $\mathcal{V}\left(\underline{\theta}_0\right) \subseteq \Theta^*$, such that $\mathbf{\Omega}_t \leq (1+r)\,\mathbf{\Omega}_t$ for all $\underline{\theta} \in \mathcal{V}\left(\underline{\theta}_0\right)$. From (5.4), we have

$$h_{st+v} = \sum_{k\geq 0}\mathbf{\Omega}_v^{v-k+1}(1,1)\,\alpha_0(v-k)+\sum_{i=1}^{q}\sum_{k\geq 0}\mathbf{\Omega}_v^{v-k+1}(1,1)\,\alpha_i(v-k)\,\varepsilon_{st+v-k-i}^{+}+\sum_{i=1}^{q}\sum_{k\geq 0}\mathbf{\Omega}_v^{v-k+1}(1,1)\,\beta_i(v-k)\,\varepsilon_{st+v-k-i}^{-}.$$

Following the same argument as in (**a**) we have

$$\sup_{\underline{\theta}\in\mathcal{V}\left(\underline{\theta}_0\right)}\frac{h_t\left(\underline{\theta}_0\right)}{h_t} \leq K + K\sum_{i=1}^{q}\sum_{k\geq 0}(1+r)^k\,\mu^{k\delta}\left|\varepsilon_{t-k-i}\right|^{\delta}$$

the Minkowski inequality entails

$$E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in\mathcal{V}\left(\underline{\theta}_0\right)}\frac{h_t^2\left(\underline{\theta}_0\right)}{h_t^2}\right\} \leq \left(K + Kq\sum_{k\geq 0}(1+r)^k\,\mu^{k\delta}E\left\{\left|\varepsilon_t\right|^{\delta}\right\}\right)^2 < \infty,$$

we obtain $E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in\mathcal{V}(\underline{\theta}_0)}\dfrac{\varepsilon_t^2}{h_t^2}\right\}<\infty$. On the other hand, we find $\frac{\partial^2 \underline{h}_{st+v}}{\partial\gamma_j(n)\partial\gamma_{j_1}(n_1)}$

$$\left.\begin{array}{r}\gamma_j(n)\gamma_{j_1}(n_1)\gamma_{j_2}(n_2)\frac{\partial^3 \underline{h}_{st+v}}{\partial\gamma_j(n)\partial\gamma_{j_1}(n_1)\partial\gamma_{j_2}(n_2)}\leq\sum\limits_{k\geq 3}k(k-1)(k-2)\mathbf{\Omega}_v^{v-k+1}\underline{\varepsilon}_{st+v-k},\\ \gamma_j(n)\gamma_{j_1}(n_1)\ \max\left(\frac{\partial^3 \underline{h}_{st+v}}{\partial\alpha_0(n)\partial\gamma_j(n)\partial\gamma_{j_1}(n_1)},\frac{\partial^3 \underline{h}_{st+v}}{\partial\alpha_i(n)\partial\gamma_j(n)\partial\gamma_{j_1}(n_1)},\frac{\partial^3 \underline{h}_{st+v}}{\partial\beta_i(n)\partial\gamma_j(n)\partial\gamma_{j_1}(n_1)}\right)\leq\sum\limits_{k\geq 2}k(k-1)\mathbf{\Omega}_v^{v-k+1}\underline{\varepsilon}_{st+v-k},\end{array}\right\}$$

for all i,j,j_1,j_2,v and others are null. Therefore, using a similar argument as in (5.7), we can show that

$$\sup_{\underline{\theta}\in\Theta^*}\frac{1}{h_{st+v}}\frac{\partial^3 h_{st+v}}{\partial\beta_i(n)\partial\gamma_j(n)\partial\gamma_{j_1}(n_1)}\leq K\sum_{k\geq 2}k(k-1)\mu^{k\delta}\sup_{\underline{\theta}\in\Theta^*}\left(\underline{1}'\underline{\varepsilon}_{st+v-k}\right)^{\delta},$$
$$\sup_{\underline{\theta}\in\Theta^*}\frac{1}{h_{st+v}}\frac{\partial^3 h_{st+v}}{\partial\gamma_j(n)\partial\gamma_{j_1}(n_1)\partial\gamma_{j_2}(n_2)}\leq K\sum_{k\geq 3}k(k-1)(k-2)\mu^{k\delta}\sup_{\underline{\theta}\in\Theta^*}\left(\underline{1}'\underline{\varepsilon}_{st+v-k}\right)^{\delta},$$

since $E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in\Theta^*}\left(\underline{1}'\underline{\varepsilon}_{st+v-k}\right)^{2\delta}\right\}<\infty$ for some $\delta>0$, we then have $E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in\Theta^*}\left|\frac{1}{h_t}\dfrac{\partial^3 h_t}{\partial\theta_i\partial\theta_j\partial\theta_k}\right|^2\right\}<\infty$. By the Cauchy–Schwarz inequality, we get

$$E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in\mathcal{V}(\underline{\theta}_0)}\left\{\left|\left(1-\frac{\varepsilon_t^2}{h_t^2}\right)\frac{1}{h_t}\frac{\partial^3 h_t}{\partial\theta_i\partial\theta_j\partial\theta_k}\right|\right\}\right\}<\infty.$$

To deal with the other terms of the sum in $\dfrac{\partial^3 l_t(\underline{\theta})}{\partial\theta_i\partial\theta_j\partial\theta_k}$ we show that, $E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in\Theta^*}\left\{\left|\frac{1}{h_t}\dfrac{\partial^2 h_t}{\partial\theta_i\partial\theta_j}\right|^{\tau}\right\}\right\}<\infty$ and $E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in\Theta^*}\left\{\left|\frac{1}{h_t}\dfrac{\partial h_t}{\partial\theta_i}\right|^{\tau}\right\}\right\}<\infty$ for any integer τ using Hölder inequality. To show (**d**), we use (5.3) to obtain the following results,

$$\begin{aligned}\frac{\partial\underline{h}_{st+v}}{\partial\theta_i(n)}-\frac{\partial\underline{\widetilde{h}}_{st+v}}{\partial\theta_i(n)}&=\sum_{k=0}^{\left[\frac{q}{s}\right]-1}\sum_{d=1}^{s}\left(\mathbf{\Omega}_v^{d+1}\right)^{t-k}\left(\frac{\partial\underline{\varepsilon}_{sk+d}}{\partial\theta_i(n)}-\frac{\partial\underline{\widetilde{\varepsilon}}_{sk+d}}{\partial\theta_i(n)}\right)+\left(\mathbf{\Omega}_v^{1}\right)^{t}\left(\frac{\partial\underline{h}_0}{\partial\theta_i(n)}-\frac{\partial\underline{\widetilde{h}}_0}{\partial\theta_i(n)}\right),\\ \frac{\partial\underline{h}_{st+v}}{\partial\gamma_i(n)}-\frac{\partial\underline{\widetilde{h}}_{st+v}}{\partial\gamma_i(n)}&=\sum_{m=0}^{t}\left(\mathbf{\Omega}_v^{1}\right)^{m}\mathbf{\Omega}_v^{n+1}\mathbf{\Omega}^{(i)}\mathbf{\Omega}_{n-1}^{1}\left(\mathbf{\Omega}_v^{1}\right)^{t-m-1}\left(\underline{h}_0-\underline{\widetilde{h}}_0\right)\\ &\quad+\sum_{k=0}^{\left[\frac{q}{s}\right]-1}\sum_{d=1}^{s}\sum_{m=0}^{tk}\left(\mathbf{\Omega}_v^{d+1}\right)^{m}\mathbf{\Omega}_v^{n+1}\mathbf{\Omega}^{(i)}\mathbf{\Omega}_{n-1}^{d+1}\left(\mathbf{\Omega}_v^{d+1}\right)^{t-m-1}\left(\underline{\varepsilon}_{sk+d}-\underline{\widetilde{\varepsilon}}_{sk+d}\right).\end{aligned}$$

Therefore, a.s.

$$\sup_{\underline{\theta}\in\Theta}\left|\frac{\partial h_{st+v}}{\partial\theta_i(n)}-\frac{\partial\widetilde{h}_{st+v}}{\partial\theta_i(n)}\right|\leq\sup_{\underline{\theta}\in\Theta}\left|\frac{\partial\underline{h}_{st+v}}{\partial\theta_i(n)}-\frac{\partial\underline{\widetilde{h}}_{st+v}}{\partial\theta_i(n)}\right|\leq Ku^{st+v}$$

and by Borel–Cantelli's lemma and Markov's inequality we have, a.s.

$$\sup_{\underline{\theta}\in\Theta}\left|\frac{\partial h_{st+v}}{\partial\gamma_i(n)}-\frac{\partial\widetilde{h}_{st+v}}{\partial\gamma_i(n)}\right|\leq\sup_{\underline{\theta}\in\Theta}\left|\frac{\partial\underline{h}_{st+v}}{\partial\theta_i(n)}-\frac{\partial\underline{\widetilde{h}}_{st+v}}{\partial\theta_i(n)}\right|\leq Ku^{st+v}$$

The second-order derivatives can be treated similarly. Therefore, a.s. $\sup_{\underline{\theta}\in\Theta}\left|\dfrac{\partial^2 h_{st+v}}{\partial\underline{\theta}\partial\underline{\theta}'}-\dfrac{\partial^2\widetilde{h}_{st+v}}{\partial\underline{\theta}\partial\underline{\theta}'}\right|<Ku^{st+v}$ and

$$\left|\frac{1}{\widetilde{h}_{st+v}}-\frac{1}{h_{st+v}}\right|\leq\frac{Ku^{st+v}}{h_{st+v}},\frac{h_{st+v}^2}{\widetilde{h}_{st+v}^2}\leq Kand\left|\frac{h_{st+v}^2}{\widetilde{h}_{st+v}^2}-1\right|\leq Ku^{st+v}$$

So, we obtain

$$\left|\frac{\partial \widetilde{l}_{st+v}}{\partial \theta_i}(\underline{\theta}_0)-\frac{\partial l_{st+v}}{\partial \theta_i}(\underline{\theta}_0)\right| = 2\left|\left(\frac{\varepsilon^2_{st+v}}{\widetilde{h}^2_{st+v}}-\frac{\varepsilon^2_{st+v}}{h^2_{st+v}}\right)\frac{1}{h_{st+v}}\frac{\partial h_{st+v}}{\partial \theta_i}+\left(1-\frac{\varepsilon^2_{st+v}}{h^2_{st+v}}\right)\left(\frac{1}{h_{st+v}}-\frac{1}{\widetilde{h}_{st+v}}\right)\frac{\partial h_{st+v}}{\partial \theta_i}\right.$$
$$\left.+\left(1-\frac{\varepsilon^2_{st+v}}{\widetilde{h}^2_{st+v}}\right)\frac{1}{\widetilde{h}_{st+v}}\left(\frac{\partial h_{st+v}}{\partial \theta_i}-\frac{\partial \widetilde{h}_{st+v}}{\partial \theta_i}\right)\right|(\underline{\theta}_0)$$
$$\leq Ku^{st+v}\left(1+Ke^2_{st+v}\right)\left\{1+\frac{1}{h_{st+v}(\underline{\theta}_0)}\frac{\partial h_{st+v}}{\partial \theta_i}(\underline{\theta}_0)\right\}.$$

The Markov inequality, assertion (**a**), and the independence between e_t and $h_t(\underline{\theta}_0)$ entail that, for all $\sigma > 0$

$$P\left(N^{-\frac{1}{2}}\sum_{t=1}^{N}\sum_{v=1}^{s}u^{st+v}\left(1+Ke^2_{st+v}\right)\left\{1+\frac{1}{h_{st+v}(\underline{\theta}_0)}\frac{\partial h_{st+v}}{\partial \theta_i}(\underline{\theta}_0)\right\}>\sigma\right)\longrightarrow 0.$$

The second part of assertion (**d**) follows in a similar fashion. To prove (**e**) we apply a central limit theorem for martingale differences, since $E_{\underline{\theta}_0}\left\{\frac{\partial l_t}{\partial \underline{\theta}}(\underline{\theta}_0)\middle|\mathcal{F}_{t-1}\right\}=0$ and $Var_{\underline{\theta}_0}\left\{\frac{\partial l_t}{\partial \underline{\theta}}(\underline{\theta}_0)\middle|\mathcal{F}_{t-1}\right\}$ exists. Hence for any $\underline{\lambda}\in\mathbb{R}^r$, the sequence $\left\{\underline{\lambda}'\frac{\partial l_t}{\partial \underline{\theta}}(\underline{\theta}_0),\mathcal{F}_t\right\}$ is a square-integrable stationary martingale difference. The central limit theorem of Billingsley [4] and the Wold–Cramér device allow us to derive the asymptotic normality result. Moreover, by Taylor series expansion for the second-order derivatives about $\underline{\theta}_0$, we have for all i and j,

$$N^{-1}\sum_{t=1}^{sN}\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}(\underline{\theta}^*_{ij})=N^{-1}\sum_{t=1}^{sN}\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}(\underline{\theta}_0)+N^{-1}\sum_{t=1}^{sN}\frac{\partial}{\partial\underline{\theta}'}\left(\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}\left(\widetilde{\underline{\theta}}_{ij}\right)\right)(\underline{\theta}^*_{ij}-\underline{\theta}_0)$$

for some for some random vector $\widetilde{\underline{\theta}}_{ij}$ such that almost surely $\left\|\underline{\theta}_0-\widetilde{\underline{\theta}}_{ij}\right\|\leq\left\|\underline{\theta}_0-\underline{\theta}^*_{ij}\right\|\leq\left\|\underline{\theta}_0-\widehat{\underline{\theta}}_{sN}\right\|$. From the strong consistency of $\widehat{\underline{\theta}}_{sN}$, the periodic ergodicity and assertion (**c**) we have, almost surely,

$$\limsup_{N\to\infty}\left\|(sN)^{-1}\sum_{t=1}^{sN}\frac{\partial}{\partial\underline{\theta}'}\left(\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}\left(\widetilde{\underline{\theta}}_{ij}\right)\right)\right\|\leq\limsup_{N\to\infty}(sN)^{-1}\sum_{t=1}^{sN}\sup_{\underline{\theta}\in\mathcal{V}(\underline{\theta}_0)}\left\|\frac{\partial}{\partial\underline{\theta}'}\left(\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}(\underline{\theta})\right)\right\|$$
$$=E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in\mathcal{V}(\underline{\theta}_0)}\left\|\frac{\partial}{\partial\underline{\theta}'}\left(\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}(\underline{\theta})\right)\right\|\right\}<\infty.$$

Therefor, since $\left\|\widehat{\underline{\theta}}_{sN}-\underline{\theta}_0\right\|\longrightarrow 0$ a.s. as $N\to\infty$, the second term $N^{-1}\sum_{t=1}^{sN}\frac{\partial}{\partial\underline{\theta}'}\left(\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}\left(\widetilde{\underline{\theta}}_{ij}\right)\right)(\underline{\theta}^*_{ij}-\underline{\theta}_0)$ converges a.s. to 0 and the first one converges to J . To complete the proof of Theorem 4.2 it suffices to apply the Slutsky lemma.■

Proof of Theorem 4.3

This theorem is a consequence of proving the same intermediate results (**i**)–(**iv**) in Lemma 5.1 linked with Theorem 4.1. The *i.i.d.*assumption on (e_t) was only used in the proof of (**ii**) and (**iii**). The proof of (**ii**) is the same but is replaced $a(v)\,e^+_{st+v-k}+b(v)\,e^-_{st+v-k}=c(e_{st+v-k-1},e_{st+v-k-2},...)$, *a.s.*, where $c(e_{t-k-1},e_{t-k-2},...)$ is a measurable function of the e_{t-u}, $u>k$. Thus

$$|a(v)|\,e^+_{st+v-k}+|b(v)|\,e^-_{st+v-k}=|c(e_{st+v-k-1},e_{st+v-k-2},...)|\,,a.s. \tag{5.8}$$

and then $|a(v)|\,E\left\{e^+_{st+v-k}\middle|\mathcal{F}_{st+v-1}\right\}+|b(v)|\,E\left\{e^-_{st+v-k}\ \mathcal{F}_{st+v-1}|\right\}=|c(e_{st+v-k-1},e_{st+v-k-2},...)|$, *a.s.* Under Assumption 2, the process $|c(e_{t-k-1},e_{t-k-2},...)|$ is constant. Therefore, (5.8) is in contradiction with $A3$, unless $a(v)=b(v)=0$

and hence (ii) follows. The proof of (**iii)** followed essentially from the fact that

$$E_{\underline{\theta}_0}\left\{l_t\left(\underline{\theta}_0\right)\right\}=E_{\underline{\theta}_0}\left\{\frac{h_t^2\left(\underline{\theta}_0\right)e_t^2}{h_t^2\left(\underline{\theta}\right)}+\log h_t^2\left(\underline{\theta}_0\right)\right\}=1+E_{\underline{\theta}_0}\left\{\log h_t^2\left(\underline{\theta}_0\right)\right\}<\infty.$$

The second equality remains valid when the independence of (e_t) is replaced by Assumption 2. The proof of Theorem 4.3 is now complete.■

Proof of Theorem 4.4

We follow the scheme of proof of Theorem 4.2, using similar arguments as in Escanciano (2009). We start by proving (**a**). In view of first derivative of $\frac{\partial l_t}{\partial \underline{\theta}}$, then by Hölder inequality and the fact that the norm is multiplicative imply that

$$\begin{aligned}&\left(E_{\underline{\theta}_0}\left\{\left\|\frac{\partial l_{st+v}}{\partial \underline{\theta}}\left(\underline{\theta}_0\right)\frac{\partial l_{st+v}}{\partial \underline{\theta}'}\left(\underline{\theta}_0\right)\right\|\right\}\right)^{2(1+\tau)/\tau}\\&\leq 4\left(E_{\underline{\theta}_0}\left\{\left|1-e_{st+v}^2\right|^{2(1+\tau)}\right\}\right)^{2/\tau}E_{\underline{\theta}_0}\left\{\left\|\frac{1}{h_{st+v}}\frac{\partial h_{st+v}}{\partial \underline{\theta}}\right\|^{2(1+\tau)/\tau}\right\}E_{\underline{\theta}_0}\left\{\left\|\frac{1}{h_{st+v}}\frac{\partial h_{st+v}}{\partial \underline{\theta}'}\right\|^{2(1+\tau)/\tau}\right\}.\end{aligned}$$

Note that under Assumption 2, $A.6$ and $A.7$, we conclude that the first expectation in (**a**) exists. Using the same argument we prove the second assertion in (**a**). The proof of the non-singularity of I is similar to the one given in Theorem 4.2, with the argument used in Theorem 4.3. By the first derivative of $\frac{\partial l_t}{\partial \underline{\theta}}$ and the law of iterated expectations we prove the second part of (**b**). The proof of (**c**). it is sufficient to prove the existence of $E_{\underline{\theta}_0}\left\{\sup\limits_{\underline{\theta}\in\mathcal{V}(\underline{\theta}_0)}\frac{\varepsilon_{st+v}^4}{h_{st+v}^4}\right\}$ where $\mathcal{V}\left(\underline{\theta}_0\right)$ is a neighborhood of $\underline{\theta}_0$, defined in the proof of Theorem 4.2. Then by Hölder inequality, we have

$$E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in\mathcal{V}(\underline{\theta}_0)}\frac{\varepsilon_{st+v}^4}{h_{st+v}^4}\right\}\leq\left(E_{\underline{\theta}_0}\left\{\left|e_{st+v}\right|^{4(1+\tau)}\right\}\right)^{1/(1+\tau)}\left(E_{\underline{\theta}_0}\left\{\sup_{\underline{\theta}\in\mathcal{V}(\underline{\theta}_0)}\left|\frac{h_{st+v}\left(\underline{\theta}_0\right)}{h_{st+v}}\right|^{(1+\tau)/\tau}\right\}\right)^{\tau/(1+\tau)},\tag{5.9}$$

where $\tau>0$. Under the Assumption 2, $A.6$ and by using the same argument as in Theorem 4.2, we prove the existence of the second expectation in the right-hand side of the inequality (5.9). Finally, using the assumption $A.7$ the conclusion follows. To show (**d**), we have by Markov inequality, the Assumption 2, $A.6$ and $A.7$

$$\begin{aligned}&P\left(N^{-\frac{1}{2}}\sum_{t=1}^{N}\sum_{v=1}^{s}u^{st+v}\left(1+Ke_{st+v}^2\right)\left\{1+\frac{1}{h_{st+v}\left(\underline{\theta}_0\right)}\frac{\partial h_{st+v}}{\partial \underline{\theta}}\left(\underline{\theta}_0\right)\right\}>\sigma\right)\\&\leq\sigma^{-1}N^{-\frac{1}{2}}\sum_{t=1}^{N}\sum_{v=1}^{s}u^{st+v}E\left\{E\left\{\left(1+Ke_{st+v}^2\right)\middle|\mathcal{F}_{st+v-1}\right\}\left\{1+\frac{1}{h_{st+v}\left(\underline{\theta}_0\right)}\frac{\partial h_{st+v}}{\partial \underline{\theta}}\left(\underline{\theta}_0\right)\right\}\right\}\\&\leq\sigma^{-1}\left(1+K\right)N^{-\frac{1}{2}}\sum_{t=1}^{N}\sum_{v=1}^{s}u^{st+v}\left(1+E_{\underline{\theta}_0}\left\{\frac{1}{h_{st+v}\left(\underline{\theta}_0\right)}\frac{\partial h_{st+v}}{\partial \underline{\theta}}\left(\underline{\theta}_0\right)\right\}\right)\longrightarrow 0.\end{aligned}$$

for all $\sigma>0$. The rest of proof is essentially the same as in Francq and Zakoian [12].

6 Empirical evidence

Now, we illustrate the $QMLE$ described in previous sections (at least for a moderate periodicity coefficients $s=5$ say), we provide some numerical results from Monte Carlo experiment. We simulate 500 independent trajectories via some specifications of $PTGARCH_s\,(1,1)$ models with length $n \in\{1000, 3000\}$ with standard $\mathcal{N}(0,1)$ and $MA(1)$ as innovations distributions, with vector $\underline{\theta}$ of parameters described in the bottom of each table below which is chosen to satisfy the SPS condition (3.1). For each trajectory, the vector $\underline{\theta}$ has been estimated with QML noted as $\widehat{\underline{\theta}}_n$. The $QMLE$ algorithm has been executed for these series under the $MATLAB8$ using $fminsearch.m$ as a minimizer function. In Tables below, the columns are correspond to the average of the parameters estimates over the 500 simulations. In order to show the performance of $QMLE$, the roots mean square error $(RMSE)$ of the each $\widehat{\underline{\theta}}_n\,(v)$, $v = 1, ...s$, (results between bracket), are reported in each table. The asymptotic distributions of $\widehat{\underline{\theta}}_n\,(v)$, $v = 1, ..., s$ over 500 simulations followed by their box plot summary are plotted after each appropriate tables.

6.1 Standard $TGARCH$ models

The first example illustrating our theoretical analysis is the standard $TGARCH\,(1,1)$ model, its vector of parameter is $\underline{\theta} = (\alpha_0, \alpha_1, \beta_1, \gamma_1)'$, chosen to subject the condition $\gamma_L = E\left\{\log\left\{\alpha_1 e_0^+ + \beta_1 e_0^- + \gamma_1\right\}\right\} < 0$ The results of simulation according to two designs for $\underline{\theta}$ are given in table(1)

	$\mathcal{N}(0,1)$		$MA(1)$	
Parameters	1000	3000	1000	3000
$\widehat{\alpha}_0$	1.0108 (0.0813)	0.9983 (0.0852)	1.0196 (0.0821)	1.0075 (0.0859)
$\widehat{\alpha}_1$	0.4965 (0.0555)	0.4984 (0.0318)	0.5080 (0.0552)	0.5098 (0.0314)
$\widehat{\beta}_1$	0.2460 (0.0464)	0.2487 (0.0263)	0.2577 (0.0453)	0.2543 (0.0257)
$\widehat{\gamma}_1$	0.1467 (0.0513)	0.1515 (0.0519)	0.1409 (0.0502)	0.1455 (0.0501)
Design (1): $\underline{\theta} = (1.0, 0.5, 0.25, 0.15)'$				
$\widehat{\alpha}_0$	0.9670 (0.0835)	0.9760 (0.0815)	0.9702 (0.0806)	0.9900 (0.0804)
$\widehat{\alpha}_1$	0.4886 (0.0867)	0.4964 (0.0332)	0.5002 (0.0599)	0.5001 (0.0327)
$\widehat{\beta}_1$	0.2470 (0.0983)	0.2426 (0.0275)	0.2536 (0.0487)	0.2509 (0.0267)
$\widehat{\gamma}_1$	0.0080 (0.0245)	0.0064 (0.0084)	0.0011 (0.0091)	0.0006 (0.0071)
Design (2): $\underline{\theta} = (1.0, 0.5, 0.25, 0.0)'$				

Table(1); Average and $RMSE$ of 500 simulations of $QMLE$ for $TGARCH\,(1,1)$

The asymptotic distribution of the sequence $\left(\sqrt{n}(\widehat{\theta}_n(i) - \theta(i))\right)_{n\geq 1}$, $i = 1, ..., 4$ followed by their box plot summary are shown in figure $Fig2$.

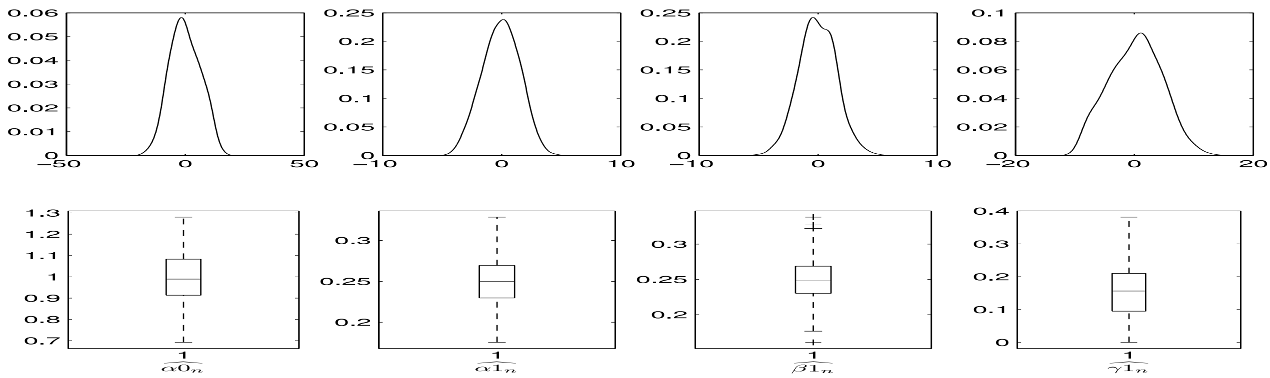


Fig2. Top panels: the asymptotic distribution of$\sqrt{n}(\widehat{\theta}_n(i) - \theta(i))$. Bottom panels: Box plot summary of $\widehat{\theta}_n(i)$, $i = 1, ..., 4$, according to design(1) of table(2)

6.2 $PTGARCH_s\,(1,1)$

The second example of our Monte Carlo experiment is devoted to estimates the periodic $TGARCH_s\,(1,1)$ with $s = 5$ and two regimes. i.e., $\varepsilon_t = h_t e_t$ and

$$h_n = \alpha_0\,(n) + \alpha_1\,(n)\,\varepsilon_{n-1}^{+} + \beta_1\,(n)\,\varepsilon_{n-1}^{-} + \gamma_1\,(n)\,h_{n-1}$$

in which $\alpha_0\,(n) = \alpha_0\,(1)\,\mathbb{I}_{\{n-5[n/5]\leq 3\}} + \alpha_0\,(2)\,\mathbb{I}_{\{n-5[n/5]>3\}}$ and similar definition for the others coefficients ($[x]$ denoting the integer part of x). This situation is raised in modelling some daily returns when we suspect the so-called "Monday effect" (opening price) of day-of-the week seasonality (see for instance Franses and Raap [16]). Its vector of parameter to be estimated is thus $\underline{\theta}' = \left(\underline{\theta}'\,(1)\,, \underline{\theta}'(2)\right)$ where $\underline{\theta}(i) = (\alpha_0\,(i)\,, \alpha_1\,(i)\,, \beta_1\,(i)\,, \gamma_1\,(i))'$, $i = 1, 2$, are chosen to ensure the SPS condition of our model. To this end, we assume that $\gamma_L = 4E\left\{\log\left\{\alpha_1\,(1)\,e_0^{+} + \beta_1\,(4)\,e_0^{-} + \gamma_1\,(1)\right\}\right\}$ $+E\left\{\log\left\{\alpha_1\,(2)\,e_0^{+} + \beta_1\,(2)\,e_0^{-} + \gamma_1\,(2)\right\}\right\} < 0$. So, the results of simulation are gathered in table (2)

	$\mathcal{N}(0,1)$		$MA(1)$	
Parameters	1000	3000	1000	3000
$\widehat{\underline{\alpha}}_0$	$1.0102, 0.4960$	$1.0019, 0.4800$	$1.0166, 0.5126$	$.0087, 0.4985$
	$(0.0102, 0.0040)$	$(0.0019, 0.0200)$	$(0.0166, 0.0126)$	$(0.0087, 0.0015)$
$\widehat{\underline{\alpha}}_1$	$0.5010, 0.2396$	$0.4997, 0.2460$	$0.5106, 0.2511$	$0.5003, 0.2501$
	$(0.0010, 0.0104)$	$(0.0003, 0.0040)$	$(0.0106, 0.0011)$	$(0.0093, 0.0071)$
$\widehat{\underline{\beta}}_1$	$0.2511, 0.4854$	$0.2508, 0.4901$	$0.2581, 0.4991$	$0.2505, 0.5009$
	$(0.0011, 0.0146)$	$(0.0008, 0.0099)$	$(0.0081, 0.0009)$	$(0.0075, 0.0039)$
$\widehat{\underline{\gamma}}_1$	$0.1441, 1.0091$	$0.1486, 1.0136$	$0.1409, 0.9960$	$0.1451, 0.9993$
	$(0.0059, 0.0091)$	$(0.0014, 0.0136)$	$(0.0091, 0.0040)$	$(0.0049, 0.0007)$
Design(1):$\underline{\alpha}_0 = (1, 0.5)$, $\underline{\alpha}_1 = (0.5, 0.25)$, $\underline{\beta}_1 = (0.25, 0.5)$ and $\underline{\gamma}_1 = (0.15, 1)$.				
$\widehat{\underline{\alpha}}_0$	$0.9712, 0.4822$	$0.9801, 0.4963$	$0.9851, 0.4870$	$0.9940, 0.4915$
	$(0.0288, 0.0578)$	$(0.0199, 0.0437)$	$(0.0249, 0.0630)$	$(0.0160, 0.0485)$
$\underline{\alpha}_1$	$0.4887, 0.2434$	$0.4956, 0.2489$	$0.5001, 0.2476$	$0.5006, 0.2489$
	$(0.0113, 0.0166)$	$(0.0044, 0.0071)$	$(0.0001, 0.0024)$	$(0.0076, 0.0031)$
$\widehat{\underline{\beta}}_1$	$0.2480, 0.4880$	$0.2486, 0.4900$	$0.2488, 0.4977$	$0.2504, 0.5007$
	$(0.0120, 0.0220)$	$(0.0024, 0.0010)$	$(0.0022, 0.0023)$	$(0.0074, 0.0007)$
$\widehat{\underline{\gamma}}_1$	$0.0039, 0.0022$	$0.0019, 0.0010$	$0.0015, 0.0069$	$0.0009, 0.0015$
	$(0.0339, 0.0622)$	$(0.0099, 0.0030)$	$(0.0315, 0.0669)$	$(0.0079, 0.0475)$

Design (2): $\underline{\alpha}_0 = (1, 0.5)$, $\underline{\alpha}_1 = (0.5, 0.25)$, $\underline{\beta}_1 = (0.25, 0.5)$ and $\underline{\gamma}_1 = (0.0, 0.0)$.
Table(2); Average and $RMSE$ of 500 simulations of $QMLE$ for $PTGARCH_7\,(1,1)$

The asymptotic distribution of $\sqrt{n}(\widehat{\theta}_n(i) - \theta(i))$ followed by its box plot summary of $\widehat{\theta}_n(i)$, $i = 1, ..., 4$ according to design(1) of table(2) are showed in figure Fig3 below.

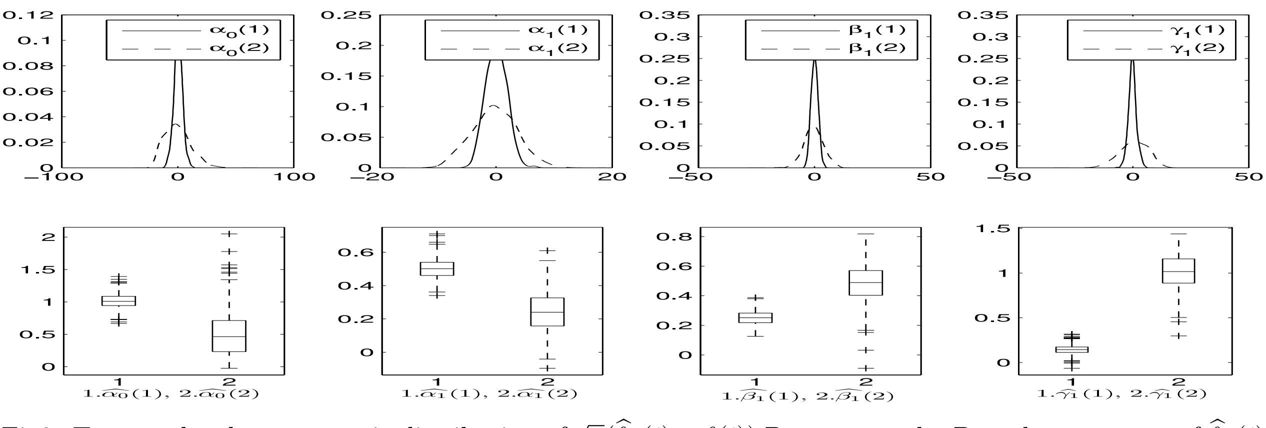


Fig3. Top panels: the asymptotic distribution of $\sqrt{n}(\widehat{\theta}_n(i) - \theta(i))$.Bottom panels: Box plot summary of $\widehat{\theta}_n(i)$, $i = 1, ..., 4$, according to design(1) of table(3)

Now, a few comments are in order. Table(1) compares the asymptotic parameters estimates and their $RMSE$ through two designs of parameters over 500 independent simulations of the standard $TGARCH(1,1)$ for sample sizes $n = 1000$ and $n = 3000$. The asymptotic values of $QMLE$ corresponding to $MA(1)$ innovation are generally overestimate, contrary to those of $QMLE$ corresponding to $\mathcal{N}(0,1)$. Also, it can be noted that there is no significant difference between the $RMSE$ corresponding to different innovations. Regarding now Table(2) where the model was simulate according to $PTGARCH_5(1,1)$ via two designs in which the parameters of the first and second regime in design (1) are such that $E\left\{\log\left\{\alpha_1(v)\,e_0^+ + \beta_1(v)\,e_0^- + \gamma_1(v)\right\}\right\} < 0$, $v = 1,2$ in contrast with design (2), the second regime is explosive in the sense that $E\left\{\log\left\{\alpha_1(2)\,e_0^+ + \beta_1(2)\,e_0^- + \gamma_1(2)\right\}\right\} > 0$, but the SPS of the model is satisfied. Also, as it can been seen that the results are in general quite satisfactory in accordance with the asymptotic theory.

7 Applications for exchange rates

The proposed model is now investigated to real financial time series. So we apply our model to two foreign exchange rates of Algerian Dinar against $U.S.$-Dollar (USD/DZD) and Euro (EUR/DZD), noted respectively $\left(y_t^{(d)}\right)$ and $\left(y_t^{(e)}\right)$ already analyzed by Hamdi and Souam [15] via a mixture periodic $GARCH$ models. This data transformed to a daily log returns $\left(r_t^{(d)} = \log\left(y_t^{(d)}/y_{t-1}^{(d)}\right) \text{ and } r_t^{(e)} = \log\left(y_t^{(e)}/y_{t-1}^{(e)}\right)\right)_{t\geq 1}$ of prices from January $3, 2000$ to September $29, 2011$, after removing the days when the market was closed (weekends, holidays,...), we provides 3055 observations supposed to be uniformly distributed on 611 weeks. The graphics of prices and their associated returns series are plotted in Fig4.

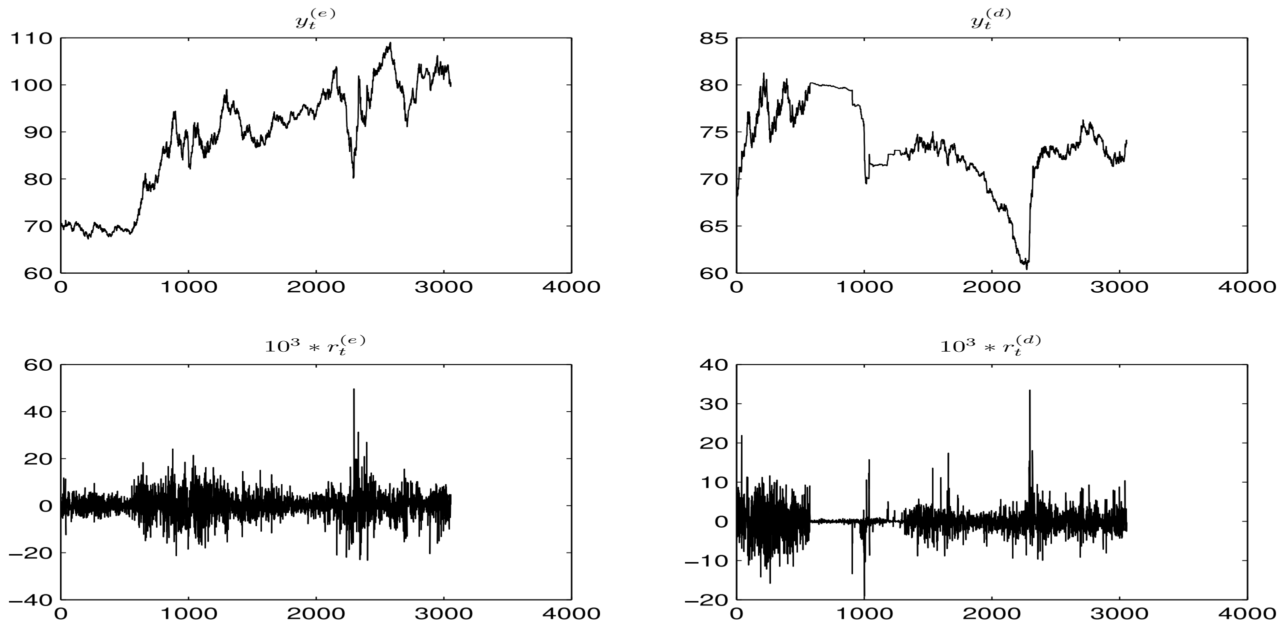


Fig 4. Top panels the prices of EUR,USD/DZ. Bottom panels their returns

Some elementary statistics of $\left(y_t^{(e)}\right)_{t\geq 1}$, $\left(y_t^{(d)}\right)_{t\geq 1}$ and their correspond returns series are summarized in table (3)

Series	means	Std.Dev	Median	Skewness	Kurtosis	Arch(300)	LBtest
$y_t^{(e)}$	88.61	11.57	91.09	-0.51	2.13	100%	99.96%
$r_t^{(e)}$	0000	0000	0000	0.400	9.00	100%	24.2%
$r_t^{(e)2}$	0000	0000	0000	0000	0000	00%	00%
$\left\|r_t^{(e)}\right\|$	0000	0000	0000	3.00	18.00	100%	100%
$y_t^{(d)}$	73.45	4.24	73.12	-0.60	3.76	100%	100%
$r_t^{(d)}$	0000	0000	0000	1.000	13.00	100%	43%
$r_t^{(d)2}$	0000	0000	0000	0000	1000	00%	00%
$\left\|r_t^{(d)}\right\|$	0000	0000	0000	3.000	2.100	100%	100%

Table(3): Descriptive statistics of the series $\left(y_t^{(e)}\right)_{t\geq 1}$, $\left(y_t^{(d)}\right)_{t\geq 1}$ and their returns

In Table (3), the column LBtest (Ljung–Box (portmanteau) test), shows that, on the one hand, at the 24.2% (resp. 43%) significance level, there is not enough evidence to reject the null hypothesis $H0$: "The residuals of $\left(r_t^{(e)}\right)_{t\geq 1}$ (resp $\left(r_t^{(d)}\right)_{t\geq 1}$) are not autocorrelated", contrary to the series $\left(r_t^{(e)2}\right)_{t\geq 1}$ (resp. $\left(r_t^{(d)2}\right)_{t\geq 1}$) which presents a significant $ARCH$ effects in its residuals. On the other hand, by the Arch(300) column, for testing $K0$: "No residuals heteroscedasticity of $\left(r_t^{(e)}\right)_{t\geq 1}$(rep. $\left(r_t^{(d)}\right)_{t\geq 1}$)", shows that trough the first three hundred lags, $K0$ should be rejected. Moreover, by examination of the sample correlations functions (ACF) of the series of returns, (see Fig 5 bellow)

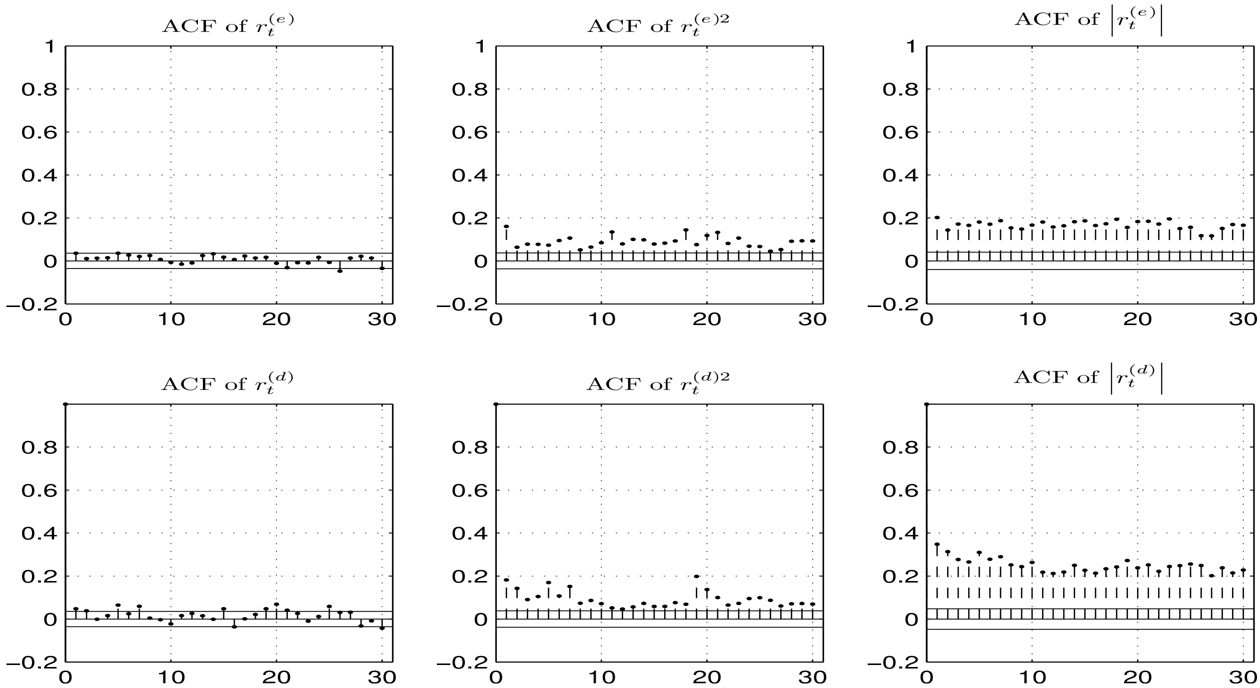


Fig5. The ACF of the returns and of their squred and absolute series

it can be observed that $\left(r_t^{(d)}\right)_{t\geq 1}$ (resp. $\left(r_t^{(e)}\right)_{t\geq 1}$) present a *Taylor-Effect* phenomena characterized by the fact that sample autocorrelations function of the absolute returns are usually larger than the squared one. Hence, the modelling of the series $\left(r_t^{(d)}\right)_{t\geq 1}$ (resp. $\left(r_t^{(e)}\right)_{t\geq 1}$) by a standard $GARCH$ models should be rejected in favor of certain asymmetric models. However, it is obvious to address the question of day-of-the-week seasonality in returns (see for instance Franses and Raap [16]). So, in order to analyze the seasonality, we fitted the following simple $PTGARCH_5\,(1,1)$ model for each series or equivalently we estimate their volatility processes $(h_t)_{t\geq 1}$ according to a period version

$$h_n = \alpha_0\,(n) + \alpha_1\,(n)\,r_{n-1}^{+} + \beta_1\,(n)\,r_{n-1}^{-} + \gamma_1\,(n)\,h_{n-1} \tag{6.1}$$

The estimated five-regimes (intraday) parameters of $\left(h_t^{(e)}\right)_{t\geq 1}$ (resp. $\left(h_t^{(d)}\right)_{t\geq 1}$) and their $RMSE$ according to (6.1) are reported in table(5)

Series		$\underline{\underline{\widehat{h}_t^{(d)}}}$				$\underline{\underline{\widehat{h}_t^{(e)}}}$		
Days	$\underline{\widehat{\alpha}}_0^{(e)}$	$\underline{\widehat{\alpha}}_1^{(e)}$	$\underline{\widehat{\beta}}_1^{(e)}$	$\underline{\widehat{\gamma}}_1^{(e)}$	$\underline{\widehat{\alpha}}_0^{(d)}$	$\underline{\widehat{\alpha}}_1^{(d)}$	$\underline{\widehat{\beta}}_1^{(d)}$	$\underline{\widehat{\gamma}}_1^{(d)}$
Sunday	0.7787	0.3744	0.5128	0.5408	1.1619	0.4394	0.3399	0.7730
	(0.0922)	(0.0817)	(0.0915)	(0.0632)	(0.0390)	(0.0033)	(0.0010)	(0.0211)
Monday	0.4231	0.5505	0.4646	0.5181	0.9158	0.3867	0.4665	0.5174
	(0.0753)	(0.0594)	(0.0474)	(0.0195)	(0.0129)	(0.0571)	(0.0764)	(0.0698)
Tuesday	0.0000	0.9077	0.3803	0.5669	1.0811	0.3144	0.3929	0.6128
	(0.0439)	(0.1409)	(0.0646)	(0.0011)	(0.0521)	(0.0701)	(0.0720)	(0.0934)
Wednesday	0.0212	0.6114	0.5092	0.5852	0.7307	0.2324	0.2522	0.4420
	(0.0408)	(0.0842)	(0.0660)	(0.0626)	(0.0246)	(0.0451)	(0.0518)	(0.0592)
Thursday	0.1485	0.7269	0.4297	0.6623	1.0706	0.3091	0.3520	0.6093
	(0.0685)	(0.0146)	(0.0721)	(0.0626)	(0.0447)	(0.0605)	(0.0742)	(0.0947)

Table(5): $QMLE$ estimates and their $RMSE$ of $\left(h_t^{(e)}\right)_{t\geq 1}$ and $\left(h_t^{(d)}\right)_{t\geq 1}$

Some elementary statistics associated with the estimated $\left(r_t^{(e)}\right)_{t\geq 1}$ (resp. $\left(r_t^{(d)}\right)_{t\geq 1}$) according to (6.1) noted $\left(\widehat{r}_t^{(e)}\right)_{t\geq 1}$ (resp. $\left(\widehat{r}_t^{(d)}\right)_{t\geq 1}$) are reported in table (6) below

Series	means	Std.Dev	Median	Skewness	Kurtosis	Arch(300)	LBtest
$\widetilde{r}_t^{(e)}$	0000	0.005	0000	0.391	8.517	100%	23.2%
$\widetilde{r}_t^{(e)2}$	0000	0.001	0000	0.002	0.001	0.01%	0.05%
$\left\lvert\widetilde{r}_t^{(e)}\right\rvert$	0000	0.002	0000	2.80	17.10	100%	99.8%
$\widetilde{r}_t^{(d)}$	0000	0000	0000	0.880	13.50	100%	45%
$\widetilde{r}_t^{(d)2}$	0000	0000	0000	0000	8590	0.01%	0.01%
$\left\lvert\widetilde{r}_t^{(d)}\right\rvert$	0000	0000	0000	3.100	1.800	100%	100%

Table(6): Descriptive statistics of the series $\left(y_t^{(e)}\right)_{t\geq 1}$, $\left(y_t^{(d)}\right)_{t\geq 1}$ and their returns

Regarding the parameters in Table (5), it is can be shown that these parameters forces the models to be SPS. The elementary statistics presented in Table (6) shows the Arch effect (resp. heteroscedasticity) in residuals of $\left(\widehat{r}_t^{(e)}\right)_{t\geq 1}$ and of $\left(\widehat{r}_t^{(d)}\right)_{t\geq 1}$ are still present at almost the same significance level as of $\left(r_t^{(e)}\right)_{t\geq 1}$ and of $\left(r_t^{(e)}\right)_{t\geq 1}$ presented in Table(3). In general, the results in Table 6 of estimate returns according to $PTGARCH_s\,(1,1)$ reveal a noticeable resemblance with the results of the real returns displayed in Table 3 and hence the capability of $PTGARCH_s\,(1,1)$ to model this data is justified.

8 Summary and conclusion

This paper investigates some probabilistic, statistical properties and empirical evidence of $PTGARCH_s$ processes. The main purpose of introducing this new class of models is twofold, the first is to extended the $TGARCH$ models with constant coefficients to time-varying one, in the sense that the coefficients are periodic with period $s \geq 1$. This specification becomes an efficient tool to analyze nonlinear and non Gaussian financial time series that is capable to capture the stylized and leverage effects and hence the asymmetric properties in the volatility process. On the other hand, from a practical point of view, the approach can be used even for datasets of moderate length.

The second aim that we wish to consider in this paper is the estimation of $PTGARCH_s$ models. Indeed, after the derivation of sufficient conditions ensuring the existence of SPS solutions, we have investigated the $QMLE$ approach for estimating the parameters of $PTGARCH_s$ model. More precisely we have shown that under very mild moment condition for the volatility process, the $QMLE$ is strongly consistent and asymptotically normal. This methodology has been illustrated by a Monte Carlo study and an application to the exchange rate of Algerian Dinar against U.S. Dollar and the single European currency ($Euro$)., showing hence its performance and its efficiency. Note in end, that the results of such nature has never appeared in the literature of asymmetric models, although the area has been considered for a long time.

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