Probabilistic Economic Theory

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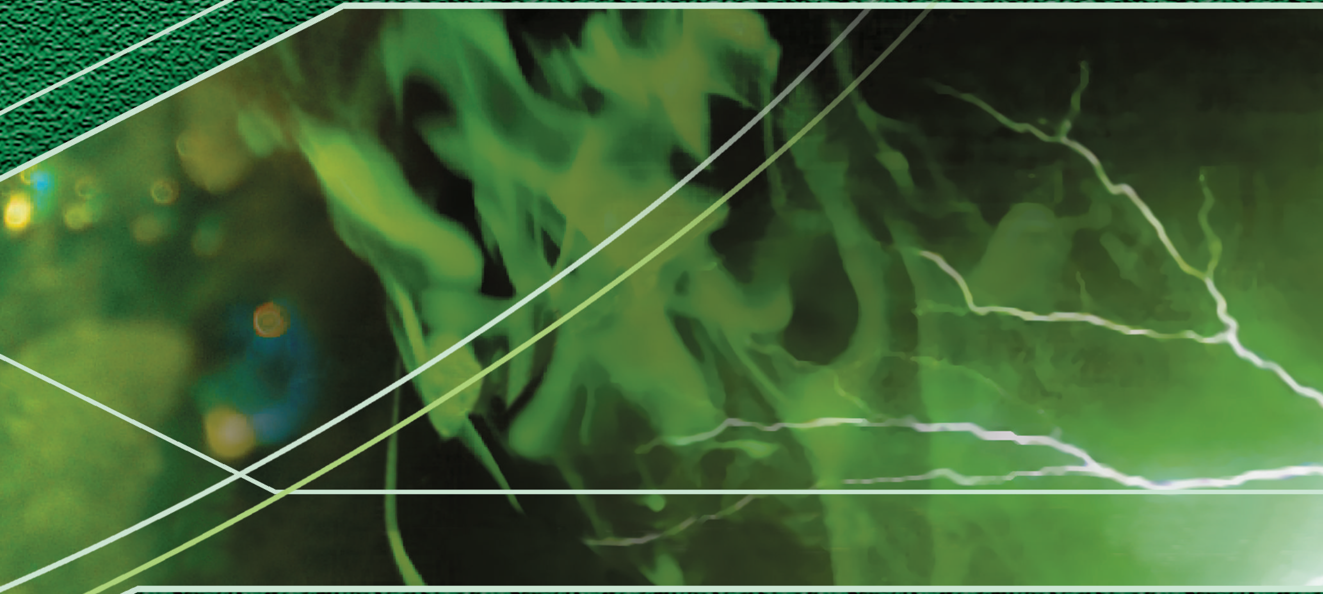

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Anatoly V. Kondratenko

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This book presents principles of physical economics, new economic discipline primarily concerned in the book with the agent-based physical modeling of the market economic systems and eventually with the elaborating of probabilistic economic theory. At the heart of physical economics and probabilistic economic theory are the well-known cornerstone concepts of classical economics, in particular the subjective theory of value, such as regularity in the sequence of market phenomena and an interdependence of those, as well as key roles of individuals' actions and social cooperation in the many-agent market processes. The main point of the concept of the physical modeling is that formal approaches and procedures of theoretical physics are used to describe these economic concepts. The obvious structural and dynamic analogy of the many-agent economic systems with the many-particle physical systems is basic to the formulation of fundamentals of the method of the agent-based physical modeling of the many-agent market economic systems in the formal economic space. It is also vital to the elaboration of the main paradigm and eventually the five general principles of physical economics, as well as of probabilistic economic theory. The uncertainty and probability principle holds a central position among of them. All the principles provide the necessary background to physical economics that include such theories as classical economy, probability economics, and quantum economy. The book provides a unique source for learning and understanding all the concepts and principles of physical economics, together with the quantitative methods of calculating and analyzing the many-good, many-agent market economies. Conceptually, the book can be viewed as an introduction to economics for physicists, expressed by means of the terms and language of physics. This book is also addressed to all those who are interested in the background of economics, foremost in the axiomatic basis of modern economic theory.

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To Olga, Ksenia and Anastasia
with love and gratitude

CONTENTS

PREFACE

I have the pleasure of bringing to the reader's notice a book that expounds on the principles of physical economics, and more specifically, probabilistic economic theory. In the broad sense, physical economics is a relatively new economic discipline concerned with the study of economic phenomena from the point of view of physics. More exactly, it is through such economic investigations that we explore methods of theoretical physics previously developed in that discipline for solving formally similar problems. Depending on the statement of the economic problem under study, those can be mechanical, statistical, thermodynamic or other methods. For this reason, there are several corresponding variants of presentation of physical economics in the literature depending on the methods applied. The two books [1, 2] can be cited as an example of the presentation of physical economics from the point of view of statistical physics. It is evident that these books primarily deal with open financial markets with huge numbers of market agents, where statistical effects obviously play one of the leading roles. The key review [3] presents a phenomenological version of physical economics in terms of classical mechanics.

As far as my book is concerned, I am interested here mainly in ordinary economies with various markets of goods and commodities. In contrast to financial markets, I believe both the deterministic and probabilistic effects to play dominant roles in such market economies. From the one side, the market agent pursues, as a rule, definite aims and explores standard work methods on the markets that, to some extent, lead to determinism on the markets. From the other side, the market agents are generally forced to work under uncertain market conditions. It means that we always have to take into account in the theory that uncertainty permanently accompanies all of the market agents' important decisions. There is no escaping the conclusion that all significant market phenomena have a probabilistic nature, too. Therefore, I study the many-agent market economic systems in this book from the point of view of classical and quantum mechanics which have been elaborated through physics to describe the deterministic and probabilistic effects in the many-particle physical systems. For this reason, I treat the term *physical economics* in the book in the narrow sense of this concept, from the point of view of classical and quantum mechanics of the many-particle systems. More exactly, physical economics here is a study of the formal agent-based physical models of the market economic systems. Finally, I define physical economics here primarily as the science of the agent-based physical modeling of the market economic systems, with the aid of methods and approaches worked out earlier in classical and quantum mechanics. When applying ideas of quantum mechanics to the many-agent economic systems, we inevitably obtain probabilistic economic theory which is understood in this book as most of physical economics. Note that all the physical economic models here are also agent-based ones. I think that there is great advantage to applying the agent-based approach of physical modeling because it makes it possible at the micro level, i.e., at the level of separate agents, to find a basis for explaining economic phenomena at the macro level, i.e., at the level of the whole economy. Analogously, quantum mechanics first explained the behavior of the separate electron in the deep pit. At the time, it undertook the explanation of the macro effects and calculation of macro quantities on the basis of the knowledge obtained there.

In essence, a new physical economic picture of the market world is drawn in the book. There is a huge number of formulas in it, and practically none of them are borrowed from economic literature. However, they all have their analogues in the picture of the physical world, expressed in theoretical physics or, even more accurately, in classical and quantum mechanics.

Physical economics is a proper economic discipline, since, in contrast to physics, the objects of its studies are the actions of real subjects of the economy but not the real subjects themselves. It addresses actions of real people in the real economic world, first of all; of buyers and sellers on the markets, focused on battling for their material interests and simultaneously achieving mutually advantageous cooperation. I follow the idea of classical economic theory in which the economy is simultaneously both the product and the process of human action. Moreover, in contrast to the physical world, both the structure of the economy as well as forms and methods of human action continuously and rapidly vary with time as a result of the general human evolution, as well as scientific and technical progress. Therefore, the economic laws should be derived from the study of practical human activities, but in no way by means of fitting of the known physical laws to the economic world. But the situation is reversed if we want to use theoretical methods of physics in search of the economic laws and to develop quantitative economic theories. The point is that physics has elaborated the enormous number of mathematical methods and apparatuses that describe the structure and dynamics of diverse physical systems, from the simple to the complex. And, there is nothing that would forbid us fruitfully applying these formal mathematical methods in economics.

I emphasize that the ideas, concepts and principles of classical economic theory constitute the foundation of physical economics in my understanding, and the approaches and methods of theoretical physics play here the role of the second plan. The task of these theoretical instruments is to give the adequate mathematical description of these ideas, concepts and principles. These help create the formal framework of the theory, as well as develop mathematical apparatus for describing the structure and behavior of market economies. Why is this possible? The point is that, in structure and properties, the many-agent market economic systems are quite similar to the many-particle physical systems. Take, for example, the polyatomic molecules. Indeed:

1. Markets consist of agents. Molecules consist of atoms.

2. Agents interact between themselves. Atoms interact between themselves.

3. Everything that markets do, the interacting agents do. Everything that molecules do, the interacting atoms do.

4. Dynamics of markets are determined by a principle of maximization (e.g. the trade maximization principle). Dynamics of molecules is determined by a principle of maximization (e.g. the least action principle).

5. Uncertainty and probability is an inherent important property of the market behavior. The same is valid for dynamics of molecules.

And this is still far from a complete enumeration of coincidences and analogies between the economic and physical systems.

It is widely-known that presently, advanced mathematical methods are applied in describing the dynamics of complex economic systems much less frequently than in physics. As far as the difference in the level of the penetration of formal mathematical methods into economics and physics is concerned, it is possible with reasonable caution to assert that this difference does not lie in the fact that in principle, mathematics cannot be widely used in economics by its very nature. Instead, physics has proven to be the more developed science mathematically in modern times for a variety of historical and technological reasons. This has been the situation for the past 300 years. Beautiful mathematical models have been created in physics during this time frame to describe dynamic phenomena in many-particle systems. Unfortunately, the same cannot be said for economic theory. It is obvious to me that, things being as they are, the correct conclusion for us now must not be to preserve the *status quo*. Nor should economists urgently develop their own unique mathematical calculations to describe economic phenomena, independent of physics. After all, why re-invent the wheel? All that is needed is to make use of some of the most salient and staggering achievements of humanity at the present time by borrowing from theoretical physics. These can be put to use for the good of the development of economic science and the global economy. This book is one of the many steps in the right direction along the proper road. Hopefully I am not wrong, although there is always that chance. There is no doubt that it will be a long road to this accomplishment, and most certainly not a fast one.

I am well aware that the very idea of using methods of theoretical physics, especially quantum mechanics, for describing economic phenomena must cause a healthy dose of skepticism from the physicists. Therefore, I emphasize that the discussion deals with the fact that only the mathematic framework of theoretical models of the respective physical systems are transferred to the physical economic models.

I incorporate into economics only the formal structural aspects of physical theories. First of all are the equations of motion for the many-particle systems, which just by themselves must not be too rigidly attached to real physical microscopic objects. Equations — they are just equations and nothing more, and if they are a beneficial descriptive tool in another science, why not make use of them? I repeat that this is just a useful mathematical object which can and should be used as a theoretical tool where it can provide benefit. For instance, in quantum mechanics wave functions and the Schrödinger equations have been successfully used for the incorporation of the uncertainty and probability principle into physics. Why, then, can we not then apply the same mathematical apparatus to the analogous uncertainty and probability principle in economic theory for purposes of mathematical description?

It is obvious that this is only an initial approximation to reality. But we are talking about modeling economic systems, and models are only models, giving only an approximate shape to the object being modeled. My physical models of economic systems also do not pretend that they are complete and precise; they can give only the approximate patterns of our market economic world, only the specific stage in our understanding of the real economic world transposed into the language of mathematics. A physical economic model is nothing other than a certain ideal, imagined construction, aimed at explaining one or more aspects of the studied phenomenon. The question is not whether it is correct or not, but whether it is useful in helping to reach a true understanding of the real economic world. Nothing more. I think that by means of this approach, some insight into the important market economy phenomena has been gained in this study.

Ten years ago I published the small book "Physical Modeling of Economic Systems: The Classical and Quantum Economies" [4]. It was my first attempt to develop an economic theory *ab initio*, and constructed an axiomatic basis of the theory from a limited set of first principles. The basic hallmarks of the theory that made it probabilistic and quantitative are as follows.

First. A careful, step-by-step development of the market agent-based physical economic models, where market agents play a main role in market phenomena.

Second. The complete integration of uncertainty and probability perspectives throughout the theory.

Third. A unifying, analytic framework that uses equations of motion in the formal price economic space to describe economy evolution in time.

For the last ten years, I have continually strived to advance the theory and to make it more clear and justified. In particular, for this purpose I developed the special mathematical apparatus, which is referred to in the book as probability economics. Still, I considerably advanced the theory by means of taking into consideration quantities of market goods as independent variables along with their prices. Due to this innovation, the economic price space was expanded up to the economic price-quantity space. During this time I also developed mathematical apparatus for describing the many-good, many-agent market economies. Despite the fact that achievements and expansions of theory mentioned above are very substantial, my new book carries the title "Probabilistic Economic Theory" and is, in essence, the second extended edition of my first book, in which I presented only very beginnings of the method of the agent-based physical modeling of economic systems and the basics of probabilistic economic theory.

In this book, the fundamental concepts of economic theory are exposed to critical rethinking for the purpose of answering such eternal questions of economic theory such as those regarding supply and demand, as well as market price and market force, market process and market equilibrium, invisible hand of market etc. I look at how all these concepts should be incorporated into economic theory and conveyed quantitatively in the same language in which physicists, chemists and other professionals in the so-called natural sciences present their theories, i.e., in the language of mathematics. In the book I presented maximally simplified models, in which only the most important special features and details of work of markets are described by means of maximally simplified mathematical apparatus. Let us stress here that the main aim of such basic models is only to reveal the essence of the studied phenomenon, not more. After this is accomplished, we can then develop the models further, including other, more sophisticated effects within them. This is the only true way of modeling science. Therefore, Chapters I–VIII are easily understood by first-year economics students. But the subsequent Chapters IX and X require an existing, thorough knowledge of physics, somewhere around the level of upper year physics courses. They only need have the slightest grasp of economic phenomena and laws of human action in the market economy, obtained, for example, in the course of reading the first chapters of this book. Generally, this book can be considered as an introduction into economics, written for physicists in standard physics terminology. The book, by the way, was initially taught as a set of lectures on economics for physics department students. If, after reading this book, a physics student has the impression that the presented physical economic models are quite simple and understandable, then I have solved a personal challenge. Indeed, I feel that the more complex the studied systems are and the phenomena within them, the simpler the model must be, taking into consideration only those effects which are of prime importance for describing the studied phenomena.

The book, as noted above, is the collection of lectures, each of which is called to answer one or several questions given above. The genre of lecture (or essay) is selected for the purpose of concentrating on the compact, clear presentation of physical economics. In it I have used a whole series of new ideas, concepts and notions for the economic theory, which arise from theoretical physics. I believe I have succeeded in avoiding the necessity of making numerous surveys and references, the like of which can be found in most other textbooks on economics and economic history. Therefore, references in the book are made only to those sources which were actually used for the fulfillment of studies, the development of models, and writing of the book. To provide convenience to students in lectures, figures and fragments of the text are reproduced several times in some chapters.

It should be emphasized once more that I borrowed ideas, concepts and notions from physics, many of which are completely consistent with the discoveries of the classical economic theories of the 19th century, the first of which being the subjective theory of value. This concerns first and foremost such important milestones in the development of classical economic theory as [5]:

1. Regularity in the sequence of market economic phenomena.
2. Exclusive and dominant roles of market agents in the market phenomena.
3. Uncertainty of the future and the probabilistic nature of all market agents' decisions.
4. Social cooperation of market agents etc.

All these aspects of human economic activities have an exceptionally important effect on market processes and determine the course of economic development. But any formal methods of providing an adequate formal description of these economic phenomena and processes at a strict mathematical level in economic theory, until now, have been absent. Necessity and expediency of borrowing by economics from physics is substantiated by the fact that theoretical physics already developed sophisticated mathematical methods to incorporate analogous concepts into formal physical models. The method of equations of motion was first, and uses systems of differential equations of the 1st and 2nd orders, but economic theory still did not.

I would like to stress here that this creative process of the transfer of the formal methods of physics into classical economic theory presents the main point of the concept of the agent-based physical modeling of the economic systems and physical economics as a whole. One can say that, in essence, physical economics is first and foremost the mathematical apparatus of classical economic theory at the contemporary level of its development. This mathematical apparatus is borrowed from theoretical physics and has therefore practically nothing to do with the mathematical apparatus of neoclassical economics.

In order not to overload the text of the book by the descriptions of the known concepts of classical economic theory upon which I rested in this investigation, I make use of complete quotations from the fundamental monograph of Ludwig von Mises [5] as epigraphs to each part and chapter of the book, with two exceptions. This method allowed me to avoid the mixing of completely different styles of the presentation in the book, which could hinder the perception of the text by readers. This is very important for me, since I have attempted to not to disappoint the readers, and to convince them of how it is fruitful to make use of achievements in theoretical physics for the development of economics. The point is not in the book's detail, but rather in its broader concept of agent-based physical modeling of economic systems, which, in my view, has enormous potential. Here lies, I think, a new and enormous field for investigation, in which an abundant harvest will be gathered for many decades yet to come. I hope that the readers will obtain a certain benefit from the acquaintance with this new physical economic perspective for economic theory.

References

1. Rosario N. Mantegna, Eugene H. Stanley. *An Introduction to Econophysics: Correlations and Complexity in Finance*. Cambridge University Press, 1999.
2. Peter Richmond, Jürgen Mimkes, and Stefan Hutzler. *Econophysics and Physical Economics*. Oxford University Press, 2013.
3. D.S. Chernavsky, N.I. Starkov, A.V. Shcherbakov. *On some problems of physical economics*. UFN, Vol. 172, N 9, pp. 1046–1066, 2002.
4. A.V. Kondratenko. *Physical Modeling of Economic Systems. Classical and Quantum Economies*. Nauka: Novosibirsk, 2005. Electronic copy available at: http://ssrn.com/abstract=1304630.
5. Ludwig von Mises. *Human Action. A Treatise on Economics*. Yale University, 1949.

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INTRODUCTION.
Probabilistic Economic Theory

> "The market economy is the social system of the division of labor under private ownership of the means of production. Everybody acts on his own behalf; but everybody's actions aim at the satisfaction of other people's needs as well as at the satisfaction of his own. Everybody in acting serves his fellow citizens. Everybody, on the other hand, is served by his fellow citizens. Everybody is both a means and an end in himself, an ultimate end for himself and a means to other people in their endeavors to attain their own ends.
>
> This system is steered by the market. The market directs the individual's activities into those channels in which he best serves the wants of his fellow men. There is in the operation of the market no compulsion and coercion. The state, the social apparatus of coercion and compulsion, does not interfere with the market and with the citizens' activities directed by the market. It employs its power to beat people into submission solely for the prevention of actions destructive to the preservation and the smooth operation of the market economy. It protects the individual's life, health, and property against violent or fraudulent aggression on the part of domestic gangsters and external foes. Thus the state creates and preserves the environment in which the market economy can safely operate... Each man is free; nobody is subject to a despot. Of his own accord the individual integrates himself into the cooperative system. The market directs him and reveals to him in what way he can best promote his own welfare as well as that of other people. The market is supreme. The market alone puts the whole social system in order and provides it with sense and meaning".
>
> Ludwig von Mises. *Human Action. A Treatise on Economics*. Page 243

Exactly how is our economic market world arranged at the deepest micro level, at the level of individuals and businesses, buyers and sellers on markets? How does the market economy function? According to what laws and rules do people make decisions about price and quantity of bought and sold goods? How does a market price eventually become established, and how can one and the same volume of goods be sold regularly each year at this price on the market? How and why does this regularity change over the course of time, and how and why do markets grow or fall? We attempt to answer these and many other similar questions in this book within the framework of a new economic discipline, namely, physical economics.

In general terms, at the conceptual or descriptive level, the answers to these questions are known from classical economic theory. They are obtained with the aid of the logical method of economic theory and they are given in this book in epigraph form to the parts and the chapters. These answers represent by themselves extensive quotations from the fundamental work of Ludwig von Mises [1]. But other questions agitate us more. Where can we search for the answers in this book, and how can all these answers of classical economic theory be expressed in terms of the natural exact sciences, i.e., in mathematical language? Therefore, in the book we attempt to develop quantitative methods of calculation for economic systems, as well as their structure and dynamics, which would make it possible to speak about the further development of quantitative economic theory.

Contemporary, real, economy is a complex dynamic system. It is therefore possible and necessary to attack the problem of studying its structure and dynamics in different ways and from different points of view. Our point of view is such that we look at the economy primarily as a collection of an enormous number of reasonably thinking and actively acting people, each of them "is not only *homo sapiens*, but no less *homo agens*" [1] simultaneously. To solve their problems and achieve their goals, these people are forced to constantly make important decisions for themselves about production, purchase and sale of goods, organization of logistics and marketing, control of other people, etc. As reasonable people, they attempt to make specific decisions that would bring more benefit and a large return for their efforts. Such rational decisions can be made only on the basis of having sufficient information regarding factors that concern their interests Therefore, people are constantly searching for and processing new, relevant market information. But we never possess completely adequate information about the things interesting us in view of the time constraints at our disposal, or in view of our obviously finite mental and technical abilities. It is our deep belief that human nature, and also the nature of market economic systems, is such that all our market decisions can only be approximate. In more technical terms, they can only be of a probabilistic nature. Furthermore, according to our vision of the market economy, all economic processes and phenomena are nothing more than the result of the actions of all players in the economy. There seems to be no escaping the conclusion that all economic processes and phenomena in the market economy are, to some extent, also probabilistic by their nature. Hence, there is only one step prior to obtaining the fundamental conclusion that the market economy is not simply a complex dynamic, but that it is still probabilistic [2–5]. Therefore, in order to be able to give a sufficiently complete description of such complex probabilistic systems, adequate economic theory must also be probabilistic to a considerable extent. For this, it is necessary to incorporate into the classical economic theory the uncertainty, as well as probability at the appropriate mathematical level, i.e., to develop probabilistic economic theory adequately in relation to the contemporary economic reality. Specifically, the present study is devoted mainly to this purpose.

In this book we have limited ourselves to the study of the probabilistic aspects of functioning markets in a sufficiently free market economy. More accurately, we study the details of supply and demand, as well as the mechanisms of the formation of prices and the establishment of equilibrium in the markets. Emphasis is given to the description of probabilistic nature of these fundamental market categories and notions. All these market concepts have been the subject of intensive investigations and critical rethinking in the book within the framework of the main paradigm of physical economics, which can be briefly formulated as follows.

All markets consist of people, buyers of some goods and sellers of other goods, simultaneously. Everything that the markets do, these people do, and it is precisely the action of all these people in the market that determine all results of the work of that market [1]. We have defined five fundamental or general principles of physical economics (and probabilistic economic theory naturally): the cooperation-oriented agent principle, the institutional and environmental principle, the dynamic and evolutionary principle, the market-based trade maximization principle, and, finally, the uncertainty and probability principle. These determine, in essence, the work of markets in our physical economic models. The cooperation-oriented agent principle speaks about the unique moving role of the market agents and significant role of social cooperation in modern market economy. The institutional and environmental principle expresses the fact that the interaction of agents with the various institutions and external environment must be taken into account simultaneously with the interaction of the agents with each other. The dynamic and evolutionary principle reflects the fact that the market behavior has, to certain extent, a deterministic character and consequently can be described with the aid of the equations of motion. The market-based trade maximization principle determines the direction of the motion of the free market as a whole under the influence of internal market forces. The uncertainty and probability principle tell us that all market phenomena are probabilistic in nature and thus help us to understand what a mathematical apparatus must do in order to adequately describe market behavior under uncertain conditions. We refer to the appropriate approach to as probabilistic economic theory, and since this theory is built by analogy with the "probabilistic physical theory" (quantum mechanics) of many-particle systems, we designated it more precisely as quantum economy [4]. Since the picture of the market economic world is built in physical economics at the micro level in approximately the same way as in physics, there is, in principle, a possibility of developing the quantitative methods of calculating the market economies by analogy with the calculation methods in physics. Consequently, it becomes possible to fruitfully use natural science concepts and speak natural science language to describe and analyze market structures and dynamics. In this book, we widely explore these great possibilities that help us to look at economic reality from the natural science points of view and discover new perspectives of economic theory for investigation and development.

In conclusion, it is natural that the direct mechanical transfer of methods and concepts of physics into the economic theory would be impossible. We only can accurately borrow and transfer general concepts and formal methods, since economic and physical phenomena are principally different in their essence and content. In this book and within the framework of this approach, the basic concepts and general principles of physical economics are developed and described. On this basis, we have developed the complex of the principally new quantitative methods of calculatio.n and analysis of many-good, many-agent market economies that we named probabilistic economic theory.

References

1. Ludwig von Mises. Human *Action. A Treatise on Economics*. Yale University, 1949.
2. Emmanual Farjoun and Moshé Machover. *Laws of Chaos: a Probabilistic Approach to Political Economy*. Verso, London, 1983).
3. Philip Ball. *Physical Modelling of Human Social Systems*. Complexus 2003; 1:190–206.
4. A.V. Kondratenko. *Physical Modeling of Economic Systems. Classical and Quantum Economies*. Nauka: Novosibirsk, 2005. Electronic copy available at: http://ssrn.com/abstract
5. K.K. Val'tukh. *Development of a Probabilistic Economic Theory*. Herald of the Russian Academy of Sciences, 2008, Vol. 98, N 1, p. 51–63.

PART A.

The Agent-Based Physical Modeling of Market Economic Systems

"In the course of social events there prevails a regularity of phenomena to which man must adjust his actions if he wishes to succeed. It is futile to approach social facts with the attitude of a censor who approves or disapproves from the point of view of quite arbitrary standards and subjective judgments of value. One must study the laws of human action and social cooperation as the physicist studies the laws of nature. Human action and social cooperation seen as the object of a science of given relations, no longer as a normative discipline of things that ought to be".

Ludwig von Mises. *Human Action. A Treatise on Economics.* Page 6

CHAPTER I.

Fundamentals of the Method of Agent-Based Physical Modeling

"For a social collective has no existence and reality outside of the individual members' actions. The life of a collective is lived in the actions of the individuals constituting its body. There is no social collective conceivable which is not operative in the actions of some individuals. The reality of a social integer consists in its directing and releasing definite actions on the part of individuals. Thus the way to a cognition of collective wholes is through an analysis of the individuals' actions".

Ludwig von Mises. *Human Action. A Treatise on Economics*. Page 43

"The whole market economy is a big exchange or market place, as it were. At any instant all those transactions take place which the parties are ready to enter into at the realizable price. New sales can be effected only when the valuations of at least one of the parties have changed".

Ludwig von Mises. *Human Action. A Treatise on Economics*. Page 231

PREVIEW.

What is the Main Point of the Concept of Agent-Based Physical Modeling?

The concept of agent-based physical modeling is based on taking the known, fundamental concepts of classical economic theory, and uniting and eventually converting them into probabilistic economic theory. It is described with the help of formal approaches and methods borrowed from theoretical physics, beginning with the method of equations of motion for many-particle physical systems. The role of the theoretical physics methods is only to provide the framework for physical economics and eventually for probabilistic economic theory. This theory is developed step-by-step with the creation of the more complicated models, each subsequent step building on the last. It includes and increases the number of concepts and principles of physical economics, and reflects on one or more of several fundamental features of the market economy. The first one, the cooperation-oriented agent principle, is the cornerstone of all the physical economic models which holds that all market phenomena have their origins in agents' actions. To put it differently, since the action of the market as a whole is a result of the actions of all the market agents and nothing other, the market agents and their actions must be at the basis of the physical economic models. In other words, according to the agent principle, all market phenomena have their origins in agents' actions.

1. The Concept of Agent-Based Physical Modeling

It is well-known that the method of conceptual modeling of economic systems has long and widely been used in economic theory. According to the new physical economic mode of thought, the main requirement for such economic models, which determines their basic predestination, is skill, with which several important basic concepts and principles must harmoniously, competently and simultaneously be incorporated into the theory. The latter is very important since, by the definition of the problem, all concepts and principles play roles that compare their significance in the economy under study.

Despite their simplicity, the first and most well-known conceptual theories of neoclassical economics, such as supply and demand (S&D below), contributed significantly to economic science. They helped economists to better understand the basic elements of the economic world, and they gave rise to graphic conceptualizations that aided the transfer of this knowledge to others, especially students. Early on, conceptual modeling had, and continues to have, great significance. This is also the case in Austrian economics, where it bears the name *method of imaginable constructions* and it is considered as the basic method in economic inquires. In theoretical physics, it is not acceptable to accentuate attention on the use of models, since theoretical physics itself can rightfully be considered as the conceptual mathematical modeling of physical systems. Specifically, in theoretical physics, the most advanced methods of theoretical modeling of complex systems have been developed. Moreover, there is the required inclination towards conducting the quantitative numerical, and, as precisely as possible, calculations of structure and properties of these models. The deep structural and dynamic analogies between the many-particle physical systems and the many-agent economic systems are exploited in this book to transfer concepts and analytical methods from theoretical physics to economics.

2. The Main Paradigm of Physical Economics

For the achievement of larger clarity, let us again mention the main idea in physical economics regarding the use of analogies with physics. For this purpose, we once more express the thought of the main paradigm of physical economics as follows: physical systems consist of atoms. As was well known long ago, all that the physical systems do, atoms do. Market systems consist of market agents, buyers and sellers. And it is also known that all that the markets do, the market agents do. Both the physical and market economic systems are the complex dynamic systems, whose dynamics are determined by interaction between the elements of the system and their interaction with the environment in the widest sense of this notion. In our view, the very existence of such structural and dynamic similarity gives rise to the possibility, in principle, of building formal, many-agent physical economic models by analogy with the theoretical models of the many-particle physical systems, for example, of polyatomic molecules. It is here that the physical economic model could include all basic concepts and principles which define the work of the economy. As a result, it could sufficiently, simply, and adequately describe both the main structural features and principal dynamic characteristics of the economy being modeled, a target at which this study is precisely aimed, since no model can immediately describe everything completely. The study of economic systems by means of physical modeling must be carried out gradually, step by step, incorporating into the model ever finer effects and properties in the way that theoretical physics has done over the course of theoretical studies of complex physical systems. Thus, physical economics borrows formal approaches and model structures from theoretical physics. In other words, physical economics uses the very body or framework of the theoretical models of the many-particle systems, but not the results of theoretical studies of the concrete real physical systems. Namely this is the essence of physical modeling of economic systems. In conclusion, the obvious structural and dynamic analogy of the many-agent economic systems with the many-particle physical systems is basic to the formulation of fundamentals of the method of the agent-based physical modeling of the many-agent market economic systems in the formal economic space, and eventually, of the five general principles of physical economics as well as probabilistic economic theory.

3. The Axioms and Principles of Physical Economics

Generally speaking, our attitude towards the problem of adequate quantitative description of the agent behavior in the market as well as market S&D and price formation is based on the two rather simple axioms of a very general character.

1. The Agent Identity Axiom.

All market agents are the same, only the supplies and demands they have different. The axiom says that all market agents share common properties, depending primarily on agent revenues and expenses, or more strictly, on supply and demand (S&D below) for traded goods and services. It is these agents' S&D that mainly determine the rational economic behavior of agents on the markets, and eventually the behavior of the whole markets. It shows a possibility of building rather common and accurate models of behavior of agents in the market, and hence the total market as a whole. It sets us on the right track for the identification and examination of the common properties in the behavior of market agents that ensure appearance of the common patterns and regularities in the course of market processes. It gives us the ability to build theoretical economic models on a fairly high scientific level by using physical and mathematical methods, which is the primary goal of physical economics and economic theory in general. We are certain that only these types of common market phenomena and processes are rightly a matter for exact scientific economic enquiry. In other words, it focuses us on building economics as an exact science in the image and after the likeness of the natural sciences.

2. The Agent Distinction Axiom.

All market agents are distinguished. The second axiom works when the first axiom fails. Thus, it defines those areas and aspects of the agents' behavior on the markets that are the subject of the studies of other sciences of more applied nature, such as marketing, behavioral science, managerial economics, psychology, policy, etc. In other words, these social sciences are concerned with the specific nuances and peculiarities in the behavior of concrete people, agents and communities in different markets and situations, etc.

Let us stress that in economics we do not study real people, but rather the real actions of these people on markets. The people can be different but market decisions and actions of these people can be the same, depending primarily on their supplies and demands on markets. It is this fact that lies in the background of the agent identity axiom.

Hence, we will sum up everything we have stated above in the form of the five general principles of physical economics and probabilistic economic theory as follows:

1. The Cooperation-Oriented Agent Principle.

The most important concept concerning markets is as follows: every market consists of market agents, buyers and sellers, all strongly interacting with each other. There are never any mysterious forces in markets. Everything that markets do, the cooperation-oriented market agents do, and therefore only the cooperation-oriented, agent-based models can provide the reasonable and justified foundation for any modern economic theory.

2. The Institutional and Environmental Principle.

Markets are never completely closed and free; all the market agents are under continuous influences and under such external institutional and environmental forces and factors as states, institutions, other markets and economies, natural and technogenic phenomena, etc. The influences, exerted by each of these forces and factors on the structure of market prices and on the market behavior, can be completely compared with the effect from inter-agent interactions. Moreover, the action of strong external institutional and environmental factors can significantly hamper the effective work of market mechanisms and even practically suppress it in a way that results in the breakdown of the market's invisible hand concept, well-known in classical economics. Therefore, the influence of institutional, environmental and other external factors must be adequately taken into account in the models, as well as simultaneously with the inter-agent interactions.

3. The Dynamic and Evolutionary Principle.

Markets are complex dynamic systems; all the market agents are in perpetual motion in search of profitable deals with each other for the sale or purchase of goods. Buyers tend to buy as cheaply as possible, and sellers want to obtain the highest possible prices. Mathematically, we can describe this time-dependent dynamic and evolutionary market process as motion in the price — quantity economic space of market agents acting in accordance with objective economic laws. Therefore, this motion has a deterministic character to some extent. This motion can and must be approximately described with the help of equations of motion;

4. The Market-Based Trade Maximization Principle.

On relatively free markets, the buyers and sellers consciously and deliberately enter into transactions of buying and selling with each other, since they make deals only under conditions in which they obtain the portion of profit that suits each of them. It is in no way compulsory that they aspire to maximize their profit in each concluded transaction, since they understand that the transactions can only be mutually beneficial. But they do attempt to increase their profit via the conclusion of a maximally possible quantity of such mutually beneficial transactions. Thus, it is possible to assert that the market as a whole strives for the largest possible volume of trade during the specific period of time. Consequently, we can make the conclusion that market dynamics can approximately be described and even approximate equations of motion for the market agents can be derived in turn by means of applying the market-based trade maximization principle to the whole economic system (more exactly, this principle is system-based).

5. The Uncertainty and Probability Principle.

Uncertainty and probability are essential parts of human action in markets. This is caused by the nature of human reasoning, as well as the fundamental human inability to accurately predict a future state of the markets. Furthermore, market outcome is the result of the actions of multiple agents, and no market is ever completely closed and free. For these reasons, all market processes are probabilistic by nature too, and an adequate description of all the market processes needs to apply probabilistic approaches and models in the economic price — quantity space. The uncertainty law results from this principle.

We assume that, from one side, these five general principles are capable of sufficiently and adequately describing the basic structural and dynamic properties of market economic systems and the market processes within them. From other side, they can be regarded as the basic pillars of physical economics, which carry on constructing step-by-step the bodies or frameworks of our physical economic models. These principles and their substantiation will be repeatedly discussed in more detail and step-by-step in this book. Concluding, let us stress that new probabilistic economic theory has been built on the basis of these principles in this book.

4. The Classical Economies

4.1. The Two-Agent Market Economies

As mentioned previously, below we will sequentially introduce into the theory the new concepts of physical modeling. They will be the building blocks in the construction of the body or framework of our models, which will also be filled step-by-step with new, concrete contents. We will start with the construction of the simplest physical economic models. In this paragraph we will create this with the use of analogies and formal methods of classical mechanics. These physical economic models will be referred to as *the classical economies*. Naturally in construction, we will use only first four principles, since only they have analogues in classical mechanics.

As we know, market agents are the buyers and sellers of goods and commodities, and as such are the major players in the market economy. They strongly interact with each other and with the institutions and the market's external environment including other market economies. They continuously make decisions concerning the prices and quantities of good, and buy or sell those in the market. All the market agents' actions govern the outcome of the market, wich is the essence of the agent principle. We believe the agents to behave to a certain extent in a deterministic way, striving to achieve their definite market goals. This means that the behavior of market agents is, in turn, governed by the strict the economic laws in the market. The fact that these laws have util now been of a descriptive nature in classical economic theory, and they have not yet been expressed in a precise mathematic language, is not of key importance in this case. What is really important is that we believe all the market agents to act according to the economic laws of social cooperation that can be approximately described with the help of the market-based trade maximization principle.

Every market agent acts in the market in accordance with the rule of obtaining maximum profit, benefit, or some other criterion of optimality. In this respect, we believe the many-agent market economic systems to resemble the physical many-particle systems where all the particles interact and move in physical space. This is also in accordance with the same system-based maximization principle, such as the least action principle in classical mechanics which is applied to the whole physical system under study. The analogous situation exists in quantum mechanics (see below in the Part F).

The main drive of our research was to take the opportunity to create dynamic physical models for market economic systems. We construct these physical economic models by analogy with physics, or more precisely by analogy with theoretical models of the physical systems, consisting of formal interacting particles in formal external fields or external environments [1]. Let us stress that these particles are fictitious; they do not really exist in nature. Therefore, the physical systems mentioned above are also fictitious and they do not exist in nature either. They are indeed only imagined constructions and served simply as patterns for constructing the physical economic models. Thus, these physical economic models consist of the economic subsystem, or simply the economy or the market. It contains a certain number of buyers and sellers, as well as its institutional and external environment with certain interactions between market agents, and between the market agents and the market institutional and external environment. Moreover, according to the dynamic and evolutionary principle we assume that equations of motion, derived in physics for physical systems in the physical space, can be creatively used to construct approximate equations of motion for the corresponding physical models of economic systems in the particular formal economic spaces.

Let us briefly give the following reasons to substantiate such an *ab initio* approach for the one-good, one-buyer, and one-seller market economy. Let price functions $p_1^D\ (t)$ and $p_1^S\ (t)$ designate desired good prices of the buyer and seller, respectively, set out by the agents during the negotiations between them at a certain moment in time t.

Analogously, by means of the quantity functions $q_1^D\ (t)$ and $q_1^S\ (t)$ we will designate the desired good quantities set out by the buyer and the seller during the negotiations in the market. Below, for brevity, we will refer to these desired values as the price and quantity quotations, which can or cannot be publicly declared by the buyer and the seller, depending on the established rules of work on the market. Note that the setting out of these quotations by the market agents is the essence of the most important market phenomenon in classical economic theory, namely the market process leading eventually to the concrete acts of choices of the market agents, being implemented by the buyer and seller through making deals (see below). Graphically, we can display these quotations as the agents' trajectories of motion in the formal economic space as will be shown below. In real market life, these quotations are discrete functions of time, but, for simplicity, we will visualize them graphically (as well as supply and demand functions, see below) as continuous linear functions or straight lines. This approximate procedure does not lead to a loss of generality, since these functions and lines are necessary to us. They are only for the illustration of the mechanism of the market work and for the most general graphic representation of the motion of the market agents in the two formal economic spaces, corresponding to the two independent variables, price P and quantity Q. We will refer to this agent motion as market behavior, for brevity, and sometimes the evolution of the economy in time. All these terms are, in essence, synonyms in this context of the discussion. And for simplicity we will call these spaces the price space and the quantity space, respectively, as well as the united space as the price-quantity space.

By setting out desired prices and quantities this way, buyers and sellers take part in the market process and act as *homo negotians* (a negotiating man) in the physical modeling, aiming to maximum satisfaction in their attempts to make a profit on the market. This is the first market equilibrium price p^{E_1} and quantity q^{E_1} at a moment in time t_1^E at which the agents' trajectories intersect, the deal takes place, and the interests of both the buyer and seller are optimally satisfied, taking inexplicitly into consideration the influence of the external environmental and institutional factors on the market in general. It is here that one can see similarity in the movement of the many-agent economic system in the price-quantity economic space (described by the buyer's trajectories $p_1^D\ (t)$, $q_1^D\ (t)$ and seller's trajectories $p_1^S\ (t)$,

$q_1^S\ (t)$) to the movement of the corresponding many-particle physical system in the physical space (described by the particles' trajectories $x_n(t)$) which is also subject to a certain physical principle of maximization. In Fig. 1, we give the graphic representation of these trajectories of agents' motion depending on the time with the help of the suitable coordinate systems of the time-price (t, P), and the time-quantity (t, Q), in the same manner as we do the construction of analogous particles' trajectories in classical mechanics. Below we will demonstrate a substantial similarity with physics that is depicted in the upper part of Fig. 1, with, the trajectory of the motion of agents in the price space (P-space below) and, in the lower part of Fig. 1 — in the quantity space (Q-space below). In the aggregate, both pictures represent the motion of market agents in the price quantity space (PQ-space below).

This agents' motion reflects the market process, which consists in changing continuously by the market agents their quotations. Note, we depicted in Fig. 1 a certain standard situation on the market, in which the buyer and the seller encountered deliberately at the moment of the time t_1 and began to discuss the potential transaction by a mutual exchange of information about their conditions, first of all the desired prices and the desired quantities of goods. During the negotiation, they continuously change these quotations until they agree on the final conditions of price p^{E_1} and quantity q^{E_1}, at the moment in time t_1^E. Such a simplest market model is applicable, for example, for the imaginable island economy in which once a year, a trade of grain occurs between a farmer and a hunter. They use the American dollar, \$. To illustrate, the situation is described below in Fig. 1. Note that in this and subsequent pictures we use arrows to indicate the direction of the agent's motion during the market process.

Up to the moment of t_1, the market has been in the simple state of rest, there were no trading in it at all. At the moment of the time t_1, there appear the buyer and the seller of grain in it, which set out their initial desired prices and quantities of grain, $p_1^D\ (t_1)$, $p_1^S\ (t_1)$, and $q_1^D\ (t_1)$, $q_1^S\ (t_1)$. Points P and V in the graphs show the position of the buyer (purchaser) and seller (vendor) at the given instant of t_1. It is natural that the desires of buyer and seller do not immediately coincide, buyer wants low price, but the seller strives for the higher price. However, both desires and needs for reaching understanding and completing

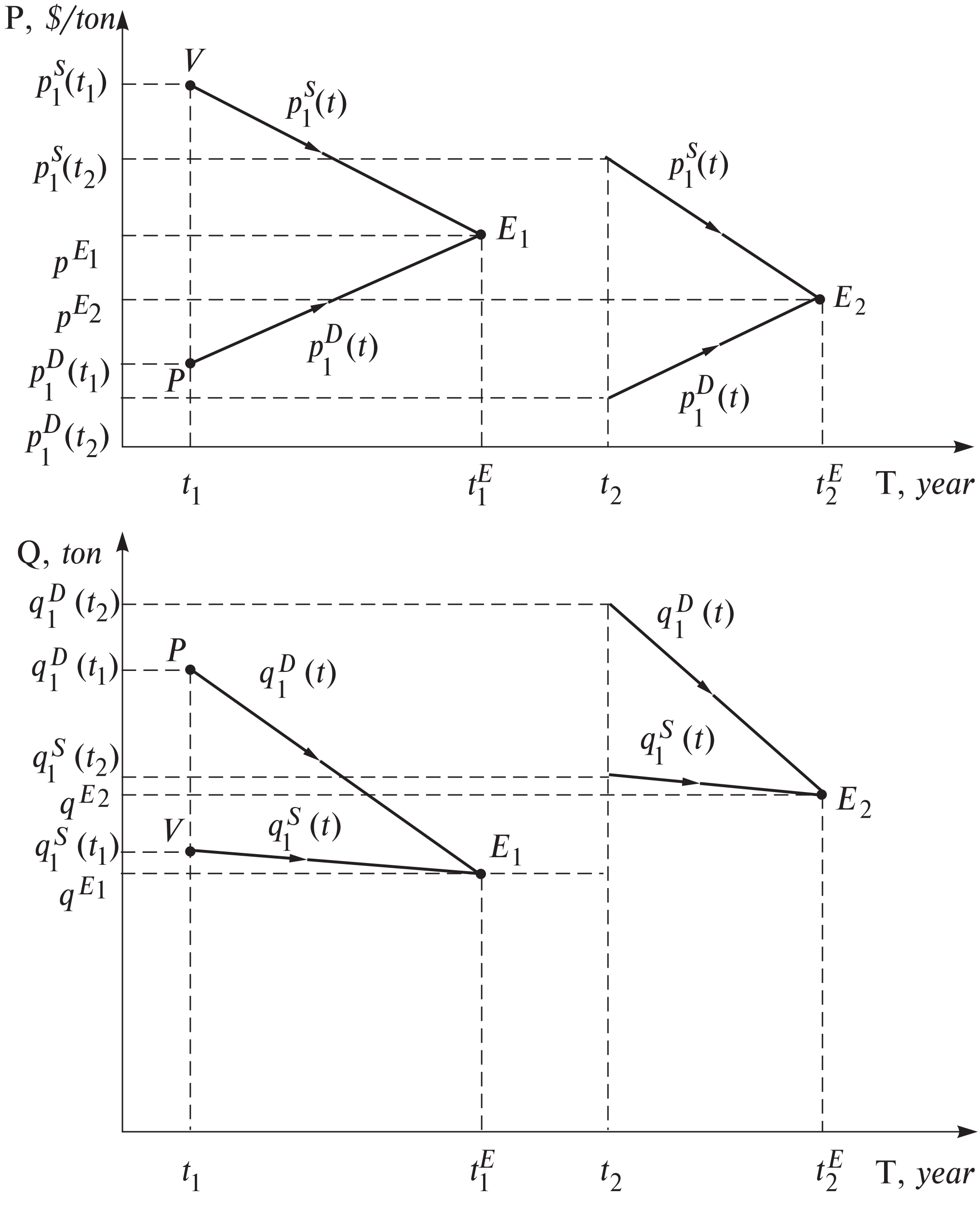

Fig. 1. Trajectory diagram displaying dynamics of the classical two-agent market economy in the one-dimensional economic price space (above) and in the economic quantity space (below). Dimension of time t is *year*, dimension of the price independent variable P is *$/ton*, and dimension of the quantity independent variable Q is *ton*.

transaction remain, otherwise the farmer and the hunter will have the difficult next year. The process of negotiations goes on, the market process of changing by the agents their quotations continues. As a result, the positions of the market agents converge and, after all, they coincide at the moment in time of t_1^E, which corresponds to the trajectories' intersection point E_1 on the graphs.

A voluntary transaction is accomplished to the mutual satisfaction. Further, the market again is immersed into the state of rest until the next harvest and its display to sale next year at the moment in time of t_2. Harvest in this season grew, therefore $q_1^S(t_2) > q_1^S(t_1)$. In this situation, the seller is, obviously, forced to immediately set out the lower starting price, $p_1^S(t_2) < p_1^S(t_1)$, while the buyer, seizing the opportunity, also reduced their price and increased their quantity of grain: $p_1^D(t_2) < p_1^D(t_1)$ and $q_1^D(t_2) > q_1^D(t_1)$. It is natural to expect in this case that the trajectories of the buyer and the seller would be slightly changed, and agreement between the buyer and the seller will be achieved with other parameters than in the previous round of trading.

Conventionally, we will describe the state of the market at every moment in time by the set of real market prices and quantities of real deals which really take place in the market. As we can see from the Fig. 1 real deals occur in the market in our case only at the moments t_1^E and t_2^E when the following market equilibrium conditions are valid (points E_i in Fig. 1):

$$p_1^D\left(t_i^E\right) = p_1^S\left(t_i^E\right) = p^{E_i}, \quad q_1^D\left(t_i^E\right) = q_1^S\left(t_i^E\right) = q^{E_i},$$
$$D_1^0\left(t_i^E\right) = S_1^0\left(t_i^E\right) = MTV\left(t_i^E\right), \; i = 1, 2. \qquad (1)$$

In this formula, we used several new notions and definitions, whose meanings need explanation. Let us make these explanations in sufficient detail in view of their importance for understanding the following presentation of physical economics. First, in contemporary economic theory, the concept of supply and demand (S&D below) plays one of the central roles. Intuitively, at the qualitative descriptive level, all economists comprehend what this concept means. Complexities and readings appear only in practice with the attempts to give a mathematical treatment to these notions and to develop an adequate method of their calculation and measurement. For this purpose, the various theories contain different mathematical models of S&D that have been developed within the framework. In these theories, differing so-called S&D functions are used to formally define and quantitatively describe S&D.

In this book, we will also repeatedly encounter the various mathematical representations of this concept in different theories, which compose physical economics, namely, classical economy, probability economics, and quantum economy.

Even within the framework of one theory, it is possible to give several formal definitions of S&D functions supplementing each other. For example, within the framework of our two-agent classical economy, we can define total S&D functions as follows:

$$D_1^0(t) \cong p_1^D(t) \times q_1^D(t), \quad S_1^0\,(t) \cong p^S(t) \times q^S(t). \tag{2}$$

Thus, we have defined at each moment of time t the total demand function of the buyer, $D_1^0(t)$, and the total supply function of the seller, $S_1^0(t)$, as the product of their price and quantity quotations. These functions can be easily depicted in the coordinate system of time and S&D [T, S&D], as it was done in Fig. 2 displaying the so-called S&D diagram. As one would expect, the S&D functions intersect at the equilibrium point E. It is accepted in such cases to indicate that S&D are equal at the equilibrium point. We consider that it is more strictly to say that equilibrium point is that point on the diagram of the trajectories, where these trajectories intersect, i.e., where the price and quantity quotations of the buyer and the seller are equal. But that in this case S&D curves intersect is the simple consequence of their definition equality of prices and quantities at the equilibrium point.

The last observation here concerns a formula for evaluating the volume of trade in the market, $MTV(t_i^E)$, between the buyer and the seller where they come to a mutual understanding and accomplishment of transaction at the equilibrium point E_i. It is clear that to obtain the trade volume (total value of all the transactions in this case), it is possible to simply multiply the equilibrium values of price and quantity that are derived from the above formula. The dimension of the trade volume is of course a product of the dimensions of price and quantity; in this example this is *\$*. The same is valid for the dimensions of the total S&D, $D_1^0(t)$ and $S_1^0(t)$.

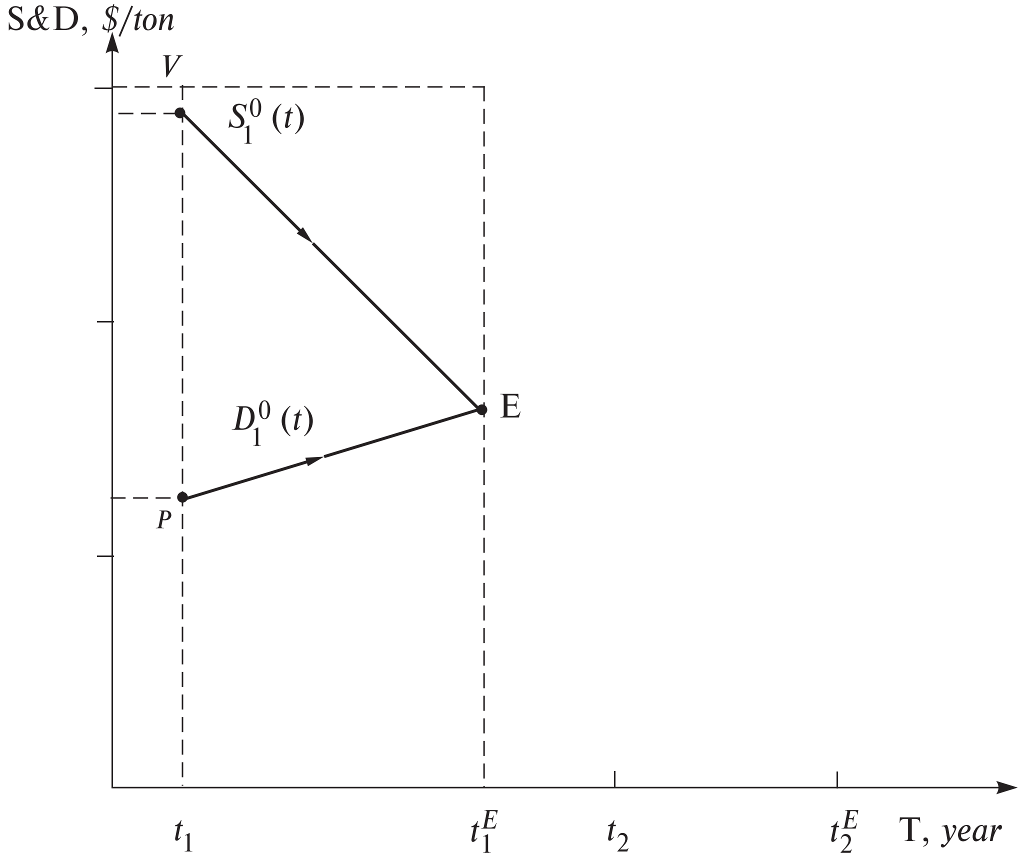

Fig. 2. S&D diagram displaying dynamics of the classical two-agent market economy in the time-S&D functions coordinate system [T, S&D], within the first time interval $[t_1, t_1^E]$.

4.2. The Main Market Rule "Sell all — Buy at all"

Having a method to more or less evaluate the price quantitatively is always advantageous, as it helps us to somewhat predict market prices. Using the main rule of work on the market is used to this end, and this strategic rule of decision making can be briefly formulated as follows: "Sell all — Buy at all". This main market rule indicates the following different strategies of market actions (action on the market is setting out quotations) for both the seller and the buyer. For the seller this strategy consists in striving to sell all the goods planned to sale at the maximally possible highest prices. Whereas for the buyer this strategy consists in the fact that it will expend all the money planned for the purchase of goods and try to purchase in this case as much as possible at the possible smallest price. Thus, the main market rule leads to the corresponding algorithms of the actions of agents on the markets, which are graphically represented in the form of agents' trajectories in the pictures. The point of intersection and the respective trade volume in the market, *MTV*, are easily found with the help of the following mathematical formulas:

$$q^{E_i} \cong q_1^S(t_i),\ D_1^0(t_i) = p_1^D(t_i) \times q_1^D(t_i),$$
$$p^{E_i} \cong D_1^0(t_i)/q_1^S(t_i) = p_1^D(t_i) \times q_1^D(t_i) / q_1^S(t_i),$$
$$MTV = p^{E_i} \times q^{E_i} = D_1^0(t_i),\ \ i = 1,\ 2. \qquad (3)$$

It is natural here to name $D_1^0(t_1)$ the total demand of buyer at the initial moment of trading. The sense of this quantity is in the fact that this is quantity of resources, planned for the purchase of goods, expressed in the money, although the dimension of this demand is the dimension of money price (*\$/ton*) multiplied by the dimension of quantity (*ton*). In our case, this is *\$*. We emphasize that, over the course of development of quantitative theory, this is very important in order to draw attention to the dimension of the used quantities and parameters, and to the normalization of the applied functions (see below).

By analogy with classical mechanics, we can treat these prices and quantity functions as the trajectories of movement of the market agents in the two-dimensional economic PQ-space as it was displayed in Fig. 3.

In principle, this representation gives nothing new in comparison with Figs. 1 and 2. Nevertheless, there is one interesting nuance here, in which the similarity of this diagram can be compared to the traditional picture in the conceptual neoclassical model of S&D. We will examine this question below. But let us now focus attention on the following nuances in the picture in Fig. 3. First, it is clearly shown by the arrows, that the buyer and the seller seemingly move towards each other on the price, with the seller reducing it, and the buyer, on the contrary, increasing it. From this, we can reflect on the illustration of normal market negotiation processes. Secondly, usually the quotations of quantities are reduced during the process of negotiations both by the buyer and by the seller. Clearly, all agents want to purchase or to sell a smaller quantity of goods at the compromise market price than at the most desired, presented at the very beginning of trading.

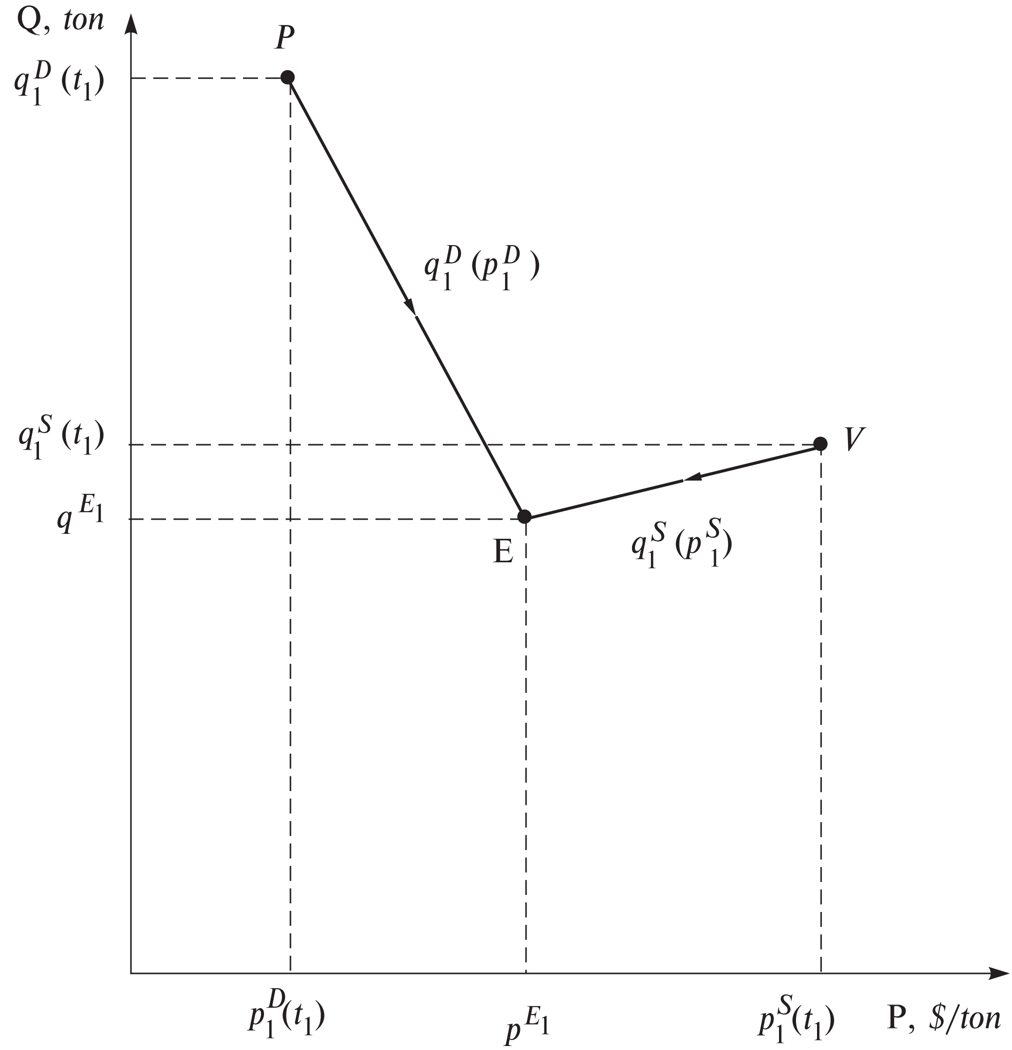

Fig. 3. Dynamics of the classical two-agent market economy in the two-dimensional economic price-quantity space within the first time interval $[t_1, t_1^E]$.

And now we turn from the simplest economy to a more developed economy, in which the farmer and hunter gradually switch from the discrete trade system (one trade per year) to the continuous trade system on the market. Generally speaking, negotiations are conducted continuously and transactions are accomplished continuously, depending on the needs of the buyer and the seller. This would continue for many years. Taking into account this new long-term outlook it is expedient to change somewhat the method of describing the work of the

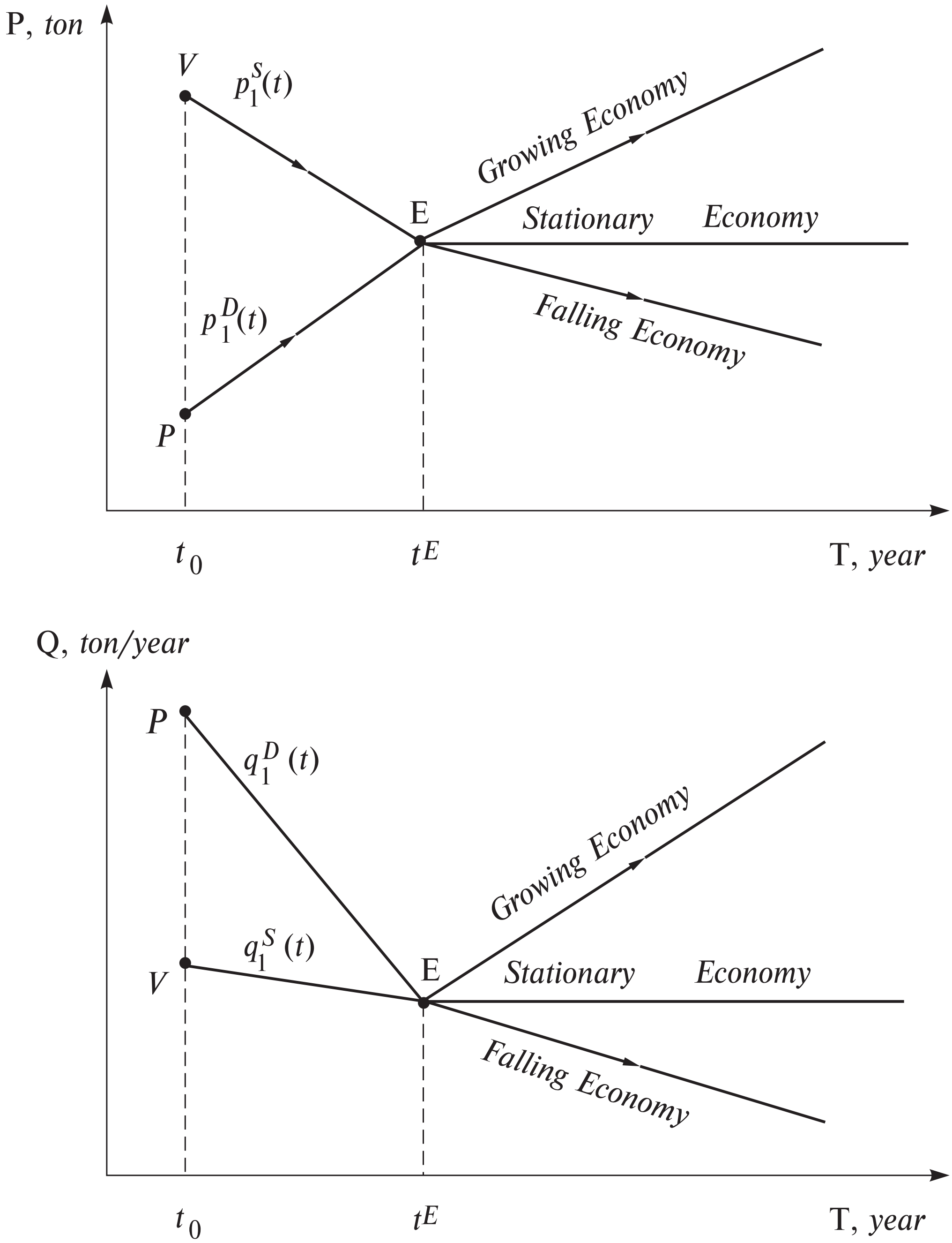

Fig. 4. The classical stationary and non-stationary two-agent market economies in the [T, P] and [T, Q] coordinate systems in the time interval $t > t^E$.

market. Namely, by quotations of a quantity of goods, it is now more convenient to represent a quantity of goods during a specific and reasonable period of time, for example year, if the discussion deals with the long-standing work of the market. In this case the dimension of a quantity would be represented by *ton/year*. We show in Figs. 4, 5 how it is possible to graphically represent the work of the market over a long span of time. We see that before the establishment of equilibrium at point E, transactions were of course accomplished, but probably did not bring maximum satisfaction to the participants in the market. This would induce agents to continue to search for long-term compromises in prices and quantities. After reaching equilibrium, the volume of trade reaches a maximum, and participants in the market therefore attempt to further support this equilibrium.

Here a fork appears in the following theory: — look at Figs. 5 and 6. If quotations cease to change, then the economy converts to a stationary state in which time appears to disappear. This is especially noticeable in Fig. 6, where this sort of stationary state is described by one point, E. We will label the economies in the stationary state simply *the stationary economies*. But if

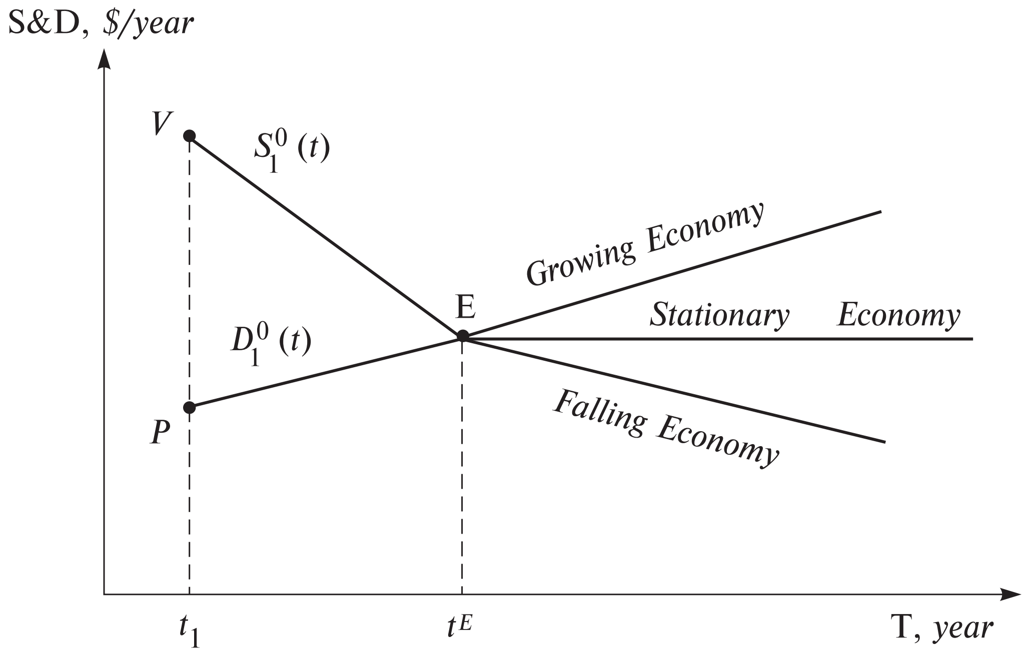

Fig. 5. The classical stationary and non-stationary two-agent market economies in the [T, S&D] — coordinate system.

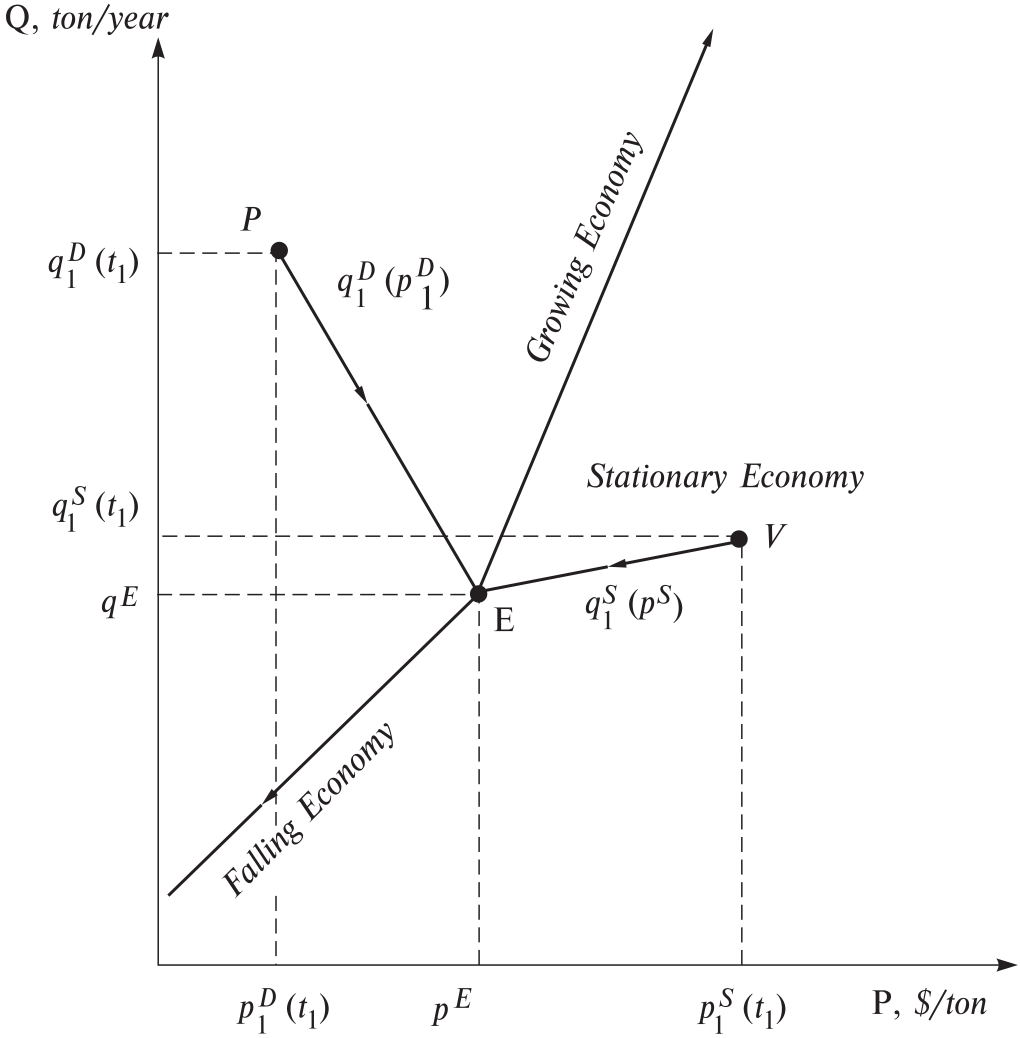

Fig. 6. The classical stationary and non-stationary two-agent market economies in the economic price-quantity space at $t > t^E$.

quotations vary with time, then the economy will be named the *time-dependent* or simply *non-stationary economies*. In Figs. 5 and 6 they are represented by two lines, which emanate from the equilibrium point E. If in this case the equilibrium quantity grows, then the economy is a growing one. But if it decreases, then the economy is falling one, which clearly is represented in Fig. 5. As a rule, in such cases, the total S&D behave similarly and this can be easily seen in Fig. 5. Let us note that their dimensions in this model have also changed, now equaling $\$ \cdot ton/year$.

4.3. The Many-Agent Market Economies

Now we will increase the level of complexity of the classical economies byexamining how it is possible to incorporate several buyers and sellers into the theory. It is understandable that each market agent will have its own trajectories in the PQ-space. In principle, they can vary greatly. There is good reason to believe that there is much similarity in the behavior of all buyers in general. The same is valid of course for all sellers. The reason is as follows. There is the intense information exchange on the market, by means of which the coordination of actions is achieved among the buyers, among the sellers, as well as among the buyers and sellers. This coordination is carried out to assist the market in reaching its maximum volume of trade, since it is precisely during the process of trading that the last point is placed in the long process of preliminary business operations: production, financing, logistics, etc. This is exactly what we would have referred to earlier as the social cooperation of the market's agents. For example, it is natural to expect that all buyers, from one side, and sellers, from other side, behave on the market in approximately the same way, since they all are guided in their behavior on the market by one and the same main rule of work on the market: "Sell all — Buy at all".

Hence it is possible to draw from the above discussion the following important conclusion: the trajectories of all buyers in the P-space will be close to each other; therefore, the totality of all buyers' trajectories can be graphically represented in the form of a relatively narrow "pipe", in which will be plotted the trajectories of all buyers. It is also possible to represent all price trajectories of the buyers by means of a single averaged trajectory, $p^D(t)$, which we will do below. We will do the same for the sellers, and their single averaged price trajectory we will designate as $p^S(t)$.

We have a completely different situation with the quantity trajectories, since each market agent can have the very different quantities, bearing in mind the fact that the behavior of the buyers' (sellers') curves can be relatively similar to each other. Nevertheless, we can establish some regularities in the behavior of the whole market, being guided by common sense and the logical method. Since the quotations of quantities are real in the classical models, we can add them in order to obtain the quantity quotations of the whole market, $q^D(t)$ и $q^S(t)$. However one should do this separately for the buyers and sellers as follows:

$$q^D(t) = \sum_{n=1}^{N} q_n^D(t), \quad q^S(t) = \sum_{m=1}^{M} q_m^S(t), \tag{4}$$

where summing up of quantity quotations is executed formally for the market, which consists of N buyers and M sellers. In this case we understand that for the whole market we can draw all the same pictures as displayed in Figs. 1–6 for the two-agent market. Thus, for instance, we can represent the dynamics of our many-agent market by the help of the following pictures in Fig. 7. In it, the dynamics of

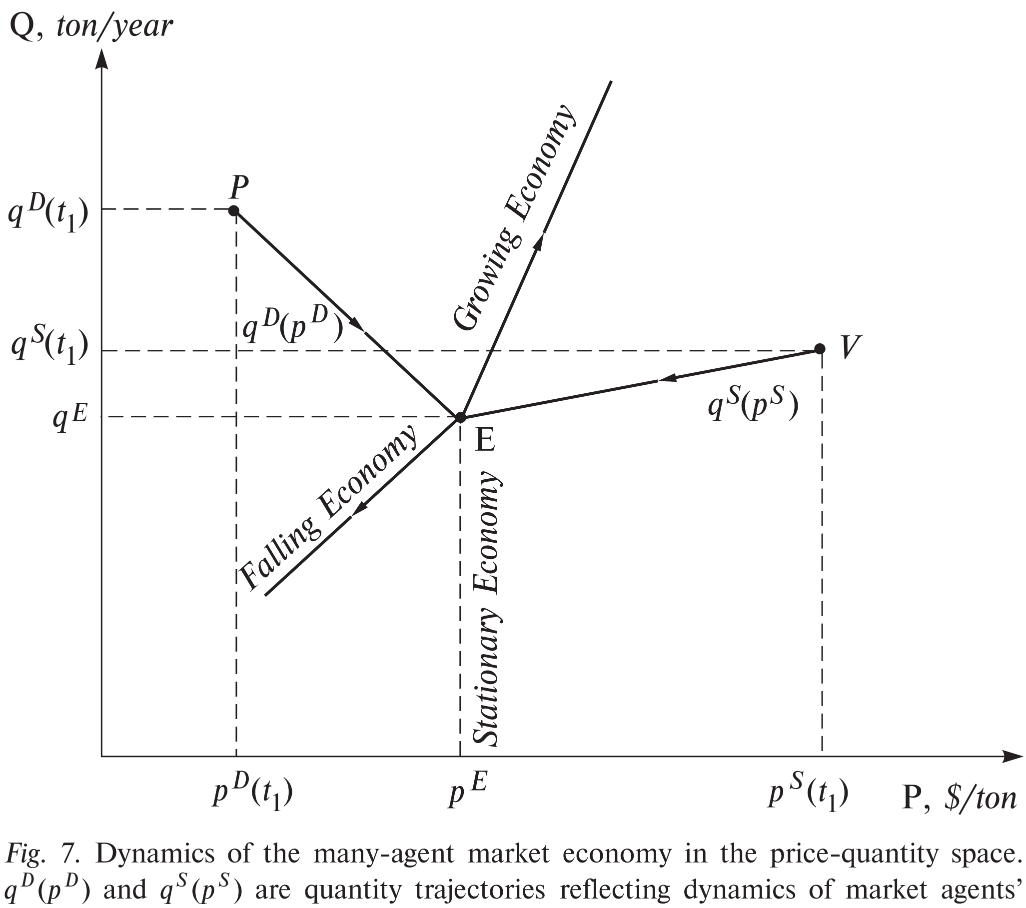

Fig. 7. Dynamics of the many-agent market economy in the price-quantity space. $q^D(p^D)$ and $q^S(p^S)$ are quantity trajectories reflecting dynamics of market agents' quotations in time up to the moment of establishment of tof establishment of the equilibrium and making transactions at the equilibrium price.

many-agent market are depicted at the moment of equilibrium (curves $q^D(p^D)$ and $q^S(p^S)$), as well as dynamics of the stationary economy (point E) and dynamics of the non-stationary growing and falling economies.

4.4. The Classical Economies *versus* Neoclassical Economies

Let us call attention to the fact that, in Figs. 3 and 6, the quotation curve of the buyer, $q_1^D\ (p^D)$, has negative slope, and the slope of the quotation curve of the seller, $q_1^S\ (p^S)$, is positive. This reflects the natural desire of the buyer to purchase more at the lower price, as far as possible, and the natural desire of the seller to sell more at the higher price, as far as possible. Specifically, it is here we reveal the visual similarity of the classical economies to the known neoclassical model of S&D. But the visual similarity of picture in Figs. 3, 6 with the corresponding famous neoclassical picture in the form of two intersected lines of S&D is only formal; economic content in them is entirely different. In classical economies, this is a graphic representation of the real market process (which really occurs on the market at a given instant), while in the neoclassical economies, this picture expresses the planned actions of market agents on the market in the future. Note that in neoclassical economics it is namely the curves $q^S(p^S)$ and $q^D(p^D)$ that are called S&D functions. The economic content of these S&D functions can be roughly expressed thus: "the market is by itself, I am by myself". If one price is on the market, then I purchase (or I sell) one quantity, and if it is another, then will I purchase (or I will sell) another quantity, and so forth. Thus, the market process is completely ignored in the neoclassical economic model. But we know that, in real market life, all market agents participate continuously in the market process, permanently changing price and quantity quotations, since each has made transaction price changes on the market. We will discuss neoclassical models in more detail in other chapters of the book. However, we will now develop an artificial classical economy that will be as similar to the neoclassical model as possible.

We will call this model *the quasi-market economy* with the "visible hand of the market" in order to distinguish it from the economies having self-organizing markets, or economies that exhibit the "invisible hand of market". In the quasi-market economy, there is a definite chief (very strict and all-seeing by definition) of the market (visible hand of the market), to whom all agents of the market for the planned period, let us say a year, must pass very detailed and reliable plans with respect to purchase and sale of goods. These plans are compiled by agents and are given to the chief in the form of tables, which are formed according to the rule stated above. If just such a price is found on the market, then I will sell a particular volume; if it is another price, then I will purchase (or sell) another, specific volume, and so forth. These agent tables are represented in Fig. 8 for simplicity in the form of continuous straight lines, a factor which does not decrease their generality in this case.

Let us note that each agent passes its plan to the chief in the form of table, and chief itself unites data of these plans and presents them in the easy-to-use shape of the two straight lines in one picture. Common sense tells us that it is most profitable for the buyer to purchase more at the minimum price. But in this scenario the seller would want to sell less. The opposite would be true for both buyer and seller at the maximum price. Graphically, this is reflected in the fact that when point

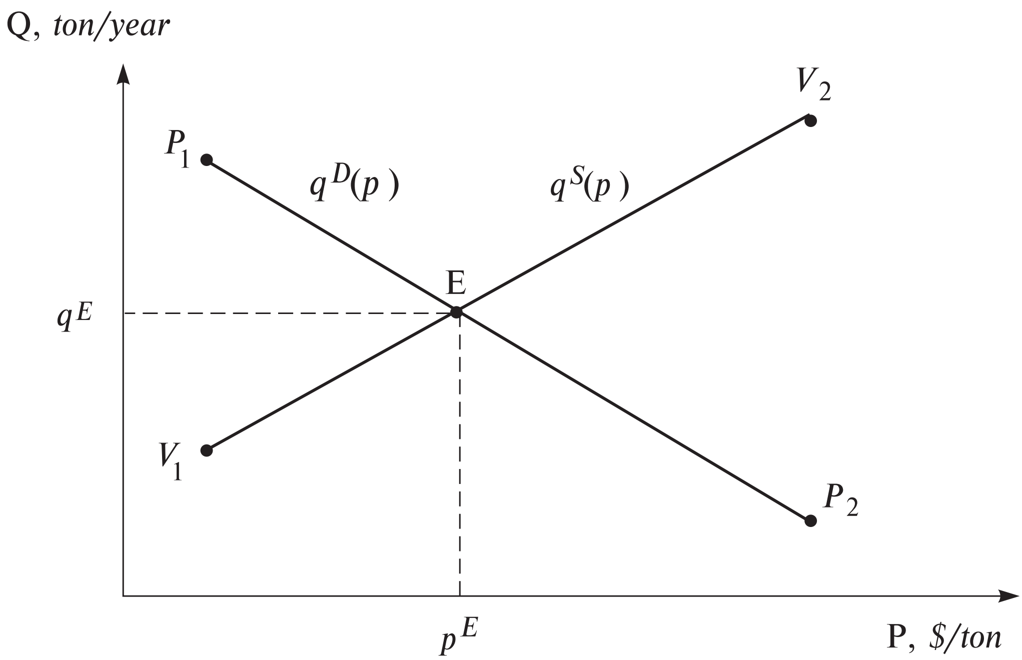

Fig. 8. The classical two-agent quasi-market economy in the economic price-quantity space.

P_1 is higher than point V_1, and the point V_2 higher than point P_2, the consequence is that the slope of the curve of the buyer is be negative, and the slope of the curve of the seller, positive. It is obvious that these two straight lines will compulsorily be crossed at the point E (p^E, q^E), where the prices and quantities of the buyer and seller coincide. Next, the chief considers that these prices and quantities reflect certain equilibrium in the market, he or she calls the equilibrium price and quantity and declares that these values of price and quantity are set for the market year. Market process is, in this case, further completely eliminated from market life, in that the decisions of the market's chief completely substitutes it during the next year. We call this model economy a quasi-market one, since plans are compiled by market agents. However, they realize them in the prices and the quantities that are essentially dictated by the market's chief.

We consider this quasi-market, stationary classical economy to be, in essence, the neoclassical model of S&D. The graphic representation of the neoclassical model economy is, by the way, the same Fig. 8, since plans in the neoclassical theory are drawn up in precisely the same way that we described above for the quasi-market economy. But in neoclassical economics, it is considered *a priori* that the market itself in some manner will carry out the role, which the chief of the market fills in the quasi-market economy. But if the market process is absent and there are no actions of agents adapting to the market, then who or what will fill this role? Moreover, if the economy is in a stationary state, then all prices and quantities are already known to all participants in the market, and their plans then are graphically reduced simply to one point: E. It is here that the neoclassical model generally lacks any sense or value.

References

1. A.V. Kondratenko. *Physical Modeling of Economic Systems. Classical and Quantum Economies*. Nauka, Novosibirsk, 2005. Electronic copy available at: http://ssrn.com/abstract=1304630.

CHAPTER II.

The Constructive Design of the Agent-Based Physical Economic Models

> "The specific method of economics is the method of imaginary constructions... Everyone who wants to express an opinion about the problems commonly called economic takes recourse to this method... An imaginary construction is a conceptual image of a sequence of events logically evolved from the elements of action employed in its formation. It is a product of deduction, ultimately derived from the fundamental category of action, the act of preferring and setting aside. In designing such an imaginary construction the economist is not concerned with the question of whether or not it depicts the conditions of reality which he wants to analyze. Nor does he bother about the question of whether or not such a system as his imaginary construction posits could be conceived as really existent and in operation. Even imaginary constructions which are inconceivable, self-contradictory, or unrealizable can render useful, even indispensable services in the comprehension of reality, provided the economist knows how to use them properly. The method of imaginary constructions is justified by its success. Praxeology cannot, like the natural sciences, base its teachings upon laboratory experiments and sensory perception of external objects... The main formula for designing of imaginary constructions is to abstract from the operation of some conditions present in actual action. Then we are in a position to grasp the hypothetical consequences of the absence of these conditions and to conceive the effects of their existence... The method of imaginary constructions is indispensable for praxeology; it is the only method of praxeological and economic inquiry".
>
> Ludwig von Mises. *Human Action. A Treatise on Economics*. Page 236

PREVIEW.

What is the Physical Economic Model?

This is the conceptual mathematical dynamic model of the many-agent economic systems in the formal space of the independent variables, prices and quantities, of all the market agents. It is built through a significant analogy with the theoretical physical models of the many-particle systems in real space, but taking consistently into account basic, specific differences in the economic and physical systems. Each model is, in essence, only an imaginary construction, which by no means completely reflects genuine reality. It is, however, capable of describing one or several basic special features of structure or functioning of the market economy in sufficiently strict mathematical language. It was primarily created to provide fresh insight into these features, and, after the addition of another model feature or interaction, to understand, precisely how it influences entire end results.

1. The Basic Concept of Physical Economic Design

By stretching a point, we can say that the basic concept of design of our physical economic models is skeuomorphism. Let us explain what this concept means in our case. As we have already mentioned repeatedly in this book, when constructing physical economic models we strive to reach a formal mathematical, linguistic and even graphical similarity to their physical prototypes. Specifically, this concerns both the structure and the dynamics, as well as the language and the methods of representation of the obtained results, including graphics. We consistently follow this basic concept of design throughout the book. Let us stress another point. Our main task in the book is the construction of economic models that, in as much as possible, highly resemble or copy the known form and custom physicists' models of many-particle systems. This facilitates understanding of the models and makes it possible to use the existing, detailed language of physics within a new economic framework. For example, the language of wave functions and probability distributions will be widely used here below, although this, of course, unavoidably leads to the appearance in the theory of a large quantity of neologisms. This may strongly hamper the reading of texts by economists, but substantially facilitates this process for specialists in the fields of natural sciences. We think that this basic design concept is quite adequate for building physics-based models of economic systems.

We begin with the requirement that the graphical scheme of the physical models must be similar to the picture of the many-particle physical systems, such as polyatomic molecules, for instance. Main elements of our physical models of economic systems are shown schematically in Fig. 1. A large sphere covers a market subsystem or

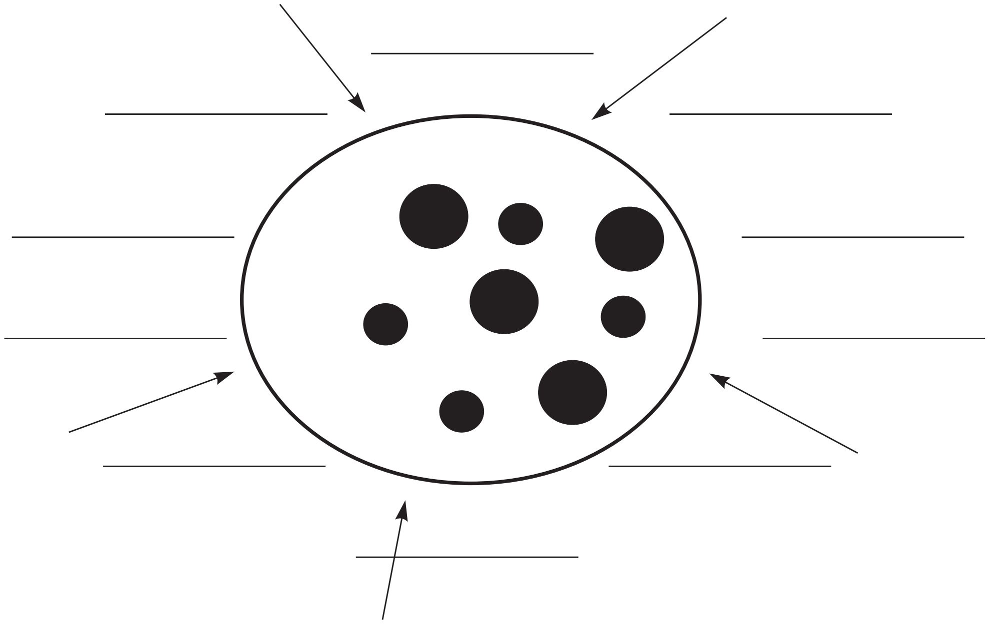

Fig. 1. The physical model scheme of an economic system: a market consisting of the interacting buyers (small dots) and sellers (big dots) who are under the influence of the internal institutions and the external environment beyond the market (covered by the conventional imaginary sphere).

economy consisting of active market subjects: buyers who have financial resources and a desire to buy goods or commodities, and sellers who have goods or commodities and a desire to sell them. They are the sellers and the buyers who form supply and demand (S&D below) in the market. Small dots inside the sphere denote buyers, and big ones denote sellers. The cross-hatched area outside the sphere represents the institutional and external environments, or more exactly, internal institutions such as the state, government, society, trade unions etc., and the external environment including other markets and economies, natural factors etc. It is evident that all elements and factors of the system influence each other; buyers compete with each other in the market for goods and sellers compete with each other for the money of buyers. Buyers and sellers interact with each other, permanently influencing each other's behavior. Institutions and the external environment influence all the eco-nomic agents, including not only businesses but also ordinary people. In other words, all the economic agents are influenced by institutional and external environments and interact with each other.

In order to develop a physical model of the economic system, it is necessary to learn to describe in an exact, mathematical way both movements (behaviors and influences) of each economic agent, i.e., buyers and sellers, the state and other institutions etc., and interactions with each other. It is the goal to derive equations of motion for market agents — the buyers and sellers — who determine the dynamics, movement, or evolution of the market system in time.

2. The Economic Multi-Dimensional Price-Quantity Space

As we already discussed above, in order to show the movement or dynamics of an economy it is necessary to introduce a formal economic space in which this movement takes place. As an example of such space we can choose a formal price space designed by the analogy with a common physical space. We choose the prices P_i of the i-th item of goods as coordinate axes: $i = 1, 2, ..., L$, where L is the number of items or goods (the bold **P** will designate below all the L price coordinates). In case there is only one good, the space is one-dimensional and represented by a single line. The coordinate system for the one-dimensional space is shown in Fig. 2.

The distance between two points in one-dimensional space p' and p'' can be for instance determined by the following:

$$|p' - p''| = \sqrt{(p'' - p')^2}. \tag{1}$$

If two goods are traded on the market ($L = 2$), the space is a plane; the coordinate system is represented in this case by two mutually perpendicular lines (see Fig. 3).

The distance between two points p' and p'' can be determined as follows:

$$|p'' - p'| = \sqrt{(p_1'' - p_1')^2 + (p_2'' - p_2')^2}. \tag{2}$$

We can build the price economic space of any dimension L in the same way. In spite of its apparent simplicity, the introduction of the

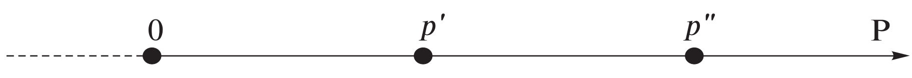

Fig. 2. The economic one-dimensional price space for the one-good market economy.

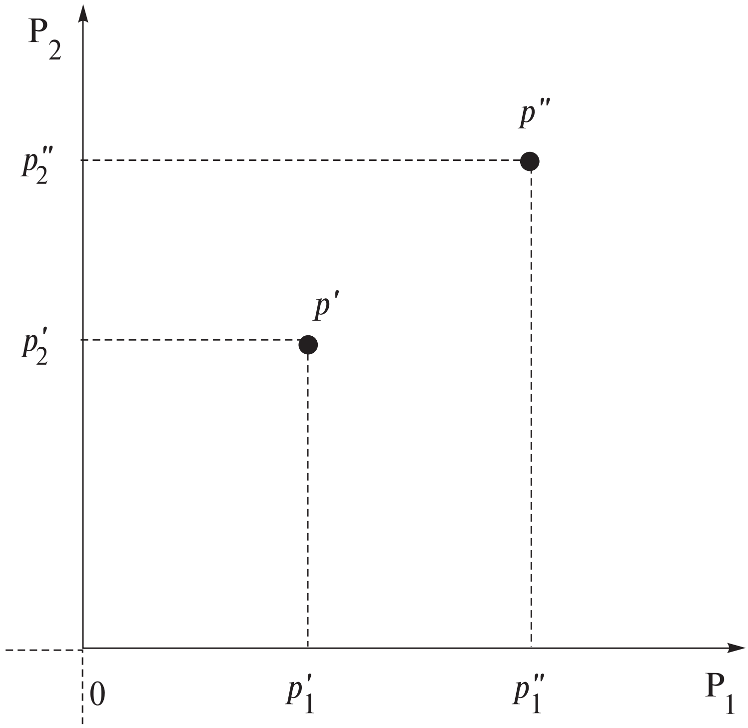

Fig. 3. The economic two-dimensional price space for the two-good market economy.

formal economic price space is of conceptual importance as it allows us to describe behavior of market agents in general mathematical terms. It represents realistic occurrences, as setting out their own price for goods at any moment of time *t* is the main function or activity of market agents. It is, in fact, the main feature or trajectory of agents' behavior in the market. Let us stress once again that it is our main goal to learn to describe these trajectories or the distributions of price probability connected with them. It is impossible to do this in a physical space. For example, we can thoroughly describe movement or the trajectory of a seller with goods in physical space, especially if they are in a car or in a spaceship. However, this description will not supply us with any understanding of their attitude towards the given goods; nor will it explain their behavior or value estimation regarding the goods as an economic agent.

Within the problem of describing agents' behavior in the market, the role of the good prices P as independent variables, or a coordinates **P** is considered here to be in many situations a unique one for market economic systems. In these cases we can study market dynamics in the economic price spaces. But market situations occur fairly often in which we need to explicitly take into account the independent good quantity variables **Q** (the bold **Q** will designate below all the L quantity coordinates) and consequently to describe economic dynamics in the economic $2 \times L$-dimensional price-quantity spaces. In these scenarios, we can imagine that the whole economic system is located in the multi-dimensional price-quantity space as it is displayed in Fig. 4. We have already used many aspects of this idea naturally when discussing classical economics. We will address any concerns in the upcoming chapters.

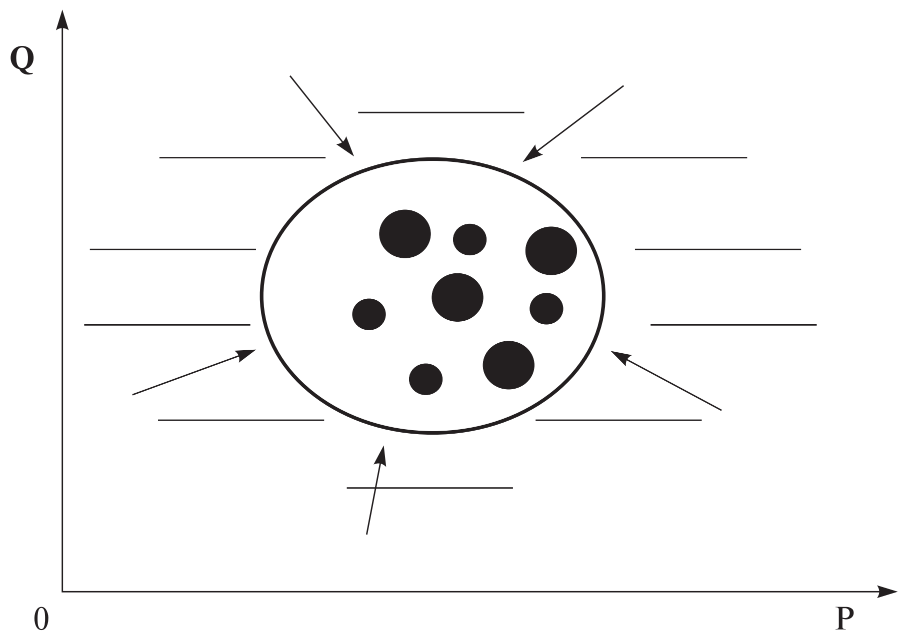

Fig. 4. The graphical model of the many-good, many-agent market economy in the economic multi-dimensional price-quantity space. It is displayed schematically in the conventional rectangular multi-dimensional coordinate system [**P**, **Q**] where, as usual, bold **P** and **Q** designate the price and quantity coordinate axes for all the goods. Again, our model economy consists of the market and the institutional and external environment. The market consists of buyers (small dots) and sellers (big dots) covered by the conventional sphere. Very many people, institutions, and natural and other factors can represent the external environment (cross — hatched area behind the sphere) of the market which exerts perturbations on market agents (pictured by arrows pointing from environment to the market).

3. The Market-Based Trade Maximization Principle and the Economic Equations of Motion

As we saw above in the example of the simplest classical economies, market agents actively make trade transactions, and there are no trade deals at all out of the equilibrium state. As the inclination of market agents' action is to make deals, we can naturally conclude that market agents and the market as a whole strive to approach an equilibrium state that can be expressed as the natural tendency of the market to reach the maximum volume of trade. This fact can serve as a guide for using the market agents' trajectories to describe their dynamics. Moreover, this fact gives us grounds to expect that equations of motion can be derived from the market-based maximization principle, used to describe these trajectories. Specifically, the main market rule "Sell all — Buy at all" can be regarded to some extent as a verbal expression of both the tendency of the market toward the trade volume maximum, and the principal ability to describe market dynamics by means of agent trajectories as solutions to certain equations of motion.

The second reason of we have confidence in creating a successful dynamic or time-dependent theory of economic systems in the economic spaces is based on the analogous dynamic theory of physical systems in physical space. We also admit that the reasonable starting point in the study of economic systems dynamics is with equations of motion for a formal physical prototype. This is in spite of the differences between the features of the economic and physical spaces and the features of the economic and physical systems. The type of equations in the spaces of both systems will be approximately the same, though the essence of the parameters and potentials in them will be completely different. It is normal in physics that one and the same equation describes different systems. For example, the equation of motion of a harmonic oscillator describes the motion of both a simple pendulum and an electromagnetic wave. Formal similarity of the equations does not mean equality of the systems which they describe.

The discipline of physics has accumulated broad experience in calculating the physical systems of different degrees of complexity with different inter-particle interactions and interactions of particles with external environments. It makes sense to try and find a way to use these achievements in finding solutions to economic problems. Should any of these attempts prove to be successful, it would establish the opportunity to do numerical research on the influence that both internal and external factors exert on the behaviours of each market agent, as well as the entire economic system's activity. This process would be done with the help of computer calculations done on the physical economic models. Theoretical economics will have acquired the most powerful research device, the opportunities of which could only be compared to the result of the discovery and exploration of equations of motion for physical systems.

The next step in developing a physical model after selecting an appropriate economic space, is the selection of a function that will assist us in describing the dynamics of an economy, such as the movement of buyers and sellers in the price space. Trajectories in coordinate physical space $x(t)$ (classical mechanics), wave functions ψ or distributions of probabilities $|\psi|^2$ (quantum mechanics), Green's functions G and S-matrices (in quantum physics), etc. are used as such functions in physics. We started above with an attempt to develop the model using trajectories in the price space $p(t)$ by analogy with the use of trajectories $x(t)$ of point-like particles used in classical mechanics. Below, this model is referred to as a classical model or simply, a classical economy. Below, we will use the term classical economy in the broad sense for designating the branch of physical modeling of many-agent economic systems with the help of methods of classical mechanics of many-particle systems. It is important to realize that each selection gives rise to its own equations of motion and, therefore, to different physical economic models. For example, if we select from these trajectory variants, then we obtain the economic Lagrange equations of motion and, therefore, the classical economies as the physical economic models. The discussion will deal with these models in detail in Chapter III. If we select wave functions, then we obtain at the output the economic Schrödinger equations of motion and, therefore, quantum economies (see Chapters IX and X). Without going into details here, let us say that both the Lagrange and Schrödinger equations appear as the result of applying the principles of maximization to the whole economic system. This is analogous to the maximization principles, which are explored in physics in obtaining the Lagrange and Schrödinger equations, respectively.

Strictly speaking, all these principles of maximization, both in economics and physics, are in essence a set of hypotheses. Their validity or effectiveness can be confirmed only via practical calculations and comparison of their results with the respective known laws and phenomena, as well as with the relevant big empirical data. But intuition suggests that this way of developing economic theory is most optimum at the present time. Since it is presently not known how to derive equations of motion in economics, borrowing existing theoretical structural models from physics is helpful. Since analogies can be drawn between the spaces and features of both physics and economics, we can use skeuomorphism and transfer the design models from the one discipline to the other.

We understand that in principle, equations of motion for economics can be derived with the aid of the market-based trade maximization principle. To be honest, we do not fully understand how this exactly works. According to some indirect signs, we can only surmise that the market-based trade maximization principle and maximization principles borrowed from physics, work in one direction. We will examine this more specifically in Chapter VIII.

References

1. A.V. Kondratenko. *Physical Modeling of Economic Systems. Classical and Quantum Economies*. Nauka, Novosibirsk, 2005. Electronic copy available at: http://ssrn.com/abstract=1304630.

PART B.
Classical Economy

"The classical economist sought to explain the formation of prices. They were fully aware of the fact that prices are not a product of the activities of a special group of people, but the result of an interplay of all members of the market society. This was the meaning of their statement that demand and supply determine the formation of prices… They wanted to conceive the real formation of prices — not fictitious prices as they would be determined if men were acting under the sway of hypothetical conditions different from those really influencing them. The prices they try to explain and do explain — although without tracing them back to the choices of the consumers — are real market prices. The demand and supply of which they speak are real factors determined by all motives instigating men to buy or to sell. What was wrong with their theory was that they did not trace demand back to the choices of the consumers; they lacked a satisfactory theory of demand. But it was not their idea that demand as they used this concept in their dissertations was exclusively determined by "economic" motives as distinguished from "noneconomic" motives. As they restricted their theorizing to the actions of businessmen, they did not deal with the motives of the ultimate consumers. Nonetheless their theory of prices was intended as an explanation of real prices irrespective of the motives and ideas instigating the consumers".

Ludwig von Mises. *Human Action. A Treatise on Economics.* Page 62

CHAPTER III.

Classical Economies in the Price Space

> "Prices are a market phenomenon. They are generated by the market process and are the pith of the market economy. There is no such thing as prices outside the market. Prices cannot be constructed synthetically, as it were. They are the resultant of a certain constellation of market data, of actions and reactions of the members of a market society. It is vain to meditate what prices would have been if some of their determinants had been different. Such fantastic designs are no more sensible than whimsical speculations about what the course of history would have been if Napoleon had been killed in the battle of Arcole or if Lincoln had ordered Major Anderson to withdraw from Fort Sumter.
>
> It is no less vain to ponder on what prices ought to be".
>
> Ludwig von Mises. *Human Action. A Treatise on Economics*. Page 395

PREVIEW.

What are the Economic Lagrange Equations?

Based on the belief that the dynamics of the many-agent market economies has to some extent a deterministic character, we derived the economic equations of motion by formal analogy with classical mechanics of the many-particle systems. As a result, we naturally obtained the economic Lagrange equations of motion describing dynamics of economic systems in time. It is fascinating that we can interpret Lagrangian as the mathematical classical representation of the market invisible hand concept.

1. Foundations of Classical Economy

The logic of the present Section is the following. With the understanding that, pursuing own various well-defined goals, the market agents behave to some extent in a deterministic way, in this Chapter we are going to outline the adequate and approximate equations of motion for the economy. In order to design a physical model and derive classical equations of motion for economic systems in the price space *ab initio*, that is with the five general principles of physical economics in mind, we first make similar approximations and assumptions needed to derive the equations of motion for physical systems. This can be found in the course of theoretical physics by Landau and Lifshitz [1, 2]. In this way we derive equations of motion for economic systems, which are similar to equations for physical systems in form, and are considered by us as an initial approximation for a physical economic model of the modelled economic system.

According to the above-stated plan of actions we could confine ourselves in this Chapter to just writing equations of motion analogous to those obtained in classical mechanics. However, we consider it useful to derive a full row of equations and to make additional comments on our actions. As we have indicated before, according to our approach to classical modeling of economic systems, every economic agent, *homo negotians*, acts not only rationally in his or her own interests, but also reasonably. They negotiate to reach a minimum price for the buyer and a maximum price for the seller, but also leave their counteragent a chance to gain profit from transactions or to achieve some other goals, economic or noneconomic in nature. Otherwise, transactions would take place only once, while all agents would prefer the continuation and stability of their business.

Besides, we presume that external forces are usually inclined to influence market operations positively, establishing common rules of play that favor gaining maximum profit, utility, trade volume, or something else for the whole economic system. Based on these assumptions, we have a firm belief that there are certain principles of optimization, and their effects on market agents result in certain rules of market behavior and certain equations of motion that are followed by all rational or reasonable players spontaneously or voluntarily. In our opinion, it is they who have the leading role in the market.

Concluding, let us repeat that we will derive below the economic Lagrange equations of motion by recognizing inexplicitly the following five general principles of physical economics:

1. **The Cooperation-Oriented Agent Principle.**
2. **The Institutional and Environmental Principle.**
3. **The Dynamic and Evolutionary Principle.**
4. **The Market-Based Trade Maximization Principle.**
5. **The Uncertainty and Probability Principle.**

It is evident that the uncertainty and probability effects begin to play significant role in classical economies only for the markets with huge numbers of agents. We do not concern ourselves with these effects within the framework of classical economy because it is much easier to study these problems within the framework of quantum economy (see next two chapters).

2. The Economic Lagrange Equations

Let us proceed to deriving equations of motion for the classical economy shown schematically in Fig. 1. We follow the same procedure as in classical mechanics [1]. To make calculations easier, we will consider here the one-good market economy, that is, only movement in one-dimensional price space with one coordinate P. Transition to a multi-dimensional case does not cause principal complications. We will consider that by the analogy with classical mechanics [1], a state of economy comprising of N buyers and M sellers and being under the influence of the environment is fully described by establishing all prices p_i and their first time t derivatives (price changing rate or velocity of movement) $\dot{p}_i = \frac{dp_i}{dt}$, where $i = 1, 2, ..., N + M$. Let $\boldsymbol{p}$ without subscripts denote the set of all prices $\boldsymbol{p}_i$ for short, similar to first and second time derivatives, i.e., for velocities p_i and accelerations $\ddot{\boldsymbol{p}}_i$ Due to their logic or by definition, equations of motion connect prices, velocities and accelerations. In classical mechanics they are second-order differential equations of time, their solution under assigned conditions at the moment t_0, $\boldsymbol{x}(t_0)$ and $\dot{\boldsymbol{x}}(t_0)$, represents the required mechanical trajectories, $\boldsymbol{x}(t)$. We are going to derive similar equations with the same view for our economic system in the price space.

By analogy with classical mechanics we assume that these equations result from the following principle of maximization (the principle of least action or the principle of stationary action in mechanics). Namely, the action S must have the least possible value:

$$S = \int_{t_1}^{t_2} L(\boldsymbol{p}, \dot{\boldsymbol{p}}, t)dt, \tag{1}$$

$$\partial S = 0. \tag{2}$$

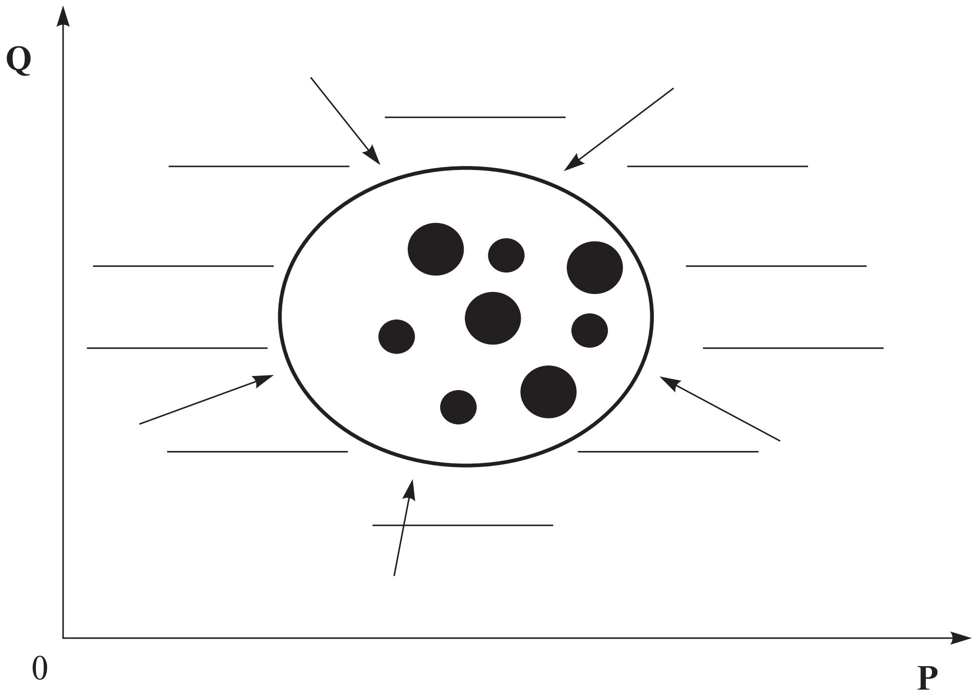

Fig. 1. Graphical model of an economy in the multi-dimensional PQ-space. It is displayed schematically in the conventional rectangular multi-dimensional coordinate system [**P**, **Q**] where **P** and **Q** designate all the agent price and quantity coordinate axes, respectively. Our model economy consists of the market and the external environment. The market consists of buyers (small dots) and sellers (big dots) covered by the conventional sphere. Very many people, institutions, as well as natural and other factors can represent the external environment (cross — hatched area behind the sphere) of the market which exerts perturbations on market agents, pictured here by arrows pointing from environment to market.

The obtained (1) and (2) lead to equations of motion or Lagrange equations [1]:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{p}_i} - \frac{\partial L}{\partial p_i} = 0 \ (i = 1, 2, \dots, N + M). \quad (3)$$

Equations of motion represent a system of second-order $N + M$ differential equations of time t for $N + M$ unknown required trajectories $p_i(t)$.

These equations employ as yet an unknown Lagrange function or Lagrangian $L\ (\boldsymbol{p}, \dot{\boldsymbol{p}}, t)$ which is to be found on the basis of research or experimental data. We will note that Lagrange functions were used in literature to solve a number of optimization problems of management science [3]. Let us emphasize that determination of the Lagrange function is the key problem that can only be solved in practice by making the data of theoretical calculation fit the experiment. It cannot be done using theoretical methods only. But what we can do quickly is to make the first obvious trial step. Here we assume that to a certain degree of approximation, the Lagrange function resembles (in appearance only!) the Lagrange function of its physical prototype, a system of $N+M$ point material particles with certain potentials. All assumptions made here can be thoroughly analyzed later at the second stage of investigation and left unchanged or made more accurate after comparison with the experimental data. Accomplishment of this stage will naturally require great effort and expense. For now, we will accept these assumptions and consider that Lagrange functions have the same form as those of their physical prototype, but all parameters and potentials of the economic system will be chosen on the basis of economic experience, not taken from the physical prototype. We consider that by adjusting parameters and potentials to the experiment we can smooth out the negative influence of assumptions made for solutions of equations of motion obtained in this particular way.

So, according to our approach, equations of motion in classical economy are nominally identical to those in the corresponding mechanical system. However, their constants and potentials will have another essence, other dimensions and other values. A great advantage of classical economies consists of the fact that mathematical solutions of these equations, analytical or numerical, have been found for a great number of Lagrange functions with different potentials. That is why it is of great help to apply them. Allow us to turn to relatively simple classical economies.

Let us consider a case of the classical economy with a single good, a single buyer, and a single seller, where environmental influence and interaction between a buyer and a seller can be described with the help of potentials. The Lagrangian of such an economy has the following form:

$$L = \frac{m_1}{2}\dot{p}_1^2 + \frac{m_2}{2}\dot{p}_2^2 - V_{12}(p_1, p_2) - U_1(p_1, t) - U_2(p_2, t). \qquad (4)$$

In (4) m_1 and m_2 are certain unknown constant values or parameters of economic agents who are the buyer and the seller respectively. The first two members of equation (4) in classical mechanics correspond to kinetic energy, and the remaining three to potential energy. Understanding the conventional character of these notions, we will use them for economy as well. Potential V_{12} (p_1, p_2) describes interaction between the buyer and the seller (it is unknown *a priori*), and potentials $U_1(p_1, t)$ and $U_2(p_2, t)$ are designed to describe environmental influence on economy. They are to be chosen with respect to experimental data according to the dynamics of the modeled economy. Lagrange equations have the following form for this type of Lagrangian:

$$\begin{cases} m_1\ddot{p}_1 = -\dfrac{\partial V_{12}(p_1, p_2)}{\partial p_1} - \dfrac{\partial U_1(p_1, t)}{\partial p_1}, \\ m_2\ddot{p}_2 = -\dfrac{\partial V_{12}(p_1, p_2)}{\partial p_1} - \dfrac{\partial U_2(p_2, t)}{\partial p_2}. \end{cases} \tag{5}$$

This system of two differential second-order equations of time t represents equations of motion for a selected classical economy. According to their form they are identical to the equations of motion of the physical prototype in physical space. In the latter system (5), the second Newton's law of classical mechanics is designated: "product of mass by acceleration equals force". And quite another matter is that potentials can be significantly different from the corresponding potentials in the physical system. We should mention once again that these potentials are to be discovered for different economies by detailed comparison of results of computation of equations of motion of economies with experimental or research data, or in other words, with data of empirical economics. At the initial stage it is natural to try to use known forms of potentials from physical theories, and we are going to do that in future. Let us note that the purchase-sale deal or transaction in the market between the buyer and the seller will take place at the time t^E when their trajectories $p_1(t^E)$ and $p_2(t^E)$ intersect: $p_1(t^E) = p_2(t^E) = p^E$, as it is shown in Figs. 2, 3 for the model grain market. The equilibrium value of price, p^E, is indeed then the real price of the good or commodity in the market, what we refer to as the market price of the good. See formulas, figures and discussions for classical economies in Chapter I.

It is interesting that a number of some common features of classical economy with equations of motion (5) are common for almost all constants m_i and potentials V_{12} and U_i. Let us consider a case where external potentials $U_i(p_i)$ do not depend on time and represent

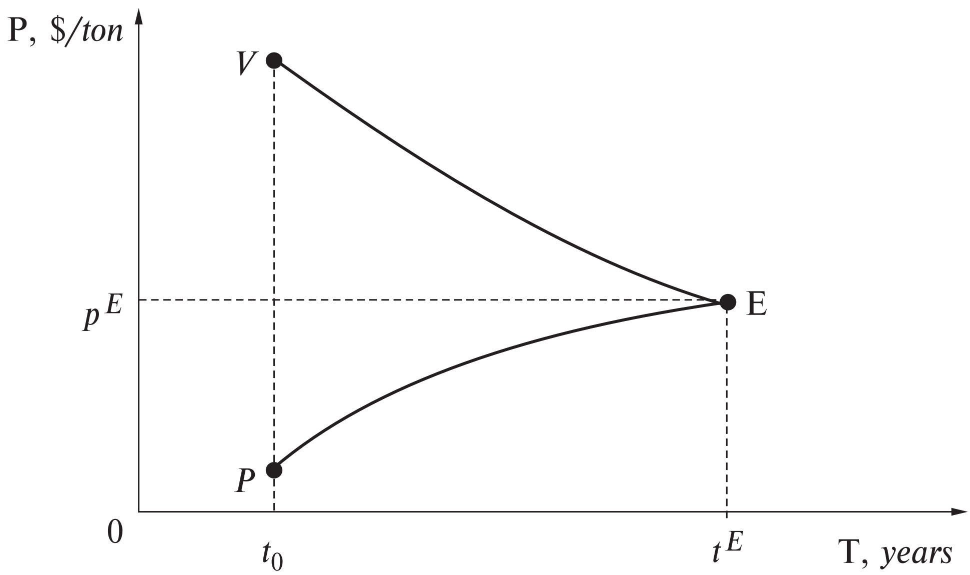

Fig. 2. The trajectory diagram showing dynamics of the classical one-good, two-agent market economy in the price-time coordinate system.

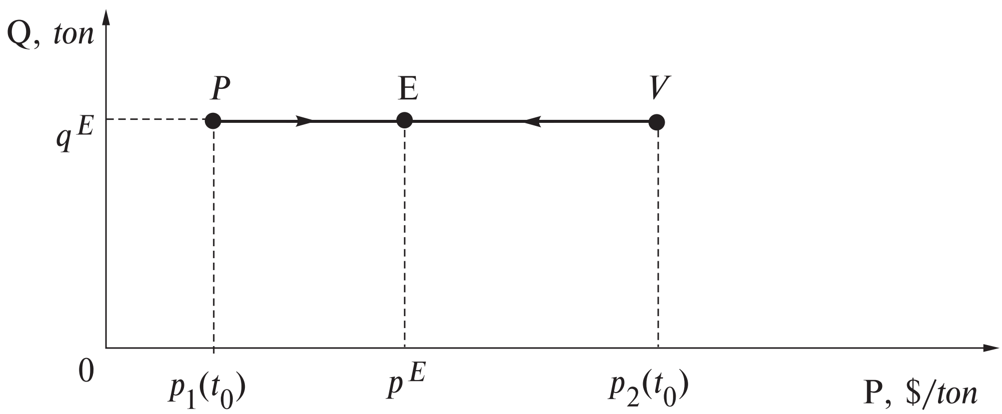

Fig. 3. Dynamics of the classical one-good, two-agent market economy in the price-quantity space, where quantity of the good traded, q, is constant. The economy is moving really in the price space.

potentials of attraction with high potential walls at the origin of coordinates that prevent economy from moving towards the negative price region. Further potential V_{12} depends only on the module of price differences of the buyer and the seller $p_{12} = |p_2 - p_1|$, namely,

$$V_{12} = V_{12}(|p_2 - p_1|).$$

We assume that potential V_{12} describes the attraction between the buyer and the seller and has its minimum at the point p_0^{12}. Then the solution of equations of motion describes movement or evolution of the entire economy as follows: the center of inertia of the whole system, introduced to theory by analogy with the center of inertia of the physical prototype, moves at a constant rate $\dot{P}$, and the internal movement, i.e., of buyers and sellers relative to each other, represents an oscillation, usually anharmonic, around the point of equilibrium p_0^{12}. This conclusion is trivially generalized for the case of an arbitrary number of buyers and sellers.

So we get classical economy with the following features:

1. Movement of the center of inertia at a constant rate signifies that if at some point of time a general price growth rate were $\dot{P}$, then this growth will continue at the same rate. In other words, this type of economy implies that prices increase at a constant rate of inflation (or rate of inflation is constant).

2. Internal dynamics of economy means that economy is oscillating near the point of equilibrium. In this case, economy is found in the equilibrium state only within an insignificant period of time, just as a mechanical pendulum is, at its lowest point, in an equilibrium state for a short period of time. Moreover, rates of changes in the relative prices of sellers and buyers are maximal at the point of equilibrium, just as for the pendulum the rate of movement is also maximal at the point of equilibrium. According to our view, oscillations of economy relative to the point of equilibrium p_0^{12} represent nothing but the economy's own business cycles, with a certain period of oscillation that is determined by solving equations of motion with specified mass m_i and potential V_{12}. These results correlate to the Walrasian cobweb model which is well known in neoclassical economics.

It is obvious that in the broad sense of the word, classical economy is the new quantitative method of describing the market economies, in which the first priority role in the establishment of market prices play the straight negotiations of buyers and sellers as to parameters of transactions. It is clear that this price formation is not intrinsic to the huge markets of contemporary economies, but is unique to the relatively small markets for the initial period of the formation of valuable market relations and corresponding markets in the distant past, when markets were small, undeveloped and by the sufficiently slow, i.e., in which the transactions were accomplished after lengthy negotiations.

3. Conclusions

In this Chapter we developed classical economies and derived the corresponding equations of motion, namely the economic Lagrange equations in the price space. Intuitively, we suppose that the applied least action principle can be treated to some extent as the market-based trade maximization principle. The relationship between these two principles becomes more clear within the framework of quantum economy (see the following Chapters). The extension of the method for the price-quantity space is straightforward therefore we will not do it here (respective formulas, figures and discussions can be found in Chapter I). Conceptually, we can regard Lagrangian as the mathematical classical representation of the market invisible hand concept. Note that, according to the institutional and environmental principle, Lagrangian include not only inter-agent interactions but also the influences of the state and other external factors on the market agents. Therefore, figuratively, we can say that the market invisible hand puts into practice simultaneously plans and decisions of both the market agents and the state, other institutions etc. As is seen from the above shown example, physical classical models or simply classical economies deserve thorough investigation, as they happen to become an efficient tool of theoretical economics. However, there are reasons to believe that quantum models where the uncertainty and probability principle is used for description of companies' and people's behavior in the market are more adequate physical models of real economic systems. Recall that probability concept was first introduced into economic theory by one of the founders of quantum mechanics, J. von Neumann, in the 40s of the XX-th century [4].

References

1. L.D. Landau, E.M. Lifshitz. *Theoretical Physics, Vol. 1. Mechanics*. Moscow, Fizmatlit, 2002.
2. L.D. Landau, E.M. Lifshitz. *Theoretical Physics, Vol. 3. Quantum Mechanics. Nonrelativistic Theory*. Moscow, Fizmatlit, 2002.
3. M. Intriligator. *Mathematical Methods of Optimization and Economic Theory*. Moscow, Airis-press, 2002.
4. J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1944.

PART C.

Probability Economics. Stationary Probabilistic Economies in the Price Space

"Natural science does not render the future predictable. It makes it possible to foretell the results to be obtained by definite actions. But it leaves unpredictable two spheres: that of insufficiently known natural phenomena and that of human acts of choice. Our ignorance with regard to these two spheres taints all human actions with uncertainty. Apodictic certainty is only within the orbit of the deductive system of aprioristic theory. The most that can be attained with regard to reality is probability".

Ludwig von Mises. *Human Action. A Treatise on Economics*. Page 105

CHAPTER IV.
Functions of Supply and Demand

> "Economics is not about things and tangible material objects; it is about men, their meanings and actions. Goods, commodities, and wealth and all the other notions of conduct are not elements of nature; they are elements of human meaning and conduct. He who wants to deal with them must not look at the external world; he must search for them in the meaning of acting men".
>
> Ludwig von Mises. *Human Action. A Treatise on Economics*. Page 92

PREVIEW.
What are Functions of Supply and Demand?

In the present Chapter the notion of supply and demand functions in the market, traditional to economics, is exposed to critical rethinking from the point of view of the uncertainty and probability principle. The Stationary Probability Model in the Price Space is developed for the description of behavior of a seller and a buyer in the price space of a one-good market in an economy being in a normal stationary state. Within the framework of the model, the terms *supply* and *demand* have changed their meaning; a new definition of the seller's supply and the buyer's demand functions is given. These functions are probabilistic in nature and they are normalized to their total supply and demand expressed in monetary units. In other words, they are the seller's and buyer's probability distributions in making a purchase/sale transaction in the market for a certain sum of money, respectively. Further, with the help of the proposed additivity and multiplicativity formulas for supply and demand, the Stationary Probability Model in the Price Space is extended to economies having many goods and many agents in the price space. With this strategy the probabilistic supply and demand functions of the whole market are constructed. As a main result of the work, we have laid the groundwork for probability economics. It is defined as a new quantitative method for description, analysis, and investigation of the model as well as real economies and markets.

1. The Neoclassical Model of Supply and Demand

An old joke in a well-known economics textbook says that creating an economist is as simple as teaching a parrot to pronounce words “supply” and “demand” (S&D below). My former managerial economics lecturer shared his own humor on this subject: If one understands the theory of S&D elasticity, you‘ve got yourself a new economics professor! These jokes reflect an important role which is played in economics by the S&D concept, the formal realization of which we will call the traditional neoclassical model of S&D. Below we will give the most widespread version of the description of this model from the textbook [1]. To start with, we will see how economics defines the demand of each individual buyer [1]. It is possible to present demand in the form of a scale or a curve showing quantity q of a product that a consumer desires, is able to buy at each given prices p, and at a certain period of time. Further, the radical property of demand consists of the following: at an invariance of all other parameters (*ceteris paribus*), reduction of price leads to the corresponding increase of the quantity demanded. And, *ceteris paribus*, the inverse is also true; an increase in price leads to the corresponding reduction of the quantity demanded. In short, there is an inverse relationship between the price p and the quantity q demanded. Economists call this inverse relationship the law of demand.

The simplest explanation of the law of demand: a high price discourages the consumer to buy, and a low price strengthens their desire to buy. The additivity rule is used to obtain the demand function of the whole market, i.e., all individual demand functions are simply summarized for obtaining the market demand function $D(p)$. The graph of the traditional demand function for a grain market is displayed in the Cartesian (P, Q)-plane in Fig. 1.

This example is intentionally taken from the textbook [1] where it has number 3–1. In order to avoid misunderstanding, we will make some remarks concerning this and all other drawings in this work. First, unlike the textbook [1], we plot price p on the horizontal axis P and quantity q on the vertical axis Q in the Cartesian (P, Q)-plane because price is an independent variable in all our theoretical constructions and conclusions. In exact sciences, an independent variable can only be

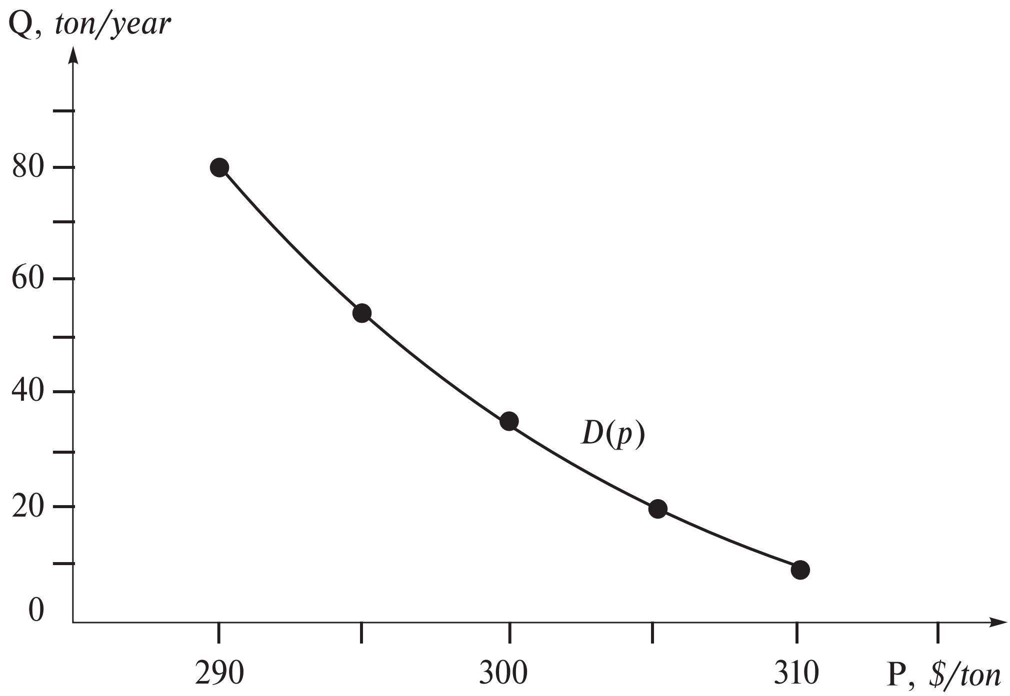

Fig. 1. Graph of the traditional neoclassical demand function *D*(*p*) for the model market of grain [1].

plotted on the horizontal axis. Second, we measure quantity of grain in metric tons (*ton*) per a year (*ton/year*), and the price in American dollars (*$*) per ton (*$/ton*).

Thus, according to the textbook [1], demand is simply the plan or intention of a buyer concerning product purchase which is expressed in the form of tables (or curves). We will discuss in detail later how adequately such tables and curves can reflect the behavior of buyers in the market, and we will now make some remarks concerning the form of representation of buyer's intentions in the given model.

First, the law of demand itself follows from neither an experiment, nor a theory; it is a statement as a whole which is consistent with common sense and elementary conclusions from real life. However, all of these conclusions are the result of observations of the behavior of real market prices and demand in the day-to-day activities of markets. In the market we only concern ourselves with real prices, real transactions, and the real sizes of these transactions. Sometimes, attention is given to total demand, but not at all to market demand functions or tables. Therefore, direct transfer of this empirical law on a quite abstract, uncertain and obscure demand function of an individual buyer is unnecessary.

In other words, the law of demand means the reflection of real market processes connected with continuous changes of S&D in the market over time. The traditional demand function is an attempt to describe a situation in the market where nothing changes. It is not a dispute about how correct or incorrect a traditional agent's demand function is. Instead, we can say that there is no basis on which to consider this model, reasonably, logically, or empirically. In principle, it is impossible to deduce a traditional agent's demand function from the data concerning the whole market. And there is no convincing empirical data, testifying that a buyer's behavior in the market is reflected by such a downward-sloping demand curve in the interval of all possible prices from zero to infinity. To understand our logic, the reader can try to draw on paper a demand curve of a buyer who wants to buy a new Mercedes car at a price of 100 000 *$/car*, or to buy shares at the stock-exchange for 100 000 *$*. We are sure that he or she will meet obstacles and recognize that there is something wrong with the traditional model. Moreover, logically it is impossible to construct a traditional function of the whole market making use of empirical data for the same reasons as that for functions of an individual agent. We will concern ourselves with this question once again in the end of Chapter.

Second, our main objection against the traditional demand function is that when real buyers enter a real market, they "keep in mind" not a concrete demand function on a whole interval of prices from zero to infinity, but a concrete desire to buy a certain quantity of demanded goods at a price acceptable for them which is near a known "yesterday's" price. This is illustrated by an example of an ordinary buyer in a consumer market, who needs a certain amount of sugar in a week – but no more and no less. It is also true for a business company in a wholesale market: it should buy exactly as many raw materials and goods as are necessary for production, without creating superfluous stocks and with delivery "just in time". Therefore, the demand function of an individual buyer can be distinct from zero only in a small interval of prices, near a known "yesterday's" market price. In order to obtain market functions it is necessary to summarize these rather narrow functions, instead of traditional functions, distinct from zero in the whole interval of prices from zero to infinity. Moreover, the fact that in the traditional model practically all authors have the demand function converging to a maximum near the zero price (some authors even have it diverging to infinity), seems, in our opinion, to be an artificial property of a person — to take the maximum “for free”. In a real market buyers do not behave like that, and in practice no life is observed in the markets near zero prices. It is a dead zone; there is neither supply nor demand there.

As for supply, here in economics everything is done by analogy with demand [1]. For an individual seller, a model known as the law of supply is found in the form of an upward-sloping curve. So, supply is defined as a graph or curve showing quantity of a product which the seller wishes and is capable of selling for in the market at each concrete price, for a variety of possible prices during a given period of time. It is assumed that there is a direct relationship between the price and quantity supplied. With a rise in prices the quantity increases respectively. Sometimes, this specific relationship is called the law of supply. Its essence is that producers make, and offer for sale, a bigger quantity of the product at high prices than at low ones. It is generally dictated by common sense. Another explanation is that, for a supplier, the price of a product represents revenue, and therefore it stimulates production and product sale. Thus, higher price promotes an intensification of production and growth in supply.

The concept of supply as well as demand is usually represented graphically, or in the form of a table. To obtain the market supply functions the additivity rule is used. In other words, a market supply function $S(p)$ results from a simple summation of the seller’s supply functions. An example is given in Fig. 2 (it is a graphic illustration of Table 3–5 from [1]).

Analogous to the case of demand, the law of supply follows from neither experiment nor theory. Such a model of the behavior of a seller in the market is simply constructed. In other words, it is defined in the model that the seller will behave in the market in exactly this manner, i.e., the seller comes to the market with his or her the strategy of behavior defined in advance, represented by a supply curve with the direct relationship of price and quantity defined along the whole interval of prices from zero to infinity. All of our objections concerning demand functions belong to supply functions of the traditional model. Here we remark that practically all authors draw market supply functions

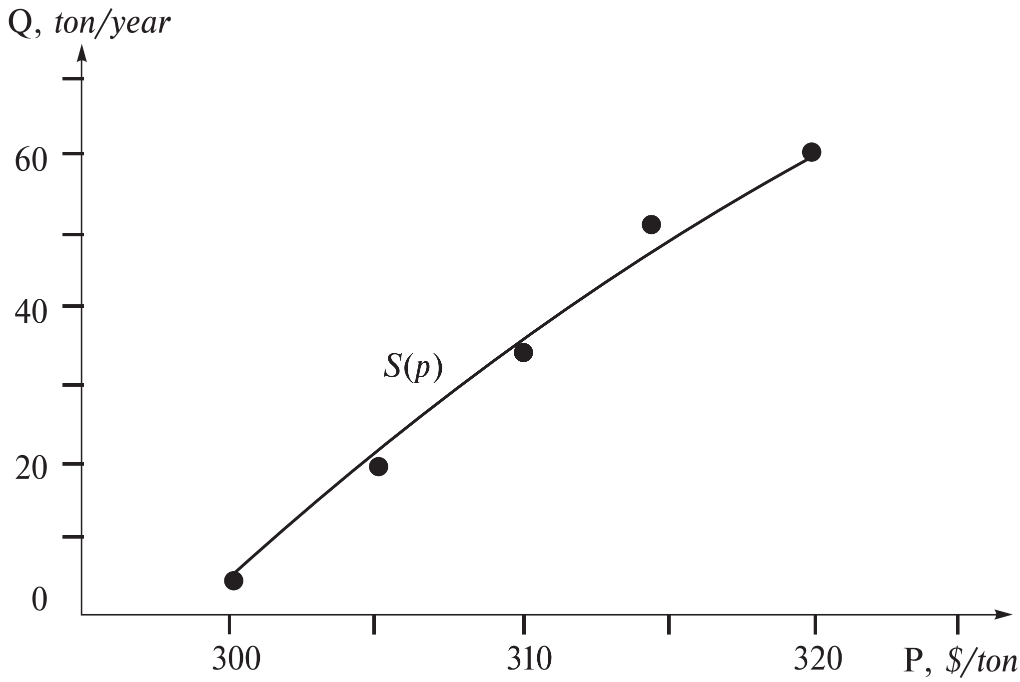

Fig. 2. Graph of the traditional neoclassical supply function S(p) for the model market of grain.

diverging to infinity at high prices which, according to the authors of these drawings, must reflect, probably, the inescapable greed of sellers, or capitalists. Actually, there is no life in the markets at very high prices. Instead, there is a dead zone here with neither demand nor supply. Since generally there are no transactions in this zone, there are no empirical observations which are able to confirm the behavior of sellers at high prices. Besides, for an infinitely large supply, one needs an infinite amount of resources which, as we know, are limited. In conclusion, remember the well-known fact that the existence of diverging functions in the theory, as a rule, indicates defects in the theory as “nature abhors a divergence”.

Now we come to the most important aspect of the traditional model. The market S&D functions are combined into one in Fig. 3 (it is Fig. 3–5 from [1]). The descending demand curve and the ascending supply curve drawn for all prices from zero to infinity should inevitably cross at some point (p^0, q^0), to which the traditional model attributes special characteristics. It is considered *a priori* that this point

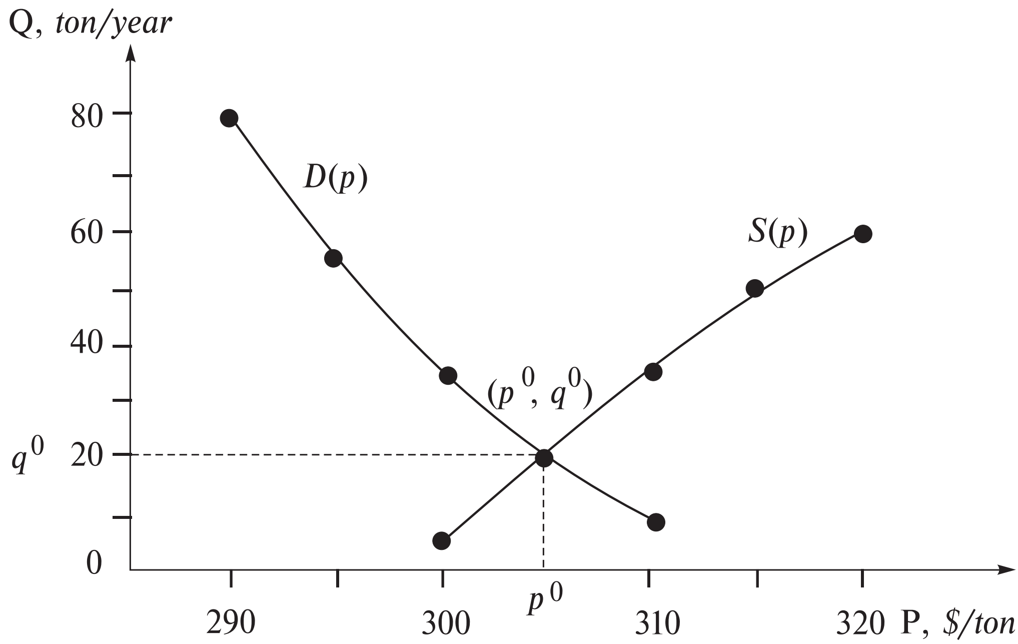

Fig. 3. The traditional neoclassical market S&D functions combined into one graph, with the neoclassical equilibrium point (p^0, q^0).

corresponds to some equilibrium state of the market, and that p^0 is a market price, i.e., the price at which all transactions in the market will be made, and the sales volume will be equal to q^0.

It is clear that this brief sketch of the model is quite rough, but as a whole, rather adequate. Detailed research of all modern treatments of this traditional model is beyond the scope of this essay, and we will discuss in detail the nature of marketprice in the following chapter (see also [2]).

2. The Stationary Probability Model in the Price Space

Now for the most complex and critical moment in the presentation of the probabilistic economic theory. We have incorporated one more aspect into the physical economic models; a fifth pillar namely, the uncertainty and probability principle. We will introduce it into the theory in order to maintain the maximum analogy with theoretical physics, and not lose key economic aspects of the behavior of agents on the market. This concerns, first of all, the process of decision making about purchase or sale of goods on the market, and also the method of mathematical representation of these decisions. Let us describe these questions in detail, using the example of our simple two-agent market economy in a stationary state as our reference point. To start,, for the purpose of reaching maximum simplicity in this stage of development of the theory, we will limit our investigation to the dynamics of the economy in the P-space. We will refer to this model below as the Stationary Probability Model in the Price Space, or the SP model for short. Let us begin with the description of the decision-making process of the buyer in the market when purchasing a particular good, which can be imagined as consisting of the following two steps.

First step. The buyer realizes a need for a good, and roughly assesses it quantitatively. They estimate how much money from their general budget they are prepared to spend for its purchase, based on their sources of financing for a certain period of time (say, for concreteness a year). At the end of this step, having refused all alternative options to spend money, he or she makes an informed choice, and formulates the initial "dot" strategy: they want to buy the good in quantity q^D at a price p^D he or she finds acceptable. Graphically, this type of dot strategy is represented by the point A in Fig. 4.

But the buyer doesn't purchase on their first day in the market. He or she understands that the exact price p^D may not be in the market on this particular day, or that the demanded good cannot be brought to the market in the necessary quantity, etc. Therefore, the buyer takes the second step in parallel with the first one.

Second step. The buyer analyzes information known to them about the market, estimates the degree of reliability and completeness of this

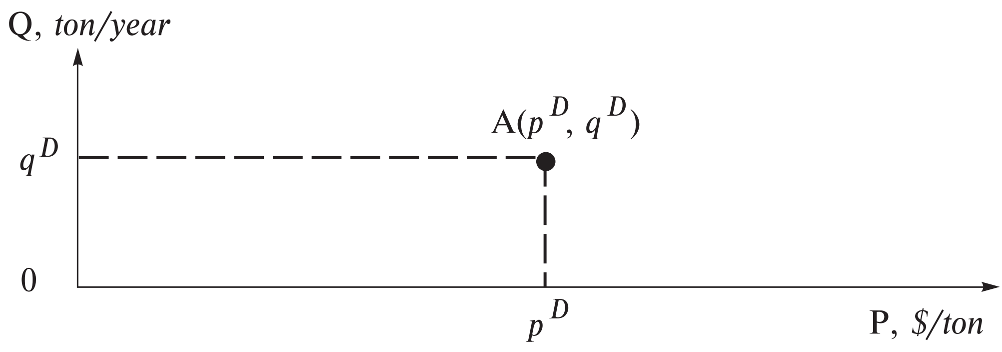

Fig. 4. Graphical representation of the buyer's dot strategy.

information, and recognizes that it is of a historical nature. Nobody knows precisely what will happen tomorrow, based on this information. The precise situation of the market is unknown, and no one possesses all the necessary information regarding the market and structure of the prices within it. This is because of a lack of such "databases" in the market for objective reasons, and owing to the limited possibilities both physical, and financial to receive and analyze such information. Buyers that they are compelled to operate in conditions of uncertainty and high risk, and that they can exert only very weak influence on these conditions. As a result, they come to the conclusion that in the market they should consider all of their future decisions and actions only as possible with a certain degree of probability. Analogously, he or she understands also that they can and should ask prices in the market only in some interval near p^D with a definite probability as a function of price p. This probabilistic aspect of people's decision-making process of is of great importance for an understanding of agent behavior in the market. The buyer thinks and acts as an oscillating man, *homo oscillans* [3]. Because of this, he or she enters the market with their unique probability strategy of behavior, which can be represented by a continuous demand function $D^*(p)$ with a maximum at price p^D. This demand function expresses the probability of making a purchase by the buyer at price p, normalized to their total quantity demanded q^D by means of taking integral along the horizontal axis p as follows:

$$\int_{-\infty}^{+\infty} D^*(p)\,dp = q^D. \tag{1}$$

Thus, in this probabilistic model demand of an individual buyer is seemingly distributed among all possible prices, but it is obvious that it is mainly concentrated within the range of some natural width near the price p^D. Uniqueness of the buyer's strategy is in the form of the demand function $D^*(p)$ which they have "chosen". In our opinion, it is a more "correct" demand function than a traditional one; it is certainly finite everywhere, and its integral over all possible prices gives, as is necessary, the quantity demanded q^D.

Clearly, this demand function also allows us to estimate the quantity demanded at a particular interval; $D^*(p)\,dp$ is the quantity demanded in the range from p to $p + dp$, and the quantity demanded $q(p, p + \Delta p)$ in the range from the price p to price $p + \Delta p$ is calculated by means of a definite integral

$$\int_{p}^{p+\Delta p} D^{*}(p)\,dp = q(p,\ p + \Delta p). \quad (2)$$

Let us take one more step and enter a new function into the model:

$$d(p) = D^{*}(p)/q^{D}. \quad (3)$$

It is obvious that this function is normalized to 1:

$$\int_{-\infty}^{+\infty} d(p)\,dp = 1. \quad (4)$$

In other words, its integral along the axis of prices is equal to 1. In our opinion, it is natural to interpret this function as a continuous distribution of the probability that the buyer will make a purchase transaction at the price p. But then the demand function $D^{*}(p)$ can be treated as a continuous distribution of probability of making the transaction by the buyer, simply normalized to the total demand q^{D}.

It is necessary to make the following remark. Usually total demand of the buyer is defined as the quantity of goods which he or she wants and can buy, for example, in pieces or tons as we saw earlier, defining total demand of the buyer as a quantity of goods demanded q^{D}. This is a standard definition of total demand in economics which seems to be not quite correct. We will explain why, using a concrete example.

Let us assume that a particular individual wants and is able to buy a new Mercedes care every year, and this for the price of one thousand dollars even though its real market price equals $100 000/*car*. Hence, his total demand equals 1 *car/year*. A second person wants and is able to buy the same Mercedes for $100 000, *every year*. Her total demand is equal to the same quantity, i.e., 1 *car/year*. It appears illogical then that extremely different buyers have the same contribution to the market demand function, and hence the same influence on the market. It is obvious that the real market power of each buyer is determined, first of all, by the quantity of money which he or she wants and is able to spend in the market. For this reason we consider it necessary to define the total demand of the buyer in terms of money, that is, as a quantity of money which he or she wants and is able to spend for purchase of goods during the year.

Then total demand of the first buyer will be equal to *\$1000/year*, the second — *\$100 000/year*, and the third who wants and is able to buy two Mercedes cars for *\$95 000–\$190 000/year*, etc. Thus, in this manner, in theory numerous needy segments of population would be practically cut off from the market. In the opposite case these people would like to and could buy expensive goods almost free of charge in large quantities, almost completely determining both the total market demand, and a market demand function that does not exist in reality. When measuring or assessing the real market demand and the real market demand function, nobody is interested in the opinion of these people. It is one thing to create a model, and it is quite another matter to deal with practical activities. In economics it is common to see where in theory something is done one way, and in practice a completely different way is used which is better suited to reality.

Thus, we will define the total demand of the buyer D^0 as a product of the good quantity q^D with price p^D, and we will naturally measure total demand in monetary units, for example, in American dollars, *\$/year*:

$$D^0 = p^D \cdot q^D. \tag{5}$$

Consequently, as a demand function we will now consider a new function $D(p)$ which is normalized to the total demand D^0:

$$D(p) = p^D \cdot D^*(p), \tag{6}$$

$$\int_{-\infty}^{+\infty} D(p)\,dp = p^D \cdot q^D = D^0, \tag{7}$$

$$d(p) = D(p)/D^0. \tag{8}$$

It is easy to see that by means of the demand function $D(p)$ it is possible to calculate a total demand at any interval of prices, for example, the integral

$$D(p, p + \Delta p) = \int_{p}^{p+\Delta p} D(p)dp \tag{9}$$

is, obviously, the total demand in the range of prices from p to $p + \Delta p$.

The integrated demand function from zero to p which we will designate with $ID(p)$ can represent some theoretical interest; and it is defined in the following way:

$$ID(p) = \int_{-\infty}^{p} D(p)\,dp. \quad (10)$$

It is apparent that the derivative of the integrated demand function, with respect to the price p is the differential demand function $D(p)$,

$$D(p) = dID(p)/dp. \quad (11)$$

We will leave detailed discussion of the properties of this function and continue the discussion of the differential demand function $D(p)$. The demand function $D(p)$ presents the distribution of probability of the buyer making a purchase transaction at the price p. It is normalized to its total demand, expressed in money, which the buyer wants and can spend for a definite good. As a case in point we take the probabilistic demand function $D(p)$ of an individual buyer in the grain market depicted in Fig. 5 as the "bell curve". From here on we will call this "buyer's bell".

It is interesting that this demand function has the dimension of *ton/year*, just like the traditional one. Therefore, they can both be formally drawn in the same system of coordinates [P, Q] and compared with each other. However, it is meaningful to do this comparison only for shapes, and not

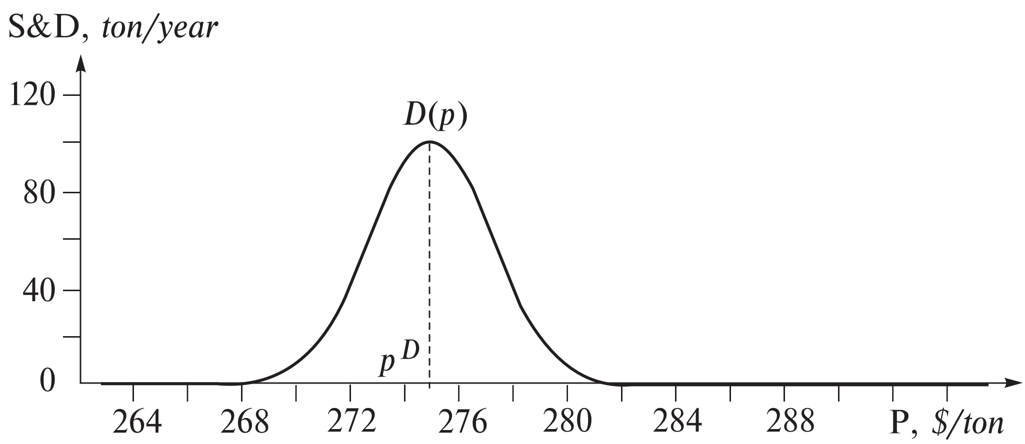

Fig. 5. The probabilistic demand function $D(p)$ of an individual buyer in the model market of grain. The buyer's price $p^D = 275{,}0$ *\$/ton*, the buyer's quantity $q^D = 2$ *ton/year*, and the buyer's bell width Γ^D is approximately equal to 5,3 *\$/ton*.

for absolute values of the functions, because these functions have very different definitions and hence different significance and values.

And now we will derive the probabilistic supply function $S(p)$ in a similar manner for the seller, wishing to sell the good in quantity q^S at a desirable price p^S. In other words, for the seller with total demand $S^0 = p^S \cdot q^S$ and the distributions of probability $S(p)$ and $s(p)$:

$$S(p) = p^S \cdot S^*(p), \quad (12)$$

$$\int_{-\infty}^{\infty} S(p)\,dp = p^S \cdot q^S = S^0, \quad (13)$$

$$s(p) = \frac{S^*(p)}{q^S} = \frac{S(p)}{S^0}. \quad (14)$$

For completeness, we will give the graphical representation of the dot strategy of the seller as the point B with coordinates p^S and q^S in Fig. 6 and of the supply function of the seller in the form of the continuous curve $S(p)$ in Fig. 7.

By means of the seller's supply function $S(p)$ it is easy to calculate the total supply at any interval of prices, using the integral

$$S(p, p + \Delta p) = \int_{p}^{p+\Delta p} S(p)\,dp, \quad (15)$$

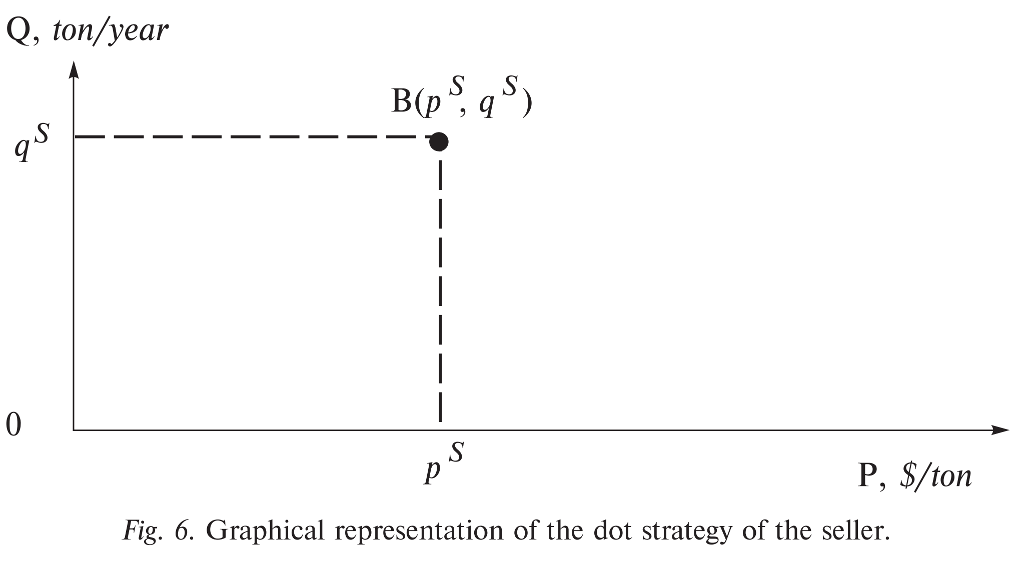

Fig. 6. Graphical representation of the dot strategy of the seller.

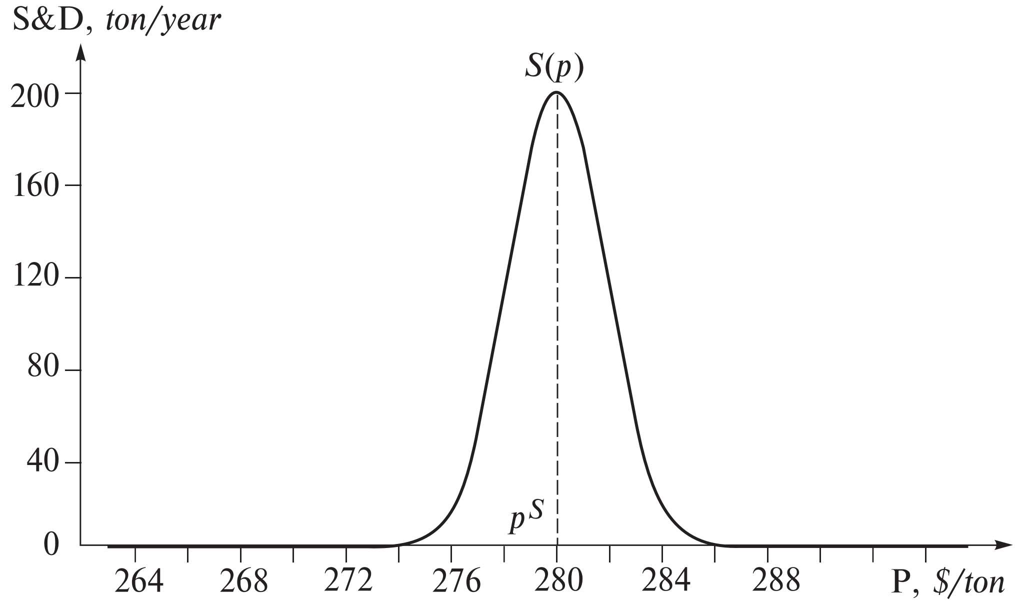

Fig. 7. The probabilistic supply function $S(p)$ of an individual seller in the model market of grain. The seller's price $p^S = 280{,}0$ *\$/ton*, the seller's quantity $q^S = 3{,}3$ *ton/year*, and the seller's bell width Γ^S is approximately equal to 4,3 *\$/ton*.

which is, obviously, the total supply in the range of prices from p to $p + \Delta p$. By analogy to demand, it is possible to define an integrated supply function $IS(p)$ by the following equation:

$$IS(p) = \int_{-\infty}^{p} S(p)\, dp. \qquad (16)$$

We will continue the discussion of the differential supply function $S(p)$. We calculate a seller's probabilistic supply function $S(p)$ in our model grain market, as exemplified in Fig. 7 as a "bell curve" that will be called below the "seller's bell".

The next step is as follows: we combine the dot strategies of the buyer and seller in Fig. 8, and their probabilistic strategies — in Fig. 9.

Thus, above we have obtained probabilistic models of the economy, or simply probabilistic economies and studied their basic special features. First of all, we see that the incorporation into the theory of the uncertainty and probability principle leads unavoidably to the probabilistic picture of markets, as well as the need for using an adequate mathematical apparatus to describe these special features.

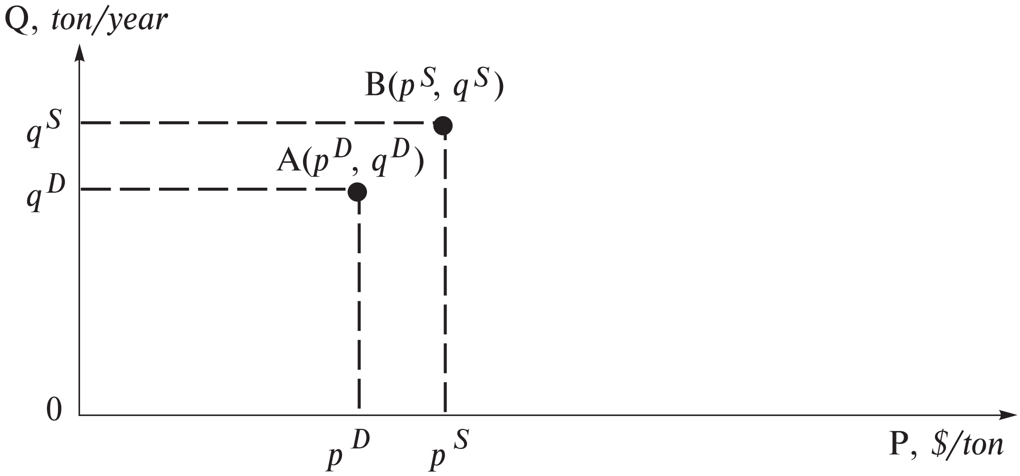

Fig. 8. Graphical representation of the market agents' dot strategies.

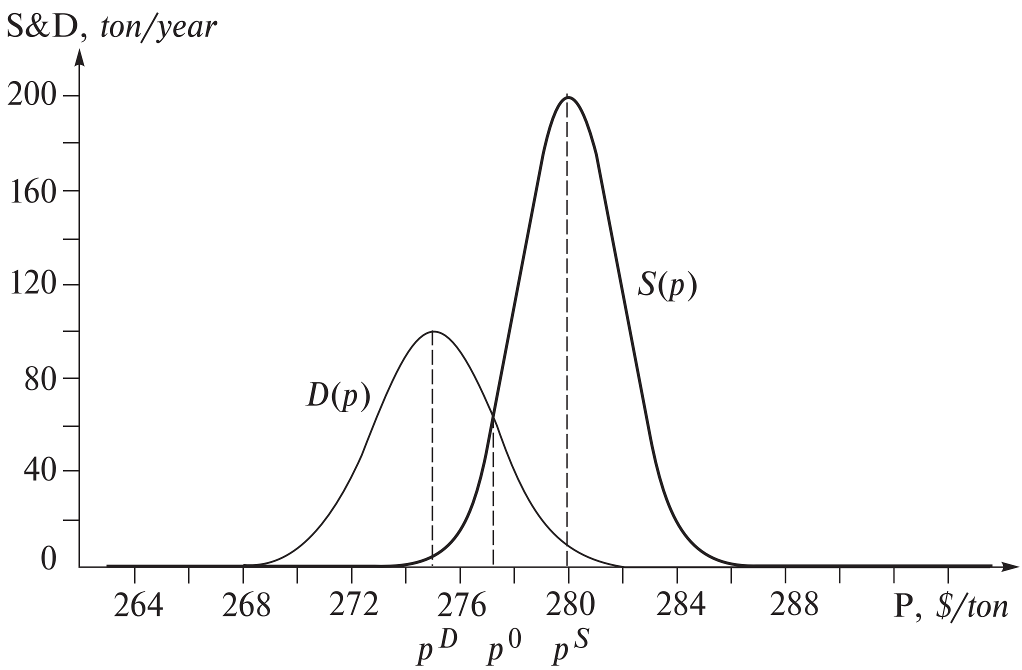

Fig. 9. The probabilistic demand function $D(p)$ of the buyer and the probabilistic supply function $S(p)$ of the seller for the model market of grain combined into one S&D diagram. These two curves intersect at an equilibrium price p^0 which equals approximately 277.2 *$/ton*.

Below, we will outline the most important elements of the mathematical apparatus for describing these kinds of probabilistic S&D functions.

Further, on the basis of common sense and real life experience we will make some remarks concerning the form of these probabilistic functions. First, from practice we know that the seller wants to sell goods as expensively as possible, and the buyer — to buy the goods as cheaply as possible. Therefore, it would appear reasonable that in the market the seller's desired price p^S is usually (but of course not always) more than the buyer's desired price p^D. Second, it is natural to suppose also that the graphs of the probabilistic functions $D(p)$ and $S(p)$ are "bell curves" with the maximum at prices p^D and p^S, respectively, which we have already used above. The simplest universal mathematical function suitable for representation of this type of probabilistic function is the Gaussian function. The Gaussian is a well-known function in mathematics and natural science. For example, it describes the normal distribution in probability theory and the wave function of the normal stationary state of the harmonic oscillator in quantum mechanics.

Using this, the agent's probabilistic S&D functions are described by two sets of formulas, respectively, as follows:

$$D(p) = D^0 \cdot g^D(p), \quad D^0 = p^D \cdot q^D, \tag{17}$$

$$g^D(p) = \sqrt{w^D/\pi} \cdot \exp\left(-w^D\left(p - p^D\right)^2\right), \tag{18}$$

$$D(p) = p^D \cdot q^D \cdot \sqrt{w^D/\pi} \cdot \exp\left(-w^D\left(p - p^D\right)^2\right), \tag{19}$$

$$\Gamma^D = \sqrt{-4 \ln 0{,}5/w^D}. \tag{20}$$

And

$$S(p) = S^0 \cdot g^S(p), \quad S^0 = p^S \cdot q^S, \tag{21}$$

$$g^S(p) = \sqrt{w^S/\pi} \cdot \exp\left(-w^S\left(p - p^S\right)^2\right), \tag{22}$$

$$S(p) = p^S \cdot q^S \cdot \sqrt{w^S/\pi} \cdot \exp\left(-w^S\left(p - p^S\right)^2\right), \tag{23}$$

$$\Gamma^S = \sqrt{-4 \ln 0{,}5/w^S}. \tag{24}$$

In these formulas $g^D(p)$ and $g^S(p)$ are Gaussians normalized to 1, and their parameters w^D and w^S determine full widths at the half maximum of their peaks Γ^D and Γ^S, respectively, which we will refer to below as natural widths.

Now let us sum up our results. Three parameters of the SP model — desirable price p^D, demanded quantity q^D, and natural width Γ^D — completely define the strategy of the buyer's behavior in the market, and three parameters — desirable price p^S, supplied quantity q^S, and width Γ^S — define the strategy of the seller's behavior. It is clear that the natural width of a curve depends on the knowledge, welfare, mentality, and other features of the agent and his or her final goals in the market. Below, we will use Gaussians for calculations and graphical representation of S&D functions. Namely, by means of these Gaussians we have calculated and plotted in Figs. 5, 7, and 9 the agent's S&D functions for the grain market with a single buyer and a single seller.

It is very important to discuss in detail Fig. 9 to gain greater insight into the market features revealed by the SP model. Firstly, it is evident that in the range of average prices where the buyer and seller bells are strongly overlapping, i.e., between the agent prices p^D and p^S, these functions behave differently. Here the demand curve is a downward-sloping one, whereas the supply curve is an upward-sloping one as in the traditional model (see Fig. 3). However, although in the traditional model such behavior of the curves is in essence a result of axioms, in the SP model it is simply an obvious consequence of that elementary fact from market life, that initially the seller wants to receive as much money as possible for their goods, and the buyer wants to pay for it as little as possible, more exactly, that $p^S > p^D$.

Besides, in this area of average prices (important for determination of the market price) both functions behave approximately as straight lines, like most S&D functions in the traditional model of economics . In this area, there is also the price p^0 at which the curves are crossed and which is considered in mainstream economics to be the market price. In the range of lower (p is less than p^D) and higher (p is more than p^S) prices, probabilistic S&D functions behave differently as compared with the traditional economic model. While traditional demand functions at very low prices and traditional supply functions at very high prices tend towards infinity, both the probabilistic functions approach zero. In real markets, the real demand provided by money is not present near zero.

In just the same way nobody supplies a madly large quantity of goods at madly high prices. Even though this sometimes happens in the market as a result of certain speculative games or bluffs, it is considered by the market only as an exceptional case, and in practice does not influence real market processes in any way. As we know from practice, only the agents working within the range of average historical market prices play a key role in the markets.

3. The Additivity Formula for Supply and Demand in Many-Agent Markets

Let us consider now a more complicated case of the many-agent market with *N* buyers and *M* sellers. Conclusions and reasoning given earlier for the case of a single buyer and a single seller are generalized very naturally here using the additivity rule discussed above. But, unlike the traditional model, we do understand why we can summarize the agent's S&D functions in the SP model: because probabilities may be summarized and have to be summarized here. And we can of course summarize the total supply and demand of all the agents to obtain the total supply and total demand in the market, respectively. In the traditional model, on the other hand, a total supply and a total demand neither of individual agents, nor of the market as a whole are properly defined at all. Moreover, the traditional market S&D functions are not rigorously defined, neither from the point of view of the economic theory, nor from the point of view of mathematics. As a consequence, the nature of these functions is rather vague (see discussion in the end of the article).

As a result, within the framework of the SP model we describe the market demand function $D(p)$ as a sum of the individual demand functions $D_n\ (p)$ by means of the additivity formula as follows:

$$D(p) = \sum_{n=1}^{N} D_n(p), \quad D^0 = \sum_{n=1}^{N} D_n^0 = \sum_{n=1}^{N} p_n^D \cdot q_n^D. \tag{25}$$

In the following formulas all designations have obvious origin and do not need additional comments:

$$D_n\ (p) = D_n^0 \cdot g_n^D(p),$$

$$g_n^D(p) = \sqrt{w_n^D/\pi} \cdot \exp\left(-w_n^D\left(p - p_n^D\right)^2\right), \tag{26}$$

$$\Gamma_n^D = \sqrt{-4\ln 0{,}5/w^D}. \tag{27}$$

We describe the market supply function $S(p)$ as a sum of the individual supply functions $S_m(p)$ using the additivity formula as follows:

$$S(p) = \sum_{m=1}^{M} S_m(p), \quad S^0 = \sum_{m=1}^{M} S_m^0 = \sum_{m=1}^{M} p_m^S \cdot q_m^S,$$

$$S(p) = \sum_{m=1}^{M} S_m(p), \tag{28}$$

We will write out the additional evident formulas without comments as well:

$$S_m(p) = S_m^0 \cdot g_m^S(p), \tag{29}$$

$$g_m^S(p) = \sqrt{w_m^S/\pi} \cdot \exp\left(-w_m^S\left(p - p_m^S\right)^2\right),$$

$$\Gamma_m^S = \sqrt{-4\ln 0{,}5/w_m^S}. \tag{30}$$

All of the formulas given and notations used have clear origins and direction. The market S&D functions are normalized to the total market S&D, expressed in currency, respectively:

$$\int_{-\infty}^{+\infty} D(p)\,dp = D^0, \qquad \int_{-\infty}^{+\infty} S(p)\,dp = S^0. \tag{31}$$

All of these equations and definitions have simple graphic representation in Figs. 10, 11, and 12 for the model market of grain used above with three buyers and three sellers. Before discussing these pictures we emphasize that all of these equations, of course, give only an initial approximation of the market functions. Applicability limits of the additivity formula for S&D and the accuracy of the calculated market functions can be estimated now only by making use of this approximation for numerous quantitative calculations for various model markets, followed by comparison of the results obtained with the known real market data.

It is easy to see in these pictures that when compared to the traditional model, the SP model brings about the formation of the probabilistic market S&D functions with a rather complicated structure. These functions are structurally similar to the complicated optical spectra of molecules, and these are in no way the straight lines of the traditional model. Because the probabilistic S&D curves consist of several bells, we shall refer below to the probabilistic market S&D functions as S&D bell towers.

As seen in Figs. 10, 11, and 12, everything that we were talking about above regarding the behavior of probabilistic individual S&D functions in the range of average prices, near zero and at very high prices, is valid for probabilistic market S&D functions too. Therefore, we will not repeat the discussion here. But let us emphasize again that all probabilistic S&D functions, are by their nature, the continuous and finite probability distributions of making transactions by buyers and sellers. We believe that their market behavior strategies are well represented by such functions.

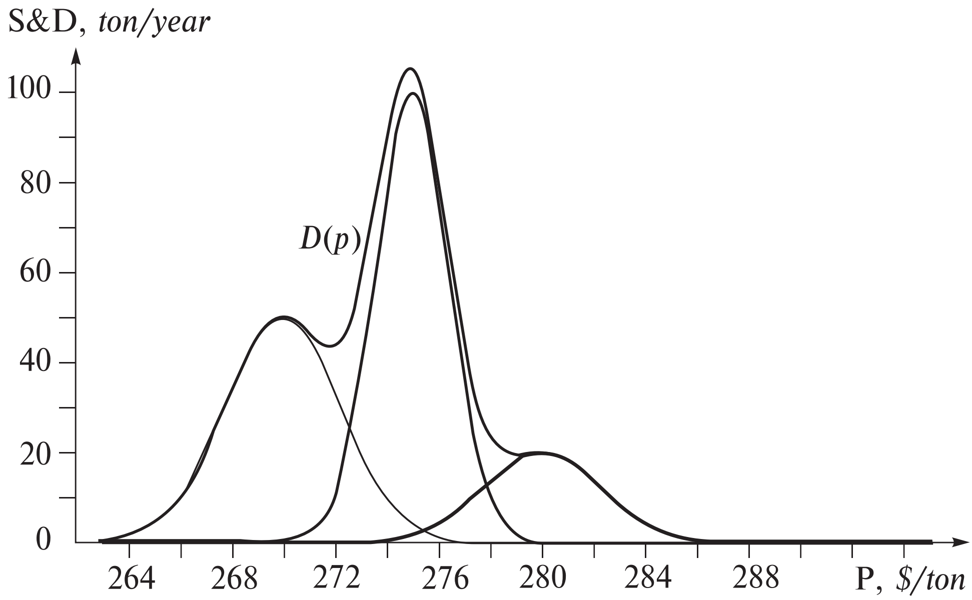

Fig. 10. The market demand function $D(p)$ as a sum of the individual demand functions $D_n(p)$ of three buyers with the following parameters: p_1^D = 270,0 *\$/ton*, q_1^D = 1,0 *ton/year*, Γ_1^D = 5,3 *\$/ton*; p_2^D = 275,0 *\$/ton*, q_2^D = 1,3 *ton/year*, Γ_2^D = 3,3 *\$/ton*; p_3^D = 280,0 *\$/ton*, q_3^D = 0,4 *ton/year*, Γ_3^D = 5,3 *\$/ton*. The demand bell tower has main maximum at price p^D which is equal approximately to 274,9 *\$/ton*.

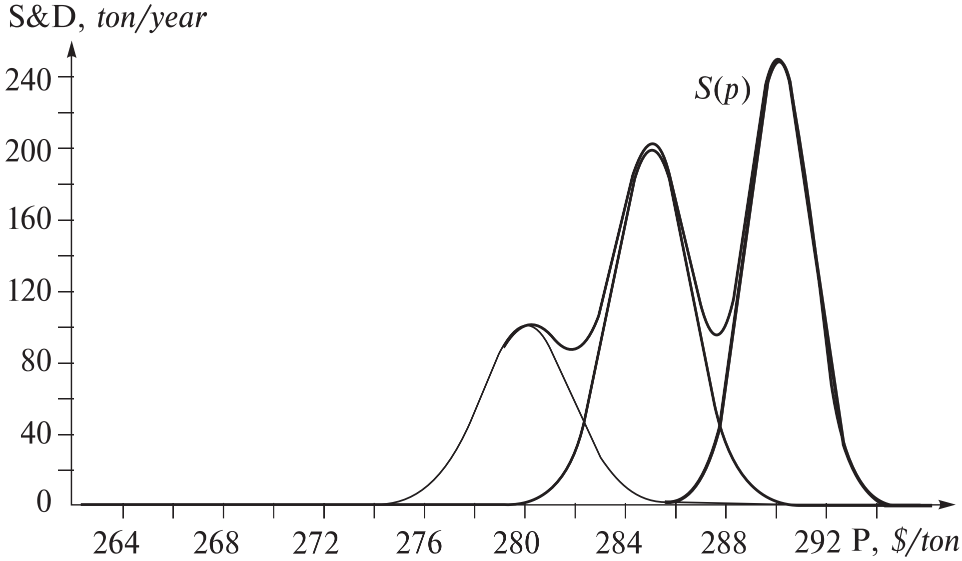

Fig. 11. The market supply function $S(p)$ as a sum of the individual supply functions $S_n(p)$ of three sellers with the following parameters: $p_1^S = 280{,}0$ *\$/ton*, $q_1^S = 1{,}6$ *ton/year*, $\Gamma_1^S = 4{,}3$ *\$/ton*; $p_2^S = 285{,}0$ *\$/ton*, $q_2^S = 2{,}8$ *ton/year*, $\Gamma_2^S = 3{,}7$ *\$/ton*; $p_3^S = 290{,}0$ *\$/ton*, $q_3^S = 2{,}8$ *ton/year*, $\Gamma_3^S = 3{,}0$ *\$/ton*. The supply bell tower has main maximum at price p^S which is equal to 290,0 *S/ton*.

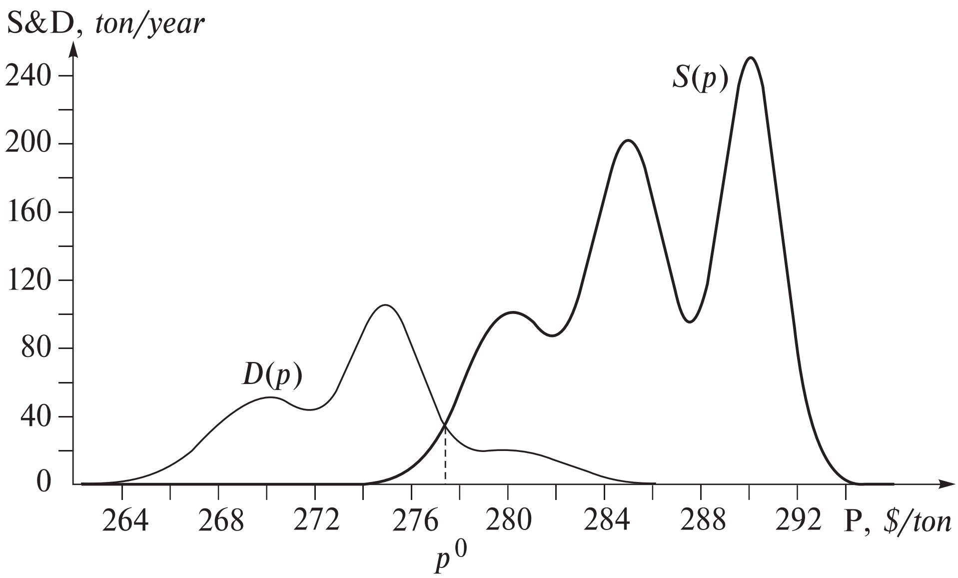

Fig. 12. The combined market S&D functions for the model market of grain with three buyers and three sellers. The S&D bell towers intersect at so-called equilibrium price p^0 which equals 277 *\$/ton*.

Therefore, we consider that the SP model of an economy describes S&D and hence describe typical situations in the market in greater detail and more adequately as compared to the traditional model of economics. It can be easily seen in Fig. 9 that, figuratively speaking, life is in full swing in the market only where both the S&D bell towers are strongly overlapping. We can call this range of prices the market life zone because practically all deals in the market are made at these prices. Note that the equilibrium price p^0 usually belongs to this range of prices too. All other price ranges, especially in the range near zero, as well as the range of very large prices we call market dead zones because no deals are made there at all. We will discuss these problems, tightly coupled with a market pricing process, in more detail elsewhere [2].

There are also strong grounds to believe that the probabilistic nature of S&D functions is one of the manifestations of a general property of human activities, namely the probability mechanism of a person's decision-making process on a concrete choice and action. Any choice and subsequent action of the person always belongs in the future. The person never precisely knows anything about future events, except that "all of us will die at some point". Nobody alive is given any ability to predict the future. Therefore it is difficult to make a reasonable choice. "Auswahl ist qual". Our future is determined by activities of all people together, both now and later on. Besides, it depends on many natural and other exogenous factors. For this reason, a person always makes decisions regarding a concrete choice taking into account probabilities of upcoming concrete events and the taking of certain steps. In this regard, we accept the approach stated in detail in the treatise [4]. We shall touch on these aspects of human action in markets once again in detail at the end of the Chapter and in [2].

4. The Multiplicativity Formula for Supply and Demand in the Many-Good Markets

Now the only step left is to expand the SP model to the most general case of any quantity of goods L in the market. These types of markets will be referred to below as the many-good markets, in contrast to the one-good markets. Obviously, we are dealing in this case with the L-dimensional P-space [3] and hence with L-dimensional S&D functions, i.e., functions of L arguments. Intuition suggests that we can use the multiplicativity rule to construct the multi-dimensional S&D functions and we can factorize these functions. In other words, we can represent the multi-dimensional market functions as products of the one-dimensional agent functions obtained above when studying one-good markets. Why can we multiply the agent functions? Because they are the partial probabilities, and we can multiply partial probabilities here to obtain a total probability.

We will consider the most general case: when the n-th buyer wants to buy all goods in the market for a definite sum of money D_n^0. According to his or her needs and criteria the buyer divides this sum of money among all goods demanded and elaborates his or her unique strategy in the market. The corresponding multi-dimensional demand function $D_n(p_1, \ldots, p_L)$ can be approximated in a factorized form using the multiplicativity formula for demand as follows:

$$D_n(p_1, \ldots, p_L) = \frac{\sum_{l=1}^{L} D_{nl}^0}{\prod_{l=1}^{L} D_{nl}^0} \prod_{l=1}^{L} D_{nl}(p_l), \tag{32}$$

where
$$D_{nl}^0 = p_{nl}^D \cdot q_{nl}^D. \tag{33}$$

In this formula Gaussians can be used for the approximate representation of the buyer's one-dimensional demand functions as follows:

$$D_{nl}(p_l) = D_{nl}^0 \cdot g_{nl}(p_l), \tag{34}$$

$$g_{nl}\ (p_l) = \sqrt{w_{nl}^D/\pi} \cdot \exp\left(-w_{nl}^D\left(p_l - p_{nl}^D\right)^2\right), \tag{35}$$

$$D_n^0 = \sum_{l=1}^{L} D_{nl}^0 = \sum_{l=1}^{L} p_{nl}^D \cdot q_{nl}^D, \tag{36}$$

In these formulas, all new parameters are clearly named: p_{nl}^D is the price, at which the n-th buyer plans to buy the l-th good in quantity q_{nl}^D, and D_{nl}^0 is the n-th buyer's total demand of the l-th good expressed in a monetary form, etc.

It is easy to check that for the n-th buyer his or her one-dimensional demand function $D_{nl}(p_l)$ is normalized to their total demand of the l-th good D^0_{nl}, and their multi-dimensional demand function $D_n(p_1, \ldots, p_L)$ is normalized to their total demand D^0_n. Then by means of the additivity formula for demand we describe the multi-dimensional market demand function as a sum of the individual multi-dimensional functions as follows:

$$D(p_1, \ldots, p_L) = \sum_{n=1}^{N} D_n(p_1, \ldots, p_L) = \sum_{n=1}^{N} \frac{\sum_{l=1}^{L} D^0_{nl}}{\prod_{l=1}^{L} D_{nl}} \prod_{l=1}^{L} D_{nl}(p_l), \tag{37}$$

$$D^0 = \sum_{n=1}^{N} D^0_n = \sum_{n=1}^{N} \sum_{l=1}^{L} D^0_{nl} = \sum_{n=1}^{N} \sum_{l=1}^{L} p^D_{nl} \cdot q^D_{nl}. \tag{38}$$

The market demand function $D(p_1, \ldots, p_L)$ is normalized to the total market demand D^0.

By the same procedure we obtain the multi-dimensional factorized supply function of the m-th seller, with the help of the multiplicativity formula for supply as follows:

$$S_m(p_1, , p_L) = \frac{\sum_{l=1}^{L} S^0_{ml}}{\prod_{l=1}^{L} S_{ml}} \prod_{l=1}^{L} S_{ml}(p_l), \tag{39}$$

where

$$S^0_{ml} = p^S_{ml} \cdot q^S_{ml}. \tag{40}$$

We can also use Gaussians to approximate the seller's one-dimensional supply functions:

$$S_{ml}(p_l) = S^0_{ml} \cdot g^S_{ml}(p_l), \tag{41}$$

$$g^S_{ml}(p) = \sqrt{w^S_{ml}/\pi} \cdot \exp\left(-w^S_{ml}\left(p_l - p^S_{ml}\right)^2\right), \tag{42}$$

$$S^0_m = \sum_{l=1}^{L} S^0_{ml} = \sum_{l=1}^{L} p^S_{ml} \cdot q^S_{ml}. \tag{43}$$

The seller's multi-dimensional supply function $S_m(p_1, \ldots, p_L)$ is normalized to his or her total supply S^0_m. Then, applying the additivity formula for supply, we obtain the multi-dimensional market supply function $S(p_1,\ldots,p_L)$ as a sum of the individual multi-dimensional supply functions as follows:

$$S(p_1,\ldots,p_L) = \sum_{m=1}^{M} S_m(p_1,\ldots,p_L) = \sum_{m=1}^{M} \frac{\sum_{l=1}^{L} S_{ml}^0}{\prod_{i=1}^{L} S_{ml}^0} \prod_{l=1}^{L} S_{ml}(p_l), \qquad (44)$$

This function is normalized to the market total supply S_0 which is calculated by means of summation as follows:

$$S^0 = \sum_{m=1}^{M} S_m^0 = \sum_{m=1}^{M} \sum_{l=1}^{L} S_{ml}^0 = \sum_{m=1}^{M} \sum_{l=1}^{L} p_{ml}^S \cdot q_{ml}^S. \qquad (45)$$

Thus, using the multiplicativity rule for S&D, we have factorized the many-good market D&S functions. That is, we have represented the multi-dimensional market functions by the products of the respective one-dimensional agent's functions, which naturally made calculations and detailed studies of markets and market processes much easier.

All these equations are, of course, approximate — limits of their applicability aren't clear — but they can be used for quantitative calculations as a first approach to reality. It is obvious that the multiplicativity rule allows us to take into account influence of one good on another good because the general budget of an agent (the total demand D_n^0 and the total supply S_m^0) is now shared for several goods. This appears to be the main effect of the "mutual influence of goods". In other words, this mechanism describes the substitution effect of goods, well-known in economics.

In conclusion we will make a non-evident remark concerning the origin of the multiplicativity formula for S&D. It is possible that the human ability to factorize appeared at some point during our evolution. In any case, human beings have become very proficient in obtaining rather simple solutions for very complicated, many-parameter and multi-dimensional problems. Very skillfully, humans are able to reduce these types of problems to a set of the simplest one-parameter and one-dimensional tasks. As an example, we can explore the problem of orientation in a real three-dimensional space. A person usually reduces this problem to the relatively simple task of finding an appropriate path consisting of a series of smaller paths. Using this same logic, an agent in the market reduces a rather difficult problem of optimal choice of his or her multi-dimensional function to a set of relatively simple problems of optimal choice of one-dimensional functions. Now, technology also gives us the ability to find solutions to many-dimensional problems, as well as analyzing their solutions in detail, but it is a topic that will be discussed in another of our articles later on.

5. The Factorization Formulas for Supply And Demand in the Many-Good, Many-Agent Markets

In Sections 2, 3, and 4 we used mainly common sense reasoning to derive the additivity and multiplicativity formulas for S&D functions in many-agent and many-good markets. Now we will briefly show that all of these formulas can be formally obtained by means of applying the factorization rule for market and agent wave functions several times over. Strictly speaking, within the more general probability framework of probability economy theory (more exactly, quantum economy) [3,9], many-agent and many-good markets are described in the multi-dimensional P-space by the total $(N+M)$-agent L-good market wave function $\Psi(\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_{N+M})$. Here, all arguments are agent price vectors in the L-dimensional P-space of the market. Now we restrict our attention to only aspects of the total market wave function $\Psi(\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_{N+M})$ that concern use of the factorization formula, used to obtain the market S&D functions with the simplest structure. As was already discussed above and in [3], we believe that the origin and structure of the agent's and market S&D functions cannot be completely understood in isolation from the phenomena of human life, such as consciousness, choice, decision and action.

So, we do the first step in applying the factorization rule and obtain the following formula for describing the approximate total market wave function as follows:

$$\Psi(\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_{N+M}) = \\ = \Psi^D(\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_N) \cdot \Psi^S(\boldsymbol{p}_{N+1}, \boldsymbol{p}_{N+2}, \ldots, \boldsymbol{p}_{N+M}). \qquad (46)$$

Thus, by means of factorization we obtain the total market wave function as a product of the buyers wave function $\Psi^D(\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_N)$ and the sellers wave function $\Psi^S(\boldsymbol{p}_{N+1}, \boldsymbol{p}_{N+2}, \ldots, \boldsymbol{p}_{N+M})$. This approximation seems therefore to be very natural, hence there is no reason to discuss it in detail.

At the second step we introduce a rather evident assumption of the model, using the factorization rule once again for the buyers' and sellers' functions:

$$\Psi^D(\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_N) = \prod_{n=1}^{N} \psi_n^D(\boldsymbol{p}_n). \tag{47}$$

$$\Psi^S(\boldsymbol{p}_{N+1}, \boldsymbol{p}_{N+2}, \ldots, \boldsymbol{p}_{N+M}) = \prod_{m=N+1}^{N+M} \psi_m^S(\boldsymbol{p}_m). \tag{48}$$

Thus, we represent the buyers' and sellers' wave functions as the product of their corresponding agent wave functions, which simplifies the market picture. This is because, upon integrating these products with respect to vectors $\boldsymbol{p}_i$, the standard market S&D functions will assume a usual form as the sums of the agent S&D functions:

$$D(\boldsymbol{p}) = \sum_{n=1}^{N} D_n(\boldsymbol{p}), \text{ where } D_n(\boldsymbol{p}) = |\psi_n^D(\boldsymbol{p})|2, \tag{49}$$

$$S(\boldsymbol{p}) = \sum_{m=1}^{M} S_m(\boldsymbol{p}), \text{ where } S_m(\boldsymbol{p}) = |\psi_m^S(\boldsymbol{p})|2\ . \tag{50}$$

Here, $\boldsymbol{p}$ is the price vector in the L-dimensional P-space of the L-good market studied. The last step, factorization of the agent functions $D_n(\boldsymbol{p})$ and $S_m(\boldsymbol{p})$, will be omitted in this case because it was described in detail in Section 4. Note that the contents of Section 5 will be discussed in detail within the framework [3] in Chapter VI (see also article [9]).

6. Comments and Discussions

Let us now make some comments and discuss the results obtained in this paper.

First. All of the agent's functions and parameters in all formulas given above are assumed to be known *a priori* for each buyer and for each seller. This is because they are considered to be the internal characteristics of each agent in the market and completely describe the agent's behavior strategy in the market. In other words, all of these agents' characteristics and hence each agent's strategy is determined by the agent themself, and so they can be considered empirical in nature. The additivity formula and the multiplicativity formula simply provide us with the tool for constructing the market S&D functions, using these agents' empirical S&D functions as building blocks. Therefore, the obtained market S&D functions should be referred to as empirical too, in contrast to the theoretical ones calculated by means of an *ab initio* method briefly discussed below.

Second. As far as the applicability limits of the SP model are concerned, the most important condition of the model's adequacy to practical activities is the condition that the economy remains in the normal stationary state [3]. In this state, by definition, there is no time dependence, there is little economic growth, sharp external shocks both from the government and other markets are absent, natural and technogenic catastrophes do not occur, etc. In this kind of state, the economy and the market S&D functions appear invariable. However, it does not mean that in the stationary state the market S&D functions are independent of each other. On the contrary, the existing strong interaction of sellers with buyers means that the market S&D functions also strongly interact with each other, but this interaction can express itself explicitly in practice only under external influence.

Third. Of course, the normal stationary state simply represents a conceivable ideal model. This design is particularly far from reality in the event of strong economic crises, or strong and rapid government interventions into markets [3, 4]. But this does not detract from its theoretical value, since the very concept of stationary states of the economy allows us, in principle, to use physical and mathematical modeling techniques for a more detailed study of economic processes and phenomena. This can be done using systematic numerical computations (so-called numerical experiments) with different parameters and different initial conditions, and comparing their results with those of the practical activities of both market agents and the market as a whole. In order to study the impact of external influences on the system, one must first deal with the case without external influence, and then one by one enter new factors into the model and study their impact on the system individually, as it is conventional to do in the natural sciences (especially physics). Note that the concept of stationary states of the economy as presented in this essay initially entered into economic theory in [3] by direct, formal transfer of the concept of stationary states from quantum mechanics. According to the economic contents within the framework of this work, it approximately corresponds to the known concept of an evenly rotating economy [4].

Fourth. As far as we know, this is the first time that the probabilistic S&D functions were entered into economic theory within the method of physical modeling of economic systems [3], whereby we proposed the method of *ab initio* calculations of the probabilistic S&D functions using more general economic principles than is used in the literature. In contrast to the theoretical probabilistic functions calculated *ab initio*, the probabilistic functions constructed in this article were obtained from the experience of practical activities, and in this paper are referred to as empirical probabilistic S&D functions. This is, as far as we know, also the first time that a probabilistic approach was used within game theory for the description of economic processes [5].

Fifth. This comment concerns the question of whether, and if so, how in practice we can measure or estimate the market S&D functions. This appears to be possible, in principle, by polling the market agents. If the number of major players in the market is small, it is enough to ask for their three parameters (price, quantity, and width) and then use the additivity and multiplicativity formulas to construct the market S&D functions. If the number of players in the market is large, it is necessary to perform the same procedure for a representative sample of agents. However, in this case we are able to measure the market S&D functions only to a certain unknown constant factor, i.e., we can directly measure only the shape of the market functions. And yet, we can still approximate this factor by normalizing these functions to the total S&D which, in turn, can be estimated rather well by using the respective historical data in the market and by sampling the agents.

Sixth. To avoid potential confusion, we make a rather important remark on the general nature of traditional and probabilistic market functions in the different variations met in our article above. It concerns the economic contents of traditional and probabilistic S&D functions. Recall the demand function defined in the tutorial given at the beginning of the article (whose definition is shared by us): “it is just the plan or intention of the buyer regarding the purchase of a product, expressed in the form of a table or curve”. In this respect, we believe that the probabilistic demand functions are economically meaningful; they contain a real intention of the buyer to make a deal for a certain amount of money, i.e., to buy a necessary product in the right quantity, either for business or for solving everyday problems. We emphasize that buyer makes their deal reasonably, knowing the market and its price structure for a large number of goods. The buyer operates only with real facts, real prices, and real needs; he or she is still trying their best to optimize costs and increase his or her profits by finding the best options in quality and prices. In other words, they behave in the marketplace as a rational agent. More exactly, he or she is acting as a person who is well oriented in the market. So his or her probabilistic demand function describes their strategy of behavior in the market quite adequately; it is finite in the range of real market prices and is close or equal to zero in the range of prices not actually observed in the market.

And what is the individual buyer demand function like in the traditional model? First, it is at its maximum or even divergent near zero, whereas in the range of average market prices it is smaller and converging to zero. Why is its maximum near zero, when this is not observed in reality in the market? Because the function is based on entirely different principles. This function is not designed to answer the question of what a real buyer thinks, but rather the question of what they would think if the price were at that level. So, naturally, the question of how many Mercedes cars the buyer would buy if the price were 1000 *$/car*, throws him or her into confusion at first. Then, after having grasped this, the buyer, of course, would confidently answer "many, very many" in accordance with the traditional model. Is there an economic meaning or a direct relationship to the real work of markets? No. Why? Because in real life there are no market prices close to zero; it is the market dead zone.

We are convinced that the subject of economic theory should be the practical activities of people in the markets. " Action is a real thing. What counts is a man's total behavior, and not his talk about planned but not realized acts" [4]. We believe that the study of people's responses to artificial questions about artificial situations, albeit quite conceivable (for example the question of how many Mercedes cars the buyer would buy free of charge), is not within the scope of economic theory. Moreover, it is known that the person's answers heavily depend on the questions themselves (this interesting issue is investigated in detail in [6] using analogies of quantum measurement theory). This is another reason why we cannot sum up the traditional agent S&D functions to obtain market functions: we cannot sum up the dreams of several people to obtain one common market fantasy. A very famous saying usually attributed to Isaac Newton, points out that "to model the madness of people is more difficult than to model the motion of planets". And fortunately, attempting it is beyond the scope of economic theory. As noted above, economic theory deals with modeling practical activities of agents in the market who behave very rationally, without disturbances. Quite another thing is the idea that the notion of rationality is relative. What one considers rational, the other does not. But this is unimportant: what matters is that everyone in the market is acting rationally and making choices according to his or her understanding of rationality. Another solution suggested to alleviate the problem of unnatural behavior of the traditional demand functions near zero (eliminate divergence at zero) was the use of the concept of utility, totally redundant in the SP model. For this reason, we fully share the author's [4] skepticism towards the concept of utility and the traditional model of S&D.

Let us again emphasize the following crucial point concerned with probabilistic economies. The probabilistic economy of the market does not have any plans, there is nothing to do in this respect with the planned economy. Instead of this, each market agent has a market behavior strategy at each moment of time, in accordance with which he or she accomplishes actions at each moment of time. Agent action on the market is first of all setting out his or her price quotations and making deals at acceptable prices. Therefore, one can roughly say that in a probabilistic economy the market is, first of all, in the very market process of establishing the market prices and selecting best agent strategies, This eventually results in rewarding the best market agents and in punishing the worse agents. The worst market agents must drop out of the market in the long run. The above stated is in good agreement with the point of view of classical economic theory, particularly, Austrian economics [4].

7. Conclusions

In this Chapter the neoclassical concept of S&D in the market, traditional in economics, is exposed to critical rethinking. The new SP model is developed to describe the market agents' behavior strategies in the P-space of a one-good market. Within the framework of the model, the very terms S&D have changed their meaning and a new definition of agents' S&D functions is given. These functions are probabilistic in nature and they are normalized to their total S&D as defined in monetary units. In other words, they are the seller's and buyer's probability distributions for making a purchase/sale transaction, respectively, in the market for a definite sum of money. The proposed additivity and multiplicativity formulas provide the necessary background for the SP model for many-agent and many-good markets. The examples of how the SP model works are given in the subsequent papers [2, 7]. Of course, as is usual in such cases, it would be very interesting to put the SP model through severe experimental and practical tests. We can nevertheless say that the SP model is an important building block for probability economics. It can be defined as a new quantitative method of describing, analyzing, and investigating any model as well as real markets and economies taking into account both the static and dynamic market processes [8] within the framework of the empirical probabilistic approach. In this regard we consider this article to be a parallel to our other papers [3] and [9], which deal with the same problems within the frameworks of the non-empirical (*ab initio*) probabilistic approach of physical economics.

In closing, we emphasize that there are strong grounds to believe that the proposed probability approach to the general problem of human action and market activities is very adequate. Moreover, we believe that during the course of his or her evolution process *homo sapiens* has been compelled to live and act in permanently uncertain conditions. For this reason, the structure of their consciousness and thinking has been organized in such a manner that it is very natural for them to consider all their plans and actions as probabilistic ones with a definite spread, or more precisely as probability distributions with definite widths. Such a probabilistic structure of consciousness and thinking is an inherent and natural feature of a person that compliments efforts to seek happiness in his or her private life and profit in business activity. Let us recall that "The most that can be attained with regard to reality is probability" [4]. Our plans, decisions and economic views are all uncertain and probabilistic in nature — just like life itself; and it is this very property that makes it so wonderful.

References

1. C.R. McConnell, S.L. Brue. *Economics. Principles, Problems, and Policies*. 14-th Edition. Irwin McGraw-Hill, 2003.
2. Anatoly Kondratenko. *Probability Economics: Market Price in the Price Space*. Electronic copy available at: http://ssrn.com/abstract=2263708. See also Chapter V.
3. Anatoly Kondratenko. *Physical Modeling of Economic Systems. Classical and Quantum Economies*. Novosibirsk: Nauka, 2005. Electronic copy available at: http://ssrn.com/abstract=1304630
4. Ludwig von Mises. *Human Action: A Treatise on Economics*. Yale University, 1949.
5. John von Neumann, Oskar Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1953.
6. Paras Mal Agrawal, Ramesh Sharda. *Quantum Mechanics and Human Decision Making*. Electronic copy available at: http://ssrn.com/abstract=1653911.
7. Anatoly Kondratenko. *Probability Economics: Market Force in the Price Space*. Electronic copy available at: http://ssrn.com/abstract=2270306. See also Chapter VI.
8. Anatoly Kondratenko. *Probability Economics: Supply and Demand, Price and Force in the Price — Quantity Space*. Electronic copy available at: http://ssrn.com/abstract=2337462. See also Chapter VII.
9. Anatoly Kondratenko. *Physical Economics: Stationary Quantum Economies in the Price — Quantity Space*. Electronic copy available at: http://ssrn.com/abstract=2363874. See also Chapter X.

CHAPTER V.
Market Price in the Price Space

> "Prices are a market phenomenon. They are generated by the market process and are the pith of the market economy. There is no such thing as prices outside the market. Prices cannot be constructed synthetically, as it were. They are the resultant of a certain constellation of market data, of actions and reactions of the members of a market society. It is vain to meditate what prices would have been if some of their determinants had been different. Such fantastic designs are no more sensible than whimsical speculations about what the course of history would have been if Napoleon had been killed in the battle of Arcole or if Lincoln had ordered Major Anderson to withdraw from Fort Sumter. It is no less vain to ponder on what prices ought to be.
>
> Any price determined on a market is the necessary outgrowth of the interplay of the forces operating, that is, demand and supply. Whatever the market situation which generated this price may be, with regard to it the price is always adequate, genuine, and real. It cannot be higher if no bidder ready to offer a higher price turns up, and it cannot be lower if no seller ready to deliver at a lower price turns up. Only the appearance of such people ready to buy or to sell can alter prices.
>
> Economics analyzes the market process which generates commodity prices, wage rates, and interest rates. It does not develop formulas which would enable anybody to compute a "correct" price different from that established on the market by the interaction of buyers and sellers".
>
> Ludwig von Mises. *Human Action. A Treatise on Economics*. Page 395

PREVIEW.
What is Market Price?

The present Chapter is concerned with the fundamental question of the nature of market pricing. This has been the focus of attention for all economic theories of value. Foundations of probability economics were developed in the previous chapter, where it was referred to as a new quantitative method for description, analysis, and investigation of both model and real economies and markets. Natural and intelligible procedures were laid down for constructing probabilistic supply and demand functions in many-agent, many-good markets of an economy currently in a normal stationary state. This was done within the framework of the Stationary Probability Model in the Price Space. In the present chapter, we examine the central issue of any economic theory using this model, namely the question of the nature of market prices. We have obtained the following main results. In the context of this model the term market price has changed its meaning. By its very nature, market prices are probabilistic. Under certain conditions they can be defined as local highs or maxima of the many-dimensional deal function, which is the product of the supply and demand functions of the many-agent and many-good market. It is shown that market prices *ceteris paribus* do not depend at all on the total market supply and demand. Particular emphasis is placed on the comparative analysis of numerical results of the model for various model markets with known empirical facts. It is also shown that market prices for one-good markets are intersection points of the supply and demand elasticity curves. At the end of the chapter we give the set of partial differential equations for the calculation of market prices in the many-agent, many-good markets.

1. Introduction

The previous Chapter laid down the probabilistic foundations for our investigation into some very important questions of economic theory: what supply and demand functions are in the market (S&D functions below), and how they can be described using mathematical and graphical techniques. We believe the Stationary Probability Model in the Price Space (the SP model below) we developed gives adequate answers to these questions. The present chapter is written to clarify one of the most intriguing questions of economic theory — what is market price? It is very difficult to find a notion more muddled up or confusing in scientific literature than that of market price. This notion has a very long economic and political history. Nevertheless, the issue of the nature of market prices has always been at the center of attention of economists, and it has continued to be of prime interest in economics. In this chapter we do not attempt to carry out a detailed historical review of the various well-known theories of famous authors because this can be easily found in economics textbooks. The essay genre which we chose for this work allows us to avoid doing such a review. For this reason we restrict ourselves to a short description of the traditional neoclassical model of economics in which market price is primarily defined as a so-called equilibrium price, i.e., the price at which the traditional S&D curves for a one-good market intersect. It is believed in this model that this price meets, to a maximal extent, the interests of the market as a whole, and therefore any deviation from it will lead to the emergence of certain market forces which will return the actual prices to this value. We adhere to another point of view. We consider all market processes, at any concrete point of time, to be completely determined as an ultimate result of concrete motives, choices, decisions, and actions of all internal (market agents) and external (institutions, other markets, etc.) participants of the market. This is particularly true for real market pricing, which determines real market prices at which a good is really sold in a market place. Therefore, we focus on the motives, choices, decisions, and actions of all participants, but mainly those of market agents, buyers and sellers. Moreover, because no market agent is able to possess all the relevant market information, and, more importantly, they have no ability to effectively process such information, they are compelled to make their choices, decisions, and actions in a probabilistic manner. In other words, they must consider that differing choices, decisions, and actions are only possibilities, and ascribe to them definite probabilities. Taking into account that all market agents' choices, decisions, and actions have a probabilistic nature, the market prices must be of a probabilistic nature too. Therefore, only probabilistic procedures and techniques can help to illuminate the market pricing phenomenon. For this reason, we use the SP model [1, 2] in this chapter to clarify the issue of the nature of market prices. Within the framework of this method, the probabilistic nature of market prices results very naturally from the probabilistic nature of the market S&D functions [1, 2]. To avoid misunderstanding, we assume the real market prices to be the results of permanent market processes, bringing about rapid changes of the market S&D functions. However, we restrict ourselves by only describing static features of the probabilistic theory of market prices. We believe that only by means of a static approach can one lay the initial reasonable groundwork for the adequate market pricing theory. We will only touch on dynamic or time-dependent aspects of the theory to clarify some details.

The structure of this Chapter is as follows. First we will briefly describe the determination mechanism of the market price in the traditional neoclassical model of economics, and then we will describe the probabilistic determination mechanism of the market price in a one-good market in an economy that exists in a normal stationary state. Then using some simple examples we will illustrate in detail how this probabilistic mechanism works in a one-good market. Particular attention will be given to the questions of independence of market prices from the total S&D and the relationship between market prices and the market S&D elasticities. The theory will be extended to a general case of many-agent and many-good markets, and finally we will sum up the main results of the work.

2. Market Price in the Neoclassical Model

In the traditional neoclassical model of economics, it is usually *a priori* assumed that the market price at which deals in the market are mainly carried out is defined by the intersection point (p^0, q^0) of the traditional market S&D curves. They are represented by an ascending curve for supply and a descending curve for demand, respectively. This is shown in Fig. 1 (also shown in Fig. 3 of the article [1], and in Fig. 3-1 of a classical textbook [3]) for the model market of grain (for more details see Chapter III and [1]). The traditional model attributes distinctive properties to this intersection point. It is believed, for example, that this point corresponds to the equilibrium state of the market, where q^0 is an equilibrium good quantity, p^0 is an equilibrium price and hence it is the market price by definition.

No doubt, the picture in Fig. 1 looks deceptively beautiful. With its help, it is easy to visualize the determination mechanism of the market price within the framework of the traditional model of economics. But this picture has several fundamental shortcomings. The most essential of them is the lack of a rigorous definition of the traditional S&D functions, both from mathematical and economic points of view. As a consequence, they have neither predictive abilities, since there is no method of strict quantitative calculation of these market functions, nor practical value, as there is no reliable empirical method of evaluation of these market functions. For more detail see our discussion in Chapter III and [1].

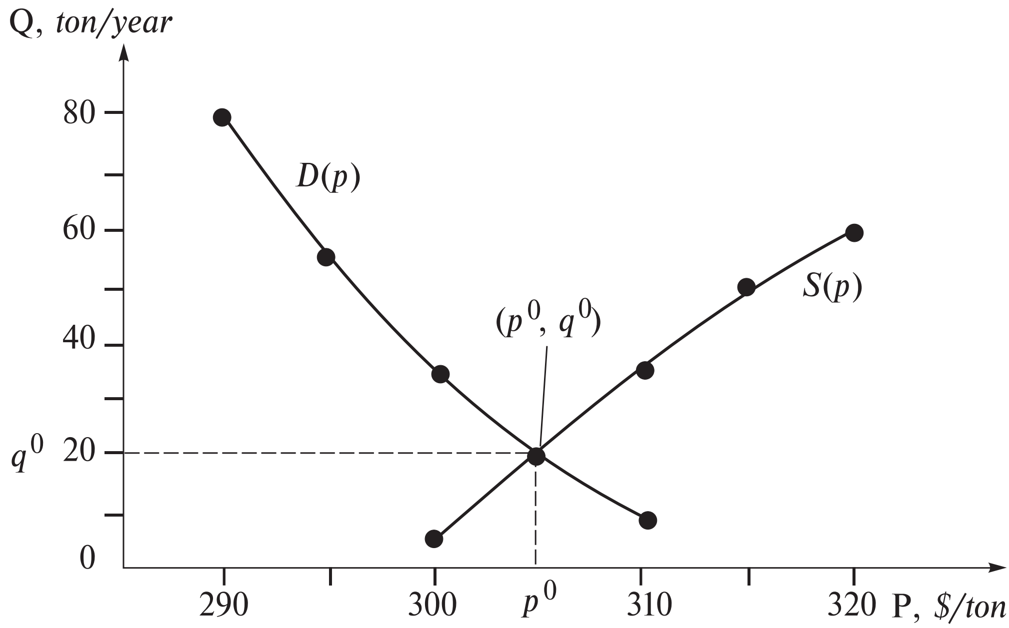

Fig. 1. The traditional market S&D functions combined in one S&D diagram for the model market of grain. The S&D curves intersect at point (p^0, q^0) which is referred to as the market equilibrium state. On a horizontal axis P the price of a ton of grain *p* is given in American dollars, and on a vertical axis Q — the amount of grain *q* in tons in a year.

Now let us emphasize again the following. It may seem strange that there is no rigorous method for extraction or evaluation of traditional S&D functions, and hence market prices from empirical data, in this fundamental model of economics. Practice gives us only actual quantities and prices at which numerous real deals in the market are carried out at any given moment of time, but not S&D curves and their equilibrium values. Likewise, there is no method for *ab initio* calculation of the traditional S&D functions and hence the market prices in economics. This picture was supposedly thought up for the explanation of known empirical S&D laws, but this picture is actually a visually incorrect representation of these laws. "It is necessary to comprehend that such pictorial or mathematical modes of representation do not affect the essence of our interpretation and that they do not add a whit to our insight. Furthermore, it is important to realize that we do not have any knowledge or experience concerning the shape of such curves. Always, what we know is only market prices — that is, not the curves but only a point which we interpret as the intersection of two hypothetical curves. The drawing of such curves may prove expedient in visualizing the problems for undergraduates. For the real tasks of catallactics they are mere byplay... It is the task of economics to deal with all commodity prices as they are really asked and paid in market transactions" [4].

From the moment of its emergence, the picture in Fig. 1 has become a real icon of economics — one of its solidified dogmas [5], and the research and study of so-called equilibrium market states have become a real *idée fixe* of mathematical economics. This idea is clearly ridiculous to all those who realize that real economies and markets represent complex dynamic systems which cannot exist in static equilibrium states at all. Mathematical economists "fail to recognize that the state of affairs they are dealing with is a state in which there is no longer any action but only a succession of events provoked by a mystical prime mover. They devote all their efforts to describing, in mathematical symbols, various "equilibria", that is, states of rest and the absence of action. They deal with equilibrium as if it were a real entity and not a limiting notion, a mere mental tool. What they are doing is vain playing with mathematical symbols, a pastime not suited to convey any knowledge" [4]. Even a wider concept of stationary states which correlates with the concept of an evenly rotated economy [4] used in the SP model of an economy represents only an imaginary construction unrealizable in real life. The market process constantly tends to bring the economy out of a stationary state due to the permanent actions of enterprising market agents, who aim to profit from their speculative actions. The concept of the stationary state is just conceived to clean out all of the market phenomena connected with dynamics, i.e., economy evolution in time [2]. "This so-called static method is precisely a proper mental tool for the examination of change. There is no means of studying the complex phenomena of action other than first to abstract from change altogether, then to introduce an isolated factor provoking change, and ultimately to analyze its effects under the assumption that other things remain equal The use of imaginary constructions to which nothing corresponds in reality is an indispensable tool of thinking. No other method would have contributed anything to the interpretation of reality" [4]. By means of such step-by-step procedures and their respective models we can understand better how our world economic is structured and how it works and evolves. We will present the General Stationary Probability Model in the Price-Quantity Space, taking into account the market processes in more detail, Chapter VI and VII, see also in [6].

3. Probabilistic Market Pricing in One-Good Markets

As we noted in Chapter III and [1], we believe that probabilistic S&D functions describe the market behavior of buyers and sellers in the price space [2] (the P-space below) of an economy more adequately than traditional ones. The use of the probabilistic S&D functions must obviously lead to the definition of the probabilistic market pricing process. Let us remind ourselves that the S&D functions in the SP model have a different significance as compared to the traditional ones, namely that they are the probability distributions of a buyer (seller) making a purchase (sale) transaction or a deal at price p, in a market of an economy existing in a normal stationary state.

Before discussing the results of the calculations of the model pertaining to one-good markets, we will recall that for a one-good market the probabilistic market demand function $D(p)$ is represented by the sum of the demand functions $D_n(p)$ of the individual buyers by means of the following additivity formula [1]:

$$D(p) = \sum_{n=1}^{N} D_n(p), \quad D^0 = \sum_{n=1}^{N} D_n^0 = \sum_{n=1}^{N} p_n^D \cdot q_n^D, \tag{1}$$

where

$$D_n(p) = D_n^0 \cdot g_n^D(p), \tag{2}$$

$$g_n^D(p) = \sqrt{w_n^D/\pi} \cdot \exp\left(-w_n^D\left(p - p_n^D\right)^2\right), \tag{3}$$

$$\Gamma_n^D = \sqrt{-4\ln 0{,}5/w_n^D}. \tag{4}$$

In the same way, we describe the probabilistic market supply function $S(p)$ as the sum of the individual supply functions $S_m(p)$ using the additivity formula as follows:

$$S(p) = \sum_{m=1}^{M} S_m(p), \quad S^0 = \sum_{m=1}^{M} S_m^0 = \sum_{m=1}^{M} p_m^S \cdot q_m^S, \tag{5}$$

where

$$S_m(p) = S_m^0 \cdot g_m^S(p), \tag{6}$$

$$g_m^S(p) = \sqrt{w_m^S/\pi} \cdot \exp\left(-w_m^S\left(p - p_m^S\right)^2\right), \tag{7}$$

$$\Gamma_m^S = \sqrt{-4\ln 0{,}5/w_m^S} . \tag{8}$$

All the formulas given and notations used have clear origins and meanings. The market S&D functions are normalized to the total market S&D, S^0 and D^0 respectively, expressed in currency:

$$\int_{-\infty}^{\infty} D(p)\,dp = D^0, \quad \int_{-\infty}^{\infty} S(p)dp = S^0. \tag{9}$$

In these formulas *N* and *M* are the numbers of buyers and sellers in the market, respectively. p_n^D and q_n^D are the desirable price and the demanded quantity of the *n*-th buyer, and p_m^S and q_m^S are the desirable price and the supplied quantity of the *m*-th seller. By definition, the probability distributions of market agents are described here by means of Gaussians (3) and (7) which are normalized to 1. Their parameters w_n^D and w_m^S determine full widths at half maximum of their peaks Γ_n^D and Γ_m^S, respectively, and are referred to below as natural widths.

Thus, in the SP model three parameters — price p_n^D, quantity q_n^D, and natural width Γ_n^D completely define the behavior strategy of the *n*-th buyer. Analogously, three parameters — price p_m^S, quantity q_m^S, natural width Γ_m^S — define the behavior strategy of the *m*-th seller in the market. The market S&D functions are represented by linear combinations of Gaussians where the coefficients are equal to the total agent S&D (D_n^0 and S_m^0, respectively) expressed in a monetary form, i.e., measured in American *$* in a definite period of time, say a year. Other details of construction of these functions can be found in Chapter III and our previous work [1]. Let us stress that the mathematical simplicity of the proposed version of the SP model provides a great scope for computational experiments, with various model markets having different features and phenomena. All computations can be easily performed by students.

In Fig. 2 an example is given of this type of probabilistic S&D functions for the model market of grain consisting of three buyers and three sellers. Note that in all our drawings we give the price of 1 ton of grain in American dollars (*$*) on the horizontal axis, and the values of

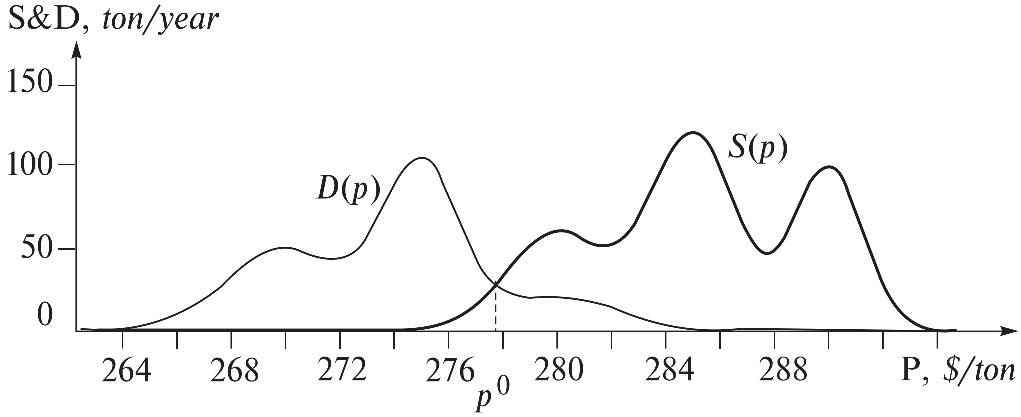

Fig. 2. The S&D functions combined into one S&D diagram for the model market of grain. The market demand function $D(p)$ is expressed as a sum of the individual demand functions $D_n(p)$ of three buyers with the following parameters: p_1^D = 270.0 *\$/ton*, q_1^D = 1,0 *ton/year*, Γ_1^D = 5,3 *\$/ton*; p_2^D = 275,0 *\$/ton*, q_2^D = 1.3 *ton/year*, Γ_2^D = 3,3 *\$/ton*; p_3^D = 280,0 *\$/ton*, q_3^D = 0,4 *ton/year*, Γ_3^D = 5,3 *\$/ton*. The demand bell tower has main maximum at price p^D which is approximately equal to 274.9 *\$/ton*. The market supply function $S(p)$ is expressed as a sum of the individual supply functions $S_n(p)$ of three sellers with the following parameters: p_1^S = 280,0 *\$/ton*, q_1^S = 1,0 *ton/year*, Γ_1^S = 4,3 *\$/ton*; p_2^S = 285,0 *\$/ton*, q_2^S = 1,7 *ton/year*, Γ_2^S = 3,7 *\$/ton*; p_3^S = 290,0 *\$/ton*, q_3^S = 1,1 *ton/year*, Γ_3^S = 3,0 *\$/ton*. The supply bell tower has its main maximum at price p^S which is equal to 285,0 *\$/ton*. The equilibrium price p^0 (the intersection point of S&D curves) is approximately equal to 277,5 *\$/ton*.

functions in *ton/year* on the vertical axis. Furthermore, the absolute values on axis Y belong only to the S&D functions and their elasticities, whereas for other functions they make only relative sense. In other words, for them only the curve forms make sense. It is also worth noting the fact that the SP model, when compared to the traditional model, brings about the formation of the probabilistic market S&D functions with a rather complicated structure. These functions are structurally similar to the complicated optical spectra of molecules, and they are in no way the straight lines of the traditional model. As probabilistic S&D curves consist of several agent bells, we will refer to the probabilistic market S&D functions as S&D bell towers. For more details see [1].

The problem which we will solve in the next sections of the Chapter is the following: how can we calculate market prices and clarify main features of the market pricing process using these probabilistic S&D functions? The answer is almost obvious. If the probability of making a purchase by a buyer at any price, p equals $D(p)$ and the probability of making a sale by a seller at the same price p equals $S(p)$, then the probability that a purchase-sale transaction (or simply a deal) will be made at price p and is equal to the product of the purchase and sale probabilities at this price p, i.e., $D(p) \cdot S(p)$. Thus, we obtain a very important result. The mechanism of making deals in the markets and the mechanism of the very market pricing process both have a probabilistic nature, market deals can be made at any price p, but with different probabilities $D(p) \cdot S(p)$. More specifically, a deal probability distribution is determined by the product of purchase and sale probability distributions, i.e., by the product of the S&D functions which will be referred to as the deal function $F(p)$:

$$F(p) = D(p) \cdot S(p). \tag{10}$$

It is also clear that the prices of local highs or maxima p^m of the deal function $F(p)$ will be the most probable prices of goods. These are naturally called *market prices* in this model as they will be dominating in the market. Thus, we obtain the major conclusion concerning the probabilistic nature of market prices and their method of their calculation by means of the known S&D functions.

Generally speaking, market deals can be arranged at all possible prices p from zero to infinity, but each of these possible deals will be made only with a definite probability which is determined by the deal function $F(p)$. The prices p^m at which the probability of arranging deals has rather sharp local highs, or maxima, can be referred to as market prices. In other words, under certain conditions the market prices are the prices of local maxima of the deal function $F(p)$.

This probabilistic market pricing mechanism within the framework of the SP model is clearly demonstrated in Fig. 3: strong overlapping of the S&D bell towers gives rise to the relatively narrow market deal function $F(p)$, exemplified in Fig. 3 as a "bell curve" that will be called the "market bell". In order to avoid misunderstanding we note that the probabilistic S&D functions each have dimension of *ton/year*, and the deal function has the dimension of (*ton/year*)^2. Therefore in Fig. 3 only the form of the curve $F(p)$ makes sense, but not its absolute values on the vertical axis (in fact, they are divided by 8). All calculations are performed for our model market of grain with three buyers and three sellers having rather narrow natural widths Γ_n^D and Γ_m^S in order to make the picture more expressive.

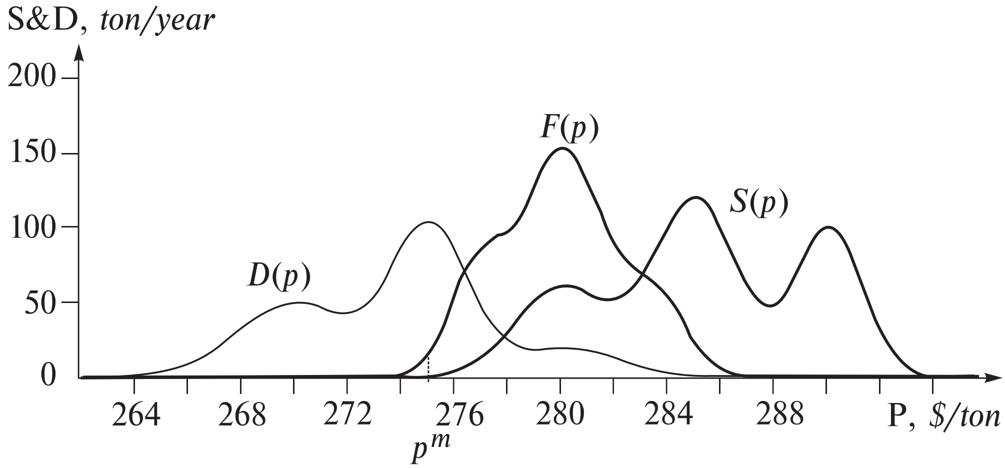

Fig. 3. The demand function $D(p)$ (to the left), supply function $S(p)$ (to the right), and deal function $F(p)$ (in the middle) in the form of the market bell combined in one picture for the model market of grain. The market price p^m corresponding to the maximum of the deal function is equal to 280,0 *$/ton*.

It is obvious that if at least one of the S&D functions, $S(p)$ or $D(p)$, presents a rather narrow curve, the respective curve $F(p)$ will also be rather narrow. In this case there is no doubt at all that the price of the local maximum of the deal function $F(p)$ or the top of the market bell can be regarded as the market price. Such situations will be discussed below in detail. In a more general case the deal functions are represented by an asymmetric, relatively wide market bell (curve) without a sharp maximum. We can say that the market prices are all prices under this market bell, and we simply call them the *market bell prices*. It may be figuratively said that life is in full swing in the market when under this market bell. We call this range of market bell prices the *market life zone* because practically all deals in the market are made at these prices. Note that the equilibrium price p^0 usually belongs to the set of market bell prices too. All other price ranges, especially a range near zero and a range of very high prices, we call the *market dead zones* because no deals are made there.

In closing, it can be said that in a broad sense the market prices are all the market bell prices which show up in the market, but with a different probability. However, under certain conditions, namely where we have a narrow market bell, the statement that a given price p is the market price means that this price p is approximately equal to the top price p^m of the market bell. In this article series we will always use the term market price in its narrow sense, unless otherwise stated.

4. Market Price and Total Supply and Demand

One of the most unexpected outcomes of the present investigation of the market pricing mechanism is the discovery that within the framework of the SP model, market prices p^m are independent from the total S&D, S^0 and D^0. This independence effect is a direct consequence of the probabilistic nature of market prices, as will be shown below for a one-good market. As a preliminary, we will briefly review the formulas for the S&D functions:

$$D(p) = D^0 d(p), \quad \int_{-\infty}^{\infty} d(p)\,dp = 1, \quad \int_{-\infty}^{\infty} D(p)\,dp = \mathrm{D}^0; \tag{11}$$

$$S(p) = S^0 s(p), \quad \int_{-\infty}^{\infty} s(p)\,dp = 1, \quad \int_{-\infty}^{\infty} S(p)\,dp = S^0. \tag{12}$$

The probability distributions $d(p)$ and $s(p)$ represent graphical forms of the S&D curves.

The deal function $F(p)$ can be presented in the following form:

$$F(p) = D^0 S^0 f(p), \quad \int_{-\infty}^{\infty} F(p)\,dp = F^0, \tag{13}$$

where

$$f(p) = d(p)s(p), \quad F^0 = \int_{-\infty}^{\infty} D(p)\,S(p)\,dp = D^0 S^0 \int_{-\infty}^{\infty} f(p)\,dp. \tag{14}$$

Thus, by definition, the overlap integral of the S&D functions F^0 gives the total deal probability in the market that, naturally, can serve as a measure of the market trading volume (or market value). For the sake of certainty, we will call F^0 a trading volume. It is obvious for one-good markets that the stronger the overlap of the S&D curves, the greater the area under the $F(p)$ curve, and hence, the greater the trading volume F^0. Fig. 4 serves as a striking illustration of this overlapping effect. Fig. 4 is a variant of Fig. 3 with an additional second supply function $S_2(p)$ which is the same as the original one $S_1(p)$ but shifted to the left by 2 *\$/ton*. It is evident that $S_2(p)$ overlaps with the demand function $D_1(p)$ more strongly and therefore their deal curve $F_2(p)$ occupies more space than the original first deal curve $F_1(p)$. Notice also that market price p^m

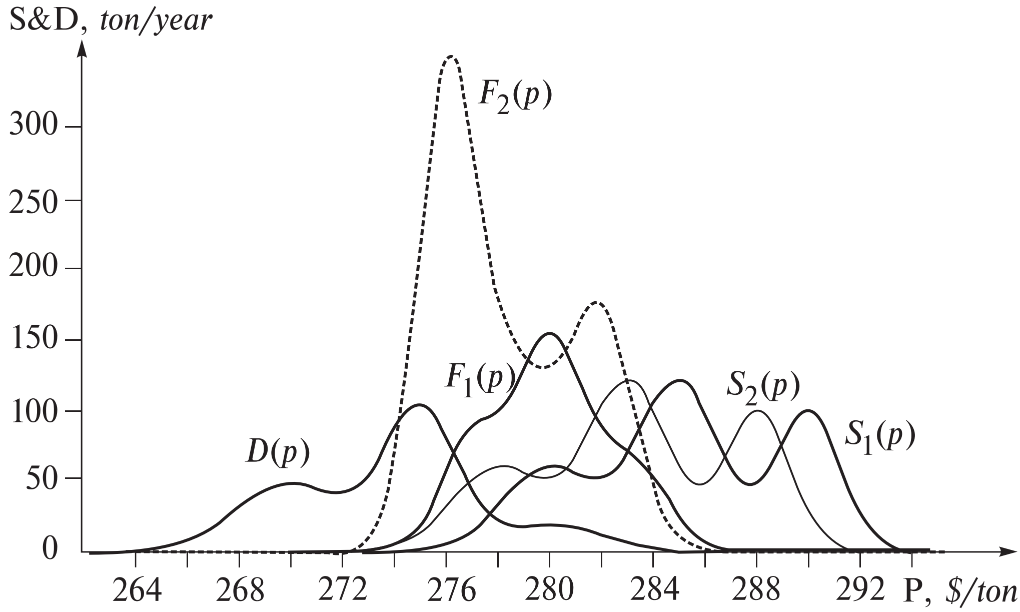

Fig. 4. Graphic illustration of the overlapping effect. If the S&D functions are brought close together, the market trading volume (it corresponds to the areas under the deal functions $F_1(p)$ and $F_2(p)$) grows.

moves to the left almost by 4 *$/ton*. The equilibrium price p^0, by the way, moves to the left by approximately 1 *$/ton* too.

We come now to the problem of relating market prices to the total S&D. It is easy to recognize that maxima of the $F(p)$ and $f(p)$ functions are common. As a consequence of this fact, market prices p^m do not directly depend either on the total demand D^0 or on the total supply S^0. It does not exclude, of course, the existence of indirect influence of the total S&D on the forms of the S&D curves (i.e., on the $d(p)$ and $s(p)$ functions) and, hence, on market prices. However, we can imagine a hypothetical case in which the stationary state of the economy changed as follows: S&D curve forms remained invariable (i.e., $d(p)$ and $s(p)$ functions unchanged) though the total S&D, S^0 and D^0, can change strongly. This is a typical example of a *ceteris paribus* situation. There is no doubt that in this case market price p^m remains the same, although the trading volume F^0 can change markedly. At first sight this effect seems to be quite strange and, at least, to contradict the conventional treatments of the empirical laws of S&D. Below we will show that it is not absolutely so. This phenomenon is displayed in Fig. 5 for our model market of grain.

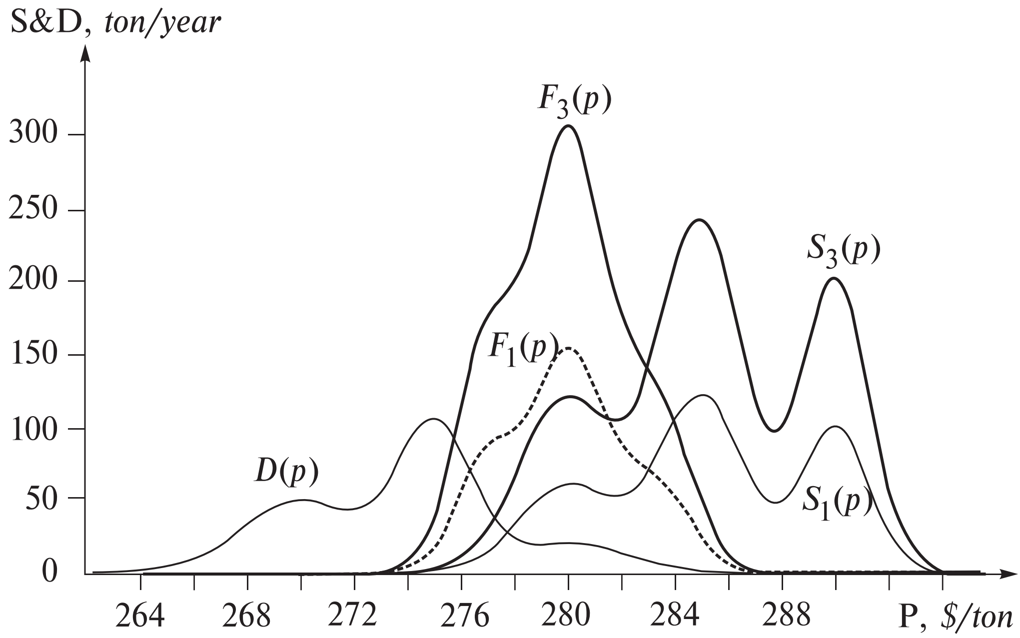

Fig. 5. Graphic illustration of the independence effect of market price p^m from the total supply $S^0(p_2^m = p_1^m)$.

Two supply functions $S_1(p)$ and $S_3(p)$ are depicted as having the same form (i.e., the same $s(p)$) but different total supplies S_1^0 and $S_3^0 = 2S_1^0$ and hence, different deal functions $F_1(p)$ and $F_3(p) = 2F_1(p)$, respectively. We can easily see in Fig. 5 that while the equilibrium price decreases somewhat ($p_3^0 < p_1^0$), the market price remains unchanged ($p_3^m = p_1^m$). The trading volume naturally increases two fold, $F_3^0 = 2F_1^0$.

In practice, this rather surprising theoretical outcome of the SP model gives a real opportunity to sell more goods, without bringing down a good price. This effect seems to be of particular importance for the central bank of a country which is interested in the issue of unsecured money on a large scale. As we see, the main concern of the central bank is to choose a necessary structure of function $s(p)$ to keep it unchanged, but gradually and evenly to increase the total supply S^0 when issuing a new portion of unsecured money. The central bank has to do it rather carefully in order not to provoke the market agent's suspicion and fear of inflation in the immediate future. In this case the Central Bank naturally counts on buyers to not react long enough to this credit expansion by change of function $d(p)$. The buyers can change total demand D^0, generally speaking, as it is necessary for them. For example, they can increase D^0 in order to acquire additional money supplied and to invest it in one way or another, thereby increasing production of goods and services. This mechanism of credit expansion is of particular value where other countries buy reserve currencies in order to accumulate state reserves. It is a real *perpetuum mobile* of economy in the hands of governments. However, there is an inescapable problem: "the governments aim at the greatest possible amount of credit expansion. Credit expansion is the governments' foremost tool in their struggle against the market economy. In their hands it is the magic wand designed to conjure away the scarcity of capital goods, to lower the rate of interest or to abolish it altogether, to finance lavish government spending, to expropriate the capitalists, to contrive everlasting booms, and to make everybody prosperous. The inescapable consequences of credit expansion are shown by the theory of the trade cycle" [4]. It is a shame, but sooner or later (and usually sooner) this *perpetuum mobile* strongly accelerates the economy such that it collapses, and a crisis ensues.

In closing, we consider the independence effect obtained in this section to give a new insight into the known phenomenon: the money depreciation rate (i.e., the speed of decreasing the price of money, or inflation) is often in practice much lower (e.g., ten times) than the money growth rate (i.e., the speed of increasing money supply) in the country.

Below, using some simple typical examples and situations from the known textbooks we will show how the SP model works with regards to the probabilistic market pricing phenomenon.

5. Monopoly of Supply

Let us suppose that there are a lot of buyers and only one seller monopolist in a one-good market. This classical model situation of monopoly of supply provides an excellent example of how the SP model works in the market. In this case, supply is described by a narrow and high Gaussian. Within the limits of very small natural widths, the monopolist's supply function $S(p)$ becomes the so-called delta function $S^0\delta(p - p^S)$ which means that, by definition, it is equal to zero everywhere, except one point p^S at which it is infinitely large. Recall that this point is the price p^S set by the monopolist. The demand function $D(p)$ is contributed to by all buyers with different financial possibilities and different market strategies, therefore the $D(p)$ curve can be rather wide. To calculate the deal function we have to multiply the obtained S&D functions. It can be easily seen that as a result for the deal function $F(p)$ we obtain the delta function too, and as a consequence, the expected output: the market price is the price p^S set by the monopolist:

$$F(p) = D(p)\, S^0 (p - p^S). \tag{15}$$

The total probability of the deals in the market, i.e., the trading volume F^0 and the respective income I_m of the monopolist are simply calculated as follows:

$$F^0 = \int_{-\infty}^{+\infty} F(p)\, dp = \int_{-\infty}^{+\infty} D(p) S^0 \delta(p - p^S)\, dp = S^0 D(p^S),$$

$$I_m = p^S F^0 = p^S S^0 D(p^S). \tag{16}$$

We obtain a very interesting formula for evaluating the trading volume and income for the monopolist in the case of the monopoly of supply. Graphically, the example is illustrated in Fig. 6.

Eq. (16) and Fig. 6 show that the monopoly of supply can hamper the effective market functioning in the interest of buyers. "Competitive prices are the outcome of a complete adjustment of the sellers to the demand of the consumers. Under the competitive price the whole supply available is sold. No part of a supply available is permanently withheld from the market. The whole economic process is conducted for the benefit of the consumers. There is no conflict between the interests of the buyers and those of the sellers, between the interests of the producers and those of the consumers The entrepreneur in his entrepreneurial capacity is always subject to the full supremacy of the consumers. It is different with the owners of vendible goods and factors of production. Under certain conditions they fare better by restricting supply and selling it at a higher price per unit. The prices thus determined, the monopoly prices, are an infringement of the supremacy of the consumers and the democracy of the market" [4]. Is it bad for the economy as a whole? Yes, because a portion of buyers remain without the opportunity to acquire these goods for business or consumption and this retards the economy growth. Thus, the very existence of monopoly of supply probably

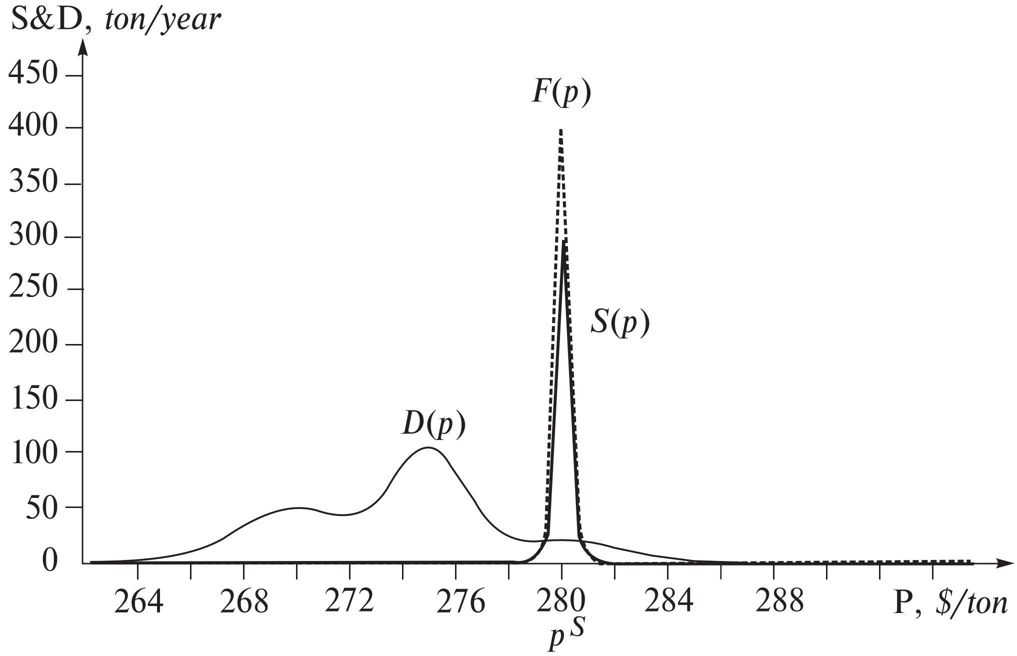

Fig. 6. Monopoly of supply in the market. The values of the deal function $F(p)$ are reduced by 15 times. Market price p^m coincides with the price p^S set by the monopolist which equals 280 *$/ton*.

leads to cases where a part of potential available for development is not used for societal benefit. Therefore, it is important to carry out reasonable regulation by the state of monopoly activity in markets in order to compel them not to set monopolistic prices. Generally speaking, a monopolist does not know the demand function $D(p)$ in detail, therefore it is a rather hard task for him or her to find out and set a higher (than competitive) monopolistic price in order to be able to take a part of supply away from the market, and nevertheless to obtain additional specific monopolistic income. Moreover, the buyers can do away with buying goods or services at monopolistic prices, and the monopolist will suffer from losses. For these and some other reasons, monopoly of supply does not always lead to monopoliatic prices and monopolistic incomes. From the point of view of the SP model, it means that the state has to provide such incentives and stimuli for buyers and monopoly that the overlap of the S&D functions will comply with the requirements and interests of economy as a whole. In other words, it means that market prices should be competitive ones.

6. Monopsony of Demand

In the case of monopsony of demand we have a market situation that is the reversal of the monopoly of supply situation. There are many suppliers (with the market supply function $S(p)$) and only one buyer. This could be the state, for instance, who sets the price p^D in the market. Monopsony of demand is described within the SP method by means of formulas

$$F(p) = D^0 \delta (p - p^D) S(p), \tag{17}$$

$$F^0 = \int_{-\infty}^{+\infty} F(p)\, dp = \int_{-\infty}^{+\infty} D^0 (p - p^D) S(p) = D^0 S(p^D). \tag{18}$$

Fig. 7 illustrates the market situation of monopsony of demand. Generally speaking, the monopsonist buyer may also abuse the market power over supplies. Therefore, the activity of the monopsonist buyer has to be regulated and directed by the state for the interests of economy as a whole towards competitive market prices.

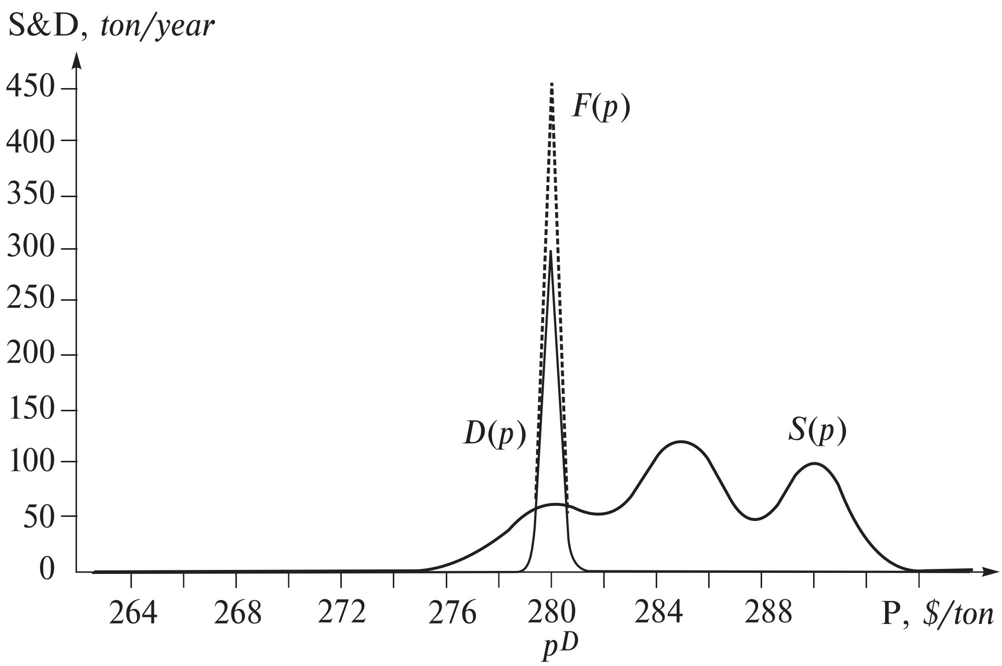

Fig. 7. Monopsony of demand in the market. The values of the deal function $F(p)$ are reduced by 40 times. Market price p^m coincides with the price p^D set by the monopsonist buyer which equals 280 *$/ton*.

7. Relationship between the Equilibrium Price and the Market Price in the One-Good Markets

The relationship between the equilibrium price p^0 and the market price p^m is of more direct interest to us. This relationship can be easily established by means of a Taylor series. Recall that in mathematics a function can often be represented aproximately by several terms that are connected with the function's derivatives at a chosen point. At that particular point we take the equilibrium price p^0 and restrict ourselves by two first terms. In other words, we represent the approximate S&D functions as straight lines in the vicinity of the equilibrium price p^0 as follows:

$$D(p) \cong a + b \cdot p, \quad S(p) \cong c + d \cdot p, \tag{19}$$

where

$$b = D'(p^0), \quad a = D(p^0) - b \cdot p^0, \;\; d = S'(p^0), \;\; c = S(p^0) - dp^0. \tag{20}$$

In these formulas $D(p^0)$ and $S(p^0)$ are the values of the S&D functions in the point p^0 in which they are equal, and $D'(p^0)$ and $S'(p^0)$ are the values of their first derivatives in p^0 . The product of these linear functions gives a parabolic curve which maximum p^m is derived in one step:

$$p^m = -\frac{a \cdot d + b \cdot c}{2b \cdot d}. \tag{21}$$

Trivial calculations lead to the equation which describes a direct relationship between the equilibrium price p^0 and the market price p^m:

$$p^m = p^0 + \frac{D(p^0)(d - |b|)}{2b|d|}. \tag{22}$$

Interpretation of the received relationship is evident: if $d > |b|$, then $p^m > p^0$; if $d < |b|$, then $p^m < p^0$. Roughly speaking, if supply is steeper than demand, market price is more than the equilibrium one; if demand is steeper than supply, market price is less than the equilibrium one. In order to make sure that this simple rule really works, it is enough to look at Figs. 3–7. If $d = |b|$, i.e., the slopes of the S&D functions are equal in magnitude but opposite in sign and the equilibrium price coincides with the market price: $p^0 = p^m$. Remember that the formulas given above are approximate, and valid only in a small vicinity surrounding the point of equilibrium p^0. To obtain more exact formulas it is necessary to enter into the theory, at the very least, the third member of a Taylor series. But qualitatively drawn conclusions about the relationship of the equilibrium price and the market price will remain the same.

8. Market Price and Elasticities of Supply and Demand

In economic literature, one could usually find observations about the existence of a relationship between market prices and S&D elasticities. However, an equation to represent it was derived for the first time only recently [2]. To be thorough, we repeat the deduction of the respective equation for a one-good market. It is well-known from mathematics that the derivative of a function is equal to zero at the point of extremum of the function. In our case it implies that at the market price p_m the derivative of the deal functions $F(p)$, i.e., the derivative function $dF(p)/dp = F'(p)$, has to be equal to zero. From here, by means of simple calculations we can easily find that the market price p_m is the solution of the following differential equation:

$$ED(p) = -ES(p), \tag{23}$$

where the market S&D elasticities, i.e., $ED(p)$ and $ES(p)$, are on the left and on the right of the equation, respectively. Below we give them the following definitions:

$$ED(p) = \frac{dD(p)}{D(p)} \bigg/ \frac{d(p)}{p} = \frac{dD(p)}{dp}\frac{p}{D(p)} = \frac{dD'(p)}{D(p)}p,$$

$$ES(p) = \frac{dS(p)}{S(p)} \bigg/ \frac{d(p)}{p} = \frac{dS(p)}{dp}\frac{p}{S(p)} = \frac{dS'(p)}{S(p)}p. \tag{24}$$

Thus, we have a very interesting result — the maximum probability of deals is reached at a price at where the S&D elasticities are equal in magnitude but opposite in sign. It is graphically illustrated in Fig. 8. Let

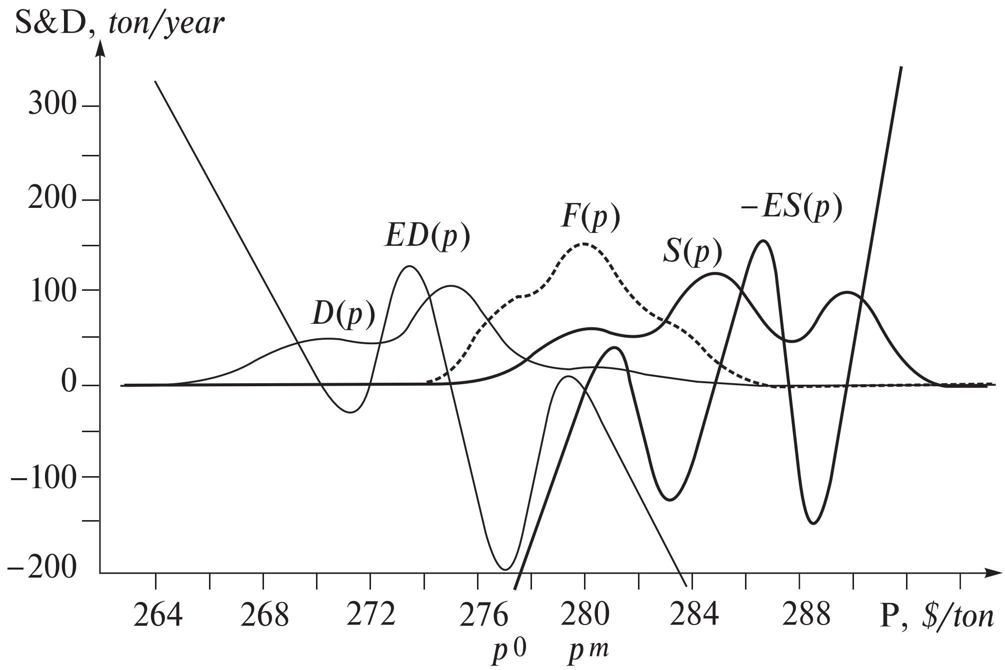

Fig. 8. Graphical illustration of the relationship of the market price p^m with the intersection point of the S&D elasticity curves $ED(p)$ (to the left) and $-ES(p)$ (to the right). Casually it turned out that the scale on axis *S&D* belongs not only to the S&D functions but also to the S&D elasticities.

us emphasize once again that the market price is not a so-called equilibrium price p^0 at which the market S&D curves intersect (i.e., they are equal in magnitude). The market price is the price p^m at which the market S&D elasticities intersect (more exactly, $ED(p)$ and $-ES(p)$). In our opinion, the fundamental result attained in [2] needs to be carefully checked, as does the SP model as a whole. For precision, notice that if the deal function has several extrema, their identification and interpretation will require some additional work.

9. Probabilistic Market Pricing in the Many-Good Markets

Recall that for calculating the agent S&D functions of many-good markets in the P-space the multiplicativity rule was used in [1]. In other words, the agent S&D functions in the P-space are represented by the products of the respective agent S&D one-good functions. Below, we are going to briefly repeat the derivation of respective set of formulas. For more detail see [1].

We will consider a very general case where the n-th buyer wants to buy all L goods in the market for a definite sum of money D_n^0. According to his or her needs and criteria the buyer divides this sum of money among all goods demanded and elaborates his or her unique strategy in the market. The corresponding multi-dimensional demand function $D(p_1, \ldots, p_L)$ can be approximately represented in a factorized form using the multiplicativity formula for demand as follows:

$$D_n(p_1, \ldots, p_L) = \frac{\sum_{l=1}^{L} D_{nl}^0}{\prod_{l=1}^{L} D_{nl}^0} \prod_{l=1}^{L} D_{nl}(p_l), \tag{25}$$

where

$$D_{nl}^0 = p_{nl}^D \cdot q_{nl}^D. \tag{26}$$

In this formula Gaussians can be used for approximate representation of the buyer's one-dimensional demand functions as follows:

$$D_{nl}(p_l) = D_{nl}^0 \cdot g_{nl}(p_l), \tag{27}$$

$$g_{nl}(p_l) = (p_l) = \sqrt{w_{nl}^D / \pi} \cdot \exp\left(-w_{nl}^D \left(p_l - p_{nl}^D\right)^2\right), \tag{28}$$

$$D_n^0 = \sum_{l=1}^{L} D_{nl}^0 = \sum_{l=1}^{L} p_{nl}^D \cdot q_{nl}^D. \tag{29}$$

All new parameters make obvious sense by the folliwing definition: p_{nl}^D is the price at which the n-th buyer plans to buy the l-th good in quantity q_{nl}^D, and D_{nl}^0 is the n-th buyer's total demand of the l-th good expressed in a monetary form, etc.

It is easy to check that for the n-th buyer his or her one-dimensional demand function $D_{nl}(p_l)$ is normalized to his or her total demand of the l-th good D_{nl}^0, and his or her multi-dimensional demand function $D_n(p_1, \ldots, p_L)$ is normalized to his or her total demand D_n^0. Then, by means of the additivity formula for demand, we describe the multi-dimensional market demand function as a sum of the individual multi-dimensional functions as follows:

$$D(p_1,\ldots,p_L)=\sum_{n=1}^{N} D_n(p_1,\ldots,p_L)=\sum_{n=1}^{N}\frac{\sum_{l=1}^{L} D_{nl}^0}{\prod_{l=1}^{L} D_{nl}^0}\prod_{l=1}^{L} D_{nl}(p_l), \tag{30}$$

$$D^0=\sum_{n=1}^{N} D_n^0=\sum_{n=1}^{N}\sum_{l=1}^{L} D_{nl}^0=\sum_{n=1}^{N}\sum_{l=1}^{L} p_{nl}^D\cdot q_{nl}^D\ . \tag{31}$$

The market demand function is normalized to the total market demand D^0.

By the same procedure we obtain the multi-dimensional factorized supply function of the m-th seller using the multiplicativity formula for supply as follows:

$$S_m(p_1,\ldots,p_L)=\frac{\sum_{l=1}^{L} S_{ml}^0}{\prod_{l=1}^{L} S_{ml}^0}\prod_{l=1}^{L} S_{ml}(p_l), \tag{32}$$

where

$$S_{ml}^0 = p_{ml}^S\cdot q_{ml}^S. \tag{33}$$

To some extent of accuracy, we can also use Gaussians as the seller's one-dimensional supply functions:

$$S_{ml}(p_l) = S_{ml}^0\cdot g_{ml}^S(p_l), \tag{34}$$

$$g_{ml}^S(p_l) = \sqrt{w_{ml}^S/\pi}\cdot\exp\left(-w_{ml}^S\left(p_l-p_{ml}^S\right)^2\right), \tag{35}$$

$$S_m^0 = \sum_{l=1}^{L} S_{ml}^0 = \sum_{l=1}^{L} p_{ml}^S\cdot q_{ml}^S. \tag{36}$$

The seller's multi-dimensional supply function $S_m(p_1,\ldots,p_L)$ is normalized to his or her total supply S_m^0. Then applying the additivity formula for supply we obtain the multi-dimensional market supply function $S(p_1, \ldots, p_L)$ as a sum of the individual multi-dimensional supply functions as follows:

$$S(p_1, \ldots, p_L) = \sum_{m=1}^{M} S_m(p_1, \ldots, p_L) = \sum_{m=1}^{M} \frac{\sum_{l=1}^{L} S_{ml}^0}{\prod_{l=1}^{L} S_{ml}^0} \prod_{l=1}^{L} S_{ml}(p_l). \tag{37}$$

This function is normalized to the market total supply S_0 which is calculated by means of summing as follows:

$$S^0 = \sum_{m=1}^{M} S_m^0 = \sum_{m=1}^{M} \sum_{l=1}^{L} S_{ml}^0 = \sum_{m=1}^{M} \sum_{l=1}^{L} p_{ml}^S \cdot q_{ml}^S. \tag{38}$$

Thus, using the multiplicativity rule for S&D we have factorized the many-good market S&D functions. That is, we represented the multi-dimensional market functions by the products of the respective one-dimensional agent's functions. This naturally made calculations and studies in detail of markets and market processes much easier.

Now we can return to our main task of obtaining a set of equations to calculate the market prices in many-good markets. Let us take rather obvious steps to achieve our goal. First of all, it is evident that the deal function $F(p)$ can be defined here in the same way as in the case of the one-good market, namely as a product of the respective multi-dimensional market S&D functions:

$$F(p_1, \ldots, p_L) = D(p_1, \ldots, p_L) * S(p_1, \ldots, p_L). \tag{39}$$

It is obvious that in our case the deal function $F(p_1, \ldots, p_L)$ is differentiable in the whole price space. It can be easily seen that the problem of finding the market prices is reduced here to finding the local maxima of the deal function in the L-dimensional P-space. It is well-known from the standard course of mathematical analysis that this task for differentiable functions is reduced in its turn to the decision of system of L partial differential equations:

$$\partial F(\boldsymbol{p})/\partial p_l, \quad l = 1, 2, \ldots, L. \tag{40}$$

This system in its turn can be easily transformed to the system of L partial differential equations expressed by means of the S&D functions as follows:

$$D(\boldsymbol{p})\,\partial S(\boldsymbol{p})/\partial p_l + \partial D(\boldsymbol{p})\,/\partial p_l S(\boldsymbol{p}) = 0, \quad l = 1, 2, \ldots, L. \tag{41}$$

In (43) and (44) for brevity we used only one bold letter $\boldsymbol{p}$ to designate all L prices p_l (now $\boldsymbol{p}$ is already the vector in the L-dimensional P-space [2]). All partial derivatives in our case are easily calculated because all differentiations are reduced to differentiation of Gaussians $g_{nl}(p_l)$ and $g^S_{ml}(p_l)$, that is carried out in a trivial way: differentiation of a Gaussian gives rise to another Gaussian.

For completeness, we note that this equation system gives extrema from which it is necessary to select local maxima interesting for us by means of obvious calculations. Not to overload this article with drawings, we have limited ourselves to the derivation of these equations and we will not show how the SP model works on the tasks thought up artificially. This will be partially done elsewhere [6, 7]. By means of MathCad programs the reader can easily find solutions to the tasks and receive expressive, multi-dimensional surfaces with maxima in the points corresponding to the market prices of all goods. And then by carrying out numerical experiments the reader can study in detail the dynamics of market price structure for all the goods when the strategies of individual agents in the market are changing. Note that by using Gaussians, students can master these computational procedures by using the described version of the SP model.

10. Comments and Conclusions

The main outcomes of this Chapter are as follows. Using the SP model we studied the central issue of any economic theory, namely the question of the nature of market prices. The result is that market prices are probabilistic by their nature. Under certain conditions they can be defined as local maxima of the multi-dimensional deal function $F(p)$ which is the product of the D&S functions of the many-agent, many-good markets. Conceptually, the deal function $F(p)$ is a probability distribution of making deals in the market. It is clear that deals can be made in a market at all possible prices, but with different probability. If the deal function $F(p)$ is represented by rather a narrow bell curve, it makes sense to refer the top or maximum of the bell to as the market prices. In a more general case, the deal functions are represented by an asymmetric relative wide market bell curve without a sharp maximum. We can say here that all market prices are the prices under this market bell, or simply the market bell prices. Figuratively, life is in full swing in the market particularly under this bell. We call this range of market bell prices *the market life zone* because practically all deals in the market are made at the respective prices.

By definition, the overlap integral of the S&D functions F^0 gives the total deal probability in the market that, naturally, can serve as a measure of a respective market value. It is shown that the market prices do not depend at all on the total market S&D, S^0 and D^0. Particular emphasis is placed on the comparative analysis of numerical results of the SP model for various model markets with known empirical facts. It is also shown that for one-good markets the market prices are intersection points of the S&D elasticity curves. A set of partial differential equations for the calculation of market prices in the many-agent, many-good markets is given at the end of the chapter.

References

1. Anatoly Kondratenko. *Probability Economics: Supply and Demand in the Price Space*. Electronic copy available at: http://ssrn.com/abstract=2250343. See also Chapter IV.
2. Anatoly Kondratenko. *Physical Modeling of Economic Systems. Classical and Quantum Economies*. Novosibirsk: Nauka, 2005. Electronic copy available at: http://ssrn.com/abstract=1304630.
3. C.R. McConnell, S.L. Brue. *Economics. Principles, Problems, and Policies*. 14-th Edition. Irwin McGraw-Hill, 2003.
4. Ludwig von Mises. *Human Action. A Treatise on Economics*. Yale University, 1949.
5. Jean-Philippe Bouchaud. *Economics Needs a Scientific Revolution* // Nature. October 2008. Vol. 455/30.
6. Anatoly Kondratenko. *Trade Maximization Principle: Market Processes, Supply and Demand Laws, and Equilbrium States*. Electronic copy available at: http://ssrn.com/abstract=2431218. See also Chapter VIII.
7. Anatoly Kondratenko. *Probability Economics: Market Force in the Price Space*. Electronic copy available at: http://ssrn.com/abstract=2270306. See also Chapter VI.

CHAPTER VI.
Market Force in the Price Space

> "It is customary to speak metaphorically of the automatic and anonymous forces actuating the "mechanism" of the market. In employing such metaphors people are ready to disregard the fact that the only factors directing the market and the determination of prices are purposive acts of men. There is no automatism; there are only men consciously and deliberately aiming at ends chosen. There are no mysterious mechanical forces; there is only the human will to remove uneasiness. There is no anonymity; there is I and you and Bill and Joe and all the rest. And each of us is both a producer and a consumer".
>
> Ludwig von Mises. Human Action. A Treatise on Economics. Page 315

PREVIEW.
What is Market Force?

Much attention will be given to the important question "what is market force" in this chapter, which is the third one in our work series devoted to probability economics. We pursue the development of probability economics, defined as a new quantitative method for description, analysis, and investigation of both model and real economies and markets. An important but broad and indefinite notion of market force in economics has been thoroughly revised from the point of view of the uncertainty and probability concept. According to the new mode of thought, knowledge of market force is best expressed in terms of the probabilistic supply and demand functions. A strict mathematical definition of market forces as the forces of supply and demand is given. Market forces defined as such have not only magnitude but also sign, therefore they can be interpreted as vectors in the multi-dimensional price space of many-good markets. We demonstrate how the agent's forces are summed up to produce the market forces of supply and demand. We have also established that under certain conditions the market prices coincide with the equilibrium prices of market forces. In other words, market prices are the prices at which market forces of supply and demand are equal in magnitude but opposite in sign by analogy with the equilibrium concept in classical mechanics.

1. Introduction

In the present chapter, we pursue the development of probability economics that was defined in the two previous chapters as a new quantitative method for description, analysis, and investigation of both the model and real economies and markets. Chapter III provided the necessary background of the Stationary Probability Model in the Price Space (the SP model below). The new and logical concept of the probabilistic market supply and demand (S&D below) functions in the price space (the P-space below) has been introduced. The probabilistic S&D functions fundamentally differ from those of the traditional model in economics. It was shown in Chapter IV that the probabilistic nature of the S&D functions leads unambiguously to the fact that the nature of market prices is also probabilistic. It was shown that within the framework of the SP model, under certain conditions, the one-good market prices can coincide with the intersection points of the S&D elasticities. More specifically, this occurs at the points at which the S&D elasticities are equal in magnitude but opposite in sign. In the present chapter, we expand upon the nature of S&D, market price, and market force which is intimately related to all the previous notions. The specific notion of market force being rather broad and indefinite in economics is the subject of extensive research and reconsideration in this article.

No doubt that normal market functioning is the process of permanent interactions between market agents who pursue well-defined goals of one kind or another. For example, a seller may seek to obtain maximal revenue, to gain market power, to achieve a stable position in a market, or to increase his or her goodwill value [3, 4]. A buyer may simply want to buy food in a supermarket at the best prices. To put it otherwise, all characters playing in a big market are market agents, both buyers and sellers. All the agents may have a perceptible effect on one another. Figuratively, we can say that one agent influences with some market force on all other agents in the market. We know at the present very little about the nature of such forces; we can neither measure them in practice nor calculate them in theory. In this regard, the present situation in economic science seems to be similar to that in physics before Newton's era more than 300 years ago. Physicists at that time knew that there were some forces among physical bodies in real space, but they did not have a sophisticated understanding of their nature. Moreover, physicists were not able to either measure or calculate them. Nevertheless, they knew enough about the features of these forces to understand how these physical forces reveal themselves in practice, and how they can be used in building houses, going under sail, and even predicting weather. We also now know a lot about the features of market forces. This information can be used to help explain how market forces appear in economic real life, and can also help further personal interests or improve human welfare.

So what we really know about market forces is that they are exerted on market agents by means of an information exchange that has an impact on an agent's decision to buy or to sell goods. In other words, it is clear that market forces are connected in one way or another with S&D in a market, and that their magnitudes determine their capacities to influence market prices. Below, we will clarify the origin of market forces and develop a method of their calculation using the SP model. To be precise, we would like to make a general remark. Not only market agents but also other economic and political participants (state, trade unions, workers, etc.) possess capacities for interaction with all market agents, and have an impact on S&D and market prices. But this process can be performed only indirectly. These entities exert their influence on the market agents by means of information interaction. Only market agents have the ability to directly to impact market prices by means of their S&D with regards to individual contributions to the market S&D functions. This leads us to demonstrate the agent's forces, as well as the mechanism for adding up the agent's forces to form the market forces.

To start with, remember that the SP model deals with economies that are in a normal stationary state [1–4] or in other words, economic situations in which all time-dependent effects are negligibly small. Probabilistically, it means that all important events (mainly deals) in the market occur in roughly the same manner, and they can be regarded as recurring, or periodic, actions (for example, every year). The SP model holds that the market S&D functions are, in essence, partial probability distributions of making deals by buyers and sellers at price $\boldsymbol{p}$. As a consequence, the total probability distribution of making deals at price $\boldsymbol{p}$ is the product of respective partial probability distributions, or in other words, the product of the market S&D functions. The abovementioned can be expressed as the following equation:

$$F(\boldsymbol{p}) = D(\boldsymbol{p}) \cdot S(\boldsymbol{p}), \tag{1}$$

where one bold letter $\boldsymbol{p}$ factually combines the prices of all the L goods traded in the market. Actually, $\boldsymbol{p}$ is a vector in the L-dimensional P-space. Of course, all prices here are independent variables. $D(\boldsymbol{p})$ is the market demand function normalized to the total market demand D^0, and $S(\boldsymbol{p})$ is the market supply function normalized to the total market supply S^0 as follows:

$$\int_{-\infty}^{+\infty} D(\boldsymbol{p})\, d\boldsymbol{p} = D^0, \qquad \int_{-\infty}^{+\infty} S(\boldsymbol{p}) d\boldsymbol{p} = S^0. \tag{2}$$

$F(\boldsymbol{p})$ is the total probability distribution of making deals which was earlier [2, 3] designated as the deal function that is normalized to the so-called trading volume F^0 in the market as follows:

$$\int_{-\infty}^{+\infty} F(\boldsymbol{p})\, d\boldsymbol{p} = \int_{-\infty}^{\infty} D(\boldsymbol{p}) S(\boldsymbol{p})\, d\boldsymbol{p} = F^0. \tag{3}$$

So, in accordance with the SP model, the S&D and deal functions of the market do not depend on time at all. Market deals can be made at any magnitude of prices but with different probabilities. Local maxima of the deal function correspond to the most probable prices in the market. Under certain conditions discussed below, when these prices dominate the market, they can, in a narrow sense, be referred to as market prices. In other cases all the prices with relatively high probabilities should, in a broad sense, be regarded as the market prices. Note that all the functions discussed above are by definition positive, continuous, finite, and differentiable at any price from $-\infty$ to $+\infty$. So, we have sketched broad outlines of the probabilistic viewpoint on the market life within the framework of the SP model of economy.

As the question of the nature of market forces cannot be regarded as trivial, we will give it careful consideration by following a step-by-step strategy. In the first section we will thoroughly explore the simplest visible model of the market with a single good, a single buyer, and a single seller. We will give a clear answer to the question of what market forces are and discuss their peculiarities. The next sections will be devoted to more complicated market situations.

2. The One-Good Markets with One-Buyer and One-Seller

Now we are dealing with the case of the one-good, one-buyer, and one-seller model market. It is evident that this model is rather far from reality, and that it is only an imaginary construction. Nevertheless, it is an extremely fruitful and productive to study the features and details of both the economic and mathematical characters that play important roles in more realistic and hence complicated models. This very simple model has proven time and again its complete reliability and productivity in our previous studies [1–4]. This case of a hypothetical model market serves only to illuminate and illustrate the origin and main features of market forces.

Further, on the basis of common sense and real life experience we will make some remarks concerning the form of the probabilistic functions discussed above [1]. First, from practice we know that a seller wants to sell goods at the highest price possible, and a buyer to buy goods as cheaply as possible. Therefore, it is reasonable that in the market, the seller's desired price p^S is usually (but not always, of course) more than the buyer's desired price p^D. Secondly, it is natural to suppose that the graphs of probabilistic functions $D(p)$, $S(p)$, and $F(p)$ are "bell curves" with maxima at prices p^D, p^S, and p^m, respectively. The simplest universal mathematical function suitable for representation of such probabilistic functions is the Gaussian function, which is used within the framework of the SP model as some basic function.

The agent's probabilistic S&D functions are described by two sets of formulas, as follows:

$$D(p) = D^0 \cdot g^D(p), \qquad D^0 = p^D \cdot q^D, \tag{4}$$

$$g^D(p) = \sqrt{w^D/\pi} \cdot \exp\left(-w^D (p - p^D)^2\right), \tag{5}$$

$$D(p) = p^D \cdot q^D \cdot \sqrt{w^D/\pi} \cdot \exp\left(-w^D (p - p^D)^2\right), \tag{6}$$

$$\Gamma^D = \sqrt{-4\ln 0{,}5\,/w^D}\ . \tag{7}$$

And

$$S(p) = S^0 \cdot g^S(p), \qquad S^0 = p^S \cdot q^S, \tag{8}$$

$$g^S(p) = \sqrt{w^S/\pi} \cdot \exp\left(-w^S(p - p^S)^2\right), \tag{9}$$

$$S(p) = p^S \cdot q^S \cdot \sqrt{w^S/\pi} \cdot \exp\left(-w^S(p - p^S)^2\right), \tag{10}$$

$$\Gamma^S = \sqrt{-4\ln 0{,}5\,/\,w^S}. \tag{11}$$

In these formulas $g^D(p)$ and $g^S(p)$ are Gaussians, normalized to 1. Their frequency parameters w^D and w^S determine so-called natural widths, or, more exactly, full widths at half maximum of their peaks Γ^D and Γ^S, respectively.

Now let us sum up the results [1]. Three parameters of the SP model — price p^D, quantity q^D, and natural width Γ^D — completely define the strategy of the buyer's behavior in the market, and three parameters — price p^S, quantity q^S, and natural width Γ^S — reflect the strategy of the seller's behavior. It is obvious that the width of a curve depends on knowledge, welfare, mentality, features of the agent, and his or her final goals in the market. Below, we will use Gaussians for calculation and graphical representation of S&D functions. All the considerations discussed above are summarized pictorially in Fig. 1. More specifically, it is by means of these Gaussians that we calculate and plot the agent's S&D functions $D(p)$ and $S(p)$. Then, using Eq. (1) we calculate and plot the deal function $F(p)$ for the model market of grain from [1] with a single buyer and a single seller. As we can see in Fig. 1, the deal function lies mainly between the S&D functions and also looks like a bell curve. The area below it is simply called a *market bell*. Evidently, in a broad sense all the market bell prices are the real market prices, whereas in a narrow sense the prices of the local maxima of the market bell are referred to as the market price. In other words, the market price p^m equals 283,5 *\$/ton* in our case.

Remember that in the traditional model of economics the market price is thought of as the so-called equilibrium price p^0. This is located at the intersection point of the S&D curves, which in our case is at 282,5 *\$/ton*. For more detail concerning the nature of the market price

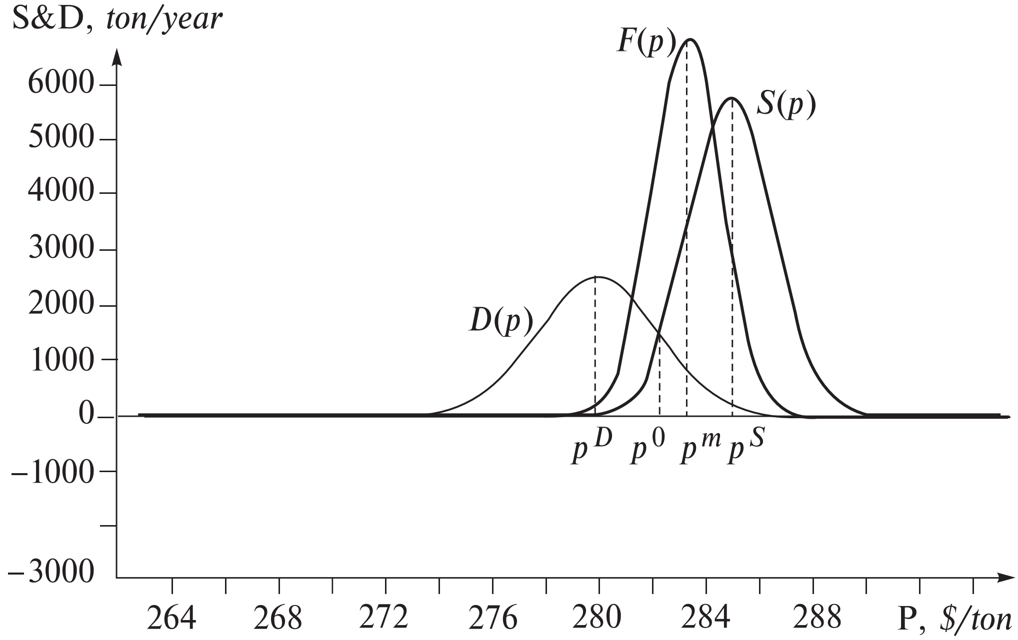

Fig. 1. Three main market functions: demand $D(p)$ (to the left), supply $S(p)$ (to the right), and deal $F(p)$ (in the middle) combined in one picture. For the sake of comparison all the magnitudes of the deal function $F(p)$ are divided by 400. Buyer's price p^D = 280,0 *\$/ton*, seller's price p^S = 285,0 *\$/ton*, market price $p^m \cong$ 283,5 *\$/ton*, equilibrium price $p^0 \cong$ 282,5 *\$/ton*.

p^m and its relationship with the equilibrium price p^0 see discussion in [2]. Within the framework of the SP model the maximum price p^m is referred to as the market price (in a narrow sense, as was discussed above), for the reason already discussed: real deals in the market are mostly fulfilled at prices near to this value. It is especially true when the deal function $F(p)$ represents a rather narrow market bell. However, there is one more reason why we study this market price with great interest: market S&D forces intersect at this market price, and, more exactly, market S&D forces are equal in magnitude but opposite in sign at this price. Below, this theorem will be proven.

It is well-known from the course of mathematical analysis that market prices, or more specifically, the maxima of the deal function $F(p)$ correspond to the points p, at which the first derivative of $F(p)$ is equal to zero:

$$dF(p)/dp = 0. \tag{12}$$

Since the deal function is the product of the S&D functions, this equation transforms into the following, slightly more complicated, equation:

$$dD(p)/dp \cdot S(p) + D(p) \cdot dS(p)/dp = 0. \tag{13}$$

It makes sense to arrange this equation in the form of an equilibrium equation as follows:

$$\frac{dD(p)/dp}{S(p)} = -\frac{dS(p)/dp}{D(p)}. \tag{14}$$

Thus, we obtain a very important outcome of the theory: market price is the price at which some characteristics of the S&D are equal in magnitude but opposite in sign. By analogy with classical mechanics, one can say that market price is at the point of equilibrium of supply and demand forces. Equality of the S&D forces resembles both in form and in content the condition for static equilibrium in classical mechanics, when two mechanical forces are equal in magnitude but opposite in sign. For this reason it makes sense to define market forces as S&D forces in accordance with Eq. (14).

Let us sum up intermediate results as follows. By definition, the buyer's demand force as function $U(p)$, and the seller's supply force as function $B(p)$, according to the following equations:

$$U(p) = D'(p)/D(p), \tag{15}$$

$$B(p) = S'(p)/S(p), \tag{16}$$

where the prime symbols indicate the first derivatives as is usual in mathematics. Now, the equilibrium equation for the market forces at the market price shows up as follows:

$$U(p) = -B(p). \tag{17}$$

Furthermore, let us call attention to the fact that in the range of medium prices, i.e., between the buyer's price p^D and the seller's price p^S, the first derivative of the demand function is negative and in contrast the first derivative of the supply function is positive in sign. It means that in this region the demand force is negative, and it is directed to the left, i.e., to the region of lower prices, more exactly, to the buyer's price p^D. This demand force mainly determines the so-called "bear" trend in the market. Therefore, for its designation we choose the letter *U* as the first letter in the word Ursus ("bear" in Latin). Opposing, the supply force is positive in the medium range of prices, and it is directed rightwards, i.e., to the side of higher prices, or, more specifically, to the seller's price p^S. This supply force mainly determines the so-called "bull" trend in the market. For this reason we designate this force by the letter *B,* as the first letter in the word Bovis ("bull" in Latin). In the region where the bull force is approximately balanced by the bear force, a major part of the deals are made in the market. This economic significance is stated by the equilibrium equation (17).

To illustrate this version of market pricing process within the framework of the SP model, the results of calculations of the market forces for our case are shown in Fig. 2 (which is in essence Fig. 1 with additional curves corresponding to the demand force $U(p)$ and supply force $B(p)$ with opposite sign). Fig. 2 is very impressive; there are two ideal straight lines of the market forces that intersect at the market price,

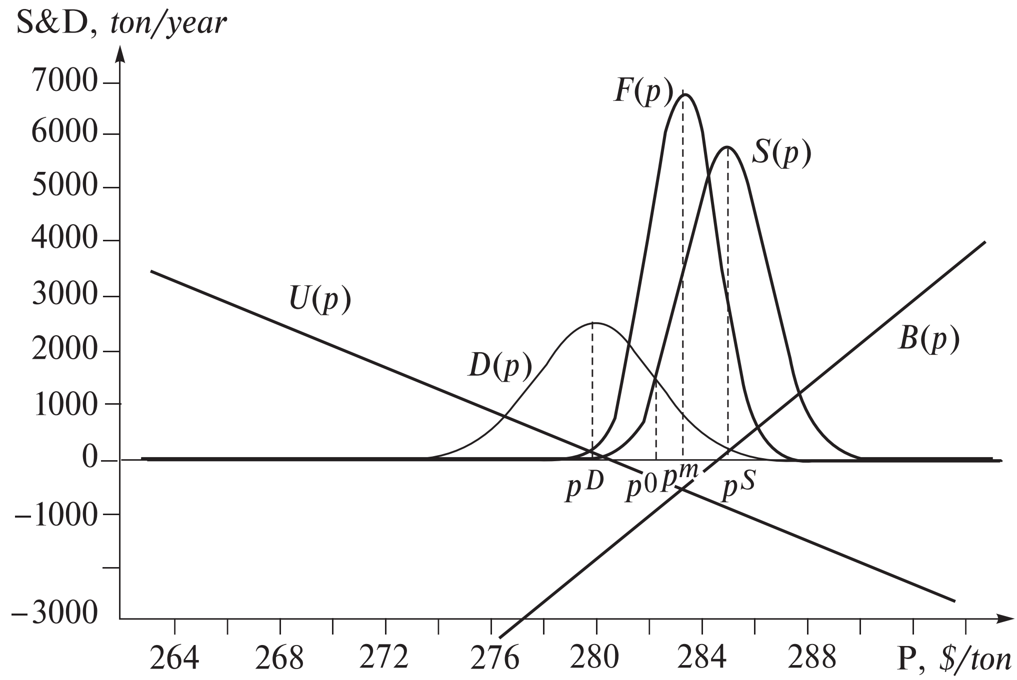

Fig. 2. Five market functions: demand $D(p)$ (to the left), supply $S(p)$ (to the right), deal $F(p)$ (in the middle), demand force $U(p)$ (below), and supply force $B(p)$ with opposite sign (below), combined in one picture. Intersection point of the D&S forces coincides with the maximum of the deal function $F(p)$, i.e., with the market price p^m.

For comparison, the magnitudes of the market forces are multiplied by 1000.

corresponding to the maximum of the deal function. It is of interest that the demand force is equal to zero at the buyer's price, p^D, where the demand is maximum, and then it increases linearly as it moves away from this price. The supply force shows the analogous behavior. These peculiarities of market forces can be easily obtained analytically, in our case by means of computing derivatives of S&D functions as follows:

$$U(p) = D'(p)/D(p) = u(p) = -2w^D(p - p^D), \tag{18}$$

$$B(p) = S'(p)/S(p) = b(p) = -2w^S(p - p^S). \tag{19}$$

So, we can see that such a surprising feature of the market forces as their linearity everywhere (i.e., at all prices), is a consequence of choosing the Gaussians as the agent's S&D functions (see Eqs. (6) and (8)). There is one more peculiarity of the market forces in our model. The linear equations describing the market S&D forces resemble in their form the forces of the quantum mechanical oscillators with frequencies $\omega^D = \alpha/\mu^D \times w^D$, $\omega^S = \alpha/\mu^S \times w^S$, and equilibrium points, p^D and p^S, respectively (for more detail see [6]).

Furthermore, we can find out the intersection point of the market forces using the obvious equality of the two forces (recall, we will use the opposite sign for supply force):

$$-2w^D \cdot (p - p^D) = 2w^S \cdot (p - p^S), \tag{20}$$

which creates an expressively simple formula for computing the market price in this model case:

$$p^m = \frac{w^D \cdot p^D + w^S \cdot p^S}{w^D + w^S}. \tag{21}$$

This equation states that the market price p^m (in a narrow sense, of course) is a frequency-weighted average of the agent prices p^D and p^S. Twice the frequency parameter is a slope of a curve (see Eq. (20)). Therefore, the larger is the slope of the buyer's (seller's) curve, the stronger is the market demand (supply) force, and hence the market price p^m will be nearer to the buyer's (seller's) price p^D (p^S).

This very important rule is exemplified most clearly by Fig. 2: the slope of the supply force is greater than the slope of the demand force, and as a consequence the market price is closer to the seller's price p^S

by 1 *$/ton* than to the buyer's price. If the slope of the supply force increases, then the market price will be practically equal to the seller's price p^S. See the case of monopoly of supply in [2], for instance, and quite to the contrary. There is one more interesting feature of this formula. The market price depends neither on the total demand nor on the total supply, in contrast with the equilibrium price p^0 which is strongly dependent on both values. This, the demand force of a market agent, is determined mainly by his or her frequency parameter. Since the notion of frequency in the given economic context is rather vague, we will express the relationship between the market price and agent prices in terms of natural widths Γ^D and Γ^S of the S&D curves as follows:

$$p^m = \frac{(\Gamma^S)^2 \cdot p^D + (\Gamma^D)^2 \cdot p^S}{(\Gamma^S)^2 + (\Gamma^D)^2}. \tag{22}$$

Now, this relationship can be described otherwise. The smaller the demand width Γ^D, the closer the market price p^m to the demand price p^D. The same is valid of course for the case of supply. It can be easily seen in Fig. 2 that supply width Γ^S is smaller than demand width Γ^D. Therefore, the market price p^m is closer to the seller's price p^S. In [2] there are two expressive examples which strengthen this rule: monopoly of supply and monopsony of demand. Both the monopolist and monopsonist have very narrow S&D functions, respectively (practically so-called special δ — functions). Therefore, market prices in these extreme cases coincide with the monopolist's and monopsonist's prices, respectively. Now we will complicate our model market by adding several buyers and sellers to the market.

3. The One-Good Markets with Several Buyers and Sellers

In this case the market S&D functions are obtained by summing up the respective agent S&D functions [1, 2], in other words, by using the following additivity formula [1, 2]:

$$D(p) = \sum_{n=1}^{N} D_n(p), \quad D^0 = \sum_{n=1}^{N} D_n^0 = \sum_{n=1}^{N} p_n^D \cdot q_n^D. \tag{23}$$

In the following formulas, all designations have an obvious origin and do not need additional comment:

$$D_n(p) = D_n^0 \cdot g_n^D(p), \tag{24}$$

$$g_n^D(p) = \sqrt{w_n^D/\pi} \cdot \exp\left(-w_n^D\left(p - p_n^D\right)^2\right),\ \Gamma_n^D = \sqrt{-4\ln 0{,}5 / w_n^D} \tag{25}$$

We describe the market supply function $S(p)$ as a sum of the individual supply functions $S_m(p)$ making use of the additivity formula [1, 2] as follows:

$$S(p) = \sum_{m=1}^{M} S_m(p), \quad S^0 = \sum_{m=1}^{M} S_m^0 = \sum_{m=1}^{M} p_m^S \cdot q_m^S. \tag{26}$$

We will also write out the additional straightforward formulas, without comment:

$$S_m(p) = S_m^0 \cdot g_m^S(p), \tag{27}$$

$$g_m^S(p) = \sqrt{w_m^S/\pi} \cdot \exp\left(-w_m^S\left(p - p_m^S\right)^2\right),\ \Gamma_m^S = \sqrt{-4\ln 0{,}5 / w_m^S}. \tag{28}$$

IIn the above definitions and equations N and M are the numbers of buyers and sellers, respectively. Other details can be found in [1, 2]. Clearly, definitions of the market forces as the S&D forces, as well as the equality of the S&D forces at the market price, retain their shapes — see Eqs. (15)–(17). The expected generalization of Eqs. (18)–(20) for a one-good, many-agent market can be easily made using a trivial mathematical treatment as follows:

$$U(p) = \sum_{n=1}^{N} U_n(p), \qquad B(p) = \sum_{m=1}^{M} B_m(p), \tag{29}$$

$$U_n(p) = D_n'(p)/D(p) = -2\,w_n^D(p - p_n^D)D_n(p)/D(p), \tag{30}$$

$$B_m(p) = S_m'(p)/S(p) = -2\,w_m^S(p - p_m^S)S_m(p)/S(p). \tag{31}$$

Thus, it is to be expected that the market demand force is a sum of the buyer's demand forces, and the market supply force is a sum of the seller's supply forces. But the definition of agent forces does not coincide here with the simple and elegant definition in the case with a single buyer and single seller, i.e.,

$$U_n(p) \neq -2\,\omega_n^D\left(p - p_n^D\right), \quad \text{if} \quad N \geq 2, \tag{32}$$

$$B_n(p) \neq -2\,\omega_m^S\left(p - p_m^S\right), \quad \text{if} \quad M \geq 2. \tag{33}$$

So, we can see that in more general cases the agent S&D forces are summed up differently than forces in classical mechanics. The "collectivization" effect can be briefly outlined as follows. Formally, it is a consequence of the probabilistic nature of the S&D functions. The same can be expressed in other words. People in the markets behave quite reasonably, communicating with each other and exchanging information on the market, being in a continuous search for generally cheaper (for buyers) or more expensive (for sellers) markets. A very important circumstance is the fact that buyers and sellers, to the largest extent and despite their seemingly irreconcilable differences, seek a structure of market prices which would suit most buyers and most sellers. That last statement is extremely important, as otherwise there would be no deals, and it would not suit everybody. Formally, the search for an optimal structure of market prices leads to the fact that the agent S&D functions are not just points in the P-space, but functions with certain natural widths. This also leads to another algebra for market forces. To make sure that it is true, it is enough to make natural widths in Eqs. (29)–(31) very small or to make frequency parameters very large, and we will see that Eqs. (30) and (31) transform into Eqs. (18) and (19) near the agent prices. It is enough to notice that with very small widths the market S&D functions are practically equal everywhere to a one concrete agent S&D function. Under these circumstances, the market S&D functions will not realistically overlap and the deal function will tend to zero (i.e., the market will die). Thus, the very fact of existence of natural widths of the agent S&D functions and of an unusual summing rule for them is a consequence of such rational and collective behavior of individual market agents.

To avoid misunderstanding, note several remarks refining the results obtained in this section. First of all, the agent S&D forces as ideal straight lines are a consequence of our choice of standard probability distributions, namely Gaussians. They would not otherwise be such ideal straight lines. Moreover, these lines are also ideal straight lines, far from their curve bells, and it is an unambiguous fact of the theory which does not deserve special attention. An artificial origin of this effect is proven by the fact that a "collectivization" mechanism of the S&D functions clears it away (Fig. 3). As soon as we add two sellers to the market (to the left and to the right of the "old" seller with number 2) it will radically change the supply force of this seller. Instead of infinity we have zero at very small and very large prices. Whereas under its bell, i.e., near price p_2^S, the market force of the seller remains a straight line, slightly changing only near the bell borders. In this case the collectivization mechanism is at work: all market agents interact with each other, competing for buyers' money and sellers' goods. Evidently, the more agents in the market the stronger the overlap among their S&D functions, and hence the stronger the collectivization effect.

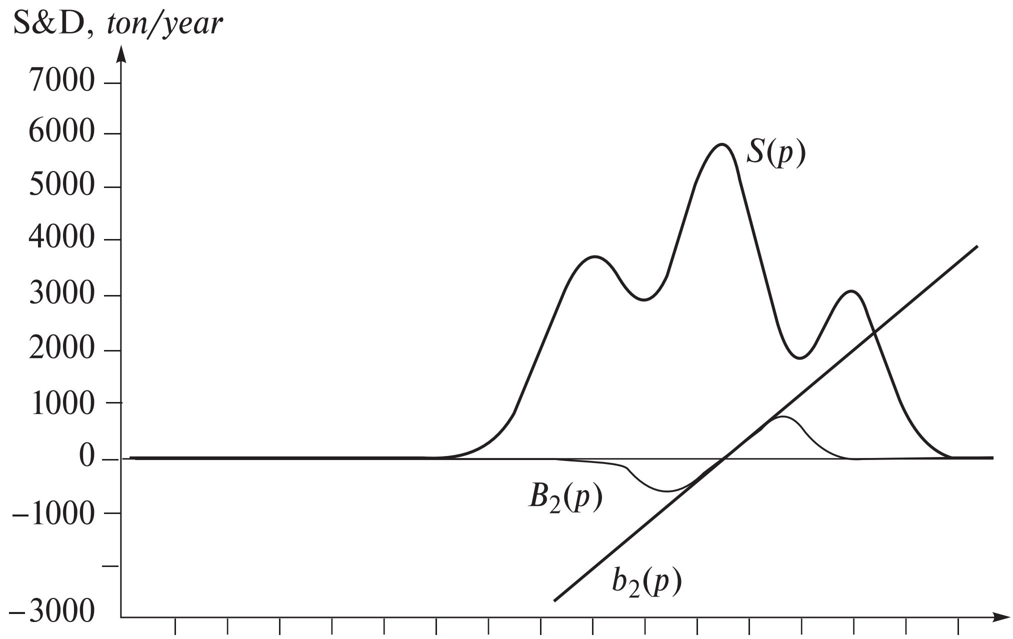

Fig. 3. Presentation of the collectivization effect. The market supply function $S(p)$ of three sellers (above), the individual second seller's supply force $b_2(p)$ (ideal straight line) as in Fig. 2, the second seller's supply force $B_2(p)$ (line under the second seller's bell) as a member of the sellers collective. For comparison, all forces are multiplied by 1000.

Let us explore Eqs. (29)–(31) in order to apply to a more complicated model market of grain consisting of several buyers and sellers, and to provide fresh insight into the features of probabilistic market forces. To start with, we consider the results of computational study of the three-buyer and three-seller market depicted in detail in Figs. 4–7. Below, we will describe some reasonable conclusions drawn from the given pictures.

Conclusion 1. It can be easily seen from Fig. 4 that the demand force $U(p)$ has a rather complicated structure, reflecting in some specific manner the structure of the demand function $D(p)$. The demand force is equal to zero at the points of local extrema of the demand function. At that price the demand force intersects the horizontal axis and changes its sign. It means that the smaller the buyer's width, the smaller the impact is on the market demand force of this buyer. Under the

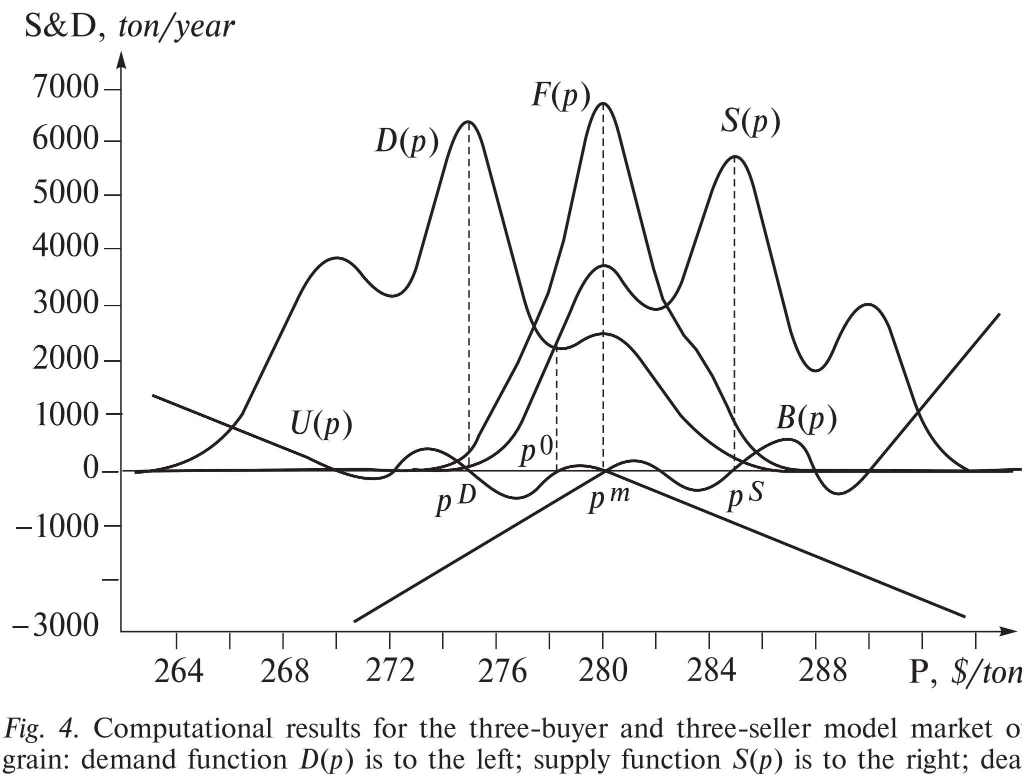

Fig. 4. Computational results for the three-buyer and three-seller model market of grain: demand function $D(p)$ is to the left; supply function $S(p)$ is to the right; deal function $F(p)$ is in the center; demand force $U(p)$ is mainly under the demand bell tower; supply force $B(p)$ is mainly under the supply bell tower. For comparison, $F(p)$ is divided by 400 whereas $U(p)$ and $B(p)$ are multiplied by 1000. Market price p^m = 280,0 *\$/ton*. Equilibrium price $p^0 \cong$ 278,2 *\$/ton*, p^D = 275,0 *\$/ton*, p^S = 285,0 *\$/ton*.

conditions with very small widths, the main role in the market price determination is played by the buyers with maximal desirable prices, p_n^D. Analogously, in this case the sellers with minimal desirable prices, p_m^S, make the most contributions to the market pricing process. Thus, we have shown that in market cases with agents having small natural widths, all important market pricing processes are conducted in only a very narrow range of prices and are determined only by buyers and sellers with the maximal and minimal prices, respectively.

Conclusion 2. Fig. 5 shows that a two-fold increase of all the seller widths, *ceteris paribus*, leads to significant changes both of supply and supply force functions. The supply curve is shown as a one-bell curve strongly resembling a Gaussian. Therefore, the supply force curve is now practically a straight line, intersecting the horizontal axis at only one point. It is interesting that the market price did not markedly change its magnitude despite such a significant increase of the widths. To avoid misunderstanding, note that the deal function now has two maxima and as a consequence, the market forces intersect now twice,

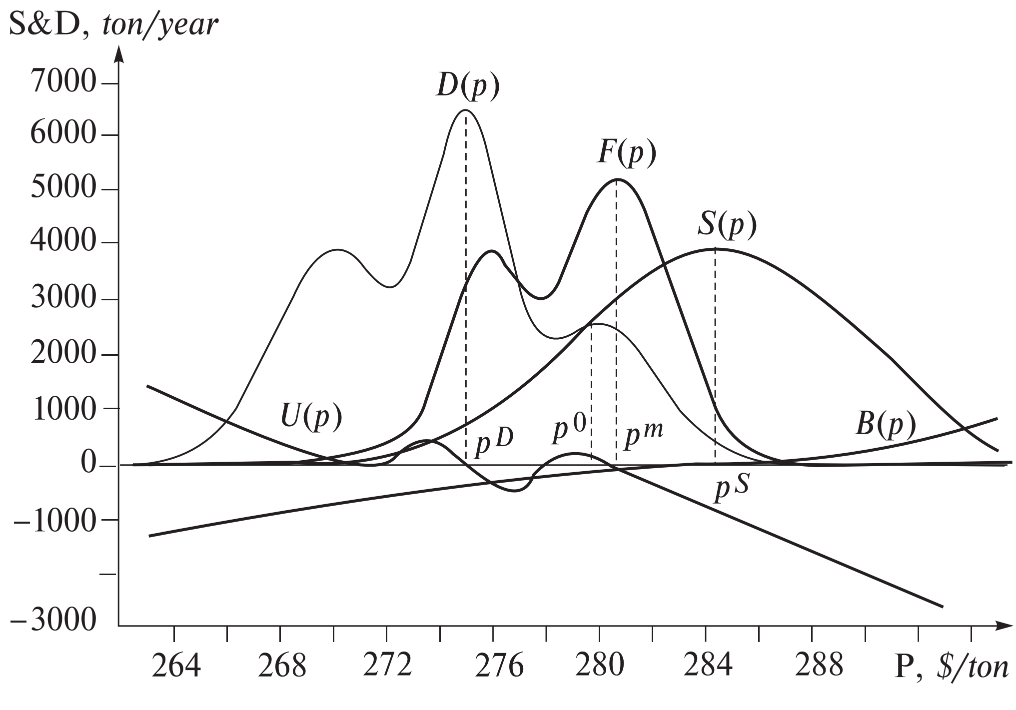

Fig. 5. In comparison to Fig. 4 only all seller widths are multiplied by 2. Market price $p^m \cong 280{,}5$ *\$/ton*, equilibrium price $p^0 \cong 279{,}5$ *\$/ton*.

and it is an artificial feature of the model. A simple increase of the number of buyers in the model will eliminate it. The same conclusion can be made for the case of increasing the buyer widths.

Conclusion 3. It is seen in Fig. 6 that a two-fold increase of all buyers and sellers widths simultaneously led to the fact that structures of the S&D functions became very simple; now they have combined into one-bell curve instead of three. Moreover, the S&D force curves became near straight lines, intersecting at the market price, p^m. The very market price did not much change. Thus, we obtained the following intermediate outcome of the work: market prices (in a narrow sense, of course) are rather stable with respect to width changes. We believe this outcome to be of great importance for practical studies of real markets. Using questionnaires and other related procedures, including reading reports of firms and corporations, we can obtain more or less exact data about desirable prices and quantities of main market agents. But to obtain reliable information about agent width is, generally speaking, a rather difficult task.

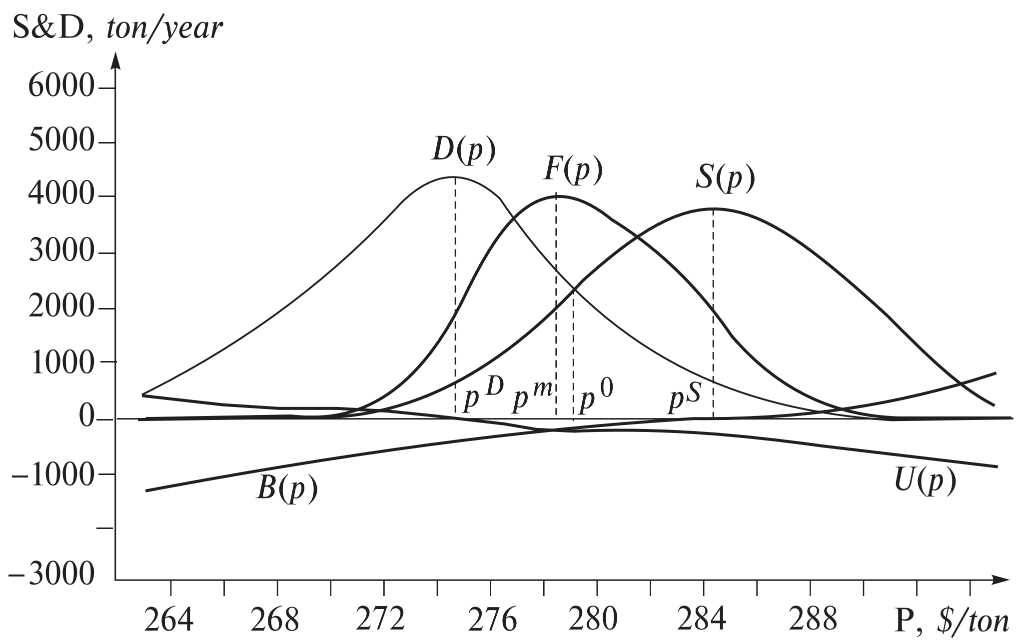

Fig. 6. In this picture all is the same as in Fig. 4 except that all the agent widths are multiplied by 2. Market price $p^m \cong 278{,}5$ *\$/ton*, equilibrium price $p^0 \cong 279{,}2$ *\$/ton*. A magnitude of market price calculated using Eq. (22) equals approximately 279,0 *\$/ton*.

Why is that so? Usual buyers do not often think about such things. Firms and corporations have this type of information in the form of possible discounts and premiums for quality or service, but these data are usually commercial secrets.

Conclusion № 4. Market prices p^m and trading volume F^0 (i.e., so-called market value) are rather strongly dependent on the total width of the market supply function. This width can be changed by sellers of their own accord, for example by concluding the cartel agreement to assign definite quotas to the sellers in order to be able "to cooperate in the substitution of a monopoly price for the competitive price" [5]. This effect is illustrated clearly in Fig. 7, where in contrast to Fig. 6, all seller prices p^S are equal to the same value 285,0 *$/ton* in accordance with the cartel agreement. We can see that the cartel agreement has led to an increase of the supply market force and the market price (from 278,5 *$/ton* to 282,1 *$/ton*), whereas trading volume has significantly decreased. The market bell (corresponding to the $F(p)$ curve) became

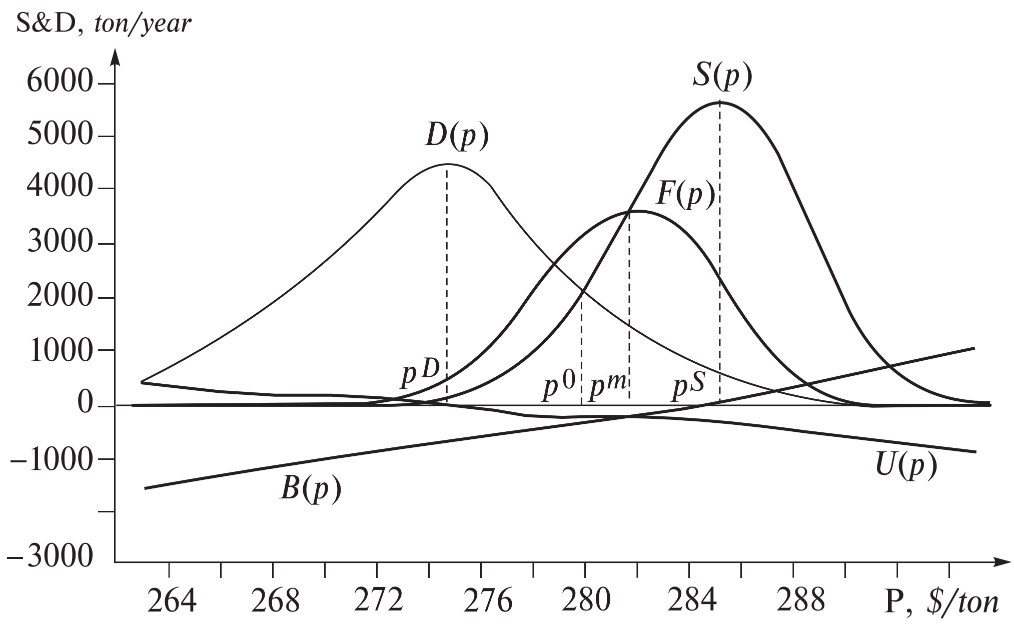

Fig. 7. Presentation of the cartel agreement effect on market price and trading volume (market value). This picture differs from the one in Fig. 6 only in that all seller prices p^S are equal to 285,0 *$/ton*. Market price $p^m \cong 282{,}1$ *$/ton.* A magnitude of market price calculated by means of Eq. (22) approximately equals 282,0 *$/ton*. The equilibrium price $p^0 \cong 279{,}8$ *$/ton.*

smaller (the square of the market bell decreased) because the overlap between the S&D curves became smaller too. The latter fact means that the number of commercial deals in the market have decreased, which is not favorable for the economy as a whole. That also means that for the sake of more profit on account of a higher monopole price, the sellers can retain part of their goods in stock, although those goods could be demanded in the market. We could say the analogous things about monopsony of demand, but we restrict ourselves to describing only the case of monopoly of supply. For more detail see [2, 5].

All considerations summarized in Figs. 4–7 and briefly reviewed above make it possible to formulate the main result. The proposed probabilistic concept of buyers and sellers in markets provided the basis on which the behavior of markets (also probabilistic, of course) could be quantitatively studied. Moreover, we have shown that using numerical experiments with different model markets it is possible to make some predictions concerning possible changes of markets by making some changes to the input of agent data. Let us make this sort of a prediction as an example. Figs. 4–7 show that if agent widths are comparable in magnitudes to the total demand curve width (which is mainly determined by the spread of the buyer prices with a large quantity demanded), the market demand curve will be a one-bell curve that can be rather well approximated by a single Gaussian with maximum price p^D and total market demand D^0. An analogous statement can be made for supply. Under these assumptions, we predict that Eqs. (22) and (23) can be used for a rather good estimation of market price p^m. We speculate now that this method of calculation of market prices could be useful in practice more often than it might be expected from its simplicity. We believe that in many cases, especially in well-organized markets, all agents have good access to historical data concerning prices and their possible spreads dependent on the state of the markets. Therefore, in practice the usual main agents in the market have similar strategies of behavior and hence similar agent functions which can lead to the following effect: the S&D functions will be one-bell curves which can be approximated by one Gaussian for demand and another Gaussian for supply. We do not know the market S&D functions in detail in this case, but we can use the simple method described above for obtaining a rather good estimation of market price. This outcome is not typically expected, and needs thorough experimental testing. Naturally, this requirement presents a considerable challenge to the economists.

4. Market Forces in the Many-Good Markets

Recall that for calculating the agent S&D functions in the many-good markets the multiplicativity formula was used in [1]. In other words, the agent S&D functions are represented by the products of the respective agent S&D one-good functions. Below we are going to briefly repeat the derivation of the respective set of formulas. For more detail see [1, 2].

We will consider the most general case in which the n-th buyer wants to buy all L goods in the market for a definite sum of money D_n^0. According to his or her needs and criteria, the buyer divides this sum of money among all goods demanded and elaborates his or her unique strategy in the market. The corresponding multi-dimensional demand function $D(p_1, \ldots, p_L)$ can be approximately represented in a factorized form using the multiplicativity formula for demand as follows:

$$D_n(p_1, \ldots, p_L) = \frac{\sum_{l=1}^{L} D_{nl}^0}{\prod_{l=1}^{L} D_{nl}^0} \prod_{l=1}^{L} D_{nl}(p_l), \tag{34}$$

where

$$D_{nl}^0 = p_{nl}^D \cdot q_{nl}^D. \tag{35}$$

In this formula Gaussians can be used for an approximate representation of the buyer's one-dimensional demand functions as follows:

$$D_{nl}(p_l) = D_{nl}^0 \cdot g_{nl}^D(p_l), \tag{36}$$

$$g_{nl}(p_l) = \sqrt{w_{nl}^D / \pi} \cdot \exp\left(-w_{nl}^D \left(p_l - p_{nl}^D\right)^2\right),$$

$$\Gamma_{nl}^D = \sqrt{-4 \ln 0{,}5 / w_{nl}^D}, \tag{37}$$

$$D_n^0 = \sum_{l=1}^{L} D_{nl}^0 = \sum_{l=1}^{L} p_{nl}^D \cdot q_{nl}^D. \tag{38}$$

In these formulas all new parameters make obvious sense by the definition: p_{nl}^D is the price, at which the n-th buyer plans to buy the l-th good in quantity q_{nl}^D, and D_{nl}^0 is the n-th buyer's total demand of the l-th good expressed in a monetary form, etc.

It is easy to check that for the n-th buyer, his or her one-dimensional demand function $D_{nl}(p_l)$ is normalized to their total demand of the l-th good D_{nl}^0, and his or her multi-dimensional demand function $D_n(p_1, \ldots, p_L)$ is normalized to their total demand D_n^0. Then, by means of the additivity formula for demand, we describe the multi-dimensional market demand function as a sum of the individual multi-dimensional functions as follows:

$$D(p_1, \ldots, p_L) = \sum_{n=1}^{N} D_n(p_1, \ldots, p_L) = \sum_{n=1}^{N} \frac{\sum_{l=1}^{L} D_{nl}^0}{\prod_{l=1}^{L} D_{nl}^0} \prod_{l=1}^{L} D_{nl}(p_l), \qquad (39)$$

$$D^0 = \sum_{n=1}^{N} D_n^0 = \sum_{n=1}^{N} \sum_{l=1}^{L} D_{nl}^0 = \sum_{n=1}^{N} \sum_{l=1}^{L} p_{nl}^D \cdot q_{nl}^D \ . \qquad (40)$$

The market demand function is normalized to the total market demand D^0.

By the same procedure we obtain the multi-dimensional factorized supply function of the m-th seller, with the help of the multiplicativity formula for supply as follows:

$$S_m(p_1, \ldots, p_L) = \frac{\sum_{l=1}^{L} S_{ml}^0}{\prod_{l=1}^{L} S_{ml}^0} \prod_{l=1}^{L} S_{ml}(p_l), \qquad (41)$$

where

$$S_{ml}^0 = p_{ml}^S \cdot q_{ml}^S. \qquad (42)$$

To some extent of accuracy, we can also use Gaussians as the seller's one-dimensional supply functions:

$$S_{ml}(p_l) = S_{ml}^0 \cdot g_{ml}^S(p_l), \qquad (43)$$

$$g_{ml}^S(p_l) = \sqrt{w_{ml}^S/\pi} \cdot \exp\left(-w_{ml}^S\left(p_l - p_{ml}^S\right)^2\right),$$

$$\Gamma_{ml}^S = \sqrt{-4\ln 0{,}5\,/w_{nl}^S}, \qquad (44)$$

$$S_m^0 = \sum_{l=1}^{L} S_{ml}^0 = \sum_{l=1}^{L} p_{ml}^S \cdot q_{ml}^S. \tag{45}$$

The seller's multi-dimensional supply function $S_m(p_1, \ldots, p_L)$ is normalized to their total supply S_m^0. Then, applying the additivity formula for supply, we obtain the multi-dimensional market supply function $S(p_1, \ldots, p_L)$ as a sum of the individual multi-dimensional supply functions as follows:

$$S(p_1, \ldots, p_L) = \sum_{m=1}^{M} S_m(p_1, \ldots, p_L) = \sum_{m=1}^{M} \frac{\sum_{l=1}^{L} S_{ml}^0}{\prod_{l=1}^{L} S_{ml}^0} \prod_{l=1}^{L} S_{ml}(p_l). \tag{46}$$

This function is normalized to the market total supply S_0 which is calculated by summing as follows:

$$S^0 = \sum_{m=1}^{M} S_m^0 = \sum_{m=1}^{M} \sum_{l=1}^{L} S_{ml}^0 = \sum_{m=1}^{M} \sum_{l=1}^{L} p_{ml}^S \cdot q_{ml}^S. \tag{47}$$

Thus, using the multiplicativity formula for S&D we have factorized the many-good market S&D functions. That is, we represented the multi-dimensional market functions by the products of the respective one-dimensional agent's functions that naturally made calculations and detailed studies of markets and market processes much easier.

Now we can return to our main task of obtaining a set of equations for calculating market prices in many-good markets. Let us take rather obvious steps towards achieving our goal. First of all, it is evident that the deal function can be defined here in the same way as in the case of a one-good market, namely as the product of the respective multi-dimensional market S&D functions:

$$F(p_1, \ldots, p_L) = D(p_1, \ldots, p_L) \cdot S(p_1, \ldots, p_L). \tag{48}$$

It is obvious that in our case the deal function $F(p_1, \ldots, p_L)$ is differentiable in the whole price space. It can also be easily seen that the problem of finding the market prices is reduced here to finding the local maxima of the deal function in the L-dimensional price space. It is well-known from the standard course of mathematical analysis, that this task for differentiable functions is reduced in turn to solving the system of L partial differential equations:

$$\partial F(\boldsymbol{p})/\partial p_l = 0, \quad l = 1, 2, \ldots, L. \tag{49}$$

This system in its turn can be easily transformed to the system of L partial differential equations expressed by means of the S&D functions as follows:

$$D(\boldsymbol{p})\cdot \partial S(\boldsymbol{p})/\partial p_l + \partial D(\boldsymbol{p})/\partial p_l \cdot S(\boldsymbol{p}) = 0, \quad l = 1,2,\ldots, L. \tag{50}$$

For brevity, in (49) and (50) we used only one bold letter $\boldsymbol{p}$ to designate all L prices p_l (now $\boldsymbol{p}$ is already a vector in the L-dimensional P-space [2]). All partial derivatives in our case are easily calculated because all differentiations are reduced to differentiation of Gaussians $g_{nl}(p_l)$ and $g^S_{ml}(p)$ in a trivial way: differentiation of one Gaussian gives rise to another Gaussian.

Furthermore, it makes sense to rewrite the system of equations (50) as the system of equilibrium equations, similar to the analogous equation for a one-good market (see Eq. (14)) as follows:

$$\frac{\partial D(\boldsymbol{p})/\partial p_l}{D(\boldsymbol{p})} = -\frac{\partial S(\boldsymbol{p})/\partial p_l}{S(\boldsymbol{p})}, \quad l = 1, 2, \ldots, L. \tag{51}$$

Let us rewrite this system once again in terms of the market forces as follows:

$$U_l(\boldsymbol{p}) = -B_l(\boldsymbol{p}), \quad l = 1, 2, \ldots, L. \tag{52}$$

That is equivalent to a simple equilibrium equation in a vector form:

$$\boldsymbol{U}(\boldsymbol{p}) = -\boldsymbol{B}(\boldsymbol{p}), \tag{53}$$

where now, by their definition, $\boldsymbol{U}(\boldsymbol{p})$ and $\boldsymbol{B}(\boldsymbol{p})$ are the S&D vector forces in the L-dimensional P-space with their components $U_l(\boldsymbol{p})$ and $B_l(\boldsymbol{p})$, respectively:

$$U_l(\boldsymbol{p}) = \frac{D(\boldsymbol{p})/\partial p_l}{D(\boldsymbol{p})}, \quad B_l(\boldsymbol{p}) = \frac{S(\boldsymbol{p})/\partial p_l}{S(\boldsymbol{p})}, \quad l = 1, 2, \ldots, L. \tag{54}$$

Thus, we have a generalized formula (17) for the calculation of market prices in the case of a one-good market so that we now have the equilibrium equation (53) of the same structure but already in the vector form, because the price space here is a multi-dimensional one. As we can see, we again obtained the full formal analogy with the equilibrium condition in classical mechanics. However, as in the case of a one-good market, the additivity rule for the S&D forces will differ in design from the respectively very simple additivity rule of classical mechanics because the S&D forces differ in nature from the forces in classical mechanics. Specifically, the market S&D forces are probabilistic and collective by their economic origin. The respective additivity rules are easily obtained from Eqs. (34)–(40):

$$U_l(\boldsymbol{p}) = \sum_{n=1}^{N} \frac{D_n(\boldsymbol{p})}{D(\boldsymbol{p})} U_{nl}(\boldsymbol{p}), \qquad l = 1, 2, \dots, L. \tag{55}$$

$$B_l(\boldsymbol{p}) = \sum_{m=1}^{M} \frac{S_m(\boldsymbol{p})}{S(\boldsymbol{p})} B_{ml}(\boldsymbol{p}), \qquad l = 1, 2, \dots, L, \tag{56}$$

where the buyers demand forces $\boldsymbol{U}_n(\boldsymbol{p})$ and sellers supply forces $\boldsymbol{B}_m(\boldsymbol{p})$ are naturally defined in the following way:

$$U_{nl}(\boldsymbol{p}) = \frac{\partial D_n(\boldsymbol{p})/\partial p_l}{D_n(\boldsymbol{p})}, \; B_{ml}(\boldsymbol{p}) = \frac{\partial S_m(\boldsymbol{p})/\partial p_l}{\partial S_m(\boldsymbol{p})}, \; l = 1, 2, \dots, L. \tag{57}$$

These equations can be written also in terms of vectors as follows:

$$\boldsymbol{U}(\boldsymbol{p}) = \sum_{n=1}^{N} \frac{D_n(\boldsymbol{p})}{D(\boldsymbol{p})} \boldsymbol{U}_n(\boldsymbol{p}), \qquad \boldsymbol{B}(\boldsymbol{p}) = \sum_{m=1}^{M} \frac{S_m(\boldsymbol{p})}{S(\boldsymbol{p})} \boldsymbol{B}_m(\boldsymbol{p}). \tag{58}$$

It can be easily shown that the vector components of the S&D forces of individual agents are to be calculated using the following formulas:

$$U_{nl}(\boldsymbol{p}) = U^*_{nl}(p_l), \qquad l = 1, 2, \dots, L, \tag{59}$$

$$B_{ml}(\boldsymbol{p}) = B^*_{ml}(p_l), \qquad l = 1, 2, \dots, L, \tag{60}$$

where we introduced the standard or natural definitions for the vector components of the one-good demand forces of individual buyers $U^*_{nl}(p_l)$ and the vector components of the one-good supply forces of individual sellers $B^*_{ml}(p_l)$ as follows:

$$U^*_{nl}(p_l) = \frac{dD_n(p_l)/dp_l}{D_{nl}(p_l)} = -2\,w^D_{nl}(p_l - p^D_{nl}), \qquad l = 1, 2, \dots, L, \tag{61}$$

$$B^*_{ml}(p_l) = \frac{dS_m\,(p_l)/dp_l}{S_{ml}(p_l)} = -2\,w^S_{ml}(p_l - p^S_{ml}), \qquad l = 1,2,\ldots,L. \qquad (62)$$

Using vector notations we get these equations in a more compact form:

$$\boldsymbol{U}_n(\boldsymbol{p}) = \boldsymbol{U}^*_n(\boldsymbol{p}), \qquad \boldsymbol{B}_m(\boldsymbol{p}) = \boldsymbol{B}^*_m(\boldsymbol{p}). \qquad (63)$$

Let us repeat once again that the developed systems of equilibrium equations give as solutions not only the market prices (corresponding to local maxima of the deal function), but also the prices at which the deal function has local minima. To select the market prices we should make a definite attempt: to calculate the magnitudes of the deal function in several points near the selected solution $\boldsymbol{p}^m$ and compare them with the magnitude $F(\boldsymbol{p}^m)$. If $F(\boldsymbol{p}^m)$ is greater than magnitudes in the vicinity of $\boldsymbol{p}^m$, it means that the selected point $\boldsymbol{p}^m$ is really the market price. If this is not the case, than it is just a local minimum of the deal function.

Although the market forces equilibrium equation (53) for discovering market prices is rather complicated, it can be easily solved by using any of the well-known computer software programs available, even for very large models and real markets. Despite the fact that the equilibrium equation is expressed in terms of multi-dimensional functions and parameters and the obtained solutions have to be presented and analyzed in multi-dimensional P-space, performing such calculations is not beyond a student's capability. It is necessary to emphasize that this simplicity means that it is possible to obtain very fast, practical solutions of rather complicated equilibrium equations, brought about by several factorizations of the market S&D functions. The implementation of the probabilistic technique (more exactly, the SP model) developed and discussed above, opened new avenues for detailed computer modeling of many-good markets. The consisted of a large number of market agents, whole industries, regional and raw material markets, intimately associated capital and labor markets, and so on.

5. Conclusions

What is a market force? Much attention was given to this important question in this chapter. The important but broad and indefinite notion of market force in economics has undergone a thorough revision from the point of view of a probability concept. According to the new mode of thinking, the knowledge of the market force is best expressed in terms of the probabilistic S&D functions. The strict mathematical definition of market forces as the forces of S&D is given. Thus, market forces have not only a magnitude but also a sign. Therefore, they can be interpreted as vectors in a multi-dimensional P-space of many-good markets [3]. We have shown the manner in which agent's forces are summed up to produce market forces of S&D. We also established that under certain conditions, market prices coincide with the equilibrium prices of market forces. In other words, market prices are prices at which market forces of S&D are equal in magnitude but opposite in sign, analogous to the equilibrium concept in classical mechanics. The obtained system of equilibrium equations is relatively simple from a computational point of view, even for very complicated many-agent, many-good model and real markets, due to applying the factorization rules [1, 2]. This feature of the developed SP model makes it possible to carry out computer calculations and modeling of markets with large numbers of market agents and traded goods.

References

1. Anatoly Kondratenko. *Probability Economics: Demand and Supply in the Price Space*. Electronic copy available at: http://ssrn.com/abstract=2250343. See also Chapter IV.
2. Anatoly Kondratenko. P*robability Economics: Market Price in the Price Space*. Electronic copy available at: http://ssrn.com/abstract=2263708. See also Chapter V.
3. Anatoly Kondratenko. *Physical Modeling of Economic Systems. Classical and Quantum Economies*. Novosibirsk: Nauka, 2005. Electronic copy available at: http://ssrn.com/abstract=1304630.
4. Anatoly Kondratenko. *Probability Economics: Supply and Demand, Price and Force in the Price — Quantity Space*. Electronic copy available at: http://ssrn.com/abstract=2337462. See also Chapter VII.
5. Ludwig von Mises. *Human Action. A Treatise on Economics*. Yale University, 1949.
6. Anatoly Kondratenko. *Physical Economics: Stationary Quantum Economies in the Price — Quantity Space*. Electronic copy available at: http://ssrn.com/abstract=2363874. See also Chapter X.

PART D.

Probability Economics. Stationary Probabilistic Economies in the Price-Quantity Space

"Value is the importance that acting man attaches to ultimate ends. Only to ultimate ends is primary and original value assigned. Means are valued derivatively according to their serviceableness in contributing to the attainment of ultimate ends. Their valuation is derived from the valuation of the respective ends. They are important for man only as far as they make it possible for him to attain some ends.

Value is not intrinsic, it is not in things. It is within us; it is the way in which man reacts to the conditions of his environment.

Neither is value in words and doctrines. It is reflected in human conduct. It is not what a man or groups of men say about value that counts, but how they act. The oratory of moralists and the pompousness of party programs are significant as such. But they influence the course of human events only as far as they really determine the actions of men".

Ludwig von Mises. *Human Action. A Treatise on Economics.* Page 96

CHAPTER VII.

Supply and Demand, Price and Force

"In the real world acting man is faced with the fact that there are fellow men acting on their own behalf as he himself acts. The necessity to adjust his actions to other people's actions makes him a speculator for whom success and failure depend on his greater or lesser ability to understand the future. Every action is speculation. There is in the course of human events no stability and consequently no safety".

Ludwig von Mises. *Human Action. A Treatise on Economics*. Page 113

PREVIEW.

What are Supply and Demand?

This chapter is the sequel to the previous chapters we have discussed above, dedicated to the elaboration of a complex of fundamentally new economic models, which we have named probability economics. In all of these models we have implemented the concept of uncertainty and probability, step-by-step. Specifically, in this paper we set out the General Stationary Probability Model in the Price-Quantity Space, which is an important generalization of the simple Stationary Probability Model in the Price Space, developed in previous chapters. This generalization is carried out as follows. For each good traded in the L good market of an economy, we treat price and quantity as two independent variables and then introduce into the theory the idea of the $(2\times L)$-dimensional "price-quantity" space by analogy with the price space of the simple model. In other words, the concept of uncertainty and probability of a buyer's and seller's choice in the market is equally applied to the price and quantity of goods. Thus, supply and demand functions of both individual agents and the market as whole are $(2\times L+1)$-dimensional surfaces within the framework of the General Stationary Probability Model in the Price-Quantity Space. The probabilistic nature of an agent's quantity choice significantly changes the meaning of supply and demand as well as significance and numerical values of supply and demand functions, market forces and prices. In addition, taking into account these quantity freedom degrees significantly expands the scope of application of the probability economics for description of market phenomena.

1. Introduction

In [1–4], we developed a probabilistic view of economy and elaborated upon a complex of fundamentally new economic models, which we refer to as probability economics. In all of these models we implemented the concept of uncertainty and probability in economic theory, step by step. Specifically, we have developed the Stationary Probability Model in the Price Space (the SP model below), describing an economy in a normal stationary state in the price space [1]. This relatively simple model has been designed to describe market situations where uncertainty in agents' price choices are relatively high, but uncertainty in agent choices of quantities is relatively small and, for this reason, can be neglected. In other words, the main point of the SP model is the following. Agent decision making concerning prices of goods bears a probabilistic nature, while agent choices of good quantities are specified beforehand.

Under normal market conditions, this assumption seems to be a rather good approximation; people really do tend to know their product requirements in quantitative terms. However, it is known that even end-users can sometimes buy more goods than they have really planned on, lured by the lower price. Even a housewife considers this possibility in her strategy of shopping in the supermarket or at a local market. Moreover, this element is used by supermarkets, which use the strategy to increase sales. Therefore, we should be able to take into account this possibility in the theory in one way or another. A second example of a market situation where it is necessary to take into account the uncertainty of a good quantity is connected with economic crises. In such cases, many buyers and sellers are compelled to institute emergency measures, and extremely adaptable many –variant purchase and production plans. For instance, pessimistic, normal and optimistic plans may be used, depending on the forecasted future state of the markets. A third example relates to the activities of the promoters and speculators in stock and financial markets. Strategies of these proactive market agents are simply aimed at extracting profits from speculative deals. In other words, a market game of those agents is made up entirely of transactions which are based on forecasting a future price and quantity structure of stock and financial markets. Obviously, in these cases we have to incorporate into the theory a possibility of taking into account the uncertainty of both prices and quantities.

Thus, with regards to every traded good in the market, every individual or market agent has two degrees of freedom in their choices and actions. Specifically, these are the price and quantity of good which the agent wants to buy or sell. These factors nearly always accompany all human choices, decisions and actions in the marketplace, as though one has been placed and moved within a two-dimensional space of price and quantity in relation to every good. And, there is always some uncertainty connected with the respective price and quantity coordinates. But if one of these two coordinates becomes unchangeable or immovable, we are left to focus solely on the other. For example, if all the agent quantities are set, and we wish to concentrate on determining good market prices, that is well described by the simple SP model. However, if agent quantities are changeable or moveable, we should take into account in the theory both degrees of freedom for every good. Through an analogy with physics, we cannot simply describe movement of the Earth around the sun as one-dimensional, using only the most important coordinate, r (sun Earth distance), because Earth definitely has three degrees of freedom: r, and two angle coordinates φ, θ.

The main object of this article is to develop such an extension of the simple SP model as to be able to take into account the quantity dimension of every agent with respect to every good traded in the market. We can achieve this objective by means of the obvious method. Bear in mind, that within the framework of the simple SP model we describe the motion of all the market agents in the L-dimensional price space (the P-subspace below) where L is the number of goods traded in the market. Now of course we must describe the motion of the agents in the $2\times L$-dimensional Price-Quantity space (the PQ-space below), with L degrees of freedom for agent prices p and L degrees of freedom for agent quantities q. Thus, the extended simple SP model will be referred to below as the General Stationary Probability Model in the

Price-Quantity Space (the GSP model below). We can conclude that the GSP model has been developed to describe the motion of market agents in an economy that is in a normal stationary state in the PQ-space.

Let us now discuss once more in brief the five principles of physical economics that also provide the basis for the construction of probability economics; the GSP model in particular. Together they constitute the most important scientific knowledge regarding the market economic world and its laws and rules. They are as follows (see Chapter I):

1. **The Cooperation-Oriented Agent Principle.**
2. **The Institutional and Environmental Principle.**
3. **The Dynamic and Evolutionary Principle.**
4. **The Market-Based Trade Maximization Principle.**
5. **The Uncertainty and Probability Principle.**

On the one hand, these five general principles of probability economics sum up our knowledge and experiences with uncertainty, probability and market processes. On the other hand, they provide the reliable basis on which the behavior of both the market agents and the market as a whole can be studied, not only rigorously and logically, but most importantly, quantitatively. The primary and most evident consequence of these principles is that all the agent prices (p) and quantities (q) for all goods are independent variables, and all the supply and demand (S&D below) functions are multi-dimensional functions of all these variables. In other words, S&D functions are now surfaces in multi-dimensional space. With these five general principles in mind, we have developed the complex of probabilistic models, i.e., probability economics. Each of the five principles is embodied in a graphic model of economic systems, enclosed in the multi-dimensional PQ-space in Fig. 1. In the next sections of the chapter we will give additional partial principles, or rules, for the particular version of probability economics, namely the GSP model.

In the closing of this section, we would like to make some remarks regarding the peculiarities of our theory. In contrast to the traditional neoclassical model of economics, there is nothing mysterious about probability economics. There are many simple and ordinary human actions in markets that converge with common sense and business or trade customs that result in particular net outcomes for the whole market.

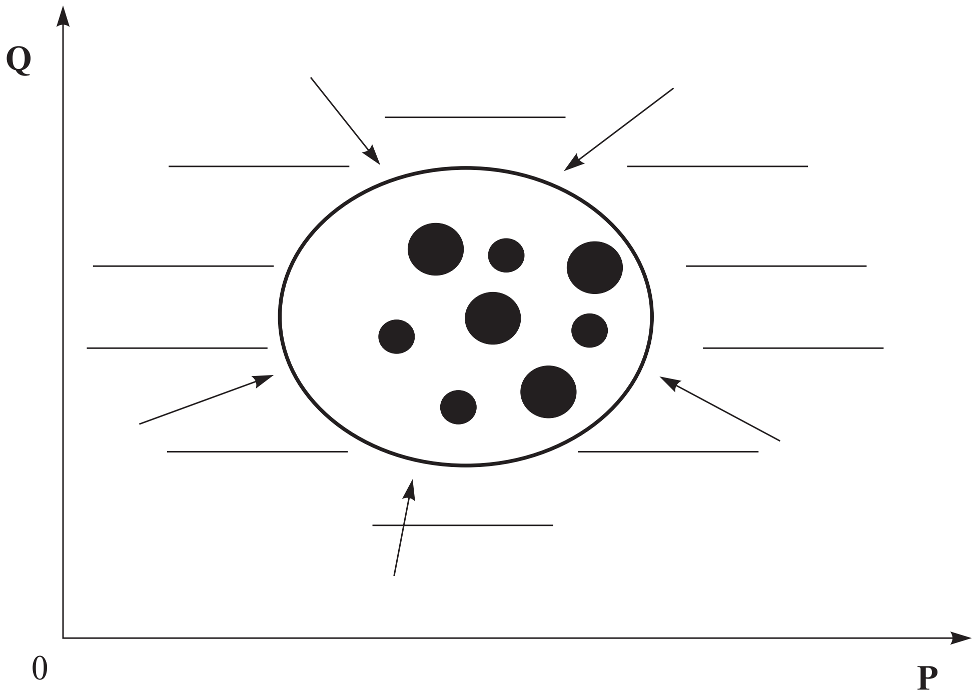

Fig. 1. Graphical model of an economy in the multi-dimensional PQ-space. It is displayed schematically in the conventional rectangular multi-dimensional coordinate system [**P**, **Q**] where **P** and **Q** designate all the agent price and quantity coordinate axes, respectively. Our model economy consists of the market and the external environment. The market consists of buyers (small dots) and sellers (big dots) covered by the conventional sphere. Very many people, institutions, and natural and other factors can represent the external environment (cross-hatched area behind the sphere) of the market which exerts perturbations on market agents (pictured by arrows pointing from environment to market).

If we can adequately describe the chosen market behavior strategy of each agent, depending on the market circumstances and his or her various interests (there is nothing we can decide for the agent), then we only need to find a method to add up all the agent strategies in order to obtain a common result for the whole market. The challenge for us has been in the work of designing the corresponding method, namely the GSP model. To avoid even the slightest hint of theory complexity at this stage, we want the model to exhibit maximum simplicity. To this end, only the smallest possible set of notions and settings should be introduced into the GSP model. The great value of the proposed GSP model resides also in the fact that the mathematical apparatus used within the framework of the present model is extremely simple; it can be easily applied by students of economics. We refer connoisseurs and lovers of physics and mathematics, including quantum mechanics, probability theory, and theory of differential equations of second order, to our other works [1, 9]. All other improvements and refinements of the present GSP model can be made later, should the need arise.

We must emphasize that it is impossible to predetermine the ideal strategy for all agents. We can, however, carry out numerical computational experiments, choosing variants of strategies for the agents and observing the market's response. Such theoretical market games, or agent-based simulative experiments, can be very useful. They make it possible to study the features of market mechanisms in great detail, and clarify some general economic phenomena. It will be clear by the end of this article that these calculations can be done very quickly on a personal computer. This holds true even for many agent, many-good markets, as all the calculations are performed making use of simple analytical functions.

2. The One-Good, One-Buyer and One-Seller Markets

2.1. Supply and Demand

The GSP model will be constructed step-by-step, successively complicating the economic system models. Let us start with the study of the simplest model economy with a one-good, one-buyer, and one-seller market. Of course, this model is very far from reality. It cannot give highly accurate quantitative results, but as long as minor details are avoided, it gives us an opportunity to concentrate on the main developments in the market's movements, and to understand how our market economic world works with regard to its basic outline, and its principle laws and rules. Fig. 1 gives us the graphical model of the studied economy with one small dot (buyer) and one big dot (seller) in the PQ-space (the (P, Q)-plane here): $\mathbf{P} = \mathrm{P}$, and $\mathbf{Q} = \mathrm{Q}$, where P and Q are price p and quantity q axes for the single good in the market. Besides the two market agents, there are other external members or actors of the economy having strong market power, which can exert influence on market agents. Some examples of these are the state, various institutions, trade unions, other markets, natural factors, etc.

We assume, as with all the stationary models, that the economy is in a normal, stationary state [1, 2]. It means, among other things, that although external influence on market agents can be strong, it is perpetual or constant in time. As we now have two independent variables, good price p and quantity q, S&D functions are functions of these two variables, namely $D(p,q)$ and $S(p,q)$. Therefore, in the rectangular three-dimensional coordinate system [P, Q, S&D], these functions can be thought of as three-dimensional surfaces that can be interpreted as spatial models of S&D functions. It is very interesting to consider all the findings discussed above in the context of the theoretical debates concerning the eternal question of economic theory: what are S&D functions in mainstream economics? At first, note that within the framework of the GSP model the perpetual question is removed from the agenda, tantalizing teachers and students. Which of the two variables — price or quantity — has to be considered as independent and plotted on the horizontal axis (abscissa) and which as dependent and plotted on the vertical axis (ordinate)? The answer is as follows: both variables are independent. For this reason, building and studying dependences of one variable on another as is done in the traditional neoclassical model of economics (see discussion in [2, 3]) seems to make no sense. Generally speaking, such dependencies can arise in theory, but only as formal dependences. For example, there could be a case where two curves depict projections of a section of S&D surfaces on the (P, Q)-plane. This curve may reflect a market process in time, perhaps by changing the main market parameters of the buyer and seller, desirable prices p^D and p^S, as well as quantities, q^D and q^S, respectively. An example of this will be given in our next article [6], where such curves graphically display the transition process of the market to an equilibrium state. These are rather artificial and technical in their construction, have a nature of intermediate character and do not have great economic meaning. And certainly these curves or dependencies in the (P, Q)-plane cannot be called S&D functions. This statement pertains equally to the p-q-dependencies in the traditional model of economics (see also discussion in [2, 3]). The meaning of introducing those into the theory is only confined to the convenience of the graphical representation for students, by whom the widely known empirical fact that free markets as usual tend to the equilibrium state is understood, and where agent prices, p^D and p^S, and quantities, q^D and q^S, level off, respectively [5].

In science, it appears that discovering something simpler than the two equations for four numbers: $p^D = p^S$, $q^D = q^S$ is a very complicated problem, which represents this well-known empirical fact. Strange as it may seem, it is nevertheless the practice in economics to express these equalities as the crossing point of the two artificial curves $q^D(p^D)$ and $q^S(p^S)$. They are referred to as S&D functions and are of questionable value for use in economic theory [5] (see also discussion in [2, 3]). Moreover, by means of this cross, the traditional model supposedly explains or even proves why this phenomenon takes place in practice, i.e., why $p^D = p^S$, $q^D = q^S$. That does not stand up under scrutiny from the point of view of normal logic, especially from the point of view of logical economic method [5]. Shortcomings of this approach are evident: one creates a false illusion of simplicity of market processes. This is because the study and description of the equilibrium state of the complex dynamic systems with huge amounts of degrees of freedom is reduced to the demonstration of one point on the PQ-plane. Furthermore, it is impossible in this way to even start building adequate quantitative methods and models, let alone to study in detail this market process. This is because there are no acting persons in the given "agentless" economic world of the traditional model, only mysterious market forces which of course are not computable. But the main argument against the traditional approach is that in practice, price and quantity are two independent variables. Therefore, they should be the same in any adequate economic models. Any function of price of quantity or quantity of the price cannot be considered as an adequate, formal, mathematical representation of the most important concept in economic theory, namely, the concept of S&D. This brings us to the point. Despite the existence of a well-developed and meaningful qualitative background of economic theory [5], there was previously no reliable quantitative mathematical foundation under the pompous building of mainstream economics. We shall continue to deal with this subject in [6, 7].

2.2. The Factorization Formula for Supply and Demand

Obviously, it is very tedious and cumbersome to carry out calculations of the two-dimensional functions and then to analyze the three-dimensional results of the calculations for one-good markets. Increasing the number of goods in a given market makes the calculations all the more complicated. For this reason, we will take the next step in the simplification of our model, which makes it the most effective method for quantitative calculations and analyses in economic theory at present. So, we *a priori* assume that in this case we can apply the factorization rule for S&D in the PQ-space with regard to the P- and Q-degrees of freedom. In other words, we assume that the two-dimensional S&D functions in the PQ-space can be accurately expressed as a product of the respective one-dimensional S&D functions, i.e., by means of the following factorization formula:

$$D(p,q) \cong C^{D} \cdot d^{P}(p) \cdot d^{Q}(q); \qquad (1)$$

$$S(p,q) \cong C^{S} \cdot s^{P}(p) \cdot s^{Q}(q) \ . \qquad (2)$$

Here $d^{P}(p)$, $s^{P}(p)$ are the respective one-dimensional price S&D functions (P-functions below) and $d^{Q}(q)$, $s^{Q}(q)$ are the respective one-dimensional quantity S&D functions (Q-functions below). Below, for an adequate, approximate representation of all of these one-dimensional functions we will find out appropriate analytical functions. Of course, C^{D} and C^{S} are normalization constants.

So, we have approximately represented the two-dimensional agent S&D functions in the PQ-plane in the form of the products of the two one-dimensional P- and Q-functions in the P- and Q-subspaces, respectively. These P- and Q-functions are one-dimensional, in this case for the one-good market. We suppose that the very possibility of such factorization comes from the mentality structure of *homo sapiens* formed during evolution. Human beings have become very proficient at coming up with fairly simple solutions for very complicated, many-parameter, and multi-dimensional problems. Very skillfully, an individual can reduce such complex to a set of the simplest one-parameter and one-dimensional tasks. It is even impossible to imagine such a super-human, able to build up plans and strategies in markets in terms of multi-dimensional functions. Thus, we believe this approximation to be rather good, since it reflects the natural line of reasoning of any person acting in markets — see also discussion in [2].

As far as forms of the one-dimensional functions are concerned, we proceed from the clear empirical fact that market agents make their decisions with the understanding of their requirements, as well as financial and productive possibilities. These requirements and possibilities are expressed roughly by the agents at the beginning of the construction of their plans in the markets, in the form of the so-called initial dot strategies; the buyer wants to buy the good in quantity q^D at a price p^D and the seller wants to sell the good in quantity q^S at a price p^S. Graphically, such initial dot strategies of the buyer and the seller can be represented in the (P, Q)-plane by the points $(p^D,\ q^D)$ and $(p^S,\ q^S)$, respectively (see Fig. 2). Then the corresponding dot S&D functions can be described by the special functions as follows:

$$d^P(p) = \delta(p - p^D),\ \ s^P(p) = \delta(p - p^S);$$
$$d^Q(q) = \delta(q - q^D),\ \ s^Q(q) = \delta(q - q^S). \tag{3}$$

In Eq. (3), $\delta(x)$ is a so-called delta function having no width at all (for more details and comments see below and [2, 3]). Furthermore, the buyer and the seller understand that they are compelled to operate under the conditions of the high uncertainty and risks of the market, and that they can exert only very weak influence on these conditions. As a result, they come to the conclusion that in the market they should consider all of their future decisions and actions only as possible with a certain degree of probability. Analogously, they also recognize that they can ask and bid prices and quantities in the market only at some intervals near p^D, p^S, q^D, q^S with definite probabilities as functions of price p and quantity q, respectively. This probabilistic aspect of the decision-making process of people is of great importance for an understanding of agent behavior in the market. The buyer and the seller think and act as an oscillating man, *homo oscillans* [1, 2]. All of these considerations can be expressed in the simpler form as follows. All the market agents understand that all of their plans and strategies can only be considered as prognoses that will be corrected in the future, in the course of the market game. This probabilistic aspect in market agent behavior is taken into account in our theory by means of the subsequent transformations of the dot S&D functions (delta functions) into the continuous bell-shaped S&D functions with maxima at the points A $(p^D,\ q^D)$ and B $(p^S,\ q^S)$ and natural widths Γ^{DP} and Γ^{SP}, respectively. These transformations, step-by step, can be easily seen in Figs. 2, 3, 4, 5, and 6 for the model market of grain (see [2] for more detail). Note that all the calculations of surfaces in the article are performed by means of the well-known program Wolfram *Mathematica*.

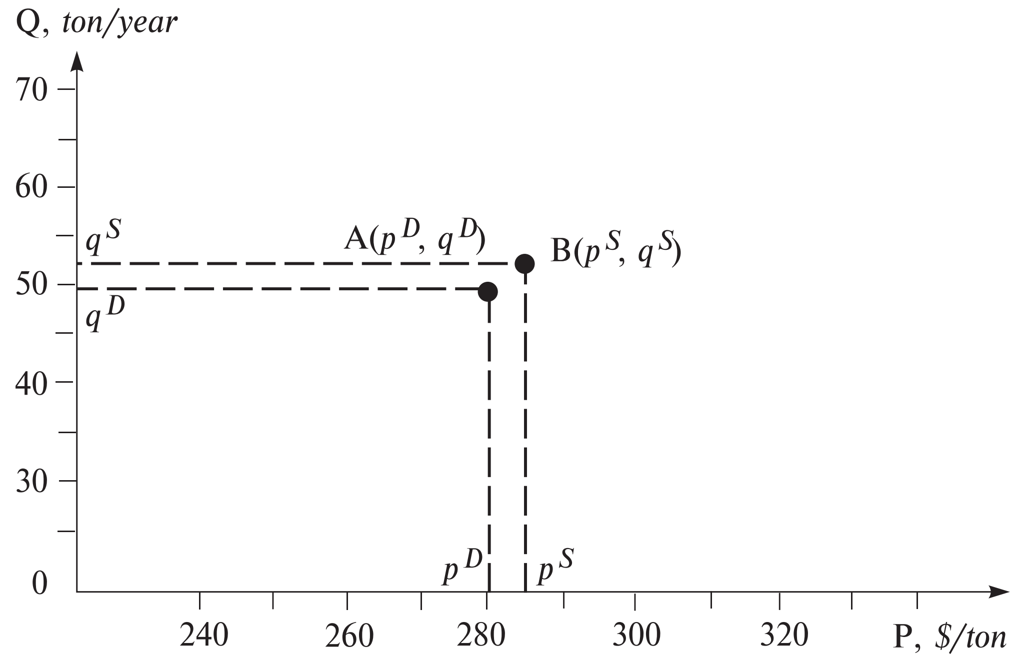

Fig. 2. Graphical representation of the dot strategies of the buyer's and seller's market behavior by the points A (p^D, q^D) and B (p^S, q^S) in the two-dimensional PQ-space (plane) of the model market of grain. p^D = 280,0 *$/ton*, q^D = 50,0 *ton/year*, p^S = 285,0 *$/ton*, q^S = 52,0 *ton/year*.

Naturally, we choose for the one-dimensional functions the following normalization rule:

$$\int_{-\infty}^{+\infty} d^P(p)dp = 1, \int_{-\infty}^{+\infty} d^Q(q)dq = 1;$$

$$\int_{-\infty}^{+\infty} s^P(p)dp = 1, \int_{-\infty}^{+\infty} s^Q(q)dq = 1. \quad (4)$$

To normalize the two-dimensional S&D functions to the total demand D^0 and total supply S^0 of the buyer and seller are as follows:

$$\int_{-\infty}^{+\infty} D(p,q)\,dp\,dq = D^0, \;\; D^0 = p^D \cdot q^D, \;\; C^D = D^0; \quad (5)$$

$$\int_{-\infty}^{+\infty} S(p,q)\,dp\,dq = S^0, \;\; S^0 = p^S \cdot q^S, \;\; C^S = S^0. \quad (6)$$

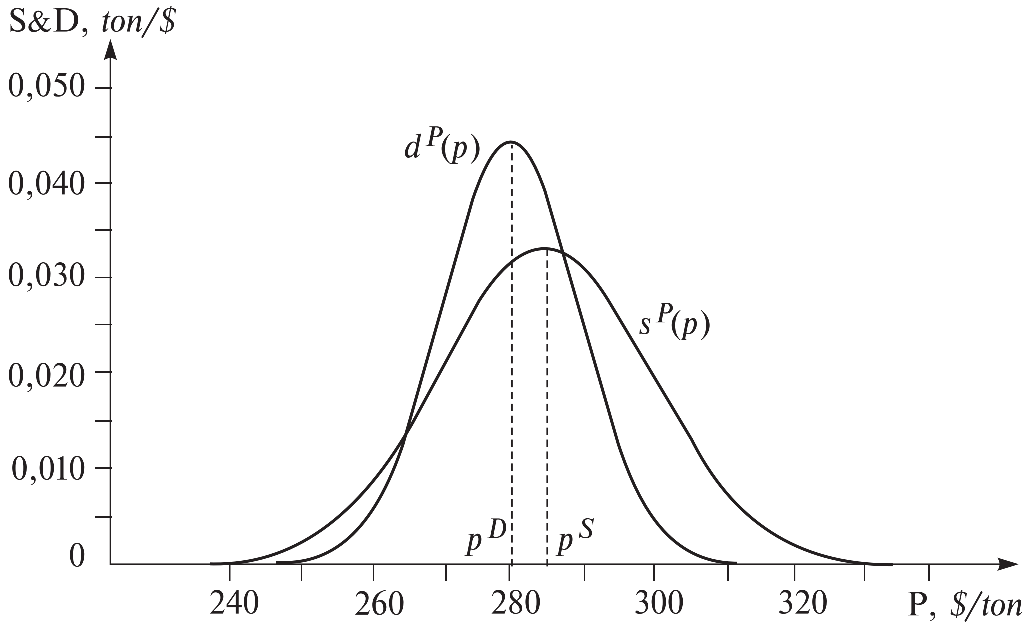

Fig. 3. Graphical representation in the conventional rectangular three-dimensional coordinate system [P, Q, S&D] of the one-dimensional S&D P-functions, $d^P(p)$ of the buyer and $s^P(p)$ of the seller, as the two-dimensional curves with maxima at prices p^D, p^S and natural widths Γ^{DP} and Γ^{SP}, respectively. $\Gamma^{DP} = 23{,}8$ *\$/ton*, $\Gamma^{SP} = 37{,}0$ *\$/ton*.

It is interesting that such normalized S&D functions $D(p,q)$ and $S(p,q)$ are dimensionless ones. Economic significance of all of the introduced above one- and two-dimensional S&D functions is evident; they are in essence the corresponding distributions of probability of the buyer and seller making a purchase transaction at the price p and quantity q. Let us stress that, by very the definition of all of these functions, the market agents themselves "choose" these functions consciously or even automatically on a subconscious level due to accumulated market experiences. Therefore it is reasonable to use so-called normal distributions or simply Gaussians [2-4] as follows:

$$d^P(p) \cong g^{DP}(p) = \sqrt{w^{DP}/\pi} \cdot \exp\left\{-w^{DP}(p - p^D)^2\right\}, \tag{7}$$

$$d^Q(q) \cong g^{DQ}(q) = \sqrt{w^{DQ}/\pi} \cdot \exp\left\{-w^{DQ}(q - q^D)^2\right\}, \tag{8}$$

$$D(p,q) \cong D^0 \cdot g^D(p,q) = D^0 \cdot g^{DP}(p) \cdot g^{DQ}(q), \tag{9}$$

$$\Gamma^{DP} = \sqrt{-4\ln 0{,}5/w^{DP}}, \quad \Gamma^{DQ} = \sqrt{-4\ln 0{,}5/w^{DQ}}, \tag{10}$$

$$s^P(p) \cong g^{SP}(p) = \sqrt{w^{SP}/\pi}\cdot \exp\left\{-w^{SP}(p-p^S)^2\right\}, \tag{11}$$

$$s^Q(q) \cong g^{SQ}(q) = \sqrt{w^{SQ}/\pi}\cdot \exp\left\{-w^{SQ}(q-q^S)^2\right\}, \tag{12}$$

$$S(p,q) \cong S^0 \cdot g^S(p,q) = S^0 \cdot g^{SP}(p)\cdot g^{SQ}(q), \tag{13}$$

$$\Gamma^{SP} = \sqrt{-4\ln 0{,}5\,/w^{SP}}, \quad \Gamma^{SQ} = \sqrt{-4\ln 0{,}5\,/w^{SQ}}. \tag{14}$$

Formulas (9) and (13) express the known relation between the Gaussian frequency parameters w^{DP}, w^{DQ}, w^{SP}, w^{SQ} with the Gaussian natural widths Γ^{DP}, Γ^{DQ}, Γ^{SP}, Γ^{SQ} (more precisely, these are full widths at half of the maximum of their peaks). Additional comments to the Eqs. (1)–(14) can be found in [2]. Generally speaking, it is not excluded that making use of some other known distributions instead of Gaussians

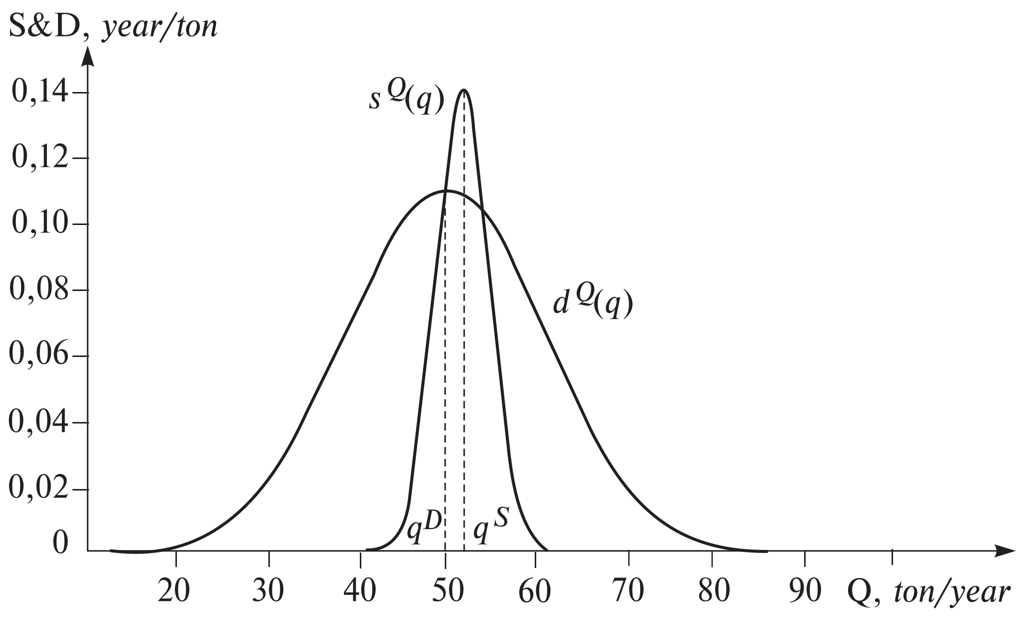

Fig. 4. Graphical representation in the conventional rectangular three-dimensional coordinate system [P, Q, S&D] of the one-dimensional S&D Q-functions, $d^Q(q)$ of the buyer and $s^Q(q)$ of the seller, as the two-dimensional curves with maxima at quantities q^D, q^S and natural widths Γ^{DQ} and Γ^{SQ}, respectively. $\Gamma^{DQ} = 26{,}4$ *ton/year*, $\Gamma^{SQ} = 6{,}8$ *ton/year.*

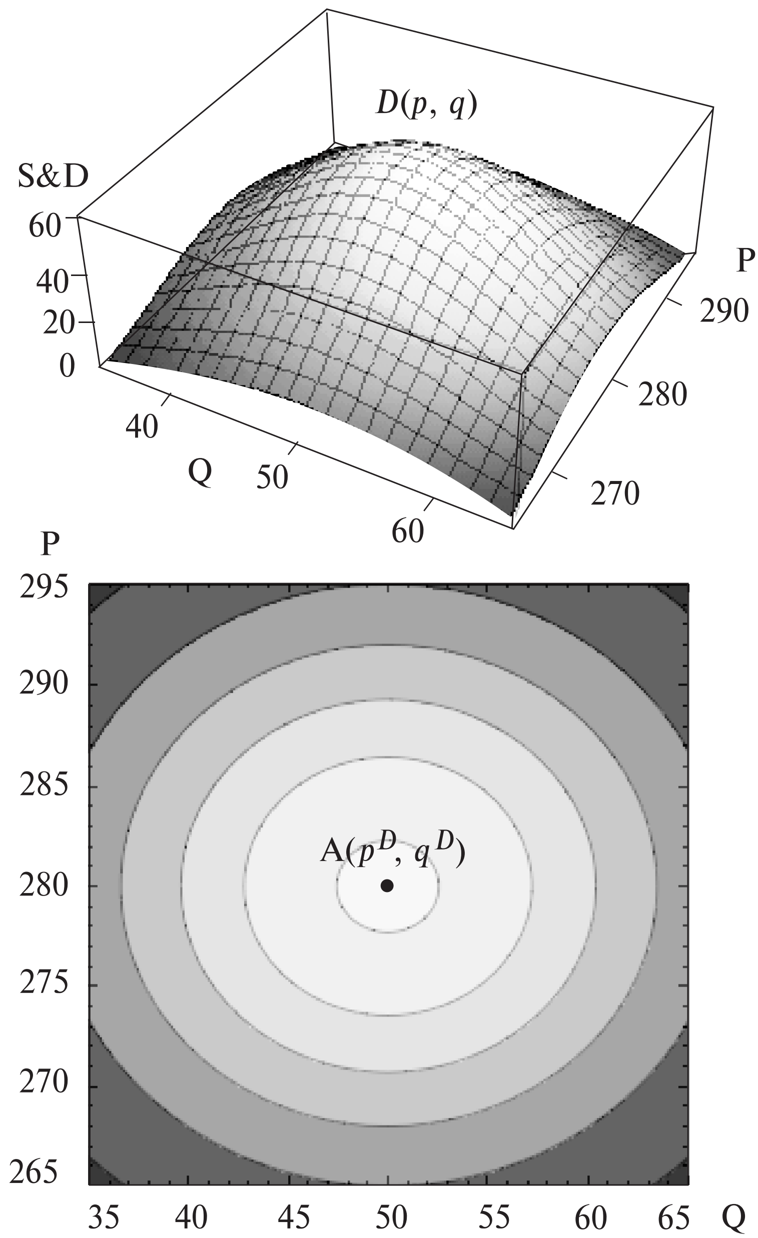

Fig. 5. Graphical representation in the conventional rectangular three-dimensional coordinate system [P, Q, S&D] of the two-dimensional demand function of the buyer as the three-dimensional surface $D(p, q)$ with maximum having the projection on the (P, Q)-plane at the point A (p^D, q^D).

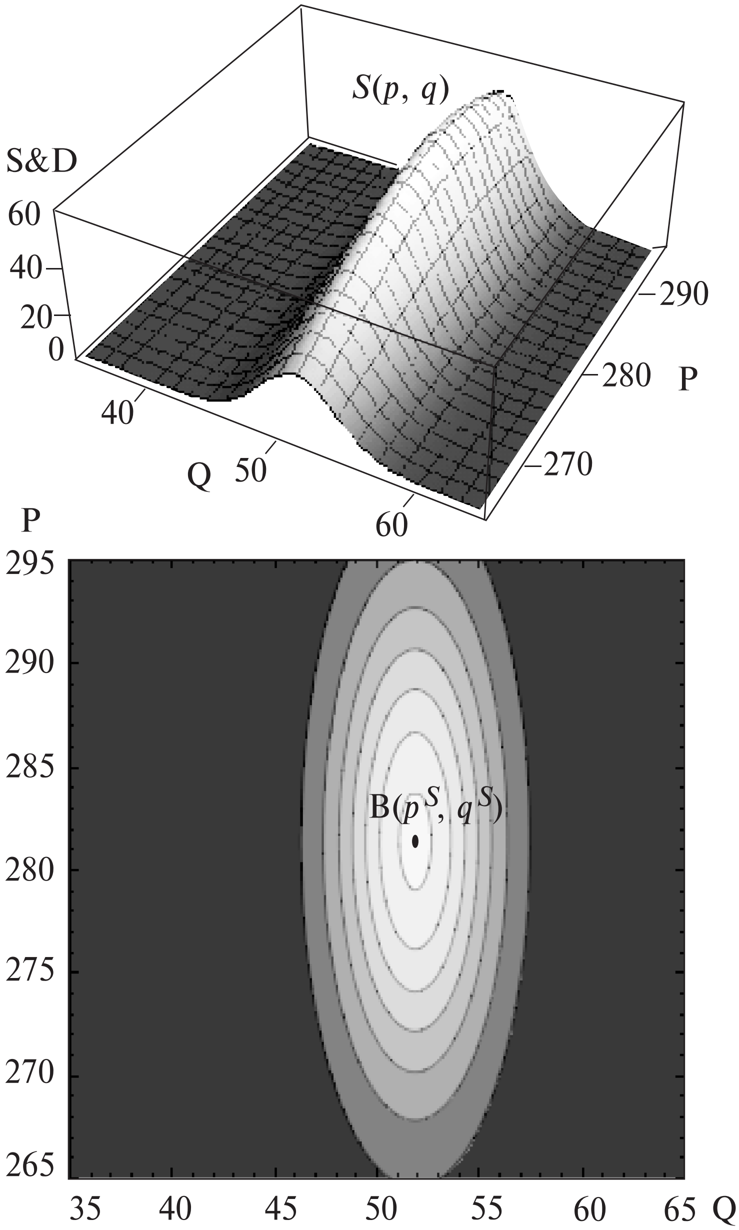

Fig. 6. Graphical representation in the conventional rectangular three-dimensional coordinate system [P, Q, S&D] of the two-dimensional supply function of seller as the three-dimensional surface $S(p,q)$ with maximum having the projection on the (P, Q)-plane at the point B (p^S, q^S).

can lead to more precise results in some concrete calculations. But at present, this type of problem set-up seems to be irrelevant. Moreover, Gaussians have an invaluable advantage — the very simple and fast ability to compute the most complicated systems. For this reason we will primarily use Gaussians in our calculations within the framework of the GSP model below. Let us call attention to the simple relationship between the S&D functions of the simple (SP) and general (GSP) models as follows:

$$\int_{-\infty}^{+\infty} D(p,q)dq = D^0 \cdot d^P(p) \equiv D(p), \tag{15}$$

$$\int_{-\infty}^{+\infty} S(p,q)dq = S^0 \cdot s^P(p) \equiv S(p), \tag{16}$$

where the SP functions described in detail in [2] are located to the right.

2.3. Market Price and Market Force

As is well-known, the most intriguing question of any economic model is the question on market price. We will thoroughly discuss all the new notions, main features and computational details for our simplest model in order not to waste time dealing with more complicated economies. As in the case of the simple SP model, we define a market price as p^m, at which the so-called deal function $F(p,q)$ reaches a maximum, and the deal function being defined by the following formula:

$$F(p,q) \equiv D(p,q) \cdot S(p,q). \tag{17}$$

Note that the economic meaning of the deal function is as follows: it describes the distribution of probability of making a deal by both the buyer and seller in the PQ-space, i.e., at price, p, and in quantity, q. Obviously, that the deal function is also now a distorted bell-shaped surface, or simply a bell (deal bell below), in the three-dimensional coordinate system [P, Q, S&D] with a maximum, projection of which on the (P, Q)-plane is in the point C (p^m, q^m). The P- and Q-coordinates of this point we refer to as the market price, p^m, and quantity, q^m. Of course, this reference has an economic meaning only in the narrow sense as the most probable values of price and quantity in markets [2].

Generally speaking, purchase/sale transactions can be made at any values of pair of arguments, price and quantity, but with different values of probability. If a surface, $F(p,q)$, is rather narrow with one distinct high (i.e., if the deal bell, $F(p,q)$, is high and narrow), than there are grounds to apply the term market price for value, p^m, in a broad sense [2]. Consider that in this case practically all the transactions are performed at this market price. Just such a case is depicted in Fig. 7 for our model market of grain.

This scenario, side by side with market price, is a possibility of great theoretical interest to quantitatively estimate the purchase/sell transaction volume, or simply trade volume, in a market during the given time period. For this purpose, introducing a new notion of trade volume operator into the theory is a natural next step. It is noted as $TV(p,q)$ and is defined as the product of the two independent variables p,q as follows:

$$TV(p,q) = p \cdot q. \tag{18}$$

It is natural in this case to use a mean value of this operator (*MTV* below) as a quantitative estimation of the trade volume as follows:

$$MTV = C^T \cdot \int_{-\infty}^{+\infty} F(p,q) TV(p,q)\, dp\, dq. \tag{19}$$

In Eq. (19), C^T is, for the time being, an indefinite normalization constant of the economic system under study. The main role of this constant is to ensure the right dimension for the trade volume that will be expressed in monetary units. It can be easily seen that in the case of the high and narrow deal bell we obtain the simpler formula for the calculation of the trade volume:

$$MTV \cong C^T \cdot F^0 \cdot p^m \cdot q^m, \tag{20}$$

where F^0 is, by definition, the overlap integral of the S&D functions:

$$F^0 = \int_{-\infty}^{+\infty} F(p,q)\, dp\, dq. \tag{21}$$

Note that we have used an analogous formula within the framework of the simple SP model for estimation of the trade volume [1, 3].

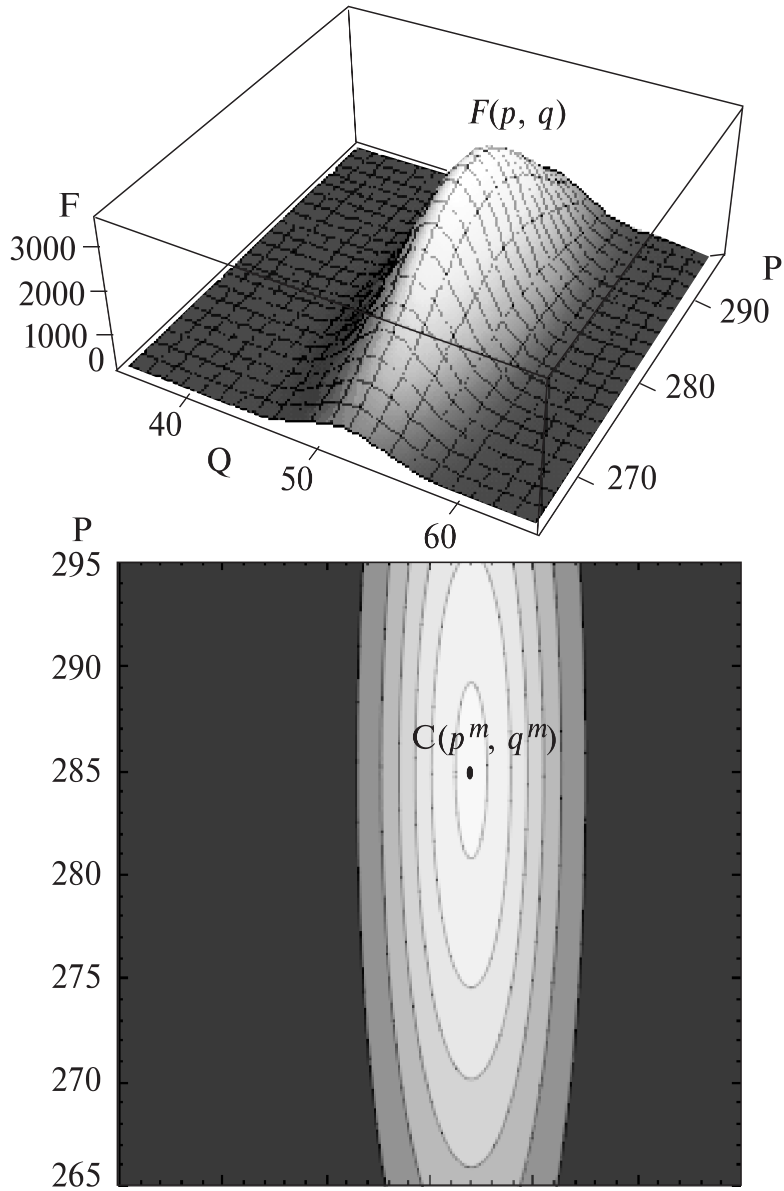

Fig. 7. Three-dimensional graphical representation in the conventional rectangular three-dimensional coordinate system [P, Q, F] of the two-dimensional deal surface $F(p,q)$ in the form of the high and narrow bell that has one maximum with the coordinates p^m and q^m in the PQ-space, i.e., with the projection in the point C (p^m, q^m) on the (P, Q)-plane. $p^m = 281{,}4$ *\$/ton*, $q^m = 51{,}9$ *ton/year*.

Let us deal now with the issue of calculating market prices and quantities. It is well-known from the standard course of mathematical analysis that extrema of a multi-dimensional function are to be determined as points on the respective surfaces at which the total differential of this function is equal to zero. In our situation, that leads to the following equation:

$$dF(p,q)=0. \tag{22}$$

This equation is clearly equivalent to a system of the two following partial differential equations:

$$\begin{aligned} &\partial F(p,q)/\partial p=0;\\ &\partial F(p,q)/\partial q=0. \end{aligned} \tag{23}$$

Expressed through the S&D functions, this system is transformed in the following manner:

$$\begin{aligned} &\partial D(p,q)/\partial p\cdot S(p,q)+D(p,q)\cdot\partial S(p,q)/\partial p=0;\\ &\partial D(p,q)/\partial q\cdot S(p,q)+D(p,q)\cdot\partial S(p,q)/\partial q=0. \end{aligned} \tag{24}$$

Furthermore, as in the case of the simple SP model [4], we introduce the following definition for market force of S&D, or, simply, S&D forces:

$$\begin{aligned} &U^P(p,q)\equiv\frac{\partial D(p,q)/\partial p}{D(p,q)}, \qquad U^Q(p,q)\equiv\frac{\partial D(p,q)/\partial q}{D(p,q)};\\ &B^P(p,q)\equiv\frac{\partial S(p,q)/\partial p}{S(p,q)}, \qquad B^Q(p,q)\equiv\frac{\partial S(p,q)/\partial q}{S(p,q)}. \end{aligned} \tag{25}$$

Now we rewrite the equation system (24) in terms of S&D forces as follows:

$$\begin{aligned} &U^P(p,q)=-B^P(p,q),\\ &U^Q(p,q)=-B^Q(p,q). \end{aligned} \tag{26}$$

It is clear that this equation system looks like the equations of equality of S&D forces in much the same fashion as the well-known equations of equality of forces at the point of an equilibrium in classical mechanics. In other words, the system of equations (26) is analogous in form to the third law of Newton. Keep in mind, the equations of equality (26) take place only in the points of maxima (extrema, to be exact) on the deal bells that we refer to as market prices and qualities. Thus, we obtain the following result: the market prices and quantities are, in essence, equilibrium prices and quantities for the S&D forces rather than the very S&D as can be expected on the basis of the traditional model. To avoid misunderstanding, we stress that the notion of stationary states of economic systems is broader than the notion of the equilibrium state. For instance, take the equilibrium state where the two equalities, discussed above, are approximately valid for four numbers: $p^D = p^S$, $q^D = q^S$, and is only a particular case of stationary states (for more details see [6]).

Substituting the definitions of S&D functions (8) and (12) for the definitions of the S&D forces (26) give rise to the following simple and distinct formulas for computing the forces:

$$U^P(p,q) = -2w^{DP}(p - p^D), \quad U^Q(p,q) = -2w^{DQ}(q - q^D);$$
$$B^P(p,q) = -2w^{SP}(p - p^S), \quad B^Q(p,q) = -2w^{SQ}(q - q^D). \tag{27}$$

Furthermore, by means of these formulas we obtain a very elegant system of linear equations for determining market prices and quantities. This method is so simple that there is no need at all to solve these equations in the common sense of the word; the two equations are fully independent of one another:

$$-2w^{DP}(p - p^D) = 2w^{SP}(p - p^S);$$
$$-2w^{DQ}(q - q^D) = 2w^{SQ}(q - q^D). \tag{28}$$

As a result, we have the following graceful formulas for calculating market prices p^m and quantities q^m:

$$p^m = (w^{DP} \cdot p^D + w^{SP} \cdot p^S)/(w^{DP} + w^{SP});$$
$$q^m = (w^{DQ} \cdot q^D + w^{SQ} \cdot q^S)/(w^{DQ} + w^{SQ}). \tag{29}$$

Thus, market prices and quantities are determined in this case by means of averaging the corresponding parameters of agents, with agent frequency parameters serving as weights. That is, these two formulas, making use of only four parameters for the buyer (p^D, q^D, w^{DP}, w^{DQ}) and four parameters for the seller (p^S, q^S, w^{SP}, w^{SQ}), determine just such a maximum point in the PQ-space ((P, Q)-plane here) in the vicinity of which deal probability takes on maximal magnitudes. As we already mentioned above, in the case when the deal bell is rather high and narrow, these maximum point coordinates can be unambiguously referred to as market price p^m and quantity q^m.

For clarity, we shall make some comments concerning the obtained results. Firstly, we refer to the fact that both the GSP and simpler SP models give the same result (29) for the market price. Of course, this is an artifact of our very simple model economy, connected mainly with making use of the Gaussians and their peculiarity; the differentiating and integrating of Gaussians gives rise to Gaussians. Applying other, more precise analytical standard functions for the representation of agent functions (for example, wave functions of anharmonic quantum oscillators, i.e., degenerated hypergeometric functions) eliminate this artifact in both models. At this point there is a second circumstance which plays a role. It is here where a simple model economy with a single buyer and single seller that the Gaussian features appear in full measure. We can infer that the influence of the choice factor of a standard function form on computational results decreases as the market agent number increases.

The second comment is concerned with the fact that the two equations (29) are independent of one another. This independence is also a consequence of the peculiarity of our model economy, in that it has only one buyer and only one seller. In model economies with a larger number of agents, relationships among market processes occurring in the P- and Q-subspaces become stronger due to the coordination and cooperation among buyers and sellers. This collective effect will be described in detail in the next section.

3. The One-Good, Many-Agent Markets

3.1. The Additivity Formula for Supply and Demand

In our opinion, there comes a time to augment the GSP model by including additional buyers and sellers. Therefore, we shall consider in this section an economic system with one good traded by the N buyers and M sellers. For the same reasons we applied to the simple SP model in [1,2], we can apply the additivity rule to construct market S&D functions, using the individual agent S&D functions as building blocks, respectively. So, we use agent demand functions to build up the market demand function $D(p, q)$, represented as a simple sum of those over all the buyers as follows:

$$D(p,q) \cong \sum_{n=1}^{N} D_n(p,q), \quad D_0 = \sum_{n=1}^{N} D_n^0 = \sum_{n=1}^{N} p_n^D \cdot q_n^D, \tag{30}$$

$$\int_{-\infty}^{+\infty} D(p,q)\, dp\, dq = D^0. \tag{31}$$

Note that, as usual in our theory, we treat the agent behavior strategies as if they are defined by the agents themselves, and we only give them an adequate mathematical form by means of formulas from the previous section. These formulas become superficially more complicated due to appearance of the buyer indexes, but in essence they remain the same. We write them out in an explicit form here just to facilitate making use of them later on:

$$D_n(p,q) = D_n^0 \cdot g_n^{DP}(p) \cdot g_n^{DQ}(q), \quad D_n^0 = p_n^D \cdot q_n^D, \tag{32}$$

$$g_n^{DP}(p) = \sqrt{w_n^{DP}/\pi} \cdot \exp\left(-w_n^{DP}(p - p_n^D)^2\right), \tag{33}$$

$$g_n^{DQ}(q) = \sqrt{w_n^{DQ}/\pi} \cdot \exp\left(-w_n^{DQ}(q - q_n^D)^2\right), \tag{34}$$

$$\Gamma_n^{DP} = \sqrt{-4 \ln 0{,}5 / w_n^{DP}}, \tag{35}$$

$$\Gamma_n^{DQ} = \sqrt{-4 \ln 0{,}5 / w_n^{DQ}}. \tag{36}$$

Analogously, we proceed with the market supply that can be approximately described through the summation of the individual supplies as follows:

$$S(p,q) \cong \sum_{m=1}^{M} S_m(p,q), \quad S_0 = \sum_{m=1}^{M} S_m^0 = \sum_{m=1}^{M} p_m^S \cdot q_m^S, \tag{37}$$

$$\int_{-\infty}^{+\infty} S(p,q)\, dp\, dq = S^0. \tag{38}$$

$$S_m(p,q) = S_m^0 \cdot g_m^{SP}(p) \cdot g_m^{SQ}(q), \quad S_m^0 = p_m^S m q_m^S, \tag{39}$$

$$g_m^{SP}(p) = \sqrt{w_m^{SP}/\pi} \cdot \exp\left(-w_m^{SP}(p - p_m^S)^2\right), \tag{40}$$

$$g_m^{SQ}(q) = \sqrt{w_m^{SQ}/\pi} \cdot \exp\left(-w_m^{SQ}(q - q_m^S)^2\right), \tag{41}$$

$$\Gamma_m^{SP} = \sqrt{-4\ln 0{,}5/w_m^{SP}}, \tag{42}$$

$$\Gamma_m^{SQ} = \sqrt{-4\ln 0{,}5/w_m^{SQ}}. \tag{43}$$

All the definitions made above for the deal function $F(p,q)$ (17) as well as for the forces of demand $U^P(p,q)$, $U^Q(p,q)$ and supply $B^P(p,q)$, $B^Q(p,q)$ (25) remain the same for many-agent markets. The same can be said about Eq. (26) for the calculation of market prices, p^m, and quantities, q^m. From here onwards, the collective effect described in detail in [4] adds complexity to formulas. So market forces are now defined as sums of the individual S&D forces over all the buyers and sellers as follows:

$$U^P(p,q) = \sum_{n=1}^{N} U_n^P(p,q), \quad U^Q(p,q) = \sum_{n=1}^{N} U_n^Q(p,q);$$

$$B^P(p,q) = \sum_{m=1}^{M} B_m^P(p,q), \quad B^Q(p,q) = \sum_{m=1}^{M} B_m^Q(p,q). \tag{44}$$

Agent forces already have a more complicated form than the simple form (27) because of the collectivization effect:

$$U_n^P(p,q) \equiv \frac{\partial D_n(p,q)/\partial p}{D(p,q)} = -2w_n^{DP}\left(p - p_n^D\right) D_n(p,q)/D(p,q),$$

$$B_m^P(p,q) \equiv \frac{\partial S_m(p,q)/\partial p}{S(p,q)} = -2w_m^{SP}\left(p - p_m^S\right) S_m(p,q)/S(p,q),$$

$$U_n^Q(p,q) \equiv \frac{\partial D_n(p,q)/\partial q}{D(p,q)} = -2w_n^{DQ}\left(q - q_n^D\right) D_n(p,q)/D(p,q),$$

$$B_m^Q(p,q) \equiv \frac{\partial S_m(p,q)/\partial q}{S(p,q)} = -2w_m^{SQ}\left(q - q_m^S\right) S_m(p,q)/S(p,q). \tag{45}$$

3.2. The System of Equations for Market Prices and Market Quantities

Eventually, the system of equations for calculation of the market prices and quantities take on the following final form:

$$S(p,q)\cdot\sum_{n=1}^{N}\left(-2w_n^{DP}\left(p-p_n^D\right)\right)\cdot D_n(p,q)=$$

$$=-D(p,q)\cdot\sum_{m=1}^{M}\left(-2w_m^{SP}\left(p-p_m^S\right)\right)\cdot S_m(p,q),$$

$$S(p,q)\cdot\sum_{n=1}^{N}\left(-2w_n^{DQ}\left(q-q_n^D\right)\right)\cdot D_n(p,q)=$$

$$=-D(p,q)\cdot\sum_{m=1}^{M}\left(-2w_m^{SQ}\left(q-q_m^S\right)\right)\cdot S_m(p,q). \qquad (46)$$

It is easy to see that the increase in the number of buyers and sellers in the market, strongly interacting amongst themselves, usually leads to the fact that the two equations become closely linked. Depending on the specific economic system, market prices and quantities calculated with these equations can, in principle, differ from the values determined in the simpler SP model, which does not take into account the probabilistic nature of the choices of the good quantities made by buyers and sellers. In other words, taking into account the probabilistic nature of the S&D functions in the Q-subspace causes market prices, p^m, to change, because market processes occurring concurrently in the P- and Q-subspaces can be strongly coupled due to interaction among agents. The extent to which they are coupled, and how much the market price changes, is determined by the specific characteristics of the investigated economic system. In some cases, when uncertainty in the choice of agent quantity is small, i.e., Γ_n^{DQ} and Γ_m^{SQ} are small, the coupling effect is minimal. However, in the opposite case, it can be substantial. It is apparent that this coupling cannot be ignored. It is impossible to analytically solve the system of equations (46) and it is often necessary to have recourse to numerical methods of computing. However, this fact does not present a real challenge because all of the functions in the system are given in an analytical form that makes all computations practically instantaneous.

4. The Many-Good, Many-Agent Markets

In this Section, we follow the road we travelled earlier, in the case of the simple SP model in [2, 4]. Bear in mind that for calculating the agent S&D functions in the many-good markets, the multiplicativity formula was used in [2]. In other words, the agent S&D functions are represented by the products of the respective agent S&D one-good functions. Below, we are going to briefly repeat the derivation of the respective set of formulas. For more details see [2–4].

We will consider the most general case where the n-th buyer wants to buy all L goods in the market for a definite sum of money, D_n^0. According to his or her needs and criteria the buyer divides this sum of money among all goods demanded and elaborates his or her unique strategy in the market. The corresponding multi-dimensional demand function, $D(p_1, \dots, p_L, q_1, \dots, q_L)$, can be approximately represented in a factorized form using the multiplicativity formula for demand as follows:

$$D_n(p_1, \dots, p_L, q_1, \dots, q_L) = \frac{\sum_{l=1}^{L} D_{nl}^0}{\prod_{l=1}^{L} D_{nl}^0} \prod_{l=1}^{L} D_{nl}(p_l, q_l), \tag{47}$$

where

$$D_{nl}^0 = p_{nl}^D \cdot q_{nl}^D. \tag{48}$$

In this formula Gaussians can be used for approximate representation of the buyer's one-dimensional demand functions as follows:

$$D_{nl}(p_l, q_l) = D_{nl}^0 \cdot g_{nl}^{DP}(p_l) \cdot g_{nl}^{DQ}(q_l), \tag{49}$$

$$g_{nl}^{DP}(p_l) = \sqrt{w_{nl}^{DP}/\pi} \cdot \exp\left(-w_{nl}^{DP}\left(p_l - p_{nl}^D\right)^2\right), \tag{50}$$

$$g_{nl}^{DQ}(q_l) = \sqrt{w_{nl}^{DQ}/\pi} \cdot \exp\left(-w_{nl}^{DQ}\left(q_l - q_{nl}^D\right)^2\right), \tag{51}$$

$$\Gamma_{nl}^{DP} = \sqrt{-4 \ln 0{,}5 / w_{nl}^{DP}}, \tag{52}$$

$$\Gamma_{nl}^{DQ} = \sqrt{-4 \ln 0{,}5 / w_{nl}^{DQ}}. \quad (53)$$

$$D_n^0 = \sum_{l=1}^{L} D_{nl}^0 = \sum_{l=1}^{L} p_{nl}^D \cdot q_{nl}^D. \quad (54)$$

In these formulas, all new parameters make obvious sense by the definition. p_{nl}^D is the price, at which the n-th buyer plans to buy the l-th good in quantity, q_{nl}^D, and D_{nl}^0 is the n-th buyer's total demand of the l-th good expressed in a monetary form, etc.

It is easy to check that for the n-th buyer, their one-dimensional demand function $D_{nl}(p_l, q_l)$ is normalized to their total demand of the l-th good, D_{nl}^0, and their multi-dimensional demand function, $D_n(p_1, \ldots, p_L, q_1, \ldots, q_L)$, is normalized to their total demand, D_n^0. Then, by means of the additivity formula for demand, we describe the multi-dimensional market demand function as a sum of the individual multi-dimensional functions as follows:

$$D(p_1, \ldots, p_L, q_1, \ldots, q_L) = \sum_{n=1}^{N} D_n(p_1, \ldots, p_L, q_1, \ldots, q_L) =$$

$$= \sum_{n=1}^{N} \frac{\sum_{l=1}^{L} D_{nl}^0}{\prod_{l=1}^{L} D_{nl}^0} \prod_{l=1}^{L} D_{nl}(p_l, q_l), \quad (55)$$

$$D^0 = \sum_{n=1}^{N} D_n^0 = \sum_{n=1}^{N} \sum_{l=1}^{L} D_{nl}^0 = \sum_{n=1}^{N} \sum_{l=1}^{L} p_{nl}^D \cdot q_{nl}^D. \quad (56)$$

The market demand function is normalized to the total market demand, D^0.

By the same procedure we obtain the multi-dimensional factorized supply function of the m-th seller with the help of the multiplicativity formula for supply as follows:

$$S_m(p_1, \ldots, p_L, q_1, \ldots, q_L) = \frac{\sum_{l=1}^{L} S_{ml}^0}{\prod_{l=1}^{L} S_{ml}^0} \prod_{l=1}^{L} S_{ml}(p_l, q_l), \quad (57)$$

where

$$S^0_{ml} = p^S_{ml} \cdot q^S_{ml}. \tag{58}$$

To some extent of accuracy, we can also use Gaussians as the seller's one-dimensional supply functions:

$$S_{ml}(p_l, q_l) = S^0_{ml} \cdot g^{SP}_{ml}(p_l) \cdot g^{SQ}_{ml}(q_l), \tag{59}$$

$$g^{SP}_{ml}(p_l) = \sqrt{w^{SP}_{ml}/\pi} \cdot \exp\left(-w^{SP}_{ml}\left(p_l - p^S_{ml}\right)^2\right), \tag{60}$$

$$g^{SQ}_{ml}(q_l) = \sqrt{w^{SQ}_{ml}/\pi} \cdot \exp\left(-w^{SQ}_{ml}\left(q_l - q^S_{ml}\right)^2\right), \tag{61}$$

$$\Gamma^{SP}_{ml} = \sqrt{-4\ln 0{,}5/w^{SP}_{ml}}, \tag{62}$$

$$\Gamma^{SQ}_{ml} = \sqrt{-4\ln 0{,}5/w^{SQ}_{ml}}, \tag{63}$$

$$S^0_m = \sum_{l=1}^{L} S^0_{ml} = \sum_{l=1}^{L} p^S_{ml} \cdot q^S_{ml}. \tag{64}$$

The seller's multi-dimensional supply function, $S_m(p_1, \ldots, p_L, q_1, \ldots, q_L)$, is normalized to his or her total supply, S^0_m. Then, applying the additivity formula for supply, we obtain the multi-dimensional market supply function, $S(p_1, \ldots, p_L, q_1, \ldots, q_L)$, as a sum of the individual multi-dimensional supply functions as follows:

$$S(p_1, \ldots, p_L, q_1, \ldots, q_L) = \sum_{m=1}^{M} S_m(p_1, \ldots, p_L, q_1, \ldots, q_L) =$$

$$= \sum_{m=1}^{M} \frac{\sum_{l=1}^{L} S^0_{ml}}{\prod_{l=1}^{L} S^0_{ml}} \cdot \prod_{l=1}^{L} S_{ml}(p_l, q_l). \tag{65}$$

This function is normalized to the market total supply, S_0, which is calculated by summing as follows:

$$S^0 = \sum_{m=1}^{M} S^0_m = \sum_{m=1}^{M} \sum_{l=1}^{L} S^0_{ml} = \sum_{m=1}^{M} \sum_{l=1}^{L} p^S_{ml} \cdot q^S_{ml}. \tag{66}$$

Thus, using the multiplicativity formula for S&D we have factorized the many-good market S&D functions. We represented the multi-dimensional market functions by the products of the respective one-dimensional agent's functions, and that made calculations and detailed studies of markets and market processes much easier.

Now we can return to our main task: obtaining a set of equations for calculating market prices in many-good markets. Let us be clear by taking rather obvious steps towards achieving our goal. First of all, it is apparent that the deal function can be defined here in the same way as in the case of a one-good market, namely, as the product of the respective multi-dimensional market S&D functions:

$$F(p_1, \ldots, p_L, q_1, \ldots, q_L) =$$
$$= D(p_1, \ldots, p_L, q_1, \ldots, q_L) \cdot S(p_1, \ldots, p_L, q_1, \ldots, q_L). \qquad (67)$$

It is obvious that in our case the deal function $F(p_1, \ldots, p_L, q_1, \ldots, q_L)$ is differentiable in the whole price space. Furthermore, it can be easily seen that the problem of finding the market prices is reduced here to finding the local maxima of the deal function in the $2{\times}L$-dimensional PQ-space. It is well-known from the standard course of mathematical analysis that this task for differentiable functions is reduced in turn to solving the system of $2{\times}L$ partial differential equations:

$$\partial F(\boldsymbol{p}, \boldsymbol{q})/\partial p_l;$$
$$\partial F(\boldsymbol{p}, \boldsymbol{q})/\partial p_l, \quad l = 1, 2, \ldots, L. \qquad (68)$$

This system, in turn, can easily be transformed to the system of $2{\times}L$ partial differential equations expressed by means of the S&D functions as follows:

$$\partial D(\boldsymbol{p}, \boldsymbol{q})/\partial p_l \cdot S(\boldsymbol{p}, \boldsymbol{q}) + D(\boldsymbol{p}, \boldsymbol{q}) \cdot \partial S(\boldsymbol{p}, \boldsymbol{q})/\partial p_l = 0;$$
$$\partial D(\boldsymbol{p}, \boldsymbol{q})/\partial p_l \cdot S(\boldsymbol{p}, \boldsymbol{q}) + D(\boldsymbol{p}, \boldsymbol{q}) \cdot \partial S(\boldsymbol{p}, \boldsymbol{q})/\partial p_l = 0, \; l = 1, 2, \ldots, L. \qquad (69)$$

For brevity, in (68) and (69) we used only one bold letter, $\boldsymbol{p}$, to designate all L prices p_l (now $\boldsymbol{p}$ is already a vector in the L-dimensional P-subspace [2]) and one bold $\boldsymbol{q}$ — all L quantities q_l (now $\boldsymbol{q}$ is already a vector in the L-dimensional Q-subspace). All partial derivatives in our case are easily calculated because all differentiations are reduced to differentiation of Gaussians $g_{nl}^{DP}(p_l)$ and others in a trivial way: differentiation of one Gaussian gives rise to another Gaussian.

Let us rewrite this system once again in terms of the market forces as follows:

$$U_l^P(\boldsymbol{p},\boldsymbol{q}) = -B_l^P(\boldsymbol{p},\boldsymbol{q});$$

$$U_l^Q(\boldsymbol{p},\boldsymbol{q}) = -B_l^Q(\boldsymbol{p},\boldsymbol{q}), \qquad l = 1, 2, \ldots, L. \tag{70}$$

That is equivalent to a simple equilibrium equation in a final vector form:

$$\boldsymbol{U}^P(\boldsymbol{p},\boldsymbol{q}) = -\boldsymbol{B}^P(\boldsymbol{p},\boldsymbol{q});$$

$$\boldsymbol{U}^Q(\boldsymbol{p},\boldsymbol{q}) = -\boldsymbol{B}^Q(\boldsymbol{p},\boldsymbol{q}), \tag{71}$$

where now, by their definition, $\boldsymbol{U}^P(\boldsymbol{p},\boldsymbol{q})$, $\boldsymbol{B}^P(\boldsymbol{p},\boldsymbol{q})$, $\boldsymbol{U}^Q(\boldsymbol{p},\boldsymbol{q})$ and $\boldsymbol{B}^Q(\boldsymbol{p},\boldsymbol{q})$ are the S&D vector forces in the $2 \times L$-dimensional price space with their components $U_l^P(\boldsymbol{p},\boldsymbol{q})$ and $B_l^P(\boldsymbol{p},\boldsymbol{q})$, respectively:

$$U_l^P(\boldsymbol{p},\boldsymbol{q}) \equiv \frac{\partial D(\boldsymbol{p},\boldsymbol{q}) / \partial q_l}{D(\boldsymbol{p},\boldsymbol{q})},$$

$$B_l^P(\boldsymbol{p},\boldsymbol{q}) \equiv \frac{\partial S(\boldsymbol{p},\boldsymbol{q}) / \partial q_l}{S(\boldsymbol{p},\boldsymbol{q})};$$

$$U_l^Q(\boldsymbol{p},\boldsymbol{q}) \equiv \frac{\partial D(\boldsymbol{p},\boldsymbol{q}) / \partial q_l}{D(\boldsymbol{p},\boldsymbol{q})},$$

$$B_l^Q(\boldsymbol{p},\boldsymbol{q}) \equiv \frac{\partial S(\boldsymbol{p},\boldsymbol{q}) / \partial q_l}{S(\boldsymbol{p},\boldsymbol{q})}, \; l = 1,\; 2, \ldots, L. \tag{72}$$

Thus, we have a generalized formula (26) for calculation of market prices in the case of a one-good market, meaning that we now have the equilibrium equation (71) of the same structure as well. Yet it is already in the vector form because the PQ-space here is a multi-dimensional one. As we can see, we have again obtained the full formal analogy with the equilibrium condition in classical mechanics. However, as in the case of a one-good market, the additivity rule for the S&D forces will differ in design from the relatively simple additivity rule of classical mechanics. This is because the S&D forces differ in nature from the forces in classical mechanics. Namely, the market S&D forces are probabilistic and collective by their economic origin. The respective additivity rules are easily obtained from Eqs. (47)–(66):

$$U_l^P(\boldsymbol{p},\boldsymbol{q}) = \sum_{n=1}^{N} \frac{D_n(\boldsymbol{p})}{D(\boldsymbol{p})} U_{nl}^P(\boldsymbol{p},\boldsymbol{q});$$

$$B_l^P(\boldsymbol{p},\boldsymbol{q}) = \sum_{m=1}^{M} \frac{S_m(\boldsymbol{p},\boldsymbol{q})}{S(\boldsymbol{p},\boldsymbol{q})} B_{ml}^P(\boldsymbol{p},\boldsymbol{q});$$

$$U_l^Q(\boldsymbol{p},\boldsymbol{q}) = \sum_{n=1}^{N} \frac{D_n(\boldsymbol{p},\boldsymbol{q})}{D(\boldsymbol{p},\boldsymbol{q})} U_{nl}^Q(\boldsymbol{p},\boldsymbol{q});$$

$$B_l^Q(\boldsymbol{p},\boldsymbol{q}) = \sum_{m=1}^{M} \frac{S_m(\boldsymbol{p},\boldsymbol{q})}{S(\boldsymbol{p},\boldsymbol{q})} B_{ml}^Q(\boldsymbol{p},\boldsymbol{q}),\ l = 1, 2, \ldots, L, \tag{73}$$

where the buyers demand forces, $\boldsymbol{U}_n(\boldsymbol{p})$, and sellers supply forces, $\boldsymbol{B}_m(\boldsymbol{p})$, are naturally defined in the following way:

$$U_{nl}^P(\boldsymbol{p},\boldsymbol{q}) \equiv \frac{\partial D_n(\boldsymbol{p},\boldsymbol{q}) / \partial q_l}{D_n(\boldsymbol{p},\boldsymbol{q})}, \quad B_{ml}^P(\boldsymbol{p},\boldsymbol{q}) \equiv \frac{\partial S_m(\boldsymbol{p},\boldsymbol{q}) / \partial q_l}{S_m(\boldsymbol{p},\boldsymbol{q})},$$

$$U_{nl}^Q(\boldsymbol{p},\boldsymbol{q}) \equiv \frac{\partial D_n(\boldsymbol{p},\boldsymbol{q}) / \partial q_l}{D_n(\boldsymbol{p},\boldsymbol{q})},$$

$$B_{ml}^Q(\boldsymbol{p},\boldsymbol{q}) \equiv \frac{\partial S_m(\boldsymbol{p},\boldsymbol{q}) / \partial q_l}{S_m(\boldsymbol{p},\boldsymbol{q})},\ l = 1, 2,\ \ldots, L. \tag{74}$$

The equations (73) can be written also in terms of vectors as follows:

$$\boldsymbol{U}^P(\boldsymbol{p},\boldsymbol{q}) = \sum_{n=1}^{N} \frac{D_n(\boldsymbol{p},\boldsymbol{q})}{D(\boldsymbol{p},\boldsymbol{q})} \boldsymbol{U}_n^P(\boldsymbol{p},\boldsymbol{q}),$$

$$\boldsymbol{B}^P(\boldsymbol{p},\boldsymbol{q}) = \sum_{m=1}^{M} \frac{S_m(\boldsymbol{p},\boldsymbol{q})}{S(\boldsymbol{p},\boldsymbol{q})} \boldsymbol{B}_m^P(\boldsymbol{p},\boldsymbol{q}),$$

$$\boldsymbol{U}^Q(\boldsymbol{p},\boldsymbol{q}) = \sum_{n=1}^{N} \frac{D_n(\boldsymbol{p},\boldsymbol{q})}{D(\boldsymbol{p},\boldsymbol{q})} \boldsymbol{U}_n^Q(\boldsymbol{p},\boldsymbol{q}),$$

$$\boldsymbol{B}^Q(\boldsymbol{p},\boldsymbol{q}) = \sum_{m=1}^{M} \frac{S_m(\boldsymbol{p},\boldsymbol{q})}{S(\boldsymbol{p},\boldsymbol{q})} \boldsymbol{B}_m^Q(\boldsymbol{p},\boldsymbol{q}). \tag{75}$$

It can be easily shown that due to the simple form of the agent S&D functions chosen (see (49) and (59)) the vector components of the S&D forces of individual agents are to be calculated using the following formulas:

$$U^P_{nl}(\boldsymbol{p},\boldsymbol{q}) = U^{P*}_{nl}(p_l), \quad B^P_{ml}(\boldsymbol{p},\boldsymbol{q}) = B^{P*}_{ml}(p_l);$$

$$U^Q_{nl}(\boldsymbol{p},\boldsymbol{q}) = U^{Q*}_{nl}(q_l), \;\; B^Q_{ml}(\boldsymbol{p},\boldsymbol{q}) = B^{Q*}_{ml}(q_l), \;\; l = 1,\, 2,\, \ldots,\, L, \qquad (76)$$

where we introduced the standard or natural definitions for the vector components of the one-good demand forces of individual buyers, $U^{P*}_{nl}(p_l)$, $U^{Q*}_{nl}(q_l)$, and the vector components of the one-good supply forces of individual sellers, $B^{P*}_{ml}(p_l)$, $B^{Q*}_{ml}(q_l)$, as follows:

$$U^{P*}_{nl}(p_l) \equiv \frac{dD_{nl}(p_l, q_l)\,/\,dp_l}{D_{nl}(p_l, q_l)} = -2w^{DP}_{nl}\left(p_l - p^D_{nl}\right),$$

$$B^{P*}_{ml}(p_l) \equiv \frac{dS_{ml}(p_l, q_l)\,/\,dp_l}{S_{ml}(p_l, q_l)} = -2w^{SP}_{ml}\left(p_l - p^S_{ml}\right),$$

$$U^{Q*}_{nl}(q_l) \equiv \frac{dD_{nl}(p_l, q_l)\,/\,dq_l}{D_{nl}(p_l, q_l)} = -2w^{DQ}_{nl}\left(q_l - q^D_{nl}\right),$$

$$B^{Q*}_{ml}(q_l) \equiv \frac{dS_{ml}(p_l, q_l)/dq_l}{S_{ml}(p_l, q_l)} = -2w^{SQ}_{ml}\left(q_l - q^S_{ml}\right), \;\; l = 1, 2, \ldots, L. \qquad (77)$$

Using vector notations we get the equations (76) in a more compact form:

$$\boldsymbol{U}^P_n(\boldsymbol{p},\boldsymbol{q}) = \boldsymbol{U}^{P*}_n(\boldsymbol{p}), \quad \boldsymbol{B}^P_m(\boldsymbol{p},\boldsymbol{q}) = \boldsymbol{B}^{P*}_m(\boldsymbol{p});$$

$$\boldsymbol{U}^Q_n(\boldsymbol{p},\boldsymbol{q}) = \boldsymbol{U}^{Q*}_n(\boldsymbol{p}), \quad \boldsymbol{B}^Q_m(\boldsymbol{p},\boldsymbol{q}) = \boldsymbol{B}^{Q*}_m(\boldsymbol{p}). \qquad (78)$$

Let us repeat once again that the developed systems of equilibrium equations (71) give as solutions not only the market prices and quantities (corresponding to local maxima of the deal function), but also the prices and quantities at which the deal function has local minima. To select the market prices and quantities we should clarify: to calculate the magnitudes of the deal function in several points near the selected solution ($\boldsymbol{p}^m$, q^m) and compare them with the magnitude $F(\boldsymbol{p}^m, q^m)$. If $F(\boldsymbol{p}^m, q^m)$ is greater than magnitudes in the vicinity of

($\boldsymbol{p}^m$, q^m), it means that the selected point ($\boldsymbol{p}^m$, q^m) properly indicates the market price. If not, than it is just a local minimum of the deal function.

Although the market forces equilibrium equation (71) for discovering market prices is rather complicated, it can be easily solved by using well-known computer software programs available, even for very large models and real markets. Despite the fact that the equilibrium equation is expressed in terms of multi-dimensional functions and parameters, and the obtained solutions have to be presented and analyzed in a multi-dimensional price space, performing such calculations is not beyond a student's competence. It is necessary to emphasize that the simplicity and ability to quickly obtain practical solutions of rather complicated equilibrium equation are brought about by factorization of the market S&D functions, i.e., by the presentation of them as the products of one-dimensional one-good agent S&D functions in the analytical form using very simple Gaussians as basic functions. Application of the probability method (more exactly, the GSP model) developed and discussed above opened new avenues for detailed computer modeling of many-good markets, consisting of a large number of market agents, whole industries, regional and raw material markets, intimately associated capital and labor markets, and so on.

5. Conclusions

This Chapter is dedicated to the elaboration of a complex of fundamentally new economic models, in which we defined probability economics. In all of these models, we implement the concept of uncertainty and probability in economic theory consistently and step by step. Specifically, in this paper we set out the GSP model which is an important generalization of the simple SP model, developed by the author in previous works. This generalization is carried out as follows. For each good traded in the L-good market of an economy, we treat price p and quantity q as two independent variables and then introduce into the theory the idea of the $(2\times L)$-dimensional PQ-space by analogy with the price space of the simple model. As a consequence, the motion of the whole economic system can be thought of as occurring in this space. In other words, the concept of uncertainty and probability regarding buyer's and seller's choices in the market is equally applied to the price and quantity of goods. Thus, S&D functions of both individual agents and the market as a whole are $(2\times L+1)$-dimensional surfaces within the framework of the GSP model. The probabilistic nature of an agent quantity choice significantly changes the meaning of S&D functions, as well as the significance and numerical values of market forces and prices. In addition, taking into account these additional quantity degrees of freedom significantly expands the scope of application of probability approaches for the description of market phenomena. Therefore, probability economics, as described in previous works, is proposed by us as a new quantitative method for the description, analysis, and investigation of both theoretical and real-world economies and markets. Like any other new theory, probability economics needs experimental or empirical verification and further refinement and development.

References

1. Anatoly Kondratenko. *Physical Modeling of Economic Systems. Classical and Quantum Economies*. Novosibirsk: Nauka, 2005. Electronic copy available at: http://ssrn.com/abstract=1304630.
2. Anatoly Kondratenko. *Probability Economics: Supply and Demand in the Price Space*. Electronic copy available at: http://ssrn.com/abstract=2250343. See also Chapter IV.
3. Anatoly Kondratenko. *Probability Economics: Market Price in the Price Space*. Electronic copy available at: http://ssrn.com/abstract=2263708. See also Chapter V.
4. Anatoly Kondratenko. *Probability Economics: Market Force in the Price Space*. Electronic copy available at: http://ssrn.com/abstract=2270306. See also Chapter VI.
5. Ludwig von Mises. *Human Action: A Treatise on Economics*. Yale University, 1949.
6. Anatoly Kondratenko. *Trade Maximization Principle: Market Processes, Supply and Demand Laws, and Equilibrium States*. Electronic copy available at: http://ssrn.com/abstract=2431218. See also Chapter VIII.
7. Anatoly Kondratenko. *Physical Economics: Stationary Quantum Economies in the Price — Quantity Space*. Electronic copy available at: http://ssrn.com/abstract=2363874. See also Chapter X.

PART E.

Probability Economics. Non-Stationary Probabilistic Economies in the Price-Quantity Space

"The market is not a place, a thing, or a collective entity. The market is a process, actuated by the interplay of the actions of the various individuals cooperating under the division of labor. The forces determining the — continually changing — state of the market are the value judgments of these individuals and their actions as directed by these value judgments. The state of the market at any instant is the price structure, i.e., the totality of the exchange ratios as established by the interaction of those eager to buy and those eager to sell. There is nothing inhuman or mystical with regard to the market. The market process is entirely a resultant of human actions. Every market phenomenon can be traced back to definite choices of the members of the market society. The market process is the adjustment of the individual actions of the various members of the market society to the requirements of mutual cooperation. The market prices tell the producers what to produce, how to produce, and in what quantity. The market is the focal point to which the activities of the individuals converge. It is the center from which the activities of the individuals radiate".

Ludwig von Mises. *Human Action. A Treatise on Economics*. Page 257

CHAPTER VIII.

The Market-Based Trade Maximization Principle: Market Processes, Supply and Demand Laws, and Equilibrium States

"The driving force of the market process is provided neither by the consumers nor by the owners of the means of production—land, capital goods, and labor — but by the promoting and speculating entrepreneurs. These are people intent upon profiting by taking advantage of differences in prices. Quicker of apprehension and farther-sighted than other men, they look around for sources of profit. They buy where and when they deem prices too low, and they sell where and when they deem prices too high. They approach the owners of the factors of production, and their competitions ends the prices of these factors up to the limit corresponding to their anticipation of the future prices of the products. They approach the consumers, and their competition forces prices of consumers' goods down to the point at which the whole supply can be sold. Profit-seeking speculation is the driving force of the market as it is the driving force of production".

Ludwig von Mises. *Human Action. A Treatise on Economics*. Page 328

"Any price determined on a market is the necessary outgrowth of the interplay of the forces operating, that is, demand and supply. Whatever the market situation which generated this price may be, with regard to it the price is always adequate, genuine, and real. It cannot be higher if no bidder ready to offer a higher price turns up, and it cannot be lower if no seller ready to deliver at a lower price turns up. Only the appearance of such people ready to buy or to sell can alter prices.

Economics analyzes the market process which generates commodity prices, wage rates, and interest rates. It does not develop formulas which would enable anybody to compute a "correct" price different from that established on the market by the interaction of buyers and sellers".

Ludwig von Mises. *Human Action. A Treatise on Economics*. Page 370

PREVIEW.

What is Market Equilibrium State?

In this Chapter we have used probability economics as a tool of unification of the market process concept of Austrian economics and the supply and demand laws and equilibrium state concepts of neoclassical economics. The market-based trade maximization principle has been proposed to serve as a connection link among of all these three concepts. In particular, we have shown that it is namely the market process that underlies the supply and demand laws' mechanisms and leads the market to the equilibrium state. It is namely the desire of market agents to maximize the market trade that is the driver of this transition market process. So, we have shown that the trade maximization principle can be regarded as one of the possible mathematical representations of this driver, and thereby the well-known invisible hand of the market. In the end, we have obtained a set of mathematical formulas that describe in detail the inverse relationships between economic efficiencies and economic uncertainties (the uncertainty law) and the agent structure of the equilibrium states of the many-agent, many-good markets. They also reveal the main features of the market transition process to these states, for instance the narrowing effect and the uncertainty relation for supply and demand.

1. Introduction

Throughout the previous four chapters we constructed probabilistic economic models that describe the stationary states of economic systems, and the most recent work brought forth the theory of stationary economies. In this chapter, an attempt is made to develop the theory of non-stationary economies. We have chosen the the market-based trade maximization principle as the basic concept of the theory. By its market-based nature, the trade maximization principle expresses mathematically the desire of market agents to make an optimal number of deals within the market to address their two most urgent problems. The first is that of organizing a cash flow sufficient enough to meet their business and human needs. The second is how to finish having obtained a reasonable profit. In order to illustrate how the developed theory can be applied to economic problems, several situational examples have been considered. Among them, three problems in particular have become our focal points in this paper. They are as follows:

First, we have described and refined in detail the known view of the market process, which is the basic concept of Austrian economics. In terms of probability economics, we have given a formal definition of the market process as *a process of conscious and purposeful changing by market agents their functions of supply and demand* (S&D below).

Second, we have thoroughly analyzed one of the main concepts of neoclassical economics, the S&D laws. We have described in detail the market process underlying the S&D laws mechanisms using trade maximization principle as a formal driver of the market process. In other words, we have clarified that it is namely the market-based trade maximization principle that provides the foundation for the S&D laws.

Third, we have examined in detail the conventional notion of the equilibrium state of the market, which is the basic concept of neoclassical economics. We have shown that it is namely through the mechanism of market process that the market comes to equilibrium. Mathematically, the application of the trade maximization principle leads the market to the equilibrium state. Figuratively, we can say that the trade maximization principle is one of the possible forms of mathematical representation of the market's invisible hand. This runs from the time of Adam Smith, guiding the market to its equilibrium state, which is the most efficient and therefore the most desirable state from the point of view of the majority of the market agents.

Thus, it can be argued that probability economics eliminates the principal conflict between Austrian economics and neoclassical economics by means of the trade maximization principle. In other words, it would appear that probability economics allows one to combine these contradictory economic theories.

2. The Non-Stationary Economies

In [1–4] we elaborated probability economics on the basis of the several relatively simple, clear concepts and principles. They are as follows:

First, we introduced the concept of stationary states [5] into the theory, enabling the clear and justified classification of states of economic systems. It is assumed that if the economy is in a normal stationary state that, roughly speaking, there is no time dependence in the economy in the mean. That is, all economic pictures averaged over a definite time period are unchangeable. More precisely, it means that all economic processes and phenomena repeat periodically, and all the agent and market S&D functions, averaged over the time period, do not change. We refer to economies being in a normal stationary state as *the stationary economies* [6]. Conversely, the economic pictures change over time in non-stationary economies. Remember that according to the ideas of Austrian economics [7], it is namely the market process that is responsible for changes of the economy over time. On this basis, we take the process of change by market agents in relation to their S&D functions as the formal definition of the market process within the framework of probability economics.

Second, we proposed the following five principles as the second cornerstone of probability economics to clarify the nature of the S&D, and develop the method of calculating the S&D functions (see Chapter I):

1. **The Cooperation-Oriented Agent Principle.**
2. **The Institutional and Environmental Principle.**
3. **The Dynamic and Evolutionary Principle.**
4. **The Market-Based Trade Maximization Principle.**
5. **The Uncertainty and Probability Principle.**

On the basis of these concept and principles, we have compiled a range of hypotheses (see [4, 6]). We have concluded that the S&D functions of each agent for each good can be quite accurately presented as the products of two continuous Gaussian functions. These functions are of the good price and quantity, respectively, with four S&D parameters, namely, the desired price and quantity of the good, as well as two widths of these functions (curves). Conceptually, these agent functions are probability distributions of deal making by the agents at a certain price and for a certain quantity of the good traded. Then, using these agent functions as building blocks, we have constructed the market S&D functions by means of the additivity formula (for agents) and the multiplicativity formula (for goods) for S&D. Eventually, all the market S&D functions show up as multi-dimensional surfaces in the formal $2\times L$-dimensional price-quantity space (below the PQ-space) with L degrees of freedom for agent prices p and L degrees of freedom for agent quantities q, and where L is the number of goods traded in the

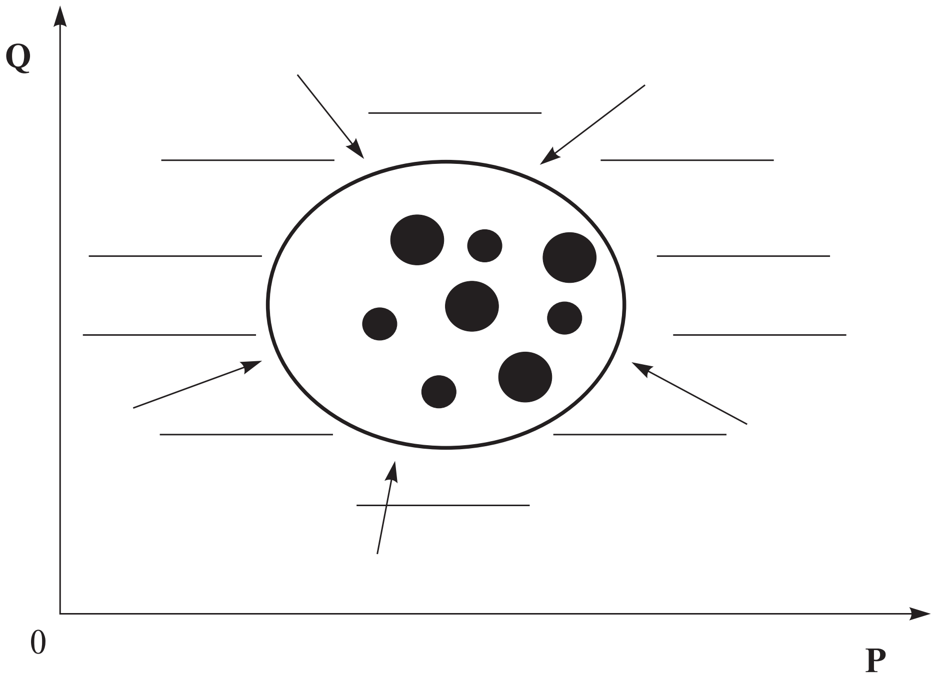

Fig. 1. Graphical model of an economy in the multi-dimensional PQ-space. It is displayed schematically in the conventional rectangular multi-dimensional coordinate system [**P**, **Q**] where **P** and **Q** designate all the agent price and quantity coordinate axes, respectively. Our model economy consists of the market and the external environment. The market consists of buyers (small dots) and sellers (big dots) covered by the conventional sphere. Very many people, institutions, as well as natural and other factors can represent the external environment (cross-hatched area behind the sphere) of the market which exerts perturbations on market agents, pictured here by arrows pointing from environment to market [4].

market. As an illustration of this, see Fig. 1 which is Fig. 1 from [4], all other details concerning this picture can be found in [4]. The probabilistic model, described briefly above, has been referred to as the General Stationary Probability Model in the Price-Quantity Space (the GSP model below). Thus, the GSP model has been constructed to describe the motion of the market agents of the economy being in a normal stationary state in the PQ-space. By means of the GSP model, we have performed a number of calculations of S&D functions for the simplest model of the economy having the market with only one buyer and one seller trading a grain. The GSP results have been presented as a series of figures (see Figs. 2–7 in [4]), with fairly detailed comments in the text of the paper and figure signatures. We have drawn the conclusion that the probabilistic mechanism underlying the formation of the S&D functions of a stationary economy has been rather well displayed by these figures.

Stationary state is a fruitful theoretical instrument for studying market work mechanisms. Of course, as is usual in theoretical disciplines such as theoretical physics, it is only an imaginable construction that represents an unchangeable market picture that is not fully realizable in practice. But it allows us to take the first, most important steps towards understanding the components within the real market world and how it works. From this unchangeable picture we can proceed further and introduce into the theory the concept of non-stationary states of an economy. As it is well known, all the businesses, large and small, and public corporations in particular, are under permanent external pressure from the side of owners and other stakeholders demanding managers to grow profits on an almost quarterly basis. This pressure causes managers to think of business growth and economic efficiency, and to therefore permanently seek new possibilities for business development. Creative actions of entrepreneurs aim not only to maintain the *status quo* in the market, but also to obtain additional profits through speculations and innovations of a different type. These entrepreneurial actions never even stop for a moment, being energized from an internal pressure from the entrepreneurs themselves. This is because the essence and the meaning of their lives are in these market processes. As a result, these external and internal pressures induce the entrepreneurs, firms and corporations to permanently reconsider their own business plans, leading to permanent changes in their S&D functions, i.e., to the perpetual market process on markets.

We think that we can extend and generalize the GSP model for describing the most important time-dependent features of the market process within the limits assigned beforehand. We will show below with a few formulas, how we can construct such a generalized probabilistic model for non-stationary states, which will then be referred to as the General Non-Stationary Probability Model in the Price-Quantity Space (below the GNP model). But before proceeding further some notions and definitions are needed.

To start with, let us introduce an intermediate notion of the slowly changing non-stationary state of the economy. Suppose that at some initial time t_0 the economy was in a normal stationary state. At the t_0

moment in time, some important changes occur in the external economic environment. Perhaps government policies are changed, a crisis occurs in neighboring economies, or perhaps something else. Evidently, under the influence of this external perturbation, the initial stationary state begins to transform into a time-dependent state which we will refer to below as a non-stationary state. Suppose further that the external changes and perturbations come about rather slowly. Moreover, the very external perturbations are rather small in magnitude. Suppose also that relaxation or adaptation of market agents to these changes occurs, in contrast, rather rapidly. Note that this last thing is possible of course only given sufficient economic liberty in the economy, as would be made possible by lenient institutes and relaxed government regulations, etc. We can expect in this case that our non-stationary state changes rather slowly over time, too. Below we will refer to these sorts of slowly changing economies as normal economies. To put it differently, a normal economy is an economy without rigid regulations, sharp external shocks, deep crises and other large negative factors retarding the normal economic development.

The notion of the normal economy correlates rather well with the notions of the progressing (expanding) and the retrogressing (shrinking) economies of Austrian economics [7]. It is these types of normal economies that are studied in this paper. Further, we will use the notion of economy in the broadest sense of the word: it is the agents themselves that constitute the market, plus the other actors of the economy such as the state, institutions of governance, rules and regulations, trade unions and employees of companies, etc. Generally speaking, these actors and institutions can also change their strategies and their rules etc. over time, so their influences on the market also depend on time. For this reason, they need to be included into the theory explicitly so that we can describe and explore the details of their structure, strategy, their impact on the market agents and the market as a whole, as well as their influence on each other, etc. However, a number of other economic disciplines investigate these problems. So far, the market is the only subject we focus on in this work. More exactly, we will try to describe a few details regarding the dynamics of the market using the notion of the market state, which is defined in this study as a set or configuration of all the agent parameters, four parameters for every market agent (see above).

So, we are now going to analyze the economy with the market in which N buyers and M sellers are trading L goods, within the framework of the GNP model in the following manner. Assume that the economy is in a non-stationary state. Assume also that all the market agents are aware of this fact. Therefore, their horizon of planning is relatively short. The buyers and sellers realize that they are often compelled to change their plans and strategies. For the sake of certainty, we assume that the horizon of planning is equal to year. Then at the beginning of each year, say at the moments of time t_1, t_2, ..., t_i, all market agents are to formulate their strategies for themselves. Namely, the buyers themselves define their demand functions $D_n(t_i, p_1, \ldots, p_L, q_1, \ldots, q_L)$ with the certain desired prices $p_{nl}^D(t_i)$, quantities $q_{nl}^D(t_i)$, widths $\Gamma_{nl}^{DP}(t_i)$ and $\Gamma_{nl}^{DQ}(t_i)$. Likewise, sellers themselves form their supply functions $S_m(t_i, p_1, \ldots, p_L, q_1, \ldots, q_L)$ with the certain desired prices $p_{ml}^S(t_i)$, quantities $q_{ml}^S(t_i)$, widths $\Gamma_{ml}^{SP}(t_i)$ and $\Gamma_{ml}^{SQ}(t_i)$.

Then, using the agent S&D functions as building blocks we construct the market functions of demand $D(t_i, p_1, \ldots, p_L, q_1, \ldots, q_L)$, supply $S(t_i, p_1, \ldots, p_L, q_1, \ldots, q_L)$ and deals $F(t_i, p_1, \ldots, p_L, q_1, \ldots, q_L)$ in the formal PQ-space. Here, prices p_l and quantities q_l of all the traded goods are independent coordinate variables (see above). So far, all S&D and deal functions are being constructed by the formulas derived prior for the GSP model, and all needed details can be found in the original paper [4]. The only difference here is that now this is done regularly, because we assume the market process to be included. This means that all plans and strategies are regularly evaluated and changed by the agents, and therefore their prices, quantities, widths and the very S&D and deal functions are changed regularly too. Formally, this means that all the functions are now discrete functions of time (or so-called *time series*). To deal with discrete functions is inconvenient, so using standard methods of approximation (e.g., the method of spline-approximation), we can make them continuous functions of time. The market functions now look like this: $D(t, p_1, \ldots, p_L, q_1, \ldots, q_L)$, $S(t, p_1, \ldots, p_L, q_1, \ldots, q_L)$ and $F(t, p_1, \ldots, p_L, q_1, \ldots, q_L)$. It is clear that these functions describe the work of the economy in real time, i.e., they describe the dynamics or the evolution of the economy over time. It is also clear that if these functions are relatively slow-changing functions of time, then the studied economy can be referred to as the normal one.

To assess the feasibility of such an assignment, one can permanently calculate the time-averaged market functions as the following integrals:

$$\boldsymbol{D}(t, p_1, \dots, p_L, q_1, \dots, q_L) = \frac{1}{(t - t_0)} \int_{t_0}^{t} \boldsymbol{D}(t, p_1, \dots, p_L, q_1, \dots, q_L)\, dt, \quad (1)$$

$$\boldsymbol{S}(t, p_1, \dots, p_L, q_1, \dots, q_L) = \frac{1}{(t - t_0)} \int_{t_0}^{t} \boldsymbol{S}(t, p_1, \dots, p_L, q_1, \dots, q_L)\, dt, \quad (2)$$

$$\boldsymbol{F}(t, p_1, \dots, p_L, q_1, \dots, q_L) = \frac{1}{(t - t_0)} \int_{t_0}^{t} \boldsymbol{F}(t, p_1, \dots, p_L, q_1, \dots, q_L)\, dt. \quad (3)$$

If these time-averaged market functions remain nearly unchanged over time, the economy can be considered stationary and one can use the ideology of the GSP model for market analysis. If they change rather rapidly over time, then the economy is non-stationary and we need to thoroughly investigate the market process, as there may be options for treatment and interpretation. If they change slowly enough then the economy can be regarded as a normal one and we can use for study the GNP model in the manner described below. So, we do have an opportunity to quantitatively assess the state of the economy at every moment in time. Moreover, by means of these time-dependent market functions we can construct a set of new economic indicators. For example, we can use the market trade function *MTV* (see [4] and below) to track functioning of the economy as a whole, in particular, to opportunely recognize the development of adverse situations in all major markets of the economy. For instance, if during the course of the year *MTV* has been growing compared with the same period a year earlier the economy is a growing one. If there has been a fall in the value of *MTV*, then the state of affairs is not good and the economy will fall into a recession.

To avoid misunderstandings, we emphasize here that we are talking not about calculating the dynamics or evolution of the market using equations of motion. Instead, we are referring to the method of empirical descriptions of real markets in real time, on the basis of plans or strategies of market agents, from all sources allegedly known to us. Bear in mind that the main task of physical economics as we laid out in [7] is to establish general equations of motion that describe the time-dependent market S&D functions.

3. The Market-Based Trade Maximization Principle and the Market Process

3.1. Mutual Market Agents' Cooperation

As far as we know, the best description of the essence of the market process is given by Ludwig von Mises in his monograph [7] (see epigraph to this chapter). It does serve as one of the starting points in this our study. For a more or less adequate description of the direction and details of the market process, in this paper we propose to use one of the possible principles of maximization for dynamic economic systems, namely, the market-based trade maximization principle as the fourth principle of physical economics (see above). To justify such a choice, and why specifically this principle can be used in economics, we present citations from our previous work on the physical modeling of economic systems [5]. "In the market economy major active players such as buyers and sellers of goods and commodities, behave to a certain extent in a deterministic way. This subordinates their behavior in the market to some strict economic laws. The fact that these laws are of a descriptive nature, and they have not yet been expressed in a precise mathematic language, is not of key importance in this case. Every rational player or market subject acts in the market in accordance with a strict rule of obtaining maximum profit, benefit, or some other criterion of optimality for themselves and in this respect, market economic systems resemble physical systems where all players, members, and elements of the system act also in accordance with some principles of maximization. According to our approach every economic agent acts not only rationally in his or her own interests, but also reasonably. They negotiate to reach a minimum price for the buyer and a maximum price for the seller, but also leave their counteragent a chance to gain profit from transactions. Otherwise, transactions would take place only once, while all agents would prefer the continuation and stability of their business. Besides, we presume that external forces influence market operations usually positively, establishing common rules of play that favor gaining maximum profit for the whole economic system. Based on these principles we have a firm belief that there is a certain principle of optimization, and its effect results in certain rules of market behavior and certain equations of motion that are followed by all rational or reasonable players spontaneously or voluntarily. In our opinion, it is they who have the leading role in the market".

Recall that, within the framework of probability economics, we formally define the market process as permanently changing the agent S&D functions. That is to say, the permanent process of change is brought on by the agents themselves, adjusting their four S&D parameters for each good traded in the market. Therefore, by definition, the market process is missing in the stationary economy. It begins when a change occurs in the external conditions of agents' work in the market, since each such change can represent both threats, or, on the contrary, new opportunities for the business. And in both cases, entrepreneurs must quickly respond to such changes. The consequence of agents adapting their S&D functions through the market process is that the economy as a whole also adapts to the new conditions. Our central objective in this paper is to clarify the mechanism and direction of this adaptation of the market and economy to the several specific changes in the external environment. Note again our assumption that before the advent of such changes in the external environment in the initial moment of time t_0, the economy was in a normal stationary state.

3.2. Market Invisible Hand

As we have argued previously in [5], to a certain extent, our economic activities are deterministic; we all want to make a profit, strive for stability, and experience repeatability of market developments. These conditions enable us to have confidence in the future and time to quietly spend the earned money to meet our needs. Within reasonable limits, the economic activity can be described as current in accordance with certain principles of maximization. The market-based trade maximization principle is chosen as a basic principle in this work, which we consider as capable in serving as the most common expression of the aspirations of the market agents. Note that it is not the task of this work to uncover the principle of maximization according to which people in the market and the market in the whole act. Conceptually, it is wrong and impossible because people consciously choose goals and means, ways and methods of action in the market. Both the goals and means as well ways and methods of human action can be comprehended only by monitoring the actions of real people, interviewing them or by engaging oneself in the market as an agent, for instance as an entrepreneur. One of the most important tasks of economic theory is to empirically identify and adequately describe some of the most important human action patterns in the market, and patterns of the market as a whole. These general patterns have been well established and described on the qualitative level in classic economic literature, particularly in [7]. We set ourselves the narrower task of describing the known patterns in mathematical language and, most importantly, constructing quantitative methods of study of the market mechanisms that underlie these patterns, using economic thought of the probability type and knowledge of the individual agent patterns.

With this aim in mind, we introduce the market-based trade maximization principle into probability economics, believing that the agents in the market consciously strive mainly for this purpose. Why do we think so? First, all the deals in the market, are, as a rule, beneficial to both sides of the deal. Roughly speaking, the more transactions there are in the market, the richer every market participant becomes, the gross domestic product of the economy increases, and the rate of the economic growth becomes higher.

Within the framework of probability economics, trade volume naturally correlates with the degree of overlap between market S&D functions [4, 5]. Moreover, the stronger they overlap, the higher the probability of making a deal as described by the deal function, and eventually, the higher the trade volume in this state of the market. It is obvious that in this market, where the S&D functions are almost the same, this maximum is theoretically achieved. But then the total market S&D need to be equal in this state, too. Obviously, it means that this state is the market equilibrium state, according to the prominent concept of neoclassical economics. Figuratively speaking, we can say that the market-based trade maximization principle can be regarded as one of the possible mathematic representations of the market's invisible hand pushing the market to the equilibrium state in the manner that maximizes the efficiency of the market. In this context, the trade volume in the market is also maximized for efficiency, and therefore, too, the maximum production in the economy as a whole, which is very good for society overall.

In conclusion, let us emphasize that in this paper we are attempting, on the one hand, to describe the rather slow restructuring of the system of market inter-agent relations that is caused by external perturbations. This restructuring happens without shocks or abrupt maneuvers by market agents. In other words, strong changes do not occur in the agent S&D functions, because most of the market agents and other market actors are inclined to retain conservative behavior in order to maintain the market *status quo*, and thereby preserve or even maximize their profits. These agents act rationally and without panic, because they have enough time to reflect and select the optimal strategy. On the other hand, one still needs to act quickly in this relaxed state. Otherwise, new external perturbations can severely hamper the market from coming close to its target equilibrium state.

4. The Market-Based Trade Maximization Principle and the Supply and Demand Laws

Qualitatively, in this Section, we will examine the most important mechanisms underlying the market process. It is common practice to describe these mechanisms by means of the empirical S&D laws. We will show below that formally, within the framework of probability economics, the trade maximization principle constitutes the theoretical foundation of the empirical S&D laws. It means that these laws have their origin in the desire of all market agents to maximize their trade volumes in order to maximize their profits in the end.

First, we briefly describe how we understand an algorithm of actions of market agents in a normal market economy in response to an external perturbation at moment in time, t_0. By convention, this algorithm can be divided into two simultaneously running market processes. The first process involves the reaction and adaptation of the market to the external perturbation. In order to understand how the market will react to external perturbation, one must, of course, deal with it at the micro level, i.e., to understand how each agent of the market will respond to the perturbation. And then, by summing all agent responses or actions, one can get the whole response of the market to the perturbation. Obviously, here each agent's response will depend on the type of perturbation, and on the player's position in the market game. Rational choices and actions of each player can be revealed by us only by using logical reasoning which has to be based on the following fact: each agent will be striving to use the new market situation with the maximum benefit for him or herself and hold his or her market position under control to avoid large losses and maybe even get some market advantage, etc. An understanding of this fact is the first building block in comprehending the market process accompanying the market perturbation, and eventually the nature of the equilibrium state (see below).

The second market process describes how market agents respond to each other's actions over time. Here each agent will seek to collect and analyze the actions of his or her competitors and customers to predict more or less correctly the direction of further movement of the markets in which he or she acts as buyer (purchases) or supplier (sells). For instance, if all or most of the agents expect a drop in prices, the market will go down. Two important rules are known from real market life that governs the response of one agent in relation to the actions of other agents. These are the S&D laws. Working out the nature of these laws will provide us with the second building block for understanding the market process accompanying market perturbation (and eventually the nature of equilibrium state, see above and below).

Now we are going to reveal a close relationship between the S&D laws and the trade maximization principle. Here it is pertinent to recall that the basis of our understanding of the market process is an awareness of the fact that there is no automaticity in the market; instead, there are conscious agent actions involved in the optimization of their S&D for achieving their goals. Therefore, we have no doubt that the basis for the S&D laws is that the market agents permanently adapt their S&D functions to changing circumstances in the market, including the processing of new market information, in order to maintain or even increase profits. They may also be striving to achieve some other market goals, for example, to increase sales, to conquer new markets, and so on.

Based on our research in the previous articles, we can give a more comprehensive or detailed picture revealing the mechanisms underlying the laws of S&D. But we will do it in this paper in a very concise way, because here we put forth a narrower objective in that we simply attempt to find a more or less acceptable mathematical description of these mechanisms. To be certain, let us again stress that no mathematical theory could make a choice or decision for the concrete person regarding their strategy in the market at any given point in time. However, in some specific cases, we believe that the most likely agent actions can be predicted and described rather correctly by using mathematical formulas. Because the general intents and purposes of agent's actions in the market of a normal economy are usually understandable and predictable, the most likely actions of agents can also be understood and expected. This allows them to be formalized and quantitatively described. However, it is important to realize here that in such cases one must accurately describe the situation in the market and outside it, i.e., to take into account all the information continuously coming to the market in reality. In theory, it should be clearly understood that we can more or less clearly describe the market process only when we have a very detailed description of all circumstances in the market and outside the market.

Before giving a detailed description of the mechanisms of market processes underlying the D&S laws, we will describe questions with well-known neoclassical economics interpretations of the laws. First, let us examine the law of demand. In its simplest form, it sounds like this: "*ceteris paribus* an increase in demand leads to higher market prices". However, as we have shown earlier [2], a simple increase of the total demand *ceteris paribus* does not change market prices at all. Essential to ensure that the market price changes is that buyers themselves will begin to increase their prices; in other words, buyers themselves will begin to change the form of their demand functions which means nothing less than the market process is triggered. Another counterexample: a new buyer comes on the market with a big total demand but with a price at much less than the existing market price. Obviously, in this case, too, the market price will not change, so nobody will be willing to sell them a product at such a low price. Again, to buy the goods, the buyer will have to improve their offer, namely, by increasing the price. Theoretically speaking, this should again bring into operation the market process.

Now, analogously, we gain an understanding of the law of supply. In the simplest form it is usually formulated as follows: "*ceteris paribus* increasing supply results in a reduction in market prices". In article [2] we have shown that the simple raising of the total supply *ceteris paribus* does not necessarily decrease market prices as in the case of demand, considered above. The second is an obvious counterexample, when a seller largely increases their own supply by a large margin. In this case too, it is clear that this does not lead to a change in the market price, since it was unlikely that there would be buyers willing to buy at the very high price. But what is it exactly that then leads to a change in the market price when the supply changes? As we have already seen before, the market process does it. Market process, in turn, as we already know, is governed by the invisible hand of the market. This leads the market to a certain ideal stationary equilibrium state, and the mathematical expression of the driver of the market process is the trade maximization principle.

As a result, we put forth here below the detailed qualitative interpretation of the empirical S&D laws on the base of the trade maximization principle in terms of overlapping market S&D functions. Suppose that the total demand for the good is becoming more than the total supply in the market. However, if there appears to be a shortage of the good, the market becomes immediately aware of this because this is immediately felt by buyers. They begin to compete among themselves for a rare resource, acting separately, each for him- or herself by offering a higher price for the good than competitors. This brings the market demand function to the point of the market supply function, which leads in turn to an increase of the market demand width and thereby to the reduction of the market demand forces. This leads again to higher market prices. Furthermore, the sellers, seeing increased demand and wanting to earn extra profit, begin to jack up their own prices. All sellers begin to act in concert with each other and coordinate their actions in such a way that the total width of the market supply is reduced, causing an increase in the supply market force. This contributes in turn to an even bigger increase in the market price. In doing so, they will dispose of more goods of their stocks or simply to produce more as it becomes profitable to them. Thus, a maximum volume of sales is achieved in the market during a given time period. This is the basic outline of the action mechanism of the law of demand.

But if the reverse situation occurs in the market at some point in time, when the total supply begins to exceed the total demand for a particular good: a situation of market surplus of a good arises. This is also quickly recognized by the sellers. Then the sellers begin to act in a piecemeal fashion, competing among themselves for money from rare buyers, offering lower prices, reducing the market supply width and thereby market supply force that puts downwards pressure on the market price even farther and buyers happily begin to strengthen cooperation. They coordinate actions to increase their market demand force, and pull the market down more, until such time as a certain equilibrium price can be reached. In such situations, the trade volume will be at the maximum, since the entire inventory will be sold. This is the detailed mechanism of the supply law.

So, we have shown in this Section that the foundation of the S&D laws is provided by the market-based trade maximization principle. Economically, the mutual desire to make profitable deals as often as possible, binds buyers and sellers together in the market. Mathematically it means that the market agents lend themselves to the requirements of the trade maximization principle or that the motion of agent adheres to the trade maximization principle. Pictorially, it shows up as a tendency of the market S&D surfaces to maximal overlapping.

5. The Market-Based Trade Maximization Principle and the Market Equilibrium State

Above we have achieved much success in understanding that the market equilibrium state comes into being due to the market process. In addition, we have laid the foundation for the understanding of the mechanism of the formation and structure of the equilibrium state. We have shown that this foundation is governed by the S&D laws, which are based on the agent-based trade maximization principle. Below we will study in greater detail the relationship between the trade maximization principle and the equilibrium state of the market. We will show that this principle governs the market process, and pushes the market in the direction of an equilibrium state. Where, by definition, the several known equilibrium equalities are reached regarding S&D prices and quantities (see below).

For clarity, in this article we consider the following concrete problem. Suppose that historically, up to some initial time, t_0, the economy has been in some inefficient but normal stationary state. For example, say that for a long time there were severe restrictions in economic activities on the part of the state which impeded the natural development of a given market. An example of this would be the setting of legislated maximum sale prices. For this reason, there would be a situation of excess supply in the market. This would mean that a portion of the products that, in principle, could be sold in the market, are not produced, the producers would not obtain sufficient profit, etc. Suppose further that at a certain time, t_0, the state withdrew these restrictions, allowing agents to freely enjoy new market freedoms. This would have to be considered in the theory as an *external perturbation* to the market. What will the market agents do in the new market environment? What will happen to the economy over time?

It is clear that the old primary state of the economy will no longer be stationary, as the market process which governs the adaptation of market agents to the new economic circumstances will have been engaged, and they will begin to change their S&D functions. Naturally, one can assume that under the influence of new factors and stimuli the economy will gradually move into some new final stationary state. A study of the details regarding this transition market process and its final outcome are our main topic of interest at this time. We expect that, in the absence of constraints on the part of the state, the economy will seek a new equilibrium stationary state that is more effective than the initial one.

Qualitatively, what will happen to the economy in the process of this transformation can be examined through the application of the S&D laws discussed above. Below, we will set the task of elaborating on an approximate mathematical quantitative description of this market process within the GNP model by formal application of the trade maximization principle to the market process of this transition.

5.1. The One-Good, One-Buyer and One-Seller Markets

5.1.1. THE MARKET TRADE FUNCTION

As usual in our papers, we begin our introduction to the theory with the fundamental concepts and principles for the simplest economic model. We assume a model economy with a market in which there is only one buyer and one seller, trading between them just a single commodity. For clarity, below we have used grain as an example. Despite the obvious limitations of this model we have seen time and again that this model allows us to establish the main features of the working mechanisms of both individual agents in the market and the market on the whole. Within the framework of probability economics the S&D functions are described by the following formulas, which are fairly simple in structure:

$$D(p,q) \cong C^S \cdot d^P(p) \cdot d^Q(q) \equiv C^D \cdot d(p,q) \qquad (4)$$

$$S(p,q) \cong C^S \cdot s^P(p) \cdot s^Q(q) \equiv C^S \cdot s(p,q). \qquad (5)$$

Here $d^P(p)$, $s^P(p)$ are the respective one-dimensional price S&D functions (P-functions below) and $d^Q(q)$, $s^Q(q)$ are the respective one-dimensional quantity S&D functions (Q-functions below). Below, for an adequate approximate representation of all these one-dimensional functions we will establish appropriate analytical functions. Of course, C^D and C^S are normalization constants. Naturally, for the one-dimensional functions we have chosen the following normalization rule:

$$\int_{-\infty}^{+\infty} d^P(p)\,dp = 1, \quad \int_{-\infty}^{+\infty} d^Q(q)\,dq = 1, \quad \int_{-\infty}^{+\infty} d\,(p,q)\,dp\,dq = 1; \qquad (6)$$

$$\int_{-\infty}^{+\infty} s^P(p)\,dp = 1, \quad \int_{-\infty}^{+\infty} s^Q(q)dq = 1, \quad \int_{-\infty}^{+\infty} s\,(p,q)\,dp\,dq = 1. \qquad (7)$$

To normalize the two-dimensional S&D functions to the total demand D^0 and total supply S^0 of the buyer and seller the process is as follows:

$$\int_{-\infty}^{+\infty} D\,(p,q)\,dp\,dq = D^0, \quad D^0 = p^D \cdot q^D, \quad C^D = D^0; \qquad (8)$$

$$\int_{-\infty}^{+\infty} S(p,q)dp\,dq = S^0, \quad S^0 = p^S \cdot q^S, \quad C^S = S^0. \qquad (9)$$

It is interesting that such normalized S&D functions $D(p,q)$ and $S(p,q)$ are dimensionless ones. Economic significance of all of the one- and two-dimensional S&D functions as introduced above is evident; they are in essence the corresponding distributions of probability of the buyer and seller making a purchase transaction at the price p and for the quantity q. Let us stress that, by the very definition of all of these functions, the market agents themselves “choose” these functions consciously or even automatically on the level of sub-consciousness due to accumulated market experiences. Therefore it is reasonable to use so-called normal distributions or simply Gaussians [2–4] as follows:

$$d^P(p) \cong g^{DP}(p) = \sqrt{w^{DP}/\pi} \cdot \exp\left\{-w^{DP}\left(p - p^D\right)^2\right\}, \tag{10}$$

$$d^Q(q) \cong g^{DQ}(q) = \sqrt{w^{DQ}/\pi} \cdot \exp\left\{-w^{DQ}\left(q - q^D\right)^2\right\}, \tag{11}$$

$$D(p,q) \cong D^0 \cdot g^D(p,q) = D^0 \cdot g^{DP}(p) \cdot g^{DQ}(q), \tag{12}$$

$$\Gamma^{DP} = \sqrt{-4\ln 0{,}5/w^{DP}}, \quad \Gamma^{DQ} = \sqrt{-4\ln 0{,}5/w^{DQ}}, \tag{13}$$

$$s^P(p) \cong g^{SP}(p) = \sqrt{w^{SP}/\pi} \cdot \exp\left\{-w^{SP}\left(p - p^S\right)^2\right\}, \tag{14}$$

$$s^Q(q) \cong g^{SQ}(q) = \sqrt{w^{SQ}/\pi} \cdot \exp\left\{-w^{SQ}\left(q - q^S\right)^2\right\}, \tag{15}$$

$$S(p,q) \cong S^0 \cdot g^S(p,q) = S^0 \cdot g^{SP}(p) \cdot g^{SQ}(p), \tag{16}$$

$$\Gamma^{\mathrm{SP}} = \sqrt{-4\ln 0{,}5/w^{SP}}, \quad \Gamma^{SQ} = \sqrt{-4\ln 0{,}5 w^{SQ}}. \tag{17}$$

Formulas (9) and (13) express the known relations between the Gaussian frequency parameters w^{DP}, w^{DQ}, w^{SP}, w^{SQ} and the Gaussian natural widths Γ^{DP}, Γ^{DQ}, Γ^{SP}, Γ^{SQ} (more precisely , these are full widths at half of the maximum of their peaks). Additional comments to the Eqs. (4)–(17) can be found in [4].

As is well-known, the most intriguing question of any economic model is the question of market price. We will thoroughly discuss all the new notions, main features and computational details for our simplest model in order not to waste time dealing with more complicated economies. As in the case of the simple SP model, we define a market price as p^m, at which the so-called deal function $F(p,q)$ reaches a maximum, with the deal function being defined by the following formula:

$$F(p,q) \equiv D(p,q) \cdot S(p,q) = D^0 \cdot S^0 \cdot f(p,q), \tag{18}$$

where, by definition,

$$f(p,q) \equiv d(p,q) \cdot s(p,q) \cong g^D(p,q) \cdot g^S(p,q). \qquad (19)$$

Note that the economic meaning of the deal function is as follows: it describes the distribution of probability of making a deal by both the buyer and seller in the PQ-space, i.e., at price, p, and in quantity, q. Obviously, that deal function is also now a distorted bell-shaped surface, or simply a deal bell, in the three-dimensional coordinate system [P, Q, S&D] with a maximum, the projection of which on the (P, Q)-plane is at the point C (p^m, q^m). The P- and Q-coordinates of this point we refer to as the market price, p^m, and the market quantity, q^m. Of course, this reference has an economic meaning only in the narrowest sense as the most probable values of price and quantity in markets [2]. Generally speaking, purchase/sale transactions can be made at any values of pair of arguments, price and quantity, but with different values of probability. If a surface, $F(p,q)$, is rather narrow with one distinct high (i.e., if the deal bell, $F(p,q)$, is high and narrow), than there are grounds to apply the term market price for value, p^m, in a broad sense [2]. Consider that in this case practically all the transactions are performed at this market price. Just such a case is depicted in Fig. 2 (which is Fig. 7 from Chapter VII) for our model market of grain.

It is naturally very interesting to have the possibility of quantitatively estimating the purchase/sell transaction volume, or simply trade volume, in a market during the given time period. For this purpose, we introduced in [4] a new notion of trade volume operator into the theory. It is noted as $TV(p,q)$ and is defined as the product of the two independent variables p, q as follows:

$$TV(p,q) = p \cdot q. \qquad (20)$$

It is natural in this case to use a mean value of this operator (MTV below) as an approximate quantitative estimation of the trade volume in the market as follows [4]:

$$MTV \equiv C^T \cdot \int_{-\infty}^{+\infty} d(p,q) \cdot TV(p,q) \cdot s(p,q)\, dp\, dq =$$

$$= C^T \cdot \int_{-\infty}^{+\infty} f(p,q) \cdot TV(p,q)\, dp\, dq. \qquad (21)$$

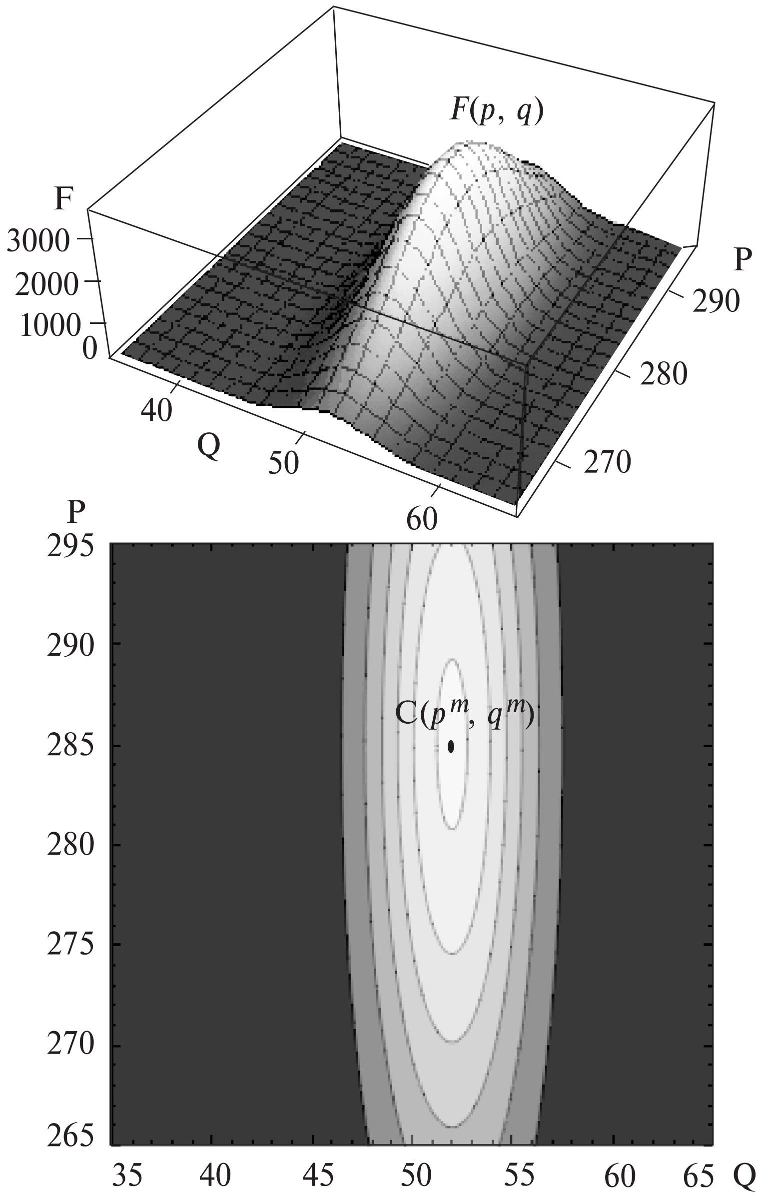

Fig. 2. Three-dimensional graphical representation in the conventional rectangular three-dimensional coordinate system [P, Q, F] of the two-dimensional deal surface $F(p,q)$ in the form of the high and narrow bell that has one maximum with the coordinates p^m and q^m in the PQ-space, i.e., with the projection in the point C (p^m, q^m) on the (P, Q)-plane. $p^m = 281{,}4$ *\$/ton*, $q^m = 51{,}9$ *ton/year*.

In Eq. (19), C^T is, for the time being, an indefinite normalization constant of the economic system under study. The main role of this constant is to ensure the right dimension for the trade volume that will be expressed in monetary units. Let us leave the detailed discussion of this constant to the other times as it is irrelevant in the present work. Instead of this, we pay attention to the fact that two functions in integrals (21) are very different in nature. If $TV(p,q)$ is an apparently slowly varying function, then $f(p,q)$ is, as a rule, the fast varying function. This fact can be used to obtain a simpler way to calculate the trade volumes. Namely, it can be easily seen that in the case of the high and narrow deal bell we obtain the simpler formula for the calculation of the trade volume in the market (the market trade function or simply trade function below):

$$MTV \cong C^T \cdot p^m \cdot q^m \cdot f^0, \tag{22}$$

where f^0 is, by definition, the following overlap integral:

$$f^0 = \int_{-\infty}^{+\infty} f(p,q)\, dp\, dq = \int_{-\infty}^{+\infty} g^D(p,q) \cdot g^S(p,q)\, dp\, dq. \tag{23}$$

Eqs. (21)–(23) are used below as a basis for the quantitative estimation of the trade functions.

It is rather difficult to calculate the trade functions in the general case. However, to do it in the case of our simple model is fairly easy since all the integrals are reduced to the known table integrals that we took out of [8]. Moreover, in order to achieve maximum simplicity and clarity, we present the final result for the simpler case where the respective S&D frequency parameters are equal ($w^{DP} = w^{SP} = w^P$, $w^{DQ} = w^{SQ} = w^Q$). This, in our view, does not significantly reduce the generality of the conclusions obtained and discussed below. This is because we are only interested in the main details of the process of transition of the market economy to the stationary equilibrium state. So, the formula for calculating the trade function MTV for our simplest model is as follows:

$$MTV\left(p^D, p^S, q^D, q^S\right) = C^T \cdot \frac{1}{4}\left(p^D + p^S\right)\left(q^D + q^S\right)\sqrt{w^P/2\pi}\,\sqrt{w^Q/2\pi}\, \times$$

$$\times \exp\left\{-\frac{1}{2} w^P \left(p^D - p^S\right)^2\right\} \exp\left\{-\frac{1}{2} w^Q \left(q^D - q^S\right)^2\right\} =$$

$$= C^T \cdot a \cdot \frac{1}{4}\left(p^D + p^S\right)\left(q^D + q^S\right) \cdot \frac{1}{\Gamma^P} \cdot \frac{1}{\Gamma^Q} \times$$

$$\times \exp\left\{-\pi a \frac{1}{(\Gamma^P)^2}\left(p^D - p^S\right)^2\right\} \exp\left\{-\pi a \frac{1}{(\Gamma^Q)^2}\left(q^D - q^S\right)^2\right\}. \quad (24)$$

If we perform similar calculations of the trade function *MTV*, starting from the more approximate formula (22), we obtain, of course, a somewhat simpler expression:

$$MTV\left(p^D, p^S, q^D, q^S\right) \cong C^T \cdot p^m \cdot q^m \cdot \sqrt{w^P/2\pi}\,\sqrt{w^Q/2\pi} \times$$

$$\times \exp\left\{-\frac{1}{2} w^P\left(p^D - p^S\right)^2\right\} \exp\left\{-\frac{1}{2} w^Q\left(q^D - q^S\right)^2\right\} =.$$

$$= C^T \cdot a \cdot p^m \cdot q^m \cdot \frac{1}{\Gamma^P} \cdot \frac{1}{\Gamma^Q} \times$$

$$\times \exp\left\{-\pi a \frac{1}{(\Gamma^P)^2}\left(p^D - p^S\right)^2\right\} \exp\left\{-\pi a \frac{1}{(\Gamma^Q)^2}\left(q^D - q^S\right)^2\right\}, \quad (25)$$

where the numeric constant a is calculated by the following simple formula:

$$a = -4 \ln 0{,}5 / 2\pi \cong 0.44, \quad (26)$$

and we have used the relationships between the S&D frequency and widths parameters (13) and (17) obtained above.

5.1.2. THE UNCERTAINTY LAW

It is very important that we can regard Eqs. (24) and (25) as the mathematical relationships between the economic efficiencies (trade volumes *MTV*) and the economic uncertainties (S&D widths Γ, see Eqs. (13), (17)). We refer to such inverse relationships between the efficiencies and uncertainties as *the uncertainty law*. The general economic sense of the uncertainty law implies generally that the economic efficiency depends inversely on uncertainty in economy. For instance, it is particularly easy to see from these relationships that the smaller the uncertainty in economic life, the higher the economic efficiency of markets and vice versa. This statement is rather widely known in real market life.

5.1.3. AGENT STRUCTURE OF THE STATIONARY EQUILIBRIUM STATE

Time has come to make another serious step in building probability economics and come to know the particulars of the agent structure of the stationary equilibrium states, i.e., structure of all agent and market S&D functions of the economy being in such states. Our market experience here above has led us to put forward the following hypothesis about the functioning of the modern real markets: the market-based trade maximization principle. In doing this, we believe that the overall driving motive of all market participants is that they strive for some reasonable trade volume maximum, which is optimal from the point of view of the majority of all participants. We have also assumed that this intuitive market movement can be mathematically described with a reasonably good degree of accuracy using the trade maximization principle. In our particular case, the application of this principle means finding the optimal values of the agent prices and quantities, p^D, p^S, q^D, and q^S, which maximize the trade function $MTV(p^D, p^S, q^D, q^S)$. It is easy to see from Eqs. (24) and (25) that the direct application of the trade maximization principle gives rise to a tendency in the market toward equalizing the S&D prices and quantities, as well as the total S&D as follows:

$$p^D = p^S = p^E, \quad q^D = q^S = q^E, \quad D^0 = S^0 = D^E = S^E. \qquad (27)$$

Eq. (27) describes the price and quantity structure of the equilibrium state or conventionally *the market equilibrium conditions* in economy.

The above-mentioned means that the agent striving for the market trade volume maximum leads the economy and market to the stationary equilibrium state, as we had expected very early on in the study. Theoretically, this transition phenomenon has been proved to be linked to certain mathematical properties of the trade function. Eq.(27) describes the basic elements of the agent structure of the market equilibrium state. The finer details of this structure will be identified below. In such stationary equilibrium states both Eqs.(24) and (25) for the trade function take the same simpler form as follows:

$$MTV\left(p^E,\ p^E,\ q^E,\ q^E\right) = C^T \cdot p^E \cdot q^E \cdot \sqrt{w^P/2\pi}\,\sqrt{w^Q/2\pi} =$$

$$= C^T \cdot a \cdot p^E \cdot q^E \cdot \frac{1}{\Gamma^P} \cdot \frac{1}{\Gamma^Q}. \quad (28)$$

Before entering into the detailed discussion of the nature of the studied transition market process, it is necessary to emphasize some very basic facts.

Firstly, the economic system can only really come into the stationary equilibrium state in the absence of strong, new, external perturbations on the market during the transition market process. Agents must adapt to the new realities of their existence and actions.

Secondly, the pure market equilibrium states, in principle, are only possible or even conceivable in the theory in the PQ-space, specifically within the framework of the GNP model, implying that price and quantity variables are independent ones. In practice, this means that in order to achieve the equilibrium state, agents should change not only their prices, but also their quantities of goods to be offered for sale or purchase. In the narrower price space (P-space) or within the SP model [1] the market cannot reach the equilibrium state, in principle.

Thirdly, *the market equilibrium conditions* (27) formally indicate, in this model case, the full equality of S&D functions that we will refer below to as *the market equilibrium equality*:

$$D(p, q) = S(p, q). \quad (29)$$

In terms of probability economics, the market equilibrium equality (29) is the most capacious expression to determine the main features of the stationary equilibrium state of the economy. It can even be taken as the mathematical description of the very concept of the market equilibrium as a market state in which the maximum coincidence or even equality of the market S&D functions is achieved.

Fourthly, it is very important also to understand that, aside from the market equilibrium equality (29), there is an additional prominent peculiarity of the transition market process to the equilibrium state. This is referred to as *the S&D narrowing effect* which we already mentioned while analyzing the nature of the S&D laws in Section 4. Clearly, this effect stems from the trade maximization principle too.

Let us now treat the S&D narrowing effect mathematically. Evidently, we can make the following important conclusions from Eqs.

(24) and (25). The smaller the S&D widths, the bigger the trade volume in the market is. In other words, the trade maximization principle leads gradually and eventually not only to equality of the market S&D functions, but also to their narrowing. In the limit, this means that both agent and market S&D functions tend in their form towards the so-called delta-functions. In other words, in the stationary equilibrium state, all the agent and market strategies become so certain that they show up as merely point strategies (pictorially, as points in the PQ-space, see [4]). This is understandable, since in the stationary equilibrium state all buyers in essence *a priori* know how many to buy and at what price, and sellers know how many to sell and at what price. In any other stationary state, the external perturbations and influences become a force to be reckoned with and agents need to act in a different way. Their actions may by not seem very effective from an economic point of view, but are quite rational in accordance with external circumstances.

So, we can make the following very important conclusion; there is a minimum of uncertainty and, as a consequence, a maximum of trade volume and hence a maximum of economic efficiency in the stationary equilibrium states of economy. By stretching a point, we could also say that there is a maximum of order and hence a minimum of entropy in the stationary equilibrium states. Moreover, as we stated above, the economies seek the stationary equilibrium states, and therefore we could say that the economies tend to have maximum order and minimum entropy in order to achieve maximum trade volume and maximum economic efficiency.

5.1.4. THE EQUILIBRIUM RULE "SELL ALL — BUY AT ALL"

It is easy to see that Eqs. (27) possess an infinite number of solutions as market equilibrium states, and each of those can theoretically be implemented in practice by the agents. Which of those will be realized depends on many specific details associated with the external circumstances in the market and then, which is also very important, on the stationary state of the economy at the initial time of t_0. However, the overall desirable market equilibrium state, or the most probable market equilibrium state, can be roughly quantified by means of the trade maximization principle as follows. As a specific example, we consider our model with the economic restrictions relieved by the state at the initial time t_0 with the following initial S&D parameters:

$$p^D \neq p^S,\ q^D \neq q^S,\ \text{i.e., } D^0 \neq S^0;$$
$$\text{for example } S^0 > D^0, \text{ with } p^D < p^S, \text{ and } q^D < q^S.$$

Just such an example is presented in Fig. 2 where the agent S&D parameters are as follows: $p^D = 280{,}0$ *\$/ton*, $q^D = 50{,}0$ *ton/year*, $D^0 = 14\,000{,}0$ *\$/ton* $p^S = 285{,}0$ *\$/ton*, $q^S = 52{,}0$ *ton/year*, $S^0 = 14\,820$ *\$/year*. As we saw in Section 3.2, the main empirical fact underlying the S&D laws is that on the thoroughly free or competitive markets, there is a strong sellers' dictate with regard to the good quantities (all goods should be sold by sellers) and there is a buyers' dictate with regard to the money amounts (the amount of money that should be spent to attain every good as originally scheduled). We believe these dictates determine together the most likely trend in the market: all the goods are sold, and all the money is spent. This can be briefly formulated as *the equilibrium rule* "Sell all — Buy at all". Literally, the equilibrium rule is interpreted in our simplest economy as follows. The seller with the total supply $S^0 = p^S q^S$ is just forced to sell all purchased or fabricated goods in quantity q^S and the buyer with the total demand $D^0 = p^D q^D$ will simply buy the goods at the full sum of money planned in advance for buying this good. At the limit, the seller simply has to adjust his or her price towards the following *equilibrium rule price*:

$$p^E \approx D^0 / q^s. \tag{30a}$$

At this equilibrium rule price, the buyer simply will not be able to resist the temptation to buy up all the goods at such a low price. As a result of this process, the maximum possible trade volume is achieved in the market. The achieved trade volume in the market, *MTV*, in monetary terms, will be equal to the amount the buyer(s) planned in advance to allocate towards this goodas follows:

$$MTV = p^E \times q^E \approx p^E \times q^S \approx D^0 / q^s. \tag{30b}$$

Thus, we used the trade maximization principle here in two steps. First we achieved equality between S&D, and then we evaluated the equilibrium price p^E and quantity q^E by means of the equilibrium rule. In our specific example, the equilibrium rule gives a reasonable estimate of the equilibrium price and quantity: $p^E = 269{,}2$ *\$/ton*, $q^E = 52{,}0$ *ton/year*, $MTV = 13\,998$ *\$/year*.

5.1.5. THE UNCERTAINTY RELATION FOR SUPPLY AND DEMAND

Remember that, within the framework of probability economics, and more specifically the GNP model, S&D functions for each market agent and for each good should be, with a sufficiently high degree of accuracy, presented as the product of two continuous Gaussian functions of price and quantity, correspondingly. Each should have four parameters, these being the desired price and quantity of the goods, as well as two widths of these price and quantity functions. It is very important to recognize that these S&D functions' widths can altogether serve as a measure of a general uncertainty in the economy. By an analogy with physics, it can be said that this is a direct consequence of the uncertainty and probability principle in economics [1–4] (see above). According to this principle, the S&D functions should be described as probability distributions in the PQ-space, but not like some trajectories in the form of dependences $p(q)$ or $q(p)$, as is the custom to do in the traditional model of neoclassical theory, for example. It is clear that this principle should be the starting point in constructing any probabilistic economic theory. It is also clear that, by an analogy with physics, in such a probabilistic theory, an uncertainty relation should also appear in one form or another.

To obtain the uncertainty relation in our model, we rewrite Eq. (28) in terms of the total S&D as follows:

$$MTV\left(p^E, p^E, q^E, q^E\right) = C^T \cdot a \cdot D^E \cdot \left(\Gamma^P \cdot \Gamma^Q\right)^{-1}, \tag{31}$$

where all parameters are apparent in their origin and meaning, and do not require further comments, and the numeric constant a is calculated by the formula (26).

Further, it is clear that in order for the formula (31) to give approximately correct numerical results, i.e, results which are consistent with the common economic sense, namely:

$$MTV\left(p^E, p^E, q^E, q^E\right) \cong D^E, \tag{32}$$

we need to take the following equality as a normalization equation for the trade function in the stationary equilibrium state:

$$C^T \cdot a \cdot \left(\Gamma^P \cdot \Gamma^Q\right)^{-1} \cong 1. \tag{33}$$

We rewrite it in the following form, more understandable and conventional in physics:

$$\Gamma^P \cdot \Gamma^Q \cong \alpha, \tag{34}$$

where the parameter α is calculated simply by:

$$\alpha = \frac{2}{\pi} \cdot (-\ln 0{,}5) \cdot C^T = a \cdot C^T. \tag{35}$$

We will refer to Eq. (34) below as *the uncertainty relation*, since it connects the S&D widths which, by their definition, reflect the uncertainties in the choice of market agents of their own desired price and quantities. The economic meaning of the uncertainty relation is fairly easy to ascertain. The widths of the price and quantity functions are not completely independent; the smaller the price function width (i.e., the less uncertainty in the choice of the price by the agents), the greater should be the widths of the quantity functions (that is, the more uncertainty in the choice of the quantity by the agents) and vice versa. More simply, if you want to sell a good at a fixed price, you cannot know in advance how much good you will sell in the market in the end. On the contrary, if you want to sell a certain quantity of a good you should be prepared for it not to be immediately clear the eventual price at which each item will really be sold in the market.

Thus, in order to achieve the stationary equilibrium state, the transition market process must go in a certain way; agents should not only narrow their S&D functions but also choose such S&D price and quantity widths in time that meet the uncertainty relation (34). Formally, it is as if the market agent were pushed by the uncertainty relation (more exactly, by the trade maximization principle, of course) to the right choice of the widths. Of course, for the real world, these theoretical conclusions can have only qualitative imperative meaning; this is only a main trend that can fall in one's favor only under the many conditions and assumptions mentioned above in this paper. Nevertheless, the practical significance of the uncertainty relation should not be understated, either. It is also well worth knowing the real-life market details of the going market process and, thereby, discovering and predicting the most likely trends in the economy. Knowledge of the uncertainty relation provides each market agent the possibility, in particular, of not fighting in vain against the market trends. Instead, it allows them to adapt to these trends by selecting a more optimal strategy, in particular the S&D widths, at a time when one must take into consideration the important market solutions.

Note, that we can consider the parameter α as the "universal" constant of the studied economy and use the following equation for calculation the unknown normalization constant C^T:

$$C^T = \alpha/a. \tag{36}$$

5.2. The Two-Good, One-Buyer, One-Seller Markets

In order to demonstrate how the equilibrium is established in many-good markets, we will carry out detailed calculations for the simplest model market case with one buyer and one seller, trading two different goods. Formulas for many-agent, many-good markets have been previously derived by us in [4]. Here, we adduce some of them for our particular case ($N = 1$, $M = 1$, $L = 2$. In brevity, indices n and m for agents are not specified in formulas explicitly, because all of them are equal to 1) as follows:

$$D(p_1, p_2, q_1, q_2) = C^D d^P(p_1, p_2)\, d^Q(q_1, q_2) \equiv C^D d(p_1, p_2, q_1, q_2), \tag{37}$$

$$S(p_1, p_2, q_1, q_2) = C^S s^P(p_1, p_2)\, s^Q(q_1, q_2) \equiv C^S s(p_1, p_2, q_1, q_2). \tag{38}$$

All price and quantity functions, i.e., functions in P- and Q-subspaces, are normalized to 1 by definition here:

$$\int_{-\infty}^{+\infty} d^P(p_1, p_2) dp_1 dp_2 = 1; \quad \int_{-\infty}^{+\infty} d^Q(q_1, q_2) dq_1 dq_2 = 1; \tag{39}$$

$$\int_{-\infty}^{+\infty} s^P(p_1, p_2) dp_1 dp_2 = 1; \quad \int_{-\infty}^{+\infty} s^Q(q_1, q_2) dq_1 dq_2 = 1. \tag{40}$$

Also, by definition, the functions $d(p_1, p_2, q_1, q_2)$ и $s(p_1, p_2, q_1, q_2)$ are S&D agent functions in the PQ-space normalized to 1:

$$\int_{-\infty}^{+\infty} d(p_1, p_2, q_1, q_2)\, dp_1 dp_2 dq_1 dq_2 = 1; \tag{41}$$

$$\int_{-\infty}^{+\infty} s\,(p_1, p_2, q_1, q_2)\, dp_1 dp_2 dq_1 dq_2 \;=\; 1. \qquad (42)$$

And the market S&D functions are normalized, as usual, to the total S&D as follows:

$$\int_{-\infty}^{+\infty} D(p_1, p_2, q_1, q_2)\, dp_1 dp_2 dq_1 dq_2 \;=\; D^0; \qquad (43)$$

$$D^0 \;=\; D_1^0 + D_2^0;\;\; D_1^0 \;=\; p_1^D \cdot q_1^D;\;\; D_2^0 \;=\; p_2^D \cdot q_2^D;\;\; C^D \;=\; D^0; \qquad (44)$$

$$\int_{-\infty}^{+\infty} S(p_1, p_2, q_1, q_2)\, dp_1 dp_2 dq_1 dq_2 \;=\; S^0; \qquad (45)$$

$$S^0 \;=\; S_1^0 + S_2^0;\;\; S_1^0 \;=\; p_1^S \cdot q_1^S;\;\; S_2^0 \;=\; p_2^S \cdot q_2^S;\;\; C^S \;=\; S^0. \qquad (46)$$

All designations in these formulas have an obvious meaning and do not require special comments. Details can be found in [4]. Further, we can conveniently express all functions through the Gaussians as follows [4]:

$$d^P(p_1, p_2) = g_1^{DP}(p_1) g_2^{DP}(p_2); \qquad d^Q(q_1, q_2) = g_1^{DQ}(q_1) g_2^{DQ}(q_2); \qquad (47)$$

$$s^P(p_1, p_2) = g_1^{SP}(p_1) g_2^{SP}(p_2); \qquad s^Q(q_1, q_2) = g_1^{SQ}(q_1) g_2^{SQ}(q_2). \qquad (48)$$

We now turn to the most interesting point. We define the trade volume operator in the manner clear from previous discussions as follows:

$$TV(p_1, p_2, q_1, q_2) \equiv p_1 \cdot q_1 + p_2 \cdot q_2. \qquad (49)$$

Accordingly, the trade function is defined as the same average value of this operator, as in the case of the market with one good:

$$MTV = \left(C^T\right)^2 \int_{-\infty}^{+\infty} d\,(p_1, p_2, q_1, q_2)\, TV(p_1, p_2, q_1, q_2) \times$$

$$\times\, s\,(p_1, p_2, q_1, q_2)\, dp_1 dp_2 dq_1 dq_2. \qquad (50)$$

Due to the fact that all the functions in our model are represented as products of Gaussians, all integrals in (50) are relatively easily computed in analytical form using the table integrals [8], although the formulas obtained are quite bulky. Therefore, here we provide results for a particular case, in which the price and quantity widths of the buyer and the seller for each good are equal, just as we did above for the case of one good on the market. Then the result becomes quite visible and economically interpretable. So,

$$MTV \cong \left(C^T\right)^2 (I_1 \times I_2 + I_3 \times I_4), \tag{51}$$

where the integrals in brackets are as follows:

$$I_1 = \frac{1}{4}\left(p_1^D + p_1^S\right)\left(q_1^D + q_1^S\right)\sqrt{w_1^P/2\pi}\,\sqrt{w_1^Q/2\pi}\,\times$$
$$\times \exp\left\{-\frac{1}{2}w_1^P\left(p_1^D - p_1^S\right)^2\right\} \times \exp\left\{-\frac{1}{2}w_1^Q\left(q_1^D - q_1^S\right)^2\right\}, \tag{52}$$

$$I_2 = \sqrt{w_2^P/2\pi}\,\sqrt{w_2^Q/2\pi}\,\times$$
$$\times \exp\left\{-\frac{1}{2}w_2^P\left(p_2^D - p_2^S\right)^2\right\} \times \exp\left\{-\frac{1}{2}w_2^Q\left(q_2^D - q_2^S\right)^2\right\}, \tag{53}$$

$$I_3 = \frac{1}{4}\left(p_2^D + p_2^S\right)\left(q_2^D + q_2^S\right)\sqrt{w_2^P/2\pi}\,\sqrt{w_2^Q/2\pi}\,\times$$
$$\times \exp\left\{-\frac{1}{2}w_2^P\left(p_2^D - p_2^S\right)^2\right\} \times \exp\left\{-\frac{1}{2}w_2^Q\left(q_2^D - q_2^S\right)^2\right\}, \tag{54}$$

$$I_4 = \sqrt{w_1^P/2\pi}\,\sqrt{w_1^Q/2\pi}\,\times$$
$$\times \exp\left\{-\frac{1}{2}w_1^P\left(p_1^D - p_1^S\right)^2\right\} \times \exp\left\{-\frac{1}{2}w_1^Q\left(q_1^D - q_1^S\right)^2\right\}. \tag{55}$$

The analysis of the formulas shows that there are many local equilibrium states of the market, where for some goods the equilibrium conditions are fulfilled, and for others they are not. But there is one unique equilibrium state in which the equilibrium conditions of the type (26) are met for all goods simultaneously. We will refer to such market equilibrium states will be referred by us to as normal, since it is clear that all of the normal economies aspire to exactly such states. So, the price and quantity structure of the normal market equilibrium state is described by *the market equilibrium conditions* as follows:

$$p_1^D = p_1^S = p_1^E,\ q_1^D = q_1^S = q_1^E,\ \ D_1^0 = S_1^0 = D_1^E = S_1^E; \tag{56}$$

$$p_2^D = p_2^S = p_2^E,\ q_2^D = q_2^S = q_2^E,\quad D_2^0 = S_2^0 = D_2^E = S_2^E. \tag{57}$$

In this case, the formula for calculating the trade volume takes the following form:

$$MTV \cong \left(C^T\right) \cdot (1/2\pi)^2 \cdot \sqrt{w_1^P w_1^Q w_2^P w_2^Q} \times \left(p_1^E q_1^E + p_2^E q_2^E\right). \tag{58}$$

To ensure that this expression (*the uncertainty law* for the given economy) gives reasonable quantitative results, we choose the natural normalization for it in the form of the following uncertainty relation for our two-good economy:

$$\left(C^T\right)^2 \cdot (1/2\pi)^2 \cdot \sqrt{w_1^P w_1^Q w_2^P w_2^Q} = 1. \tag{59}$$

Taking into account these relations, we obtain the final formulas for the trade volume as follows:

$$MTV = p_1^E q_1^E + p_2^E q_2^E, \tag{60}$$

or

$$MTV = D^0 = D_1^E + D_2^E = S^0 = S_1^E + S_2^E. \tag{61}$$

For the interpretation of the uncertainty relation for the two-good market, we rewrite it in terms of the S&D widths in such a manner:

$$\Gamma_1^P \Gamma_1^Q \Gamma_2^P \Gamma_2^Q = \alpha^2. \tag{62}$$

This relation is amenable to the most natural interpretation. Instead of one Eq. (32) in the case of the one-good market we have in fact for the two-good market the two analogous uncertainty relations for each good separately:

$$\Gamma_1^P \Gamma_1^Q = \alpha,\quad \Gamma_2^P \Gamma_2^Q = \alpha. \tag{63}$$

To avoid misunderstandings we emphasize that the parameters б and C^T are different for different economies and markets, by their meanings and definitions. In principle, they can be determined empirically by adjusting numerical results of the theory under the experiment, but we have to understand that their numerical values do not play an important role in applied economic research. They are interesting only from a theoretical or a fundamental point of view.

5.3. The Many-Good, Many-Agent Markets

Before going into a detailed description of the agent structure of the equilibrium states of the many-agent and many-good markets, we naturally generalize the formula (21) for these markets as follows:

$$MTV \equiv C^T \cdot \int_{-\infty}^{+\infty} d(\boldsymbol{p},\boldsymbol{q}) \cdot TV(\boldsymbol{p},\boldsymbol{q}) \cdot s(\boldsymbol{p},\boldsymbol{q})\, d\boldsymbol{p}\, d\boldsymbol{q} =$$

$$= C^T \cdot \int_{-\infty}^{+\infty} f(\boldsymbol{p},\boldsymbol{q}) \cdot TV(\boldsymbol{p},\boldsymbol{q})\, d\boldsymbol{p}\, d\boldsymbol{q}. \qquad (64)$$

By definition, functions in Eq. (64) are as follows:

$$D(\boldsymbol{p},\boldsymbol{q}) \equiv D^0 d(\boldsymbol{p},\boldsymbol{q}),\ S(\boldsymbol{p},\boldsymbol{q}) \equiv S^0 s(\boldsymbol{p},\boldsymbol{q}),\ f(\boldsymbol{p},\boldsymbol{q}) \equiv d(\boldsymbol{p},\boldsymbol{q}) \cdot s(\boldsymbol{p},\boldsymbol{q}). \qquad (65)$$

For brevity, in (64) and (65) we use only one bold letter, $\boldsymbol{p}$, to designate all L prices p_l (now $\boldsymbol{p}$ is already a vector in the L-dimensional P-subspace) and one bold $\boldsymbol{q}$ — all L quantities q_l (now $\boldsymbol{q}$ is already a vector in the L-dimensional Q-subspace). Generalization of formulas (20) and (49) for the trade volume operator can also be evidently done as follows:

$$TV(\boldsymbol{p},\boldsymbol{q}) \equiv \boldsymbol{p} \cdot \boldsymbol{q} = \sum_{l=1}^{L} p_l \cdot q_l. \qquad (66)$$

The same can be said about the approximate formula (22):

$$MTV \cong C^T \cdot \left(\sum_{l=1}^{L} p_l^m \cdot q_l^m \right) \cdot f^0, \qquad (67)$$

where f^0 is by definition the following S&D overlapintegral:

$$f^0 = \int_{-\infty}^{+\infty} d(\boldsymbol{p},\boldsymbol{q}) \cdot s(\boldsymbol{p},\boldsymbol{q})\, d\boldsymbol{p}\, d\boldsymbol{q}. \qquad (68)$$

The results, obtained earlier for the one-buyer, one-seller, two-good markets, allow us to understand what is happening in similar economic circumstances on many-agent, many-good markets in general cases. Naturally assume that in order to maximize the trade volume, market agents seek to get closer to the normal market equilibrium state, in which all agent prices and quantities are approximately equalizing. Among other things, this also means that as the equilibrium state is approached, the market S&D functions become similar to Gaussians. In other words, we believe that, near the normal equilibrium state, all buyers behave as a single buyer, and all sellers — as a single seller. In this case, the market S&D functions can be obtained approximately by multiplying the conventional price and quantity Gaussians for each good with the prices p_l^D, p_l^S and quantities q_l^D, q_l^S respectively. The price and quantity structure of the normal market equilibrium state can be described by *the market equilibrium conditions* as follows

$$p_l^D = p_l^S = p_l^E,\ q_l^D = q_l^S = q_l^E,\ D_l^0 = S_l^0 = D_l^E = S_l^E;\ l = 1, \dots . L. \qquad (69)$$

Obviously, in this case *the market equilibrium equalities* are now valid, too, which assert that there are equalities of both the total S&D and the market S&D functions in the normal equilibrium states as follows:

$$D^0 = S^0,\ D(\boldsymbol{p}, \boldsymbol{q}) \equiv S(\boldsymbol{p}, \boldsymbol{q}). \qquad (70)$$

And to calculate the trade function we can naturally apply the following approximate formula:

$$MTV \cong \sum_{l=1}^{L} p_l^E \cdot q_l^E. \qquad (71)$$

So we now have the natural generalization of the equilibrium equations obtained above in previous sections for the more general case of many-agent, many-good markets. Concluding the section note that we can also use the main market rule “Sell all — Buy at all” discussed above to estimate equilibrium prices, p_l^E, and quantities, q_l^E, in this case of many-agent, many-good markets.

6. Conclusions

To summarize, in this chapter we have used probability economics as a tool of unification of the market process concept of Austrian economics, and the supply and demand laws and equilibrium state concepts of neoclassical economics. The market-based trade maximization principle has been proposed to serve as a connection link among of all these three concepts.

In conclusion, note that conceptually, probability economics can be regarded as a phenomenological version of probabilistic economic theory developed in this book. This theory is designed to give the mathematical description of such well-known fundamental concepts of classical economic thought as S&D, market price and market force, market process and market equilibrium and so forth. Below, in Part F, we will show that probability economics can serve the base for the construction of the more advanced *ab initio* version of probabilistic economic theory: quantum economy. The latter has the more developed mathematical apparatus, namely equations of motion, for quantitative studies of dynamic phenomena in the economic systems.

References

1. Anatoly Kondratenko. *Probability Economics: Supply and Demand in the Price Space*. Electronic copy available at: http://ssrn.com/abstract=2250343. See also Chapter IV.
2. Anatoly Kondratenko. *Probability Economics: Market Price in the Price Space*. Electronic copy available at: http://ssrn.com/abstract=2263708. See also Chapter V.
3. Anatoly Kondratenko. *Probability Economics: Market Force in the Price Space*. Electronic copy available at: http://ssrn.com/abstract=2270306. See also Chapter VI.
4. Anatoly Kondratenko. *Probability Economics: Supply and Demand, Price and Force in the Price — Quantity Space*. Electronic copy available at: http://ssrn.com/abstract=2337462. See also Chapter VII.
5. Anatoly Kondratenko. *Physical Modeling of Economic Systems. Classical and Quantum Economies*. Nauka, Novosibirsk, 2005. Electronic copy available at: http://ssrn.com/abstract=1304630.
6. Anatoly Kondratenko. *Physical Economics: Stationary Quantum Economies in the Price — Quantity Space*. Electronic copy available at: http://ssrn.com/abstract=2363874. See also Chapter X.
7. Ludwig von Mises. *Human Action. A Treatise on Economics*. Published 1949 by Yale University.
8. I.S. Gradshteyn and I.M. Ryzhik. T*able of Integrals, Series and Products*. Nauka, Moscow, 1971.

PART F.
Quantum Economy

"Thus we are led to the insight that dealing with the uncertain conditions of the unknown future-that is, speculation-is inherent in every action, and that profit and loss are necessary features of acting which cannot be conjured away by any wishful thinking. The procedures adopted by those economists who are fully aware of these fundamental cognitions may be called the logical method of economics as contrasted with the technique of the mathematical method.

Ludwig von Mises. *Human Action. A Treatise on Economics.* Page 250

CHAPTER IX.

Quantum Economies in the Price Space

"In elementary physics, scientists know they cannot pretend they have certainty anymore; actually, uncertainty about which events happen to a particle is deeply rooted in the laws of quantum physics. The best way to express knowledge about a particle is through wave functions that describe the probability of its possible states, depending on the infinitely many sequences of interactions in which it may or may not participate. Of course, when it appears that a particle effectively interacted with other particles in an observable way, then this new knowledge corresponds to an update in the wave function that describes the particle (or set of particles); the technical term is that the wave packet collapses. In quantum physics, information has a cost, because you may only extract useful information through interaction, which not only costs you energy, but also modifies the system in ways that void other information.

Interestingly, at the same times that quantum physics revolutioned physics, thinkers of the Austrian school also revolutioned economics, by expliciting the notions of uncertainty, information costs, potential interactions, knowledge and use of knowledge, etc. Yet most economists are blissfully unaware of it — and make strenuous efforts to remain ignorant, because they invest their life in supporting and being supported by political intervention in the economy, which is systematically based upon the premise that the right information is magically available at no cost to the political planners.

All the neo-classical, keynesian and marxist models of the macro-economy neglect the intrinsic role of information in the nature of human action. The models that try to predict the future with nice simple numbers, aggregates. These aggregates, that are claimed to synthetize information about some economic activity, are static projections that by their very scalar nature cannot express the wide variety of possible outcomes depending on a growing series of interactions that are not known. Those who claim there is a determinism that makes for only one possible outcome neglect the relevance of future events; they deny not just the costs of acquiring information about relevant possible events, they also deny the very value of information. Those who don't claim determinism but still claim they know enough to define policies that must be enforced are

> dishonest or delusioned with visions of grandeur, thinking they are outside and above the crowd they claim to rule. Actually they are inside of it and at the bottom of it.
>
> If economics has to have any future as a science, it will probably be as Quantum Economics: a science describing human interrelations in terms of uncertain discrete transactions, where information matters, where it has a value and a cost; where it is acquired by interaction, and used by interaction; where it is unknown until the interaction happens, and where it is uncertain when the interaction will happen and what it will be, but where it is certain that the interaction will eventually happen, leading to an inevitable according reduction of the packet wave".
>
> François-René Rideau, 2007

In this chapter we used as epigraph thesis of French programmer François-René's Rideau, published by him in his blog LiVE JOURNAL in 2007. Thereby we give him credit for discovering the deep analogy between Austrian economics and quantum mechanics.

PREVIEW.

What are the Economic Schrödinger Equations?

In the previous Chapter we derived the economic Lagrange equations of motion that describe deterministic features of the economic dynamics in the price space. We understand that uncertainty and probability can permanently and strongly influence market agents' decision making that can result in strong probability effects in the dynamics of the economic systems. In this chapter, we take explicitly into account the impact of these effects on the dynamics of many-agent market economic systems by analogy with quantum mechanics of the formal many-particle systems. As an outcome of the above work, we now have the economic Schrödinger equations of motion in the price space. Roughly speaking, those are the quantum extension of the economic Lagrange equations of motion and we can regard the respective Hamiltonian as the mathematical quantum representation of the market invisible hand concept in the price space.

1. Introduction

The contemporary markets, as a rule, are very large, well developed and rather fast-paced, in which a fundamental role is played, from one side, by the high-rate exchange of a large volume of the relevant market information. But from other side, at the present the market agents do not have in their disposal the reliable quantitative methods of processing and forecasting the course of events on the market over the reasonable long term. This leads to the fact that, on the contemporary markets, market agents are compelled to permanently work under conditions of large uncertainty and to make decisions that will be primarily probabilistic in nature. Therefore, adequate and precise quantitative economic theories must include the concepts of uncertainty and probability. As we know, in the real space such probabilistic phenomena in physical systems are described with the aid of "probabilistic physical theory" (although this is a term is not really used in physics), namely quantum mechanics, using the wave functions and Schrödinger equations. Below, we explore an analogous approach to further develop probabilistic economic theory, more exactly, quantum economy in the price space found in [1].

Recall that the principal point in developing a physical model is the selection of a function which will help to describe economy dynamics, or more specifically, the movement of buyers and sellers in price space. For this type of function in classical economy we have chosen the agent's trajectory in price space $p(t)$. Here, we suppose implicitly that the behavior or movement of the market agent results in the establishment of a price for goods and commodities at every point of time by negotiations or information exchange both between economic agents and agents with the external environment. It appears that economic agents adjust their trajectories to each other based on their principal concerns for their own and common profit that leads to some determination in their behavior or movement in the price space. In the previous Chapter III, we obtained at least non-contradictory classical economies with the economic Lagrange equations of motion.

In the present chapter, we will speak of another quantum method used for describing movement or the behavior of market agents. This method is based on the use of a probabilistic approach to describe movements of market agents by means of wave functions and related to that, distributions of sale transactions probabilities. Agents, displaying this type of behavior, will be referred to below as *homo oscillans* (an oscillating man). The quantum method used to describe market agents' movement leads to the probabilistic mechanism of forming of functions of supply and demand (S&D below) as well as a market price at which sale transactions are carried out in the market. Quantum equations of motion and interpretation of their solutions for quantum economies will, of course, be different from those in classical economies. It should be noted that there is an analogous situation in physics.

2. Foundations of Quantum Economy

2.1. The Time-Dependent Economic Schrödinger Equation

When deriving equations of motion for quantum economy we will use the same scenario as when we derived equations of motion for classical economy; we will make the same assumptions as the ones for equations of motion for physical systems — prototypes in quantum mechanics in physical space. Because of this, we get the same equations of motion for economy in the price space. Future investigations will analyse the assumptions we have made, and gradually refine the model. This will be done by way of detailed calculations of equations solutions, by the use of different parameters and interaction potentials, and by detailed comparison with experimental data or those observed in practice.

When deriving equations of motion, we follow the procedure described in the famous book by Landau and Lifshitz [2]. First of all, let us assume that economy can be described to a desired degree of accuracy by a certain function in price space $\Psi(\boldsymbol{p})$, where the bold letter $\boldsymbol{p}$ as before designates a set of all price coordinates of all economic agents. The squared module of this function determines distribution of price probabilities: $|\Psi|^2\, d\boldsymbol{p}$ stands for a probability that economic agents will have prices in element $d\boldsymbol{p}$ of the price space. Functions Ψ in quantum mechanics are called wave functions. So, it is natural to introduce a normalization condition for finite quantities in the whole space:

$$\int |\Psi|^2\, d\boldsymbol{p} = 1. \tag{1}$$

It is also natural to determine that for the economy consisting of two independent economies or sub-economies 1 and 2, total wave function of economy is a product of sub-economy wave functions:

$$\Psi_{21}(\boldsymbol{p}_1, \boldsymbol{p}_2) = \Psi_1(\boldsymbol{p}_1)\Psi_2(\boldsymbol{p}_2). \tag{2}$$

By analogy with quantum mechanics for physical systems we consider that the wave function of economy does not only describe economy behavior at a given point of time t_0, but also determines its behavior or evolution in all future moments of time. It means that along with some other assumptions that are not considered in this paper, the equations of motion for economy can be written in the form of the so-called wave equation as follows:

$$i\alpha \frac{\partial \Psi}{\partial t} = \hat{H}\Psi. \quad (3)$$

In (3) $\hat{H}$ there is a certain linear Hermitian operator called a Hamilton operator, or a Hamiltonian, in quantum mechanics. If this Hamiltonian is known, then wave equation (3) determines the wave functions of quantum economy at any point of time. Generally speaking, the Hamilton operator is unknown for economic systems, and this problem is for future consideration. In (3) i is a usual complex variable, and α is a certain constant which is also unknown. If in quantum mechanics this is a universal or Planck's constant $\hbar$, in quantum economics this is just a parameter which may be different for various economies, and its economic sense will be determined someday.

Hence, we have obtained the required equations of motion for economy in the wave equation form, with unknown constants α and Hamilton operator $\hat{H}$. This equation describes economy dynamics in time, so let us call it the time-dependent economic Schrödinger equation (the time-dependent Schrödinger equation in physics).

2.2. The Stationary Economic Schrödinger Equation

Let us introduce the notion of stationary states for particular cases of quantum economies under the influence of constant external fields, i.e., which do not depend explicitly on time. According to the definition, stationary states of economy are so called *eigenfunctions* of the Hamilton operator, which in this case does not depend on time either. In other words, stationary states are the solutions of the stationary economic Schrödinger equations of motion (the stationary Schrödinger equation in quantum mechanics):

$$\hat{H}\Psi_n = E_n\Psi_n. \quad (4)$$

In Eq. (4) E_n denotes eigenvalues of the Hamilton operator, or *energy eigenvalues* as they are called in quantum mechanics, or just the energy of states *n*. The time-dependent equations of motion are easily integrated for stationary states (or *eigenstates*):

$$i\alpha \frac{\partial \Psi_n}{\partial t} = \hat{H}\Psi_n = E_n \Psi_n,$$

$$\Psi_n = \exp\left(-\frac{i}{\alpha} E_n t\right)\Psi_n(p), \tag{5}$$

where $\Psi_n(\boldsymbol{p})$ designates functions of prices only, i.e., they do not depend on time. Note that in contrast to classical economy prices $\boldsymbol{p}$ are independent variables in quantum economy, just as in probability economics. $\Psi_n(\boldsymbol{p})$ are also eigenfunctions of the Hamiltonian:

$$\hat{H}\Psi_n(\boldsymbol{p}) = E_n \Psi_n(\boldsymbol{p}). \tag{6}$$

Eq. (6) is also called stationary economic Schrödinger equation of motion (the stationary Schrödinger equation in physics). Eq. (5) shows the time dependence of stationary states. Every wave function can be presented as a linear combination of stationary states:

$$\Psi = \sum_n a_n \exp\left(-\frac{i}{\alpha} E_n t\right)\Psi_n(\boldsymbol{p}) \tag{7}$$

where squared expansion coefficients $|a_n|^2$ denote probabilities of different states of the system.

Below, though with rare exception, we will consider only the ground or normal state of economy with the least energy E_0 where its index 0 will be omitted. The probability distribution in the stationary state is determined by $|\Psi|^2$:

$$|\Psi(\boldsymbol{p})|^2 = |\psi(\boldsymbol{p})|^2. \tag{8}$$

In other words, probability distribution does not depend on time.

If the Hamiltonian of economy represents a sum of two parts, e.g. a sum of Hamiltonians of buyers' and sellers' sub-economies:

$$\hat{H} = \hat{H}_1(\boldsymbol{p}_1) + \hat{H}_2(\boldsymbol{p}_2), \tag{9}$$

where one of them comprises only price coordinates $\boldsymbol{p}_1$, and the other — only price coordinates $\boldsymbol{p}_2$, so eigenfunctions of the Hamilton operator of the entire economy can be written in the following form:

$$\psi_{21}(\boldsymbol{p}_1, \boldsymbol{p}_2) = \psi_1(\boldsymbol{p}_1)\psi_2(\boldsymbol{p}_2), \tag{10}$$

$$E_{12} = E_1 + E_2, \tag{11}$$

$$\hat{H}\psi_{12}(\boldsymbol{p}_1, \boldsymbol{p}_2) = E_{12}\psi_{12}(\boldsymbol{p}_1, \boldsymbol{p}_2),$$

$$\hat{H}_1(\boldsymbol{p}_1)\psi_1(\boldsymbol{p}_1) = E_1\psi_1(\boldsymbol{p}_1),$$

$$\hat{H}_2(\boldsymbol{p}_2)\psi_2(\boldsymbol{p}_2) = E_2\psi_2(\boldsymbol{p}_2).$$

So ψ_1 and E_1 are the eigenfunction and eigenenergy of the buyers' sub-economy, ψ_2 and E_2 are the eigenfunction and eigenenergy of sellers' sub-economy. It is evident that in this case we consider interaction between buyers and sellers insignificant. To avoid misunderstandings we need to remember that although we use the word 'energy' for economic systems, this notion is not related in any way to physical systems and has a different dimension in comparison to the real one. The notion is only used to make a parallel between studied economy and physical prototype more convenient. The essence of energy in economic theory is to be revealed in the future. So if (10) takes place, then

$$|\psi_{12}(\boldsymbol{p}_1, \boldsymbol{p}_2)|^2 = |\psi_1(\boldsymbol{p}_1)|^2 \, |\psi_2(\boldsymbol{p}_2)|^2. \tag{12}$$

3. The One-Good, One-Buyer, One-Seller Markets

Let us go further and assume that if there is an economy with a single good (the case of the one-dimensional price space), the Hamiltonian of the one-buyer, one-seller market economy with price coordinates p_i $(i = 1, 2)$ formally looks like a Hamiltonian of the physical system for two material particles in physical space (physical prototype of the economy), namely:

$$\hat{H} = -\frac{\alpha^2}{2m_1}\frac{\partial^2}{\partial p_1} - \frac{\alpha^2}{2m_2}\frac{\partial^2}{\partial p_2} - V_{12}(p_1, p_2) - U_1(p_1) - U_2(p_2). \tag{13}$$

In Eq. (13) m_1 and m_2 are certain economic masses (*a priori* unknown) of the buyer and the seller. Of course, they have nothing to deal with masses in physics. They have different dimensions and these can only be determined by experiment. Operators $\frac{\partial^2}{\partial p_1}$ and $\frac{\partial^2}{\partial p_2}$ are second-order operators of differentiation with respect to the price coordinates. Potential $V_{12}(p_1, p_2)$ is designed to describe the interaction between the buyer and the seller, and $U_1(p_1)$ and $U_2(p_2)$ are introduced into the theory to describe the influence of the external environment on economic agents. With the Hamiltonian selected using these means, equations of motion for our economy take the following form:

$$\left(-\frac{\alpha^2}{2m_1}\frac{\partial^2}{\partial p_1}-\frac{\alpha^2}{2m_2}\frac{\partial^2}{\partial p_2}-V_{12}(p_1,p_2)-U_1(p_1)-U_2(p_2)\right)\psi_{12}(p_1,p_2)=$$
$$=E_{12}\psi_{12}(p_1,p_2). \tag{14}$$

If we neglect the interaction between the buyer and the seller in comparison with the influence of the external environment, then the equations of motion are divided into a system of two independent equations:

$$\begin{cases}\left(-\frac{\alpha^2}{2m_1}\frac{\partial^2}{\partial p_1}-U_1(p_1)\right)\psi_1(p_1)=E_1\psi_1(p_1), & (15)\\ \left(-\frac{\alpha^2}{2m_2}\frac{\partial^2}{\partial p_2}-U_2(p_2)\right)\psi_2(p_2)=E_2\psi_2(p_2); & (16)\end{cases}$$

$$\psi_{12}(p_1,p_2)=\psi_1(p_1)\psi_2(p_2), \tag{17}$$

$$E_{12}=E_1+E_2. \tag{18}$$

Within the approximate self-consistent field approach [2], interaction V_{12} can be also considered to acquire an analogous system of related equations:

$$\begin{cases}\left(-\frac{\alpha^2}{2m_1}\frac{\partial^2}{\partial p_1}-U_1(p_1)-\tilde{V}_{12}(p_1)\right)\psi_1(p_1)=E_1\psi_1(p_1), & (19)\\ \left(-\frac{\alpha^2}{2m_2}\frac{\partial^2}{\partial p_2}-U_2(p_2)-\tilde{V}_{21}(p_2)\right)\psi_2(p_2)=E_2\psi_2(p_2). & (20)\end{cases}$$

In Eqs. (19) and (20) modified potentials $\widetilde{V}_{12}(p_1)$ and $\widetilde{V}_{21}(p_2)$ depend on the solutions $\psi_1(p_1)$ and $\psi_2(p_2)$. Note that all the equations obtained above can be generalized for systems consisting of many buyers and sellers in a straightforward manner.

3.1. Agent Probability Distributions And The Deal Function

Let us now introduce into the theory the important notions of probability distributions for the buyer and seller as follows:

$$d(p) = |\psi_1(p)|^2, \quad s(p) = |\psi_2(p)|^2. \tag{21}$$

We assume below a priori that the probability of arranging the deal by the agents at price p, $f(p)$, is directly proportional to the product of these two agent probability probabilities as follows:

$$f(p) = d(p) \cdot s(p). \tag{22}$$

This function $f(p)$ is by its very nature, correct to the normalization constants, the deal probability distributions, or simply the deal function $F(p)$ (see below).

On the basis of our experience in probability economics, we can expect that the many-agent market economies behave in the price space analogously to the polyatomic molecules in physical space. On these grounds, we suppose that, in the first approximation, we can describe the economic wave functions by means of the wave functions of the respective harmonic oscillators.

Based on practical market activity experience and common sense, we can conclude that interaction potential between the buyer and the seller must correspond to the “attraction” between them as the buyer and the seller gradually move towards each other along the price, trying to find an optimal market price according to the win-win market rule.

Figs. 1 and 2 present the typical agent probability distributions, bases on our experience in probability economics (see previous Chapters IV–VII).

It is evident that for these functions that external potential should be infinite when $p \leq 0$, a finite one when p is positive, and tends to zero when $p \to \infty$.

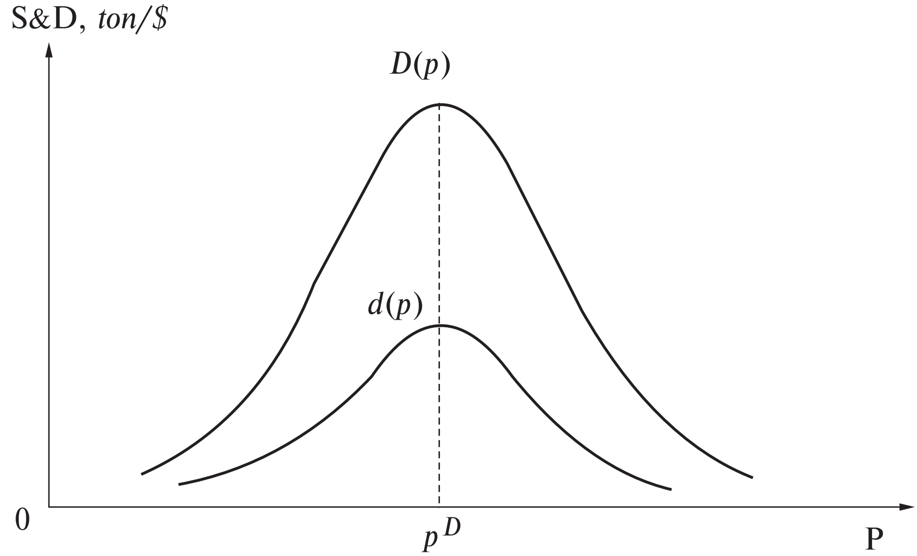

Fig. 1. The suggested form of the probability distribution of the buyer $d(p)$ and the corresponding demand function $D(p)$ in the normal state of the economy. Both functions are without nodes with one maximum at point p^D.

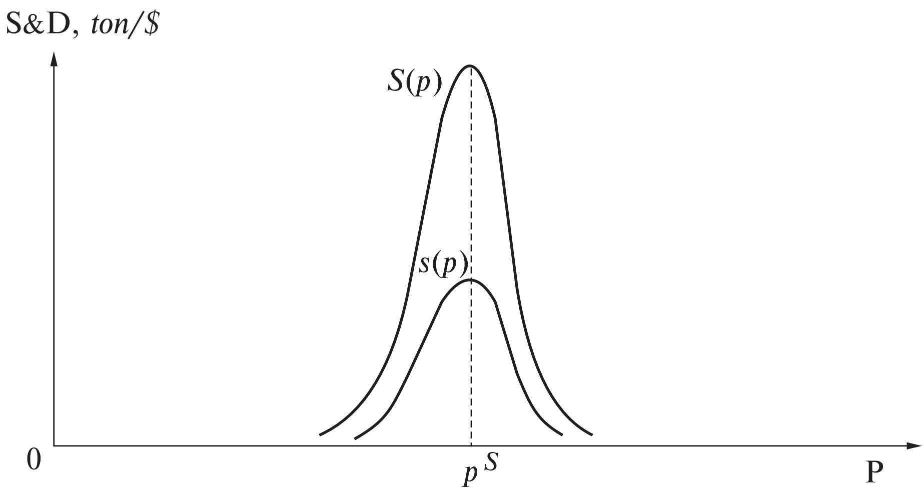

Fig. 2. The suggested form of probability distribution of the seller $s(p)$ and corresponding supply function $S(p)$ in the normal state of economy. Both functions are without nodes with one maximum at point p^S.

3.2. Supply and Demand Functions

Below we will try to answer the question on how the agent probability distributions of quantum economy are connected with empirical data. First, there are those with the functions of supply and demand (S&D below) that can be basically measured by methods of empirical economics. Here we need to make some assumptions. First of all, we assume *a priori* that the buyer's financial means and willingness to buy some goods for a sum of money D_0 (by definition, it is the total demand of the buyer) can be described by the previously specified probability distribution $d(p)$. Our experience in investigations of stationary states in probability economics, as well as common sense and intuition, suggest that the typical agent probability distribution for the buyer, $d(p)$, 'must' have the form as in Fig. 1: at zero, at infinity and in the region of negative prices, it equals zero. It is evident that it has one maximum at the most probable purchase price p^D. Analogously, we consider *a priori* that the seller's behavior in the market is completely determined by the seller's resources and desire to sell some goods for a sum of money S_0 (by definition, it is the total supply of the seller). Let us also assume that the probability that he or she will sell their goods at price p is described by seller's probability distribution $s(p)$. Experience in probability economics suggests that the typical distribution $s(p)$ behaves similarly to the distribution $d(p)$ (see Fig. 2). When $p \le 0$ and $p \to \infty$ the function tends to zero, and in the positive prices region has, it appears, a single maximum at the most probable sale price p^S.

Summing up, we can say that quantum economy (or the probabilistic approach) describes the buyers' and sellers' behavior in the market as that of thinking, evaluating, hesitating and wavering people. That is *homo oscillans* (a wavering man), and this sort of person does not feel sure at what price they should trade at in the market, but he or she has a probability distribution for the price at which the deal should be made.

Further, we detrie the buyer's demand function $D(p)$ and the seller's supply function $S(p)$ as products of the distribution functions $d(p)$ and $s(p)$ by the total demand D_0 and the total supply S_0 respectively. Formally, all the stated above has the following form:

$$D(p) = D_0 d(p), \quad S(p) = S_0 s(p); \tag{23}$$

$$\int_{-\infty}^{\infty} D(p)dp = D_0, \quad \int_{-\infty}^{\infty} S(p)dp = S_0. \tag{24}$$

It is natural that the form of these S&D functions is congruent to the form of the corresponding probability distributions (see Figs. 1 and 2). As it is seen in the figures, these functions take the form of straight lines in the medium price region. This form is typically used in neoclassical economic theory to show the supply and demand functions. Specifically, and only in this medium region, the slope of the probabilistic demand function is negative, and the slope of the probabilistic supply function is positive as in neoclassical economic theory.

3.3. Probabilistic Market Pricing Mechanism

In our present model, we have an elementary economic system consisting of a single buyer with the demand function $D(p) = D_0 d(p)$ and a single seller with the supply function $S(p) = S_0 s(p)$. So the question is: what is the price of a transaction between them? Or in other words, at what price will the transaction be made? We consider that the mechanism of making transactions and the mechanism of market pricing has a probabilistic nature; sales transactions can be made at any price but with a different probability. We consider it natural that if the probability of making a purchase by the buyer at price p equals $D(p)$, and the probability of making a sale by the seller at price p equals $S(p)$, then the probability that a purchase-sale transaction will be effected at price p, the deal function $F(p)$ is a product of the purchases and sales probabilities:

$$F(p) = D(p)S(p) = D_0 S_0 f(p), \tag{25}$$

where

$$f(p) = d(p)s(p). \tag{26}$$

It is evident that if one of the functions $D(p)$ or $S(p)$ represents a rather narrow probability distribution, then the deal function $F(p)$ will be narrow as well. In this case, a market price can be denoted by price p^m at which the probability of making a purchase-sale transaction is maximal, or deal function $F(p)$ has maximum.

It is easy to understand that as functions $F(p)$ and $f(p)$ are congruent (see (25)), they have their maximum at one and the same price p^m.

Thus, we can conclude that market price explicitly depends neither on total demand D_0 nor on total supply S_0 . Now we see that the market price in this quantum model is determined only by behavioral peculiarities of the demand and supply functions. All the above stated is graphically presented in Fig. 3 where it is clearly seen how the market price p^m is formed as a compromise value between the buyer's low price p^D and the seller's high price p^S, as a result of trading, or by negotiation and exchange of information between the buyer and the seller.

Another problem of interest is what quantity of goods will really be bought or sold in the market at the certain S&D functions. However, this task is beyond the scope of the present model and its solution will depend on special rules of market activities specified out of the present model. Nevertheless, it is natural to suggest that the trade volume in the market, *MTV*, can be defined in our model as follows:

$$MTV = C \cdot \int_{-\infty}^{\infty} F(p)dp = C \cdot \int_{-\infty}^{\infty} D(p)\, S(p)\, dp, \qquad (27)$$

where C is just a normalization constant. So, we obtained the natural result that the trade volume in the market is directly proportional to the overlap integral between the S&D functions. This makes it possible, among other things, to conclude that both the time-dependent and stationary economic Schrödinger equations derived above formally mathematically represent, to some extent, action of the market-based

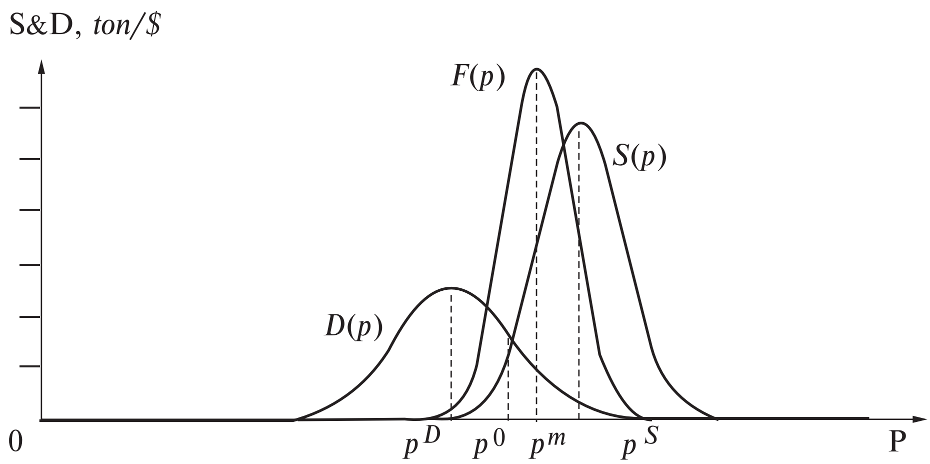

Fig. 3. Graphs of probabilistic mechanism.

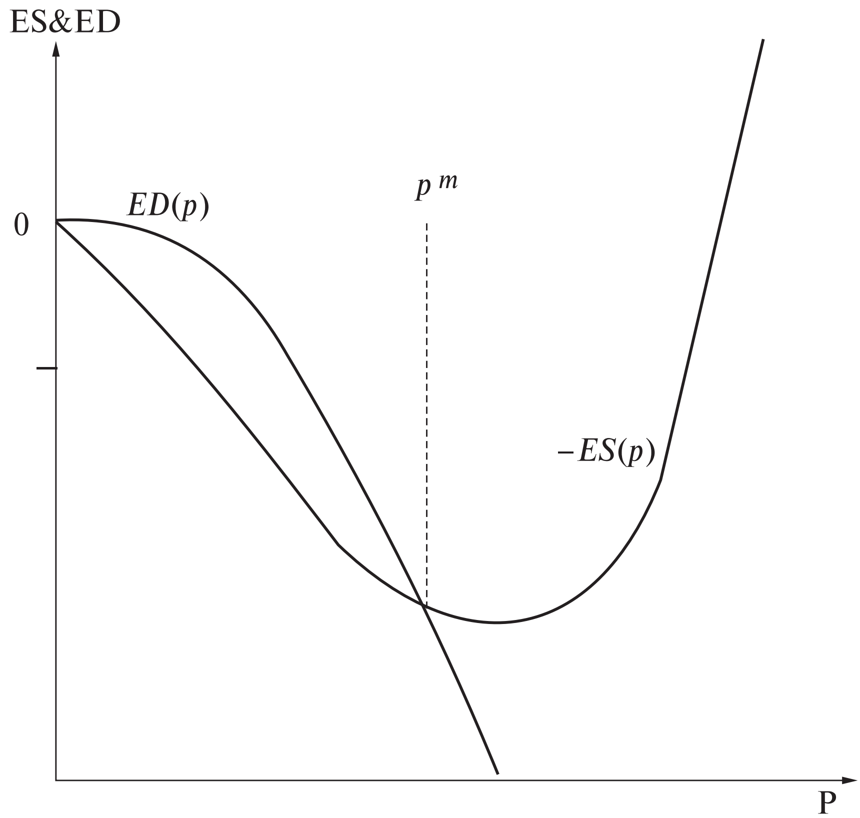

Fig. 4. Graphs illustrated the result of the theory that market price p_m is at the intersection point of the demand ($ED(p)$) and supply ($ES(p)$) elasticity curves (the last one has an opposite sign).

trade maximization principle within the framework of quantum economy in the price space (see Chapter VIII and variational principle in quantum mechanics [2]).

3.4. Market Price and Elasticities of Supply And Demand

There are certain observations in scientific literature about the relationship between a market price and demand and supply elasticities, but the type of this relationship was not expressed explicitly. Within the framework of the quantum model, such a relationship is found trivially. It is known that the derivative of a function equals zero at the maximum point. With regards to our case it means that the derivative function $F(p)$, $dF(p)/dp \equiv F'(p)$, must equal zero at maximum point p^m. With the help of simple calculations we can find that the required price p^m is a solution of the following equation

$$\frac{D'(p)}{D(p)}p = -\frac{S'(p)}{S(p)}p, \tag{28}$$

where demand and supply elasticities are on the left and on the right — $ED(p)$ and $ES(p)$ respectively:

$$ED(p) \equiv \frac{dD(p)}{dp}\frac{p}{D(p)} \quad \text{and} \quad ES(p) \equiv \frac{dS(p)}{dp}\frac{p}{S(p)}. \tag{29}$$

We have obtained a rather interesting result where the maximum probability of transaction is achieved at the market price p^m when the demand and supply elasticities are equal in absolute value but opposite in sign. This is illustrated in Fig. 4.

4. The Many-Agent Market Economies

Let us consider an economic system comprising of an arbitrary number of different buyers N and an arbitrary number of sellers M. For this kind of economy we make several assumptions analogous to those made for the elementary economies described above, namely:

1. The economy is in the ground or normal stationary state where there are no strong external perturbations that are time dependent.

2. The buyers' demand function $D(p)$ and sellers' supply function $S(p)$ are directly proportional to probability distributions $d(p)$ and $s(p)$ of conducting sales transactions at price p by buyers and sellers in the market:

$$D(p) = D_0 d(p) \quad \text{and} \quad S(p) = S_0 s(p); \tag{30}$$

$$\int_{+\infty}^{+\infty} d(p)\,dp = 1, \quad \int_{+\infty}^{+\infty} s(p)\,dp = 1; \tag{31}$$

$$\int_{+\infty}^{+\infty} D(p)\,dp = D_0, \quad \int_{+\infty}^{+\infty} S(p)\,dp = S_0, \tag{32}$$

where D_0 and S_0 are the total demand and supply in the market, and the demand function $D(p)$ and supply function $S(p)$ represent the probability distribution of conducting purchases and sales transactions at price p respectively.

3. Probability that the transaction will truly be conducted at the price p, $F(p)$, is a product of the two probabilities:

$$F(p) = D(p)S(p). \tag{33}$$

Reasoning analogous to the previous one made for the elementary system comprising a single buyer and a single seller shows that sales transactions are conducted in the market mainly at the price range near p^m, corresponding to the maximum probability of function $f(p) = d(p)s(p)$ or to the maximum of product of the demand and supply functions: $F(p) = D(p)S(p)$.

Using the same method we can show that point p^m corresponds to the point of intersection of the demand elasticity function $(ED(p))$, and the supply elasticity function, taken with the opposite sign $(-ES(p))$.

The graphically described situation above is presented in the same figures as for the case with a single buyer and a single seller economy (see Figs. 1–4).

It is notable that if slopes of the demand and supply functions are approximately equal to each other or, to be more precise, if $D'(p) \cong -S'(p)$, e.g. $D(p)$ and $S(p)$ are linear functions with slopes being equal in value but opposite in sign, then point p^m coincides with the point p^0 of intersection of demand $D(p)$ and supply $S(p)$ functions (see Fig. 3) as follows:

$$D(p^0) = S(p^0). \tag{34}$$

Hence, in this case transactions will be mainly conducted in the market near intersection p^0 of the demand and supply functions. This point is often regarded as a point of equilibrium in neoclassical economic theory and in this theory this price determines all transactions in the market. Amusingly, that in the medium price range, that is near the so-called equilibrium price p^0, the graphical quantum economic and neoclassical pictures are practically the same. Both graphs are represented by the two intercrossing lines, S&D curves (see Fig. 3). It looks like the neoclassical model is just one vague fragment of the more developed and advanced quantum economic model in the price space.

For practical use we can introduce one more assumption into the model.

4. If there is a relatively weak interaction V between buyers and sellers, then the S&D functions for the whole system may be calculated as a sum of the demand and supply functions of individual buyers and sellers:

$$D(p) \cong \sum_{i=1}^{N} D_i(p) = \sum_{i=1}^{N} D_0^i d_i(p), \tag{35}$$

$$S(p) \cong \sum_{j=1}^{M} S_j(p) = \sum_{j=1}^{M} S_0^j d_j(p). \tag{36}$$

5. Conclusions

In this Chapter we have developed quantum models of some economic systems by drawing a strict analogy with quantum mechanical theory of the analogous physical systems. There are some advantages in these quantum economic models. On the one hand, they make it possible to simultaneously consider the influence exerted by the interaction of the market agents on the behaviour of the whole economic system, which is a major subject in neoclassical economic theory. It also takes into account the interaction of the environment, government, society and other institutions with economic agents, a matter commonly investigated by institutional economic theory. On the other hand, they make it possible to establish equations of motion of economy that allow the description of the economic system's evolution in time, which, conceptually, is the essence of the main paradigm of evolutionary economics. Note that we can treat Hamiltonian as mathematical quantum representation of the market invisible hand concept. Note that, in accordance with the institutional and environmental principle, we take into account both the inter-agent interactions and interactions of the market agents with institutional and environmental factors. Therefore, we can figuratively say that that the market invisible hand executes orders not only of the market agents but also of the state, other institutions etc. For instance, the work of the market invisible hand can break down because of bad state regulations.

References

1. Anatoly Kondratenko. Physical Modeling of Economic Systems. Classical and Quantum Economies. Nauka, Novosibirsk, 2005. Electronic copy available at: http://ssrn.com/abstract=1304630.
2. L.D. Landau, E.M. Lifshitz. *Theoretical Physics, Vol. 3. Quantum Mechanics. Nonrelativistic Theory*. Moscow, Fizmatlit, 2002.

CHAPTER X.

Quantum Economies in the Price-Quantity Space

> "Striking empirical regularities suggest that at least some social order is not historically contingent, and is perhaps predictable from first principles. The role of markets as mediators of communication and distributed computation, which underlie the collective processes of price formation and allocation of resources, and the emergence of the social institutions that support those functions, are quintessentially economic phenomena. Yet the notions of markets' communication or computational capacities, and the way differences in those capacities account for the stability and historical succession of markets, may naturally be part of the physical world with its human social dynamics".
>
> J. Doyne Farmer, Martin Shubik, and Eric Smith. *Is Economics the Next Physical Science?* Physics Today 58: 37-42 (2005)

PREVIEW.

What is Quantum Economy?

In the previous chapter we validated the very principle possibility of building economic probabilistic techniques and approaches, by analogy with quantum physics, more specifically those designating formal systems of material points in physical space. In this fashion, we derived the equations of motion for economic systems in the form of Schrödinger equations of quantum mechanics, and also related the wave functions of stationary states with the functions of supply and demand as for both the agents in the market and the market as a whole. Economic Schrödinger equations are, in form, similar to the Schrödinger equations for physical systems, but in contrast to those, they are written in the formal economic the price space. In this chapter, we do attempt to move forward in building quantum economy by accounting for uncertainty and probability in the choice of market agents. This is based not only on desired price (leading to Schrödinger equations in price space), but also on the desired quantities of goods supplied for sale by sellers or designated for purchase by buyers in the market. Taking into account uncertainty and probability of quantities, we are led to Schrödinger equations in the formal economic prices-quantity space, and to a more complex structure for functions of supply and demand. A more narrow challenge in this work was to develop a method of calculating supply and demand functions from some general or first principles (*ab initio*) and solving approximate equations of motion for the economic systems. In the end, we put forward a number of assumptions or hypotheses Based on these, we built the general Hamiltonian of economic systems with a set of several unknown constants. We believe this Hamiltonian to be suitable for practical calculations of wave functions and the functions of supply and demand for economic systems. In conclusion, within the framework of a simpler model of coupled quantum harmonic oscillators, we performed a number of calculations of wave functions and respective supply and demand functions for the simple economic system. This consisted of a single buyer and a single seller, buying and selling grain on the market.

1. The Axioms of Physical Economics

This chapter is concerned with the further elaboration on the method used for the physical modeling of economic systems, the foundations of which we laid out about 10 years ago in a book [1]. The final aim of this approach is to develop a probabilistic economic theory by analogy with physics, which we often call in this book *physical economics*. The driving motive for the development of physical economics was the realization of the empirical fact that in both the behavior of market agents and in the functioning of markets you can often observe in practice certain patterns and regularities [2]. The existence of such patterns and regularities can serve as circumstantial evidence that, in turn, market agents in some way interact amongst themselves, according to some economic laws that fully define the dynamics or evolution of economic systems (below, simply *economy*) in time, provided that external influences on the market remain unchanged. Thus, we believe that the dynamics of the economy are substantially deterministic, or defined by some economic laws, which are objective in nature, and in this regard the economies are very similar to the physical systems. So it makes sense to try to simulate or model the economies on the physical systems, with the necessary caution and scientific culture; you cannot for a moment forget about the existence of disparities between economic and physical systems. A simple mechanical transfer of formulas and models from physics to economics on a "one-to-one" rule is not acceptable.

As we stated in Chapter I, our approach to the problem of the physical modeling of economic systems, to the development of physical economics and eventually to the development of probabilistic economic theory is based on the two axioms of a very general character. In virtue of their importance to the quantum economy developed in this chapter let us to discuss them once again here.

1. The Agent Identity Axiom. *All market agents are the same, only the supplies and demands they have different*. The axiom says that all market agents share common properties, depending primarily on agent revenues and expenses, or more strictly, on supply and demand (S&D below) for traded goods and services. It is these agents' S&D that mainly determine the rational economic behavior of agents on the markets, and eventually the behavior of the whole markets. It shows a possibility of building rather common and accurate models of behavior of agents in the market, and hence the total market as a whole. It sets us on the right track for the identification and examination of the common properties in the behavior of market agents that ensure appearance of the common patterns and regularities in the course of market processes. It gives us the ability to build theoretical economic models on a fairly high scientific level by using physical and mathematical methods, which is the primary goal of physical economics and economic theory in general. We are certain that only these types of common market phenomena and processes are rightly a matter for exact scientific economic enquiry. In other words, it focuses us on building economics as an exact science in the image and after the likeness of the natural science.

2. The Agent Distinction Axiom. *All market agents are distinguished. The second axiom works when the first axiom fails*. Thus, it defines those areas and aspects of the agents' behavior on the markets that are the subject of the studies of other sciences of more applied nature, such as marketing, behavioral science, managerial economics, psychology, policy, etc. In other words, these social sciences are concerned with the specific nuances and peculiarities in the behavior of concrete people, agents and communities in different markets and situations, etc.

Thus, as far as we can see, there is no problem with the principle possibility of adequately describing the rational behavior of market agents and of the market as a whole in physical and mathematical language. The problem may be simply that, to paraphrase a famous saying usually attributed to Isaac Newton, "to model the madness of people is more difficult than to model the motion of planets". See also discussions in [3, 4]. And fortunately, it is beyond the scope of economic theory, which deals with modeling practical activities of agents in the market and who behave very rationally, without disturbances. For clarity, we repeat that what we do in our work is only the study of common properties and similarities in the behavior of market agents in the market, depending primarily on S&D. This, in particular, is the subject of physical modeling, using adequate mathematical apparatus.

Let us now describe and discuss once again some important premises and rules on which is built the physical model of an economy. The first and most important common element of all our physical models is that all markets consist of agents. Therefore, the starting point in constructing them is that an economic system is conventionally represented as buyers and sellers together in an imaginary price-quantity space (the PQ-space below) as shown graphically in Fig. 1. More exactly, our model economy consists of the market and the external environment. The market consists of buyers (small dots) and sellers (big dots) covered by the conventional sphere. Generally speaking, very many people, institutions, and natural and other factors can represent the external environment (cross-hatched area behind the sphere) of the market which exerts perturbations on market agents (pictured by arrows pointing from environment to the market). For completeness, we remember that the multi-dimensional PQ-space is an artificial, imaginary construct designed for illustrative purposes, to demonstrate the analogy with the physical system, located in real spatial space. In fact, it is just a unit or set consisting of a price and a quantity for each trading good, which in probability economics and physical economics are treated as independent variables (see details in [5, 6]). It is interesting that, figuratively speaking, according to the first axiom, all the market agents, including all the people, reside in the different regions in the PQ-space, forming different clusters in accordance with their S&D, similar to the picture from real life where people live in different town districts in accordance to their revenues and expenses.

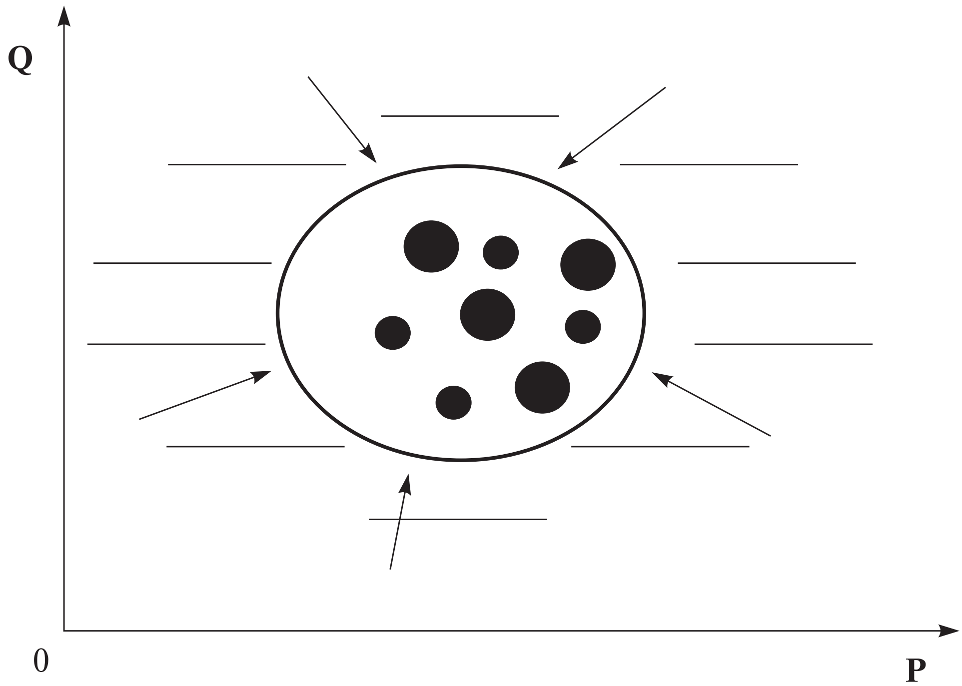

Fig. 1. Graphical model of an economy in the multi-dimensional PQ-space displayed schematically in the conventional rectangular multi-dimensional coordinate system [**P**, **Q**] where **P** and **Q** designate all the agent price and quantity coordinate axes, respectively.

The second most important common element of all our physical models is the concept of uncertainty and probability, constantly accompanying any economic activities of people, especially the work of agents in the market. An awareness of the resemblance of this concept in physics and economics gave us a clear indication of the direction in which we should go in our search and research. We have discussed this concept in economics in great detail in [5, 6], which was dedicated to the development of empirical probabilistic economic theory, or simply probability economics.

2. The Principles of Physical Economics

For the sake of clarity, note that our work on the development of probabilistic economic theory and the presentation of its results are structured as follows. All of our research on this topic is conducted simultaneously in two parallel directions of probabilistic economic theory, namely empirical and non-empirical, that we conventionally refer to as probability economics and quantum economy, respectively. Recall that probability economics is how we refer to the complex of the developed probabilistic economic models, which in essence are phenomenological or empirical, based quite firmly on established empirical facts. The cornerstone of quantum economy is the equations of motion for the whole economic system, derived from the first principles (*ab initio*). Both branches of probabilistic economic theory are based on the five general principles of physical economics formulated in [5] and in Chapter I. It makes sense to describe and discuss them once again here because they will be extensively used below when deriving the economic equations of motion in the price-quantity space.

1. The Cooperation-Oriented Agent Principle. The most important concept concerning markets is as follows: every market consists of market agents, buyers and sellers, all strongly interacting with each other. There are never any mysterious forces in markets. Everything that markets do, the cooperation-oriented market agents do, and therefore only the cooperation-oriented, agent-based models can provide the reasonable and justified foundation for any modern economic theory.

2. The Institutional and Environmental Principle. Markets are never completely closed and free; all the market agents are under continuous influences and under such external institutional and environmental forces and factors as states, institutions, other markets and economies, natural and technogenic phenomena, etc. The influences, exerted by each of these forces and factors on the structure of market prices and on the market behavior, can be completely compared with the effect from inter-agent interactions. Moreover, the action of strong external institutional and environmental factors can significantly hamper the effective work of market mechanisms and even practically suppress it in a way that results in the breakdown of the market's invisible hand concept, well-known in classical economics. Therefore, the influence of institutional, environmental and other external factors must be adequately taken into account in the models, as well as simultaneously with the inter-agent interactions.

3. The Dynamic and Evolutionary Principle. Markets are complex dynamic systems; all the market agents are in perpetual motion in search of profitable deals with each other for the sale or purchase of goods. Buyers tend to buy as cheaply as possible, and sellers want to obtain the highest possible prices. Mathematically, we can describe this time-dependent dynamic and evolutionary market process as motion in the price-quantity economic space of market agents acting in accordance with objective economic laws. Therefore, this motion has a deterministic character to some extent. This motion can and must be approximately described with the help of equations of motion.

4. The Market-Based Trade Maximization Principle. On relatively free markets, the buyers and sellers consciously and deliberately enter into transactions of buying and selling with each other, since they make deals only under conditions in which they obtain the portion of profit that suits each of them. It is in no way compulsory that they aspire to maximize their profit in each concluded transaction, since they understand that the transactions can only be mutually beneficial. But they do attempt to increase their profit via the conclusion of a maximally possible quantity of such mutually beneficial transactions. Thus, it is possible to assert that the market as a whole strives for the largest possible volume of trade during the specific period of time. Consequently, we can make the conclusion that market dynamics can approximately be described and even approximate equations of motion for the market agents can be derived in turn by means of applying the market-based trade maximization principle to the whole economic system (more exactly, this principle is system-based);

5. The Uncertainty and Probability Principle. Uncertainty and probability are essential parts of human action in markets. This is caused by the nature of human reasoning, as well as the fundamental human inability to accurately predict a future state of the markets. Furthermore, market outcome is the result of the actions of multiple agents, and no market is ever completely closed and free. For these reasons, all market processes are probabilistic by nature too, and an adequate description of all the market processes needs to apply probabilistic approaches and models in the economic price-quantity space. The uncertainty law results from this principle.

On the one hand, these five general principles of physical economics sum up our knowledge and experiences with uncertainty, probability, and market processes. On the other hand, they provide the reliable basis on which the behavior of both the market agents and the market as whole can be studied, not only rigorously and logically, but most importantly, quantitatively. The primary and most evident consequence of these principles is that all the agent prices, p, and quantities, q, for all goods are independent variables, and all the S&D functions are multi-dimensional functions of all these variables. In other words, S&D functions are now surfaces in multi-dimensional space. With these five general principles in mind, we have developed the complex of probabilistic models, i.e., probability economics. Each of the five principles is embodied in a graphical model of economic systems enclosed in the multi-dimensional PQ-space in Fig. 1. For more detail see [5]. Concluding, let us stress that the probabilistic S&D functions of probability economics are just constructed from experience of practical activities therefore they are referred to as empirical probabilistic S&D functions. The main task of probability economics is to produce market S&D functions using the known S&D functions for all the market agents as input data and to give market prices and quantities as output data.

As far as the quantum economy is concerned, on the one hand, its models are also based on the same general principles of probability economics — see above. On the other hand, scientific exploration of quantum economy has as its primary goal the discovery and solving of equations of motion for economic systems. This enables us to describe the system dynamics, including the evolution of the system in time, on the basis of certain first principles from an *ab initio* standpoint. In other words, quantum economy searches for equations of motion which enable us to find *ab initio* both agent and market S&D functions; stationary at first, but eventually, time-dependent. Obtained in this way, as solutions to equations of motion, the S&D functions serve as theoretical or *ab initio* S&D functions. This is opposed to the empirical S&D functions of probability economics, where market agents would themselves choose their S&D functions. Obviously, the potential success of such an undertaking in this investigation would be a major contribution to the development of economic theory. Recall that in [1] we laid out the foundations of the physical approach to modeling of economic systems where all models were built in the P-subspace of the PQ-space. In that work, it was assumed that the prices of goods are independent variables, and the quantities of goods required by buyers or offered for sale by sellers, are fixed.

In this chapter we extend the possibilities of quantum economy by increasing in number the degrees of freedom of the economic system. Here we will assume independent variables to be not only prices but also the quantities of goods, and this will dramatically increase the scope of physical modeling of economies. Bear in mind that we previously applied this type of generalization in probability economics in [5].

To conclude this section, it is worthwhile to stray a little from the presentation and to dwell on the significance of the term *physical economics*, borrowed from [3] and used here as a general neologism. It combines the terms probability economics as well as classical economy and quantum economy, previously coined by us in [1] for the classification of classical and quantum models of economies, and related theories and approaches. We feel the term *physical economics* to be very appropriate, and well-suited to the purposes of our study since it is very capacious and precise in its content. Its use allows us to focus on the economic content of our physical economic models and stress their orientation towards a theoretical quantitative description of the practical experiences in markets. Using this term, we show our focus on the development of economic theory, but not as an attempt to turn it into one of the special courses of physics and, especially not as a focus on physics development.

Let us now make some comments on another branch of physical economics, namely econophysics. We believe that there are two main reasons that create barriers to obtaining serious results and forging ahead in econophysics research.

First, econophysics studies are not agent-based, with all its consequences, according to the cooperation-oriented agent principle. In this case, econophysicists do research back-to-front when compared with physics method. Without serious analysis of the micro level structure of the economy it is not possible to build an adequate reliable model of the economy at the macro level. One must first take into account the importance of the process of choice, agent action, and inter-agent interaction in the economy, amongst other things. Attempts, to use sophisticated mathematical methods of theoretical physics within the framework of the traditional neoclassical, and often unjustifiably agentless, models cannot render significant results. All such attempts are often restricted by the glass bead game, i.e., complex mathematical exercises. Second, econophysics studies are too heavily weighted toward statistical phenomena in the economy to the detriment of the impact of uncertainties and probabilities in the actions of market agents, which are very important aspects according to the uncertainty and probability principle.

To save time and space in the presentation of this chapter' material, we are going to stay as close as possible to the text of our previous book on this theme [1]. During the course of presentation we will make only the most necessary comments pertaining to the theory's expansion and clarifying made assumptions and approximations. So, the main problem facing now us is to produce equations of motion that described the dynamics of our model economy (see Fig. 1) with the number of buyers N, number of sellers M and number of goods L. These equations of motion are intended for the quantitative calculation of the main characteristics of the markets, namely, the market S&D functions. Moreover, we want to get these equations on *ab initio* theoretical grounds which are based on some fairly common or universal principles, even if they are yet poorly defined and not very clear. In our view, the most sensible thing we can do at this stage to achieve this goal is to simply traverse the path that physics took at that point of development where it was faced with a similar challenge. Incorporating into theoretical physics the concept of uncertainty and probability resulted in the discovery of quantum mechanics.

After all, we are still just modeling economic systems; trying to find an imaginable economic system, in the formal PQ-space, the dynamic properties of which are reminiscent of that of a similar imaginable physical systems. There are parallels between the two systems, mainly of the infinitely small points that do not exist in nature but that represent particular values intrinsic to each discipline.

Establishing the exact form of fully consistent equations of motion for economic systems is not yet on the agenda for economic theory. Although their likely form will be based on physics equations that were constructed for physical systems, much more work must be done before we can establish equations for our discipline

3. The Time-Dependent Economic Schrödinger Equations in the Price-Quantity Space

Thus, we are going to construct physical quantum economy by analogy with quantum mechanics following the path laid out in the famous course of theoretical physics [7]. So first of all, we must assume or accept *a priori* that our model economy can be described with a sufficient degree of accuracy by a certain $(N+M)$-agent wave function Ψ in the $(2\times L)$-dimensional PQ-space. For the purposes of convenience in writing the formulas, we introduce the following notations. Bold letter $\boldsymbol{p}$ designates the set of $L\times(N+M)$ price coordinates of all market agents for all goods traded in the market. Bold letter $\boldsymbol{q}$ will serve as a single designation for all relevant $L\times(N+M)$ market agents quantity coordinates for all goods. Finally, the letter t designates the independent variable of time. By means of these definitions, our wave function of the economy is written as $\Psi(\boldsymbol{p},\boldsymbol{q},t)$. By definition, the square of the wave function, $|\Psi(\boldsymbol{p},\boldsymbol{q},t)|^2\, d\boldsymbol{p}\, d\boldsymbol{q}$, has meaning of probability that, at time t, the market agents will have prices in the interval from $\boldsymbol{p}$ to $\boldsymbol{p}+d\boldsymbol{p}$, and quantities q in the range from $\boldsymbol{q}$ to $\boldsymbol{q}+d\boldsymbol{q}$ in the P- and Q-subspaces, correspondingly. Because, by its nature the wave function is finite everywhere over the PQ-space, it is natural to normalize it to 1:

$$\int \Psi(\boldsymbol{p},\boldsymbol{q},t)|^2 d\boldsymbol{p}\, d\boldsymbol{q} = 1, \tag{1}$$

where integration is understood throughout the PQ-space.

By analogy with quantum mechanics of physical systems we will assume that our wave function of economy describes not only its behavior and properties at some initial time t_0, but also determines its behavior and properties at any time t in the future. This means that, together with a few other assumptions of a general character (see [7], for example), the equation of motion of the economy can be written in the form of the time-dependent Schrödinger equation:

$$i\alpha \frac{\partial \Psi(\boldsymbol{p},\boldsymbol{q},t)}{\partial t} = \hat{H}\Psi(\boldsymbol{p},\boldsymbol{q},t). \tag{2}$$

In this equation, i is the usual complex variable, α is a characteristic parameter different for each economy (formal analogue of the well-known in physics reduced Planck's constant $\hbar$, but for obvious reasons does not have anything to do with it), and $\hat{H}$ is a certain Hermitian Hamilton operator, or simply, *Hamiltonian*. If the Hamiltonian is known, then the Schrödinger equation (2) determines the wave function of the economy at any given time. In this sense, the Hamiltonian can be interpreted as a mathematical quantum representation of the so-called invisible hand of the market, a concept usually attributed to Adam Smith. So we have the equation of motion for the economy in the form of the time-dependent Schrödinger equation with an unknown constant α and unknown Hamiltonian of the economy. We leave the question concerning a choice of this unknown data to discuss in the following sections. And now, by analogy with physics, we will introduce into the theory one of the most important concepts in physical economics: the concept of stationary states of the economy as *eigenstates* of the Hamiltonian. We will derive the corresponding so-called stationary Schrödinger equation of economy in the next section.

4. The Stationary Economic Schrödinger Equations in the Price-Quantity Space

For the case of an economy under the influence of the external perturbations or fields unchangeable in time, we will introduce the concept of a stationary state of the economy by using the definition as follows. A stationary state with integer number n of the economy are the so-called eigenstates of its Hamiltonian, which is also perceived to be independent of time. These eigenstates, in turn, are solutions of the stationary Schrödinger equation, having that view in the wave function representation:

$$\hat{H}\Psi_n(\boldsymbol{p},\boldsymbol{q},t) = E_n\Psi_n(\boldsymbol{p},\boldsymbol{q},t). \tag{3}$$

In Eq.(3), E_n designates the eigenvalues of the Hamiltonian or eigenvalues of energy, as they are called in quantum mechanics. For convenience, at this stage we will continue our use of terminology from quantum mechanics instead of creating new words for quantum economy. We must be aware, of course, that these notions in quantum mechanics and quantum economy are unrelated in content; our point particles in the PQ-space are imaginary objects and have nothing to do with physical particles. However, we assume that the dynamics of those imaginary economic particles, which correspond to the market agents, can be described by equations, similar in spirit and structure to the Schrödinger equations, originally derived for the description of motion of electrons in real space.

The time-dependent Schrödinger equation is easily solved for the stationary states:

$$i\alpha \frac{\partial \Psi(\boldsymbol{p},\boldsymbol{q},t)}{\partial t} = \hat{H}\Psi_n(\boldsymbol{p},\boldsymbol{q},t) = E_n\Psi_n(\boldsymbol{p},\boldsymbol{q},t),$$

$$\Psi_n(\boldsymbol{p},\boldsymbol{q},t) = \exp\left(-\frac{i}{\alpha}E_n t\right)\psi_n(\boldsymbol{p},\boldsymbol{q}). \tag{4}$$

It is easy to see that the time-independant wave functions $\psi_n(\boldsymbol{p},\boldsymbol{q})$ are also eigenfunctions of a Hamiltonian as follows:

$$\hat{H}\psi_n(\boldsymbol{p},\boldsymbol{q}) = E_n\ \psi_n(p,q). \tag{5}$$

Eq.(5) is also commonly called the stationary Schrödinger equation. Eq.(4) describes the exponential dependence of the stationary states of time. Any wave function of the economy can be represented as a linear combination of the respective stationary wave functions as follows:

$$\Psi(\boldsymbol{p},\boldsymbol{q},t) = \sum_n a_n \exp\left(-\frac{i}{\alpha}E_n t\right)\psi_n(\boldsymbol{p},\boldsymbol{q}), \tag{6}$$

where the square of the decomposition coefficient $|a_n|^2$ determines the weight of the n-th stationary state in the total wave function. With a few exceptions below, we will consider only the ground or normal stationary state of economy, i.e., the state with the number $n = 0$ having the lowest energy E_0, and let down the index 0. In brief, the economy, being in such normal stationary state, will be referred to below as the *stationary economy*. Probability distribution in a stationary economy, as it can be easily guessed, is stationary, and does not depend on time:

$$|\Psi(\boldsymbol{p},\boldsymbol{q},t)|^2 = |\psi(\boldsymbol{p},\boldsymbol{q})|^2. \tag{7}$$

Thus, we have derived that the equation of motion for a stationary economy is the stationary Schrödinger equation as follows:

$$\hat{H}\psi(\boldsymbol{p},\boldsymbol{q}) = E\psi(\boldsymbol{p},\boldsymbol{q}). \tag{8}$$

The square of the wave function, $|\psi(\boldsymbol{p},\boldsymbol{q})|^2$, will be referred to below as the *total market function*. It is easy to see that it is also normalized to 1, as the original wave function of the system, $\Psi(\boldsymbol{p},\boldsymbol{q},t)$. Thus, the problem now is the fact that the Hamiltonian for economic systems remains unknown. Generally speaking, it has to be determined in the future in much the same way as was done in physics from the time of Newton up to the mid-1920's. However, it should be noted that, in a sense, quantum mechanics has been lucky in that the form of interaction forces remained the same as they were in classical mechanics. Physicists simply had to rewrite their equations in the operator representation. In quantum economy, we have to go through this stage of discovering all the market interaction forces, and eventually discerning the more or less precise Hamilton operator. In principle we are able to do this, but only after undergoing the same slow process that physics was held to for 300–400 years. This timeframe is briefly described by the winged words, "It is through the interplay of observation, prediction and comparison that the laws of nature are slowly clarified" (see the discussion in [5]). All that we can do today is to go by way of the successive approximations. First, we do not set ourselves the task of deriving accurate solutions of equations of motion, but the task of getting only their pretty approximate solutions, by means of which we can calculate the S&D functions with acceptable accuracy, describing basic market phenomena and processes known from the experience of people in the market. This whole experience has been generalized by us briefly as five general principles of physical economics and eventually of probabilistic economic theory in [5] as set out above.

5. The Hypotheses of Quantum Economy

Based on the principles below, we will discuss premises and assumptions regarding the form of agent and market functions, as well as the basic properties of the economic Hamiltonian. We have successively captured them as assumptions or hypotheses as follows:

1. The Total Market Function Factorization Hypothesis. We reason that, although the interaction between buyers and sellers can be quite strong, most of this interaction can be adequately taken into account. This is possible even if we assume the function $\psi(\boldsymbol{p},\boldsymbol{q})$ to have a factorized form, for example, as a product of two wave functions, each depending solely on buyers and sellers coordinates, respectively, which can be written formally as follows:

$$\psi(\boldsymbol{p}^D, \boldsymbol{q}^D, \boldsymbol{p}^S, \boldsymbol{q}^S) \cong \psi^D(\boldsymbol{p}^D, \boldsymbol{q}^D) \times \psi^S(\boldsymbol{p}^S, \boldsymbol{q}^S). \tag{9}$$

In Eq. (9), the symbol $\boldsymbol{p}^D$ represents all price variables of all the buyers on all goods and symbol $\boldsymbol{p}^S$ is the same for all the sellers. Symbol $\boldsymbol{q}^D$ designates, in turn, all quantity variables of all the buyers for all the goods, and $\boldsymbol{q}^S$ is the same for all the sellers. The wave function of the economic subsystem of the market buyers, $\psi^D(\boldsymbol{p}^D, \boldsymbol{q}^D)$, will be referred to as the market buyers function, and the wave function of the economic subsystem of the market sellers — the market sellers function. Naturally these functions are also normalized to 1.

2. The Market Agents Functions Factorization Hypothesis. Buyers can strongly interact among themselves, competing for rare goods, putting on pressure and causing price to go up. However, we admit that much of this interaction can be properly taken into account, even if we use the market buyers function in the following factorized form:

$$\psi^D (\boldsymbol{p}^D, \boldsymbol{q}^D) \cong \prod_{n=1}^{N} \psi_n^D (\boldsymbol{p}_n^D, \boldsymbol{q}_n^D), \tag{10}$$

where the symbol $\boldsymbol{p}_n^D$ denotes the price vector of n-th buyer in the L-dimensional P-subspace, i.e., $\boldsymbol{p}_n^D$ is the set of all L prices of n-buyer for all L goods and symbol $\boldsymbol{q}_n^D$ represents all L quantity variables of the n-th buyer for all L goods. Similarly, sellers can strongly interact among themselves, entering the competition for the cash of buyers and putting downwards pressure on prices. Nevertheless, in this case, we also admit that much of this interaction can be properly described by the market sellers functions factorized in the following manner:

$$\psi^S (\boldsymbol{p}^S, \boldsymbol{q}^S) \cong \prod_{m=1}^{M} \psi_m^S (\boldsymbol{p}_m^S, \boldsymbol{q}_m^S), \tag{11}$$

where the symbols $\boldsymbol{p}_m^S$ and $\boldsymbol{q}_m^S$ have obvious meaning, analogous to the buyers symbols. Thus, we approximately presented the functions of subsystems of buyers and sellers in the form of simple products of the wave functions of individual buyers and sellers, respectively (one-agent approximation below). As we shall see below, this greatly simplifies market pictures and calculations of market S&D functions. Naturally, the wave functions of individual agents, or simply one-agent functions, are also normalized to 1.

3. The One-Agent Functions Factorization Hypothesis. The next step toward simplifying the structure of wave functions is almost obvious. We believe that one-agent functions can approximately obtained making use of the following multiplicativity formulas for S&D [5, 8–10]:

$$\psi_n^D(\boldsymbol{p}_n^D, \boldsymbol{q}_n^D) \cong \prod_{l=1}^{L} \psi_{nl}^D\ (p_{nl,} q_{nl}), \tag{12}$$

$$\psi_m^S(p_m^S, q_m^S) \cong \prod_{l=1}^{L} \psi_{ml}^S\ (p_{ml,} q_{ml}), \tag{13}$$

where all the designations are of obvious meaning, and do not require additional comments. Thus, we presented the one-agent functions $\psi_n^D(\boldsymbol{p}_n^D, \boldsymbol{q}_n^D)$ and $\psi_m^S(\boldsymbol{p}_m^S, \boldsymbol{q}_m^S)$ of buyers and sellers in the form of products of the one-good functions $\psi_{nl}^D\ (p_{nl,} q_{nl})$ and $\psi_{ml}^S\ (p_{ml,} q_{ml})$ of the corresponding agents, respectively (one-good approximation below). These one-good functions are also to be normalized to 1.

4. The One-Good Functions Factorization Hypothesis. Ournext step in the procedure to simplify the structure of wave functions as much as possible is also evident. We assume that with a fairly good degree of accuracy, we can also factorize the one-good functions of agents by the following:

$$\psi_{nl}^D\ (p_{nl}, q_{nl}) \cong \psi_{nl}^{DP}(\ p_{nl}) \times \psi_{nl}^{DQ}(q_{nl}), \tag{14}$$

$$\psi_{ml}^S\ (p_{ml}, q_{ml}) \cong \psi_{ml}^{SP}(p_{ml}) \times \psi_{ml}^{SQ}(q_{ml}), \tag{15}$$

where on the right side are the price and quantity one-good wave functions of agents, normalized to 1. Thus, we expressed one-good functions in the form of products of the respective one-variable wave functions of price and quantity (one-variable approximation below) which will be referred to below as basic wave functions.

Here we will summarize regarding the use of factorization hypotheses for wave functions of the economic system. First we presented the total market function as a product of the one-agent functions, and ultimately as a product of the one-agent, one-good, one-variable functions. This we call the *one-agent*, *one-good*, *one-variable approach*. We discussed in detail in [5, 8–10] some possible reasons to use this approach in probability economics. Here, note that they are all relevant in the case of physical economics too, naturally. The basic idea behind all these justifications is that our real economic world reflects the real people within it, as well as their consciousness, ways of thinking, and decision-making. In particular, one of the most remarkable human actions on the market is the ability to reduce all complex multi-dimensional tasks to one-dimensional tasks and solutions. This is despite the fact that he or she has to interact with a huge number of counterparties and process lots of complex market information. Details can be found in [5, 8–10]. We are going to move on towards further simplification of the problem's solution and introduce some new hypotheses into the theory.

5. The S&D Functions Probabilistic Nature Hypothesis. The concept of S&D plays a key role in economic theory. Bear in mind that, according to the most general definition, S&D are just plans or intentions of buyers and sellers that they want and can achieve in the market. For a more detailed description of these notions in mathematical language, different authors may generate different pictures or formulas. We adhere to the probabilistic view of S&D and have developed an appropriate theory, which we started investigating in paper [1]. S&D functions are defined in physical economics in much the same way as in probabilistic economics. For instance, S&D functions of each individual agent are, by definition, agent probability distributions in the PQ-space. This would have to be calculated by integration of the total market functions on all the coordinates of all other agents, which results in the need to carry out very complicated calculations. However, if we take advantage of market functions factorization hypotheses, i.e., formulas (10)–(15) of the one-agent, one-good, one-variable approach, then the integration becomes quite a trivial affair. As a result, we get a very simple and clear definition of agent S&D functions, $D_n\ (p, q)$ and $S_m\ (p, q)$ as follows:

$$D_n(\boldsymbol{p}, \boldsymbol{q}) \equiv D_n^0 \cdot |\psi_n^D(\boldsymbol{p}, \boldsymbol{q})|^2, \tag{16}$$

$$S_m(\boldsymbol{p}, \boldsymbol{q}) \equiv S_m^0 \cdot |\psi_m^S(\boldsymbol{p}, \boldsymbol{q})|^2. \tag{17}$$

Note that, beginning with the Eq. (16), we use bold letters $\boldsymbol{p}$ and $\boldsymbol{q}$ without upper characters D and S just for designation of L-dimensional price and quantity vectors in the P- and Q-subspaces. Evidently, the combination of those will give a vector $\boldsymbol{r}$ in the PQ-space. D_n^0 and S_m^0 are here the normalization constants. The integrals of the left parts of Eq. (16) and (17) are equal to these constants, the meaning of which we discuss a little bit later.

It makes sense here to give some explanations about the above definition and economic significance of the D&S functions in our theory. The economic Schrödinger equations make it possible to calculate the wave functions of economy, squares of modules of which have the following economic significance: they are probability distributions of making purchase/sales deals by agents at a given price and for a given quantity of goods. In other words, the wave functions represent only some preliminary assessments and plans of market agents, forming together the chosen strategy of agents' behavior on the market. In practice, however, we are dealing with the functioning of complicated market mechanisms with their inputs, outputs, and rules. For a description of the market work, we need to use special concepts, notions, and tools; the main ones being the concept of S&D. However, there is a problem here resulting from the fact that S&D functions are absent in the economic Schrödinger equations; they must still be designed, making use of wave functions as building blocks in such a way that they are able to adequately reflect or describe the market functioning and its results for market prices, the volume of transactions etc. It is appropriate here to draw an analogy with quantum mechanics, as follows. The Schrödinger equation in it only gives the wave functions of stationary states. Physicists, for example, might be interested, say, in the results of particular scattering experiments in colliders. In order to be able to describe the scattering processes in detail — perhaps to calculate threshold energy of particle or resonances and the respective cross sections — we must first develop a scattering theory and derive formulas for calculating scattering cross sections. These would need to adequately describe the conditions of the scattering experiment under study.

Thus, as a matter of fact, the formulas (16), (17) can be regarded as definitions of the agent S&D functions in physical economics. In order to obtain the market S&D functions, we need simply to add up the corresponding agent S&D functions, as it is done in quantum mechanics to calculate the distributions of mass or charge of many-particle systems in space. To emphasize the analogy with physics, we introduce into the theory S&D operators using the following definitions:

$$\hat{D}(\boldsymbol{p}^D, \boldsymbol{q}^D) = \sum_{n=1}^{N} D_n^0 \cdot \delta(\boldsymbol{p} - \boldsymbol{p}_n^D)\,\delta(\boldsymbol{q} - \boldsymbol{q}_n^D), \tag{18}$$

$$\hat{S}(\boldsymbol{p}^S, \boldsymbol{p}^S) = \sum_{m=1}^{M} S_m^0 \cdot \delta(\boldsymbol{p} - \boldsymbol{p}_m^S)\,\delta(\boldsymbol{q} - \boldsymbol{q}_m^S). \tag{19}$$

Then, as usual in quantum mechanics, we define the market S&D functions as average values of these operators as follows:

$$D(\boldsymbol{p},\boldsymbol{q}) \equiv \int_{-\infty}^{+\infty} |\psi(\boldsymbol{p}^D, \boldsymbol{q}^D, \boldsymbol{p}^S, \boldsymbol{q}^S)|^2 \cdot \hat{D}(\boldsymbol{p}^D, \boldsymbol{q}^D)\, d\boldsymbol{p}^D d\boldsymbol{q}^D =$$

$$= \sum_{n=1}^{N} D_n^0 \int_{-\infty}^{+\infty} |\psi(\boldsymbol{p}^D, \boldsymbol{q}^D, \boldsymbol{p}^S, \boldsymbol{q}^S)|^2\, \delta(\boldsymbol{p} - \boldsymbol{p}_n^D)\,\delta(\boldsymbol{q} - \boldsymbol{q}_n^D)\, d\boldsymbol{p}_n^D d\boldsymbol{q}_n^D =$$

$$= \sum_{n=1}^{N} D_n^0 \cdot |\psi_n^D(\boldsymbol{p},\boldsymbol{q})|^2, \tag{20}$$

$$S(\boldsymbol{p},\boldsymbol{q}) \equiv \int_{-\infty}^{+\infty} |\psi(\boldsymbol{p}^D, \boldsymbol{q}^D, \boldsymbol{p}^S, \boldsymbol{q}^S)|^2 \cdot \hat{S}(\boldsymbol{p}^S, \boldsymbol{q}^S)\, d\boldsymbol{p}^S d\boldsymbol{q}^S = (\boldsymbol{p}^S, \boldsymbol{q}^S)\, d\boldsymbol{p}^S d\boldsymbol{q}^S =$$

$$= \sum_{m=1}^{M} S_m^0 \int_{-\infty}^{+\infty} |\psi(\boldsymbol{p}^D, \boldsymbol{q}^D, \boldsymbol{p}^S, \boldsymbol{q}^S)|^2 \cdot \delta(\boldsymbol{p} - \boldsymbol{p}_m^S)\,\delta(\boldsymbol{q} - \boldsymbol{q}_m^S)\, d\boldsymbol{p}_m^S d\boldsymbol{q}_m^S =$$

$$= \sum_{m=1}^{M} S_m^0 \cdot |\psi_m^S(\boldsymbol{p},\boldsymbol{q})|^2. \tag{21}$$

When deriving formulas (20) and (21), we used all the factorization hypotheses described above. So we got the following additivity formulas for the market D&S functions which are exceptionally simple in their structure and economic meaning:

$$D(\boldsymbol{p},\boldsymbol{q}) \equiv \sum_{n=1}^{N} D_n(\boldsymbol{p},\boldsymbol{q}), \tag{22}$$

$$S\ (\boldsymbol{p},\boldsymbol{q}) \equiv \sum_{m=1}^{M} S_m(\boldsymbol{p},\boldsymbol{q}). \tag{23}$$

The full formal analogy with the quantum mechanics with respect to the mass and charge distributions noted above is clear.

Now we will take another obvious step in detailing the structure of the agent S&D functions, which sheds more light on a probabilistic mechanism of their formation. To do this we substitute equations (12)–(15) into definitions (16) and (17) and carry out the necessary simple calculations, writing out the result regarding the demand from buyers, as follows:

$$D_n(\boldsymbol{p},\boldsymbol{q}) = D_n^0 \cdot \psi_n^D(\boldsymbol{p},\boldsymbol{q}) \cong D_n^0 \cdot |\prod_{l=1}^{L} \psi_{nl}^D\ (p_l, q_l)|^2 =$$

$$= D_n^0 \cdot \prod_{l=1}^{L} \psi_{nl}^D| (p_l, q_l)|^2 =$$

$$= D_n^0 \cdot \prod_{l=1}^{L} D_{nl}\ (p_l, q_l) \,/\, D_{nl}^0 = D_n^0 \cdot \frac{\prod_{l=1}^{L} D_{nl}(p_l, q_l)}{\prod_{l=1}^{L} D_{nl}^0}. \tag{24}$$

In Eq. (24), we used another new definition, this time for the one-agent, one-good demand function $D_{nl}\ (p_l, q_l)$ with normalization constant D_{nl}^0, the meaning of which we will discuss later:

$$D_{nl}(p_l, q_l) \equiv D_{nl}^0 \cdot |\ \psi_{nl}^D(p_l, q_l)|^2 = D_{nl}^0 \cdot |\ \psi_{nl}^{DP}(p_l)|^2 \cdot |\ \psi_{nl}^{DQ}(q_l)|^2. \tag{25}$$

It is in absolutely the same way that we get a similar formula for the supply of sellers:

$$S_m\ (\boldsymbol{p},\boldsymbol{q}) = S_m^0 \cdot \frac{\prod_{l=1}^{L} S_{ml}(p_l, q_l)}{\prod_{l=1}^{L} S_{ml}^0}, \tag{26}$$

$$S_{ml}(p_l, q_l) \equiv S_{ml}^0 \cdot |\ \psi_{ml}^S(p_l, q_l)|^2 = S_{ml}^0 \cdot |\ \psi_{nl}^{DP}(p_l)|^2 \cdot |\ \psi_{nl}^{DQ}(q_l)|^2. \tag{27}$$

Eqs. (24)–(27) we refer to as multiplicativity formulas for S&D. To simplify the look of these formulas, we introduce one more definition for squares of modules for one-agent, one-good, one-variable wave functions, normalized to 1, as follows:

$$d_{nl}^{P}(p_l) = |\psi_{nl}^{DP}(p_l)|^2, \quad \int_{-\infty}^{+\infty} d_{nl}^{P}(p_l)\, dp_l = 1, \tag{28}$$

$$d_{nl}^{Q}(q_l) = |\psi_{nl}^{DQ}(q_l)|^2, \quad \int_{-\infty}^{+\infty} d_{nl}^{Q}(q_l)\, dq_l = 1, \tag{29}$$

$$s_{ml}^{P}(p_l) = |\psi_{ml}^{SP}(p_l)|^2, \quad \int_{-\infty}^{+\infty} s_{nl}^{P}(p_l)\, dp_l = 1, \tag{30}$$

$$s_{ml}^{Q}(q_l) = |\psi_{ml}^{SQ}(q_l)|^2, \quad \int_{-\infty}^{+\infty} s_{ml}^{Q}(q_l)\, dq_l = 1. \tag{31}$$

In terms of those *s* and *d* functions, which we will call below basic functions, one-agent, one-good S&D functions are now written as follows:

$$D_{nl}(p_{l,} q_l) = D_{nl}^{0} \cdot d_{nl}^{P}(p_l) \cdot d_{nl}^{Q}(q_l), \tag{32}$$

$$S_{ml}(p_{l,} q_l) = S_{ml}^{0} \cdot s_{ml}^{P}(p_l) \cdot s_{ml}^{Q}(q_l). \tag{33}$$

6. The Basic Functions Gaussian Form Hypothesis. In describing the foundations of probabilistic economic theory [5] we hed already held a detailed discussion on the possible form of basic functions, the most suitable to describe the typical strategies of agents on the market with an acceptable degree of accuracy. We came to the conclusion that the best we can make in this respect at present is to use one-dimensional, two-parameter Gaussians. Thus, the basic functions of the S&D will be approximately represented as follows:

$$D_{nl}(p_l, q_l) = D_{nl}^{0} \cdot g_{nl}^{DP}(p_l) \cdot g_{nl}^{DQ}(q_l), \tag{34}$$

$$g_{nl}^{DP}(p_l) = \sqrt{w_{nl}^{DP}/\pi} \cdot \exp(-w_{nl}^{DP}(p_l - p_{nl}^{D}))^2, \tag{35}$$

$$g_{nl}^{DQ}(q_l) = \sqrt{w_{nl}^{DQ}/\pi} \cdot \exp(-w_{nl}^{DQ}(q_l - q_{nl}^{D}))^2, \tag{36}$$

$$\Gamma_{nl}^{DP} = \sqrt{-4\ln 0{,}5/w_{nl}^{DP}}, \tag{37}$$

$$\Gamma_{nl}^{DQ} = \sqrt{-4\ln 0{,}5/w_{nl}^{DQ}}. \tag{38}$$

Similar formulas for supply are as follows:

$$S_{ml}(p_l, q_l) = S_{ml}^{0} \cdot g_{ml}^{SP}(p_l) \cdot g_{ml}^{SQ}(q_l), \tag{39}$$

$$g_{ml}^{SP}(p_l) = \sqrt{w_{ml}^{SP}/\pi} \cdot \exp(-w_{ml}^{SP}(p_l - p_{ml}^{S}))^2, \tag{40}$$

$$g_{ml}^{SQ}(q_l) = \sqrt{w_{ml}^{SQ}/\pi} \cdot \exp(-w_{ml}^{SQ}(q_l - q_{ml}^{S}))^2, \tag{41}$$

$$\Gamma_{ml}^{SP} = \sqrt{-4\ln 0{,}5/w_{ml}^{SP}}, \tag{42}$$

$$\Gamma_{ml}^{SQ} = \sqrt{-4\ln 0{,}5/w_{ml}^{SQ}}. \tag{43}$$

All formulas and symbols have evident meaning and do not require special comments. Each Gaussian is defined by two parameters; a frequency parameter w and price p or quantity q, at which the Gaussian reaches its maximum value. Instead of the frequency parameter w, we will often use the so-called natural width Γ, which is, by definition, a full width at half maximum of the Gaussian. Other details can be found in [5].

7. The Total S&D Hypothesis. Thus, in quantum economy we decided to use the same basic functions as in probability economics where they are constructed according to another logic. Probability economics has the goal to put forth the most plausible and the most simple descriptions of behavior strategies of agents on the market. This gives us the following tip when trying to choose the normalization constants and their interpretation in quantum economy. Just as in probability economics, we will assume that they are the respective total agent S&D. For instance, D_{nl}^{0} is the total demand of the n-th agent on the l-th good, which is determined by the natural formula:

$$D_{nl}^{0} = p_{nl}^{D} \cdot q_{nl}^{D}, \tag{44}$$

and total demand of the n-th agent on all goods, D_n^0, is defined, of course, as the sum of the total one-good demands:

$$D_n^0 = \sum_{l=1}^{L} D_{nl}^{0} = \sum_{l=1}^{L} p_{nl}^{D} \cdot q_{nl}^{D}. \tag{45}$$

Continuing to follow this logic, we get that total market demand, D^0, defined naturally as the integral of the market demand function $D(\boldsymbol{p},\boldsymbol{q})$ over the PQ-space is as follows:

$$D^0 \equiv \int_{-\infty}^{+\infty} D(\boldsymbol{p},\boldsymbol{q})\, d\boldsymbol{p}\, d\boldsymbol{q} = \sum_{n=1}^{N} D_n^0 = \sum_{n=1}^{N}\sum_{l=1}^{L} D_{nl}^0 = \sum_{n=1}^{N}\sum_{l=1}^{L} p_{nl}^D \cdot q_{nl}^D. \quad (46)$$

Completely similar in structure and meaning, formulas for supply take place such that we give them without special comments:

$$S_{ml}^0 = p_{ml}^S \cdot q_{ml}^S. \quad (47)$$

$$S_m^0 = \sum_{l=1}^{L} S_{ml}^0 = \sum_{l=1}^{L} p_{ml}^S \cdot q_{ml}^S. \quad (48)$$

$$S^0 \equiv \int_{-\infty}^{+\infty} S(\boldsymbol{p},\boldsymbol{q})\, d\boldsymbol{p}\, d\boldsymbol{q} = \sum_{m=1}^{M}\sum_{l=1}^{L} S_{ml}^0 = \sum_{m=1}^{M}\sum_{l=1}^{L} p_{ml}^S \cdot q_{ml}^S. \quad (49)$$

8. The Coupled Quantum Harmonic Oscillators Hypothesis. Now let us regress a few steps, back to our basic problem of trying to establish the Hamiltonian for economy. To this end, we draw attention to the formulas (14) and (15) for basic wave functions. Based on the formulas for the corresponding basis functions of S&D (34)–(43), we can easily infer that the basic wave functions are Gaussians too:

$$\psi_{nl}^{DP}(p_{nl}) = \sqrt[4]{\frac{w_{nl}^{DP}}{\pi}} \cdot \exp\left(-\frac{1}{2} w_{nl}^{DP}(p_{nl} - p_{nl}^D)^2\right), \quad (50)$$

$$\psi_{nl}^{DQ}(q_{nl}) = \sqrt[4]{\frac{w_{nl}^{DQ}}{\pi}} \cdot \exp\left(-\frac{1}{2} w_{nl}^{DQ}(q_{nl} - q_{nl}^D)^2\right), \quad (51)$$

$$\psi_{ml}^{SP}(p_{ml}) = \sqrt[4]{\frac{w_{ml}^{SP}}{\pi}} \cdot \exp\left(-\frac{1}{2} w_{ml}^{SP}(p_{ml} - p_{ml}^S)^2\right), \quad (52)$$

$$\psi_{ml}^{SQ}(q_{ml}) = \sqrt[4]{\frac{w_{ml}^{SQ}}{\pi}} \cdot \exp\left(-\frac{1}{2} w_{ml}^{SQ}(q_{ml} - q_{ml}^S)^2\right). \quad (53)$$

It is not difficult to recognize markers in these functions of the one-dimensional wave functions for quantum harmonic oscillators, being in normal stationary states. Indeed, the Hamiltonian and the wave function of the normal state of the one-dimensional quantum harmonic oscillator is given by well-known two-parameter formulas [7]:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{\mu\omega^2}{2}(x - x_0)^2. \tag{54}$$

$$\psi(x) = \sqrt[4]{\frac{m\omega}{\hbar\,\pi}}\exp\left(-\frac{\mu\omega^2}{2\hbar}(x - x_0)^2\right). \tag{55}$$

The energy of this state is determined by the famous formula:

$$E_0 = \frac{1}{2}\hbar\,\omega, \tag{56}$$

where μ is the mass of the physical particle, $\hbar$ is the well-known reduced Planck's constant, ω is the frequency of the oscillator, x_0 is a point at which the potential energy is minimal, and the wave function of the oscillator reaches the maximum value. Thus, if we assume that there is such a relationship between the frequency parameters w and ω:

$$w = \frac{\mu\omega}{\alpha}, \tag{57}$$

then we can formally consider basic wave functions of our economy to be wave functions of quantum harmonic oscillators with the masses μ, the frequencies ω and the price and quantity values of coordinates, in which the functions are maximal. Leaving the discussion of the meanings of these parameters, we will only comment that the words mass, frequency and energy are used here only for the sake of convenience. They have nothing to do, of course, with physical masses, frequencies of vibrations or the energy of physical systems.

Intuition suggests that the noticed similarity or analogy with physics has deep roots. Analysis of this analogy provides insight into the nature and structure of economic systems. It gives us the chance to formally treat basic wave functions of market agents (50)–(53) as wave functions of oscillators and the agents themselves as a set of oscillators in the PQ-space. As a consequence, all of this provides reason enough to regard the whole market as the system of the coupled basic one-dimensional quantum harmonic oscillators. The total Hamiltonian of the market can be expressed, step-by-step, as the sum of Hamiltonians of all individual oscillators in the natural following manner:

$$\hat{H} = \sum_{n=1}^{N} \hat{H}_n^D + \sum_{m=1}^{M} \hat{H}_m^S, \tag{58}$$

the Hamiltonian of each oscillator is broken down into the simple sum of the respective one-good Hamiltonians as follows:

$$\hat{H}_n^D = \sum_{l=1}^{L} \hat{H}_{nl}^D, \quad \hat{H}_m^S = \sum_{l=1}^{L} \hat{H}_{ml}^S, \tag{59}$$

and every such Hamiltonian is represented, in turn, as the sum of the so-called basic price and quantity Hamiltonians:

$$\hat{H}_{nl}^D = \hat{H}_{nl}^{DP}(p_l) + \hat{H}_{nl}^{DQ}(q_l)\ , \quad \hat{H}_{ml}^S = \hat{H}_{ml}^{SP}(p_l) + \hat{H}_{ml}^{SQ}(q_l), \tag{60}$$

and we can, ultimately, portray the total Hamiltonian of the system as a double sum of the basic Hamiltonians as follows:

$$\hat{H} = \sum_{nl}^{NL} \hat{H}_{nl}^{DP}(p_{nl}) + \sum_{ml}^{ML} \hat{H}_{ml}^{SP}(p_{ml}) + \sum_{nl}^{NL} \hat{H}_{nl}^{DQ}(q_{nl}) + \sum_{ml}^{ML} \hat{H}_{ml}^{SQ}(q_{ml}). \tag{61}$$

All designations in these formulas are obvious and do not require comments. Basic Hamiltonians are, evidently, Hamiltonians of the quantum harmonic oscillators. We will write them in the usual format (54):

$$\hat{H}_{nl}^{DP}(p_{nl}) = -\frac{\alpha^2}{2\mu_{nl}^{DP}} \frac{\partial^2}{\partial p_{nl}^2} + \frac{\mu_{nl}^{DP}\omega_{nl}^{DP2}}{2}(p_{nl} - p_{nl}^D)^2, \tag{62}$$

$$\hat{H}_{nl}^{DQ}(q_{nl}) = -\frac{\alpha^2}{2\mu_{nl}^{DQ}} \frac{\partial^2}{\partial q_{nl}^2} + \frac{\mu_{nl}^{DQ}\omega_{nl}^{DQ2}}{2}(q_{nl} - q_{nl}^D)^2, \tag{63}$$

$$\hat{H}_{ml}^{SP}(p_{ml}) = -\frac{\alpha^2}{2\mu_{ml}^{SP}} \frac{\partial^2}{\partial p_{ml}^2} + \frac{\mu_{ml}^{SP}\omega_{ml}^{SP2}}{2}(p_{ml} - p_{ml}^D)^2, \tag{64}$$

$$\hat{H}_{ml}^{SQ}(q_{ml}) = -\frac{\alpha^2}{2\mu_{ml}^{SQ}} \frac{\partial^2}{\partial q_{ml}^2} + \frac{\mu_{ml}^{SQ}\omega_{ml}^{SQ2}}{2}(q_{ml} - q_{ml}^S)^2. \tag{65}$$

The energy of such a system is found by means of the additivity rule:

$$E_0 = \frac{1}{2}\sum_{nl}^{NL} \alpha\omega_{nl}^{DP} + \frac{1}{2}\sum_{ml}^{ML} \alpha\omega_{ml}^{SP} + \frac{1}{2}\sum_{nl}^{NL} \alpha\omega_{nl}^{DQ} + \frac{1}{2}\sum_{ml}^{ML} \alpha\omega_{ml}^{SQ}. \qquad (66)$$

All members on the right-side of the equation should be of the same dimension. This means that all frequencies, both the ones for price as well as quantity, must also be of the same dimension. Therefore, the dimensions of the price and quantity masses must be different, which is not surprising as the dimensions of independent price and quantity variables are different. Let us repeat, that the word energy is used only for the convenience of drawing the analogy with physics. Economic energies have nothing to do with physical energy, and they are just eigenvalues of the economic Hamiltonians. Therefore, we will not discuss the meaning of economic energy. Moreover, we cannot ourselves clarify this meaning at the present time; it can, in principle, be done later after numerous calculations have been performed and they have been matched with empirical data.

To avoid misunderstanding, we emphasize that the fact that the total Hamiltonian of the system is represented as a sum of many basic Hamiltonians, does not mean that all basic oscillators are independent or free. Generally speaking, it is quite the contrary. In most cases, all market agents strongly interact with each other, and, therefore, all oscillators are strongly coupled. To put it another way, all parameters of all oscillators must be derived, strictly speaking, by solving the Schrödinger equation for the whole economic system. Then, the subsequent approximation represents the total Hamiltonian as a sum of Hamiltonians of independent basic oscillators. Strong interaction among agents, resulting in a strong coupling of oscillators, can manifest itself only when, for example, one of the frequencies or the masses change. In this case, due to the strong interaction of oscillators, parameters of other oscillators also will change. Concluding this section, note that within the framework of this model every agent can be imaged as a bundle of coupled basic oscillators. In turn, the whole economic system can be regarded as a set of these agent bundles.

Note finally that the system of quantum harmonic oscillators is not the simplest physical system that may serve as a physical prototype for the economic systems. For example, we can instead use a simpler system of infinite potential wells to model economic systems. It is well-known that for a one-dimensional infinite well potential $V(x)$ with width $a = x_2 - x_1$ (if $x < x_1$ or $x > x_2$, then the potential $V(x)$ is equal to infinity, and inside the well this potential is equal to 0). The Hamiltonian and wave function of the normal state are given by the following two-parameter formula [7]:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x). \tag{67}$$

$$\psi(x) = \sqrt{\frac{2}{a}} \cdot \sin\left(\frac{\pi}{a}\left(x - x^m + \frac{a}{2}\right)\right), \tag{68}$$

where x^m is a coordinate, in which the wave function $\psi(x)$ reaches its maximum value:

$$x^m = \frac{1}{2}(x_1 + x_2). \tag{69}$$

In other words, x^m is the middle of the well. The energy of this state is given by the well-known formula:

$$E_0 = \frac{\pi^2 \hbar^2}{2ma^2}. \tag{70}$$

Interestingly, in this case the wave function depends only on the size and position of the well, neither having dependency on the mass of the particle, nor on the Planck's constant. We believe that when applying it to economic models, the model of strongly coupled infinite potential wells is not so far from reality than would appear at first glance. Sinusoidal basic functions could be well-suited for purposes not only qualitatively correct, but sufficiently accurate for quantitative descriptions of the agents' behavior in the market in a fairly wide variety of cases.

9. The Economic Mass Hypothesis. As we can see, in the oscillators model discussed above, the challenge is in determining the three parameters for each basic oscillator: economic mass μ, frequency ω, and desired price p or quantity q, which altogether define all the most interesting properties of the entire economic system. If we know them, we can calculate all the S&D functions, both of every agent and of the market. The main advantage of this model is the ability to assign each parameter to a particular agent and good that allows the correct economic interpretation of these parameters to be sought out. It is quite simple to find the meaning of frequencies, since they are directly correlated with natural widths of the basic S&D functions. This correlation is easily derived by means of the formulas (37), (38), (42), (43) and $\omega = \dfrac{\alpha}{\mu} w$ as follows:

$$\Gamma = \sqrt{-4\ln 0{,}5\,\alpha/\mu\omega}. \tag{71}$$

This formula implies that frequency, in physical economics, is a formal parameter of the theory, which determines the width of the basic S&D functions. In principle, these frequencies must be obtained by solving the Schrödinger equation for the system as in quantum mechanics of molecules. In the simplest case of the oscillator model of an economy, they can be specified and preassigned to the agents as constant parameters according to their behavior strategies in the market. Less clear in the theory is the economic meaning of the masses, but formal similarities of these masses and the full S&D, observed above, allows us to put forward the following hypothesis. For the masses we can use the total basic S&D by means of the following definitions:

$$\mu_{nl}^{DP} \cong \gamma^{P} \cdot D_{nl}^{0}, \quad \mu_{nl}^{DQ} \cong \gamma^{Q} \cdot D_{nl}^{0}, \tag{72}$$

$$\mu_{ml}^{SP} \cong \gamma^{P} \cdot S_{ml}^{0}, \quad \mu_{ml}^{SQ} \cong \gamma^{Q} \cdot S_{ml}^{0}. \tag{73}$$

Here γ are some normalization constants, the primary purpose of which is to ensure the correct dimension of both price and quantity masses. And then, using the additivity formulas derived above, we can get completely natural sums as definitions of the masses of agents, subsystems of buyers and sellers, and, eventually, of the whole economic system as follows:

$$\mu_{n}^{D} = \sum_{l=1}^{L} \mu_{nl}^{DP}, \quad \mu_{m}^{S} = \sum_{l=1}^{L} \mu_{ml}^{SP}, \tag{74}$$

$$\mu^{D} = \sum_{n=1}^{N} \mu_{n}^{D}, \quad \mu^{S} = \sum_{m=1}^{M} \mu_{m}^{S}, \tag{75}$$

$$\mu = \mu^{D} + \mu^{S}. \tag{76}$$

All designations in these formulas are clear in origin and do not require special comments.

10. The Economic Hamiltonian Hypothesis. The total value of the above results, concerning the system of oscillators, lies in the fact that these results point the way to a method of constructing the total Hamiltonian of economic systems. More specifically, these results, along with intuition suggest that a Hamiltonian of an economic system in the PQ-space has the same general form as Hamiltonians of the similar physical systems of material points in real space. More exactly, there are good grounds to believe that, for the sake of physical modeling of economic systems, we can use the following economic Hamiltonian:

$$\hat{H} = \sum_{nl}^{NL}\left(-\frac{\alpha^2}{2\mu_{nl}^{DP}}\frac{\partial^2}{\partial p_{nl}^2} - \frac{\alpha^2}{2\mu_{nl}^{DQ}}\frac{\partial^2}{\partial q_{nl}^2}\right) + \sum_{ml}^{ML}\left(-\frac{\alpha^2}{2\mu_{ml}^{SP}}\frac{\partial^2}{\partial p_{ml}^2} - \frac{\alpha^2}{2\mu_{ml}^{SQ}}\frac{\partial^2}{\partial q_{ml}^2}\right) +$$

$$+ \sum_{nm}^{NM} V_{nm}^{SD} + \sum_{n\neq n'}^{NN} V_{nn'}^{DD} + \sum_{m\neq m'}^{MM} V_{mm'}^{SS} + \sum_{n=1}^{N} U_n^D + \sum_{m=1}^{M} U_m^S, \qquad (77)$$

where all the designations are clear and do not require special comments. Pairwise potentials (V) are intended to describe the interaction among the agents, and the potentials U are introduced to describe the impact of the external environment on agents. At the moment we do not have a sufficiently clear idea of what all these inter-agent forces and external potentials are. We can only make some additional assumptions with the following charcteristics.

From experience, we know that buyers and sellers tend to travel towards each other until the market reaches the so-called equilibrium state, in which prices and quantities offered by the buyers, on the one hand, and the sellers, on the other hand, are equal [5, 6]. This means that the force, acting between a buyer and a seller, is attractive [1]. Therefore, we will assume that it formally looks like the force of attraction between the two point particles with masses. The foregoing notions can be represented by the following formula:

$$V_{nm}^{SD} = \left\{\gamma_l^{SDP}\cdot\frac{\mu_{nl}^{DP}\cdot\mu_{ml}^{SP}}{|p_n - p_m|} + \gamma_l^{SDQ}\cdot\frac{\mu_{nl}^{DQ}\cdot\mu_{ml}^{SQ}}{|q_n - q_m|}\right\}, \qquad (78)$$

where γ_l^{SDP} and γ_l^{SDQ} are constants, which should be determined by means of comparison with empirical data, they also serve as the normalization constants to ensure the correct dimension of forces. By definition, all of them, or at least most of them have values greater than 0.

Now let us consider the possible view of potentials that describe the interaction between the two buyers $V_{nn'}^{DD}$ and between the two sellers $V_{mm'}^{SS}$. From experience, we know that buyers, interacting with each other by competing for a rare resource, exert an upward pressure on prices, according to the well-known law of demand. Metaphorically speaking, they behave in much the same way as the same charged particles. So, it makes sense to attempt using the following sums to describe the inter-buyer potentials $V_{nn'}^{DD}$:

$$V_{nn'}^{DD} = \sum_{l=1}^{L} \left\{ \gamma_l^{DDP} \cdot \frac{\mu_{nl}^{DP} \cdot \mu_{ml}^{DP}}{|p_{nl} - p_{ml}|} + \gamma_l^{DDQ} \cdot \frac{\mu_{nl}^{DQ} \cdot \mu_{ml}^{DQ}}{|q_{nl} - q_{ml}|} \right\}, \tag{79}$$

where all or a substantial part of the γ_l^{DDP} and γ_l^{DDQ} have magnitudes greater than zero, as a rule. They are also not known *a priori*, and shall be determined empirically by comparing the results of calculations with empirical data. In addition, these constants are also intended to ensure the correct dimension of potentials. A similar formula is taken by definition to describe the interaction of two sellers as follows:

$$V_{mm'}^{SS} = \sum_{l=1}^{L} \left\{ \gamma_l^{SSP} \cdot \frac{\mu_{ml}^{SP} \cdot \mu_{ml}^{SP}}{|p_{ml} - p_{ml}|} + \gamma_l^{SSQ} \cdot \frac{\mu_{ml}^{SQ} \cdot \mu_{ml}^{SQ}}{|q_{ml} - q_{ml}|} \right\}, \tag{80}$$

where all, or at least, most of the constants γ_l^{SSP} and γ_l^{SSQ} have values less than zero, i.e., they are negative, as a rule. Basis for this claim follows immediately from the empirical law of supply. Sellers, competing for cash of buyers, have usually to reduce prices. In other words, the interaction between sellers at have an effect on prices, causing them to decrease, analogous to the interaction of unlikely charged particles in physical systems.

With regard to the interaction of agents with the external environment, of course, all the potentials are chosen *ad hoc*. That is, in each case, it could be either taxes or excise duties or a natural disaster in the form of drought, etc., and the corresponding potentials shall be treated separately. The only thing we can say right now is that in order to describe these interactions, one can use the potentials that depend solely on the coordinates of a single particle, such as local potentials [1]:

$$U_n^D = \sum_l^L \left\{ U_{nl}^{DP}(p_{nl}) + U_{nl}^{DQ}(q_{nl}) \right. , \quad (81)$$

$$U_m^S = \sum_l^L \left\{ U_{ml}^{SP}(p_{ml}) + U_{ml}^{SQ}(q_{ml}) \right. , \quad (82)$$

where all the designations are clear and do not require comments. In dealing with specific tasks, one must naturally choose those potentials in a simple analytical form with some constants.

Thus, for starting *ab intio* calculations of a stationary economy's wave functions, we have proposed to use the Hamiltonian that was built to resemble of the Hamiltonian of the related physical systems for material points in the real space, with small variations specific to economic systems. The great advantage of this Hamiltonian is the fact that it contains only the constants α and γ. The total wave function can initially be completely factorized, for example, in the form products of the corresponding basic functions $\psi_{nl}^{DP}(p_{nl})$, $\psi_{nl}^{DQ}(q_{nl})$. It is precisely these functions that should be determined by solving the Schrödinger equations using any standard method of quantum mechanics. For example, we can pointedly suggest that the most appropriate is the self-consistent field method [1, 7], in combination with the method of Roothaan in quantum chemistry. If we use Gaussians as basis functions, , then, to *ab initio* calculate stationary states of economic systems within the framework of the Rootaan's method, we can rather quickly and easily adapt the well-known GAUSSIAN-type programs which have been successfully used for many years in quantum chemistry of atoms, molecules, and solids.

At the present, this is as far as we can go with quantum economy, treated in the narrow sense as the method of *ab intio* wave function calculation of the stationary model economies, by means of deriving solutions of the economic Schrödinger equations for these economies. However, it is enough to start with such calculations for various stationary model economies seeking out ranges of possible values for all the model constant parameters α and γ as well as possible external potentials U.

6. Numerical Calculations for the Model of Coupled Quantum Harmonic Oscillators

As stated above, in order to be able to obtain *ab initio* solutions to economic Schrödinger equations, one must first create a rather complicated computer program from the outset. Therefore, in the present work we restrict our consideration to the solutions of the Schrödinger equations for the approximate model of oscillators. Currently, this is enough for our purpose because the main aim of the present paper is to demonstrate only the very fact that it is possible to obtain S&D functions by solving the Schrödinger equation for economic systems. As a stationary quantum economy, we will choose the simplest system with one buyer, one seller operating in the market with a single traded good, namely, grain. In the oscillators model, the wave functions depend only on frequency parameters *w* and equilibrium

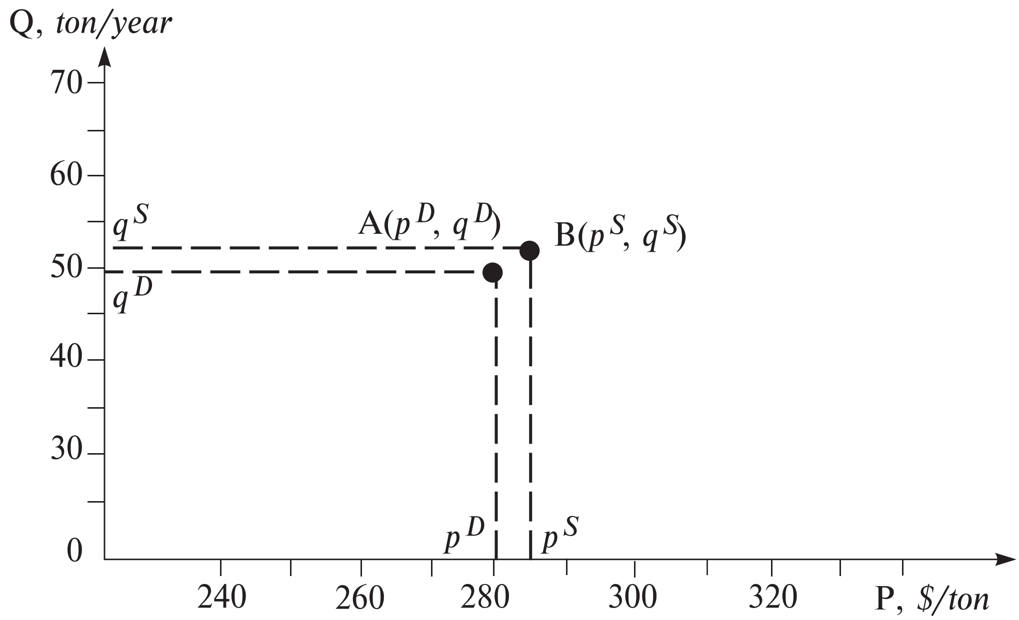

Fig. 2. Graphical representation of the dot strategies of the buyer's and seller's market behavior by the points A (p^D, q^D) and B (p^S, q^S) in the two-dimensional PQ-space (plane) of the model market of grain. p^D = 280,0 *\$/ton*, q^D = 50,0 *ton/year*, p^S = 285,0 *\$/ton*, q^S = 52,0 *ton/year*.

values of p or q coordinates for each basic oscillator. Nevertheless, for greater certainty we will assume that in our particular model, all other constants are simply equal to 1:

$$\alpha = 1, \quad \gamma^P = 1, \quad \gamma^Q = 1.$$

The results of the calculations are shown in Figs. 2–6, where captions contain all the information concerning the details of the calculations. Additional comments to the pictures can be found in Chapter VII and article [5], where these calculations were carried out for the first time within the framework of probability economics. Readers interested in greater detail in the economic meaning of the results obtained, we refer to this article. In the present Chapter we restrict our investigation to considering, discussing and demonstrating the principal ability of quantum economy to obtain the S&D functions from first principles by solving economic Schrödinger equations.

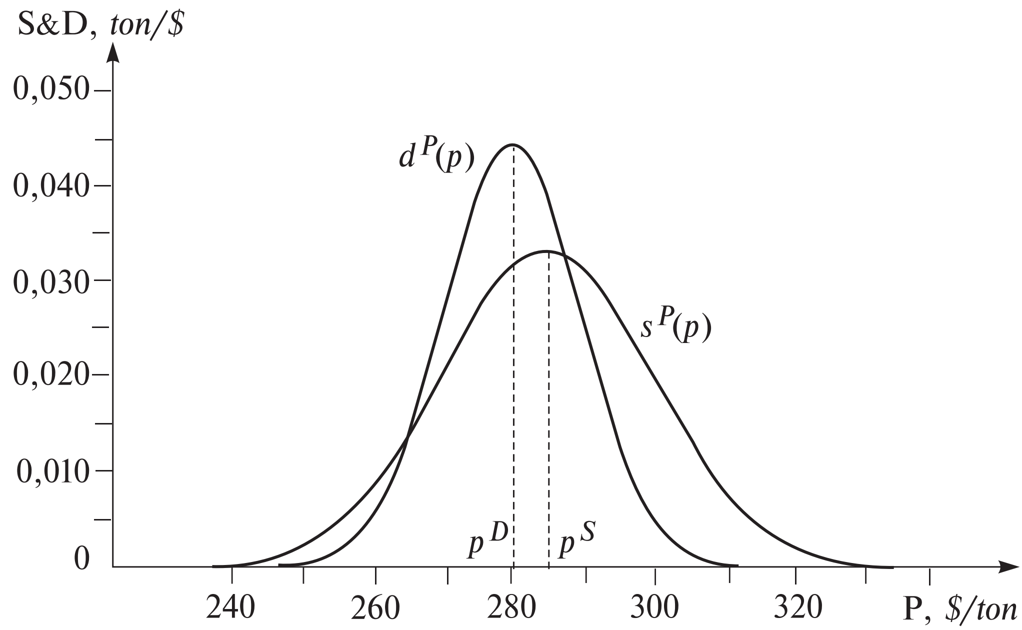

Fig. 3. Graphical representation in the conventional rectangular three-dimensional coordinate system [P, Q, S&D] of the one-dimensional S&D P-functions, $d^P(p)$ of the buyer and $s^P(p)$ of the seller in our model market of grain (see Fig. 2), as the two-dimensional curves with maxima at prices $p^D = 280{,}0$ *\$/ton*, $p^S = 285{,}0$ *\$/ton* and natural widths $\Gamma^{DP} = 23{,}8$ *\$/ton*, $\Gamma^{SP} = 37{,}0$ *\$/ton*, respectively.

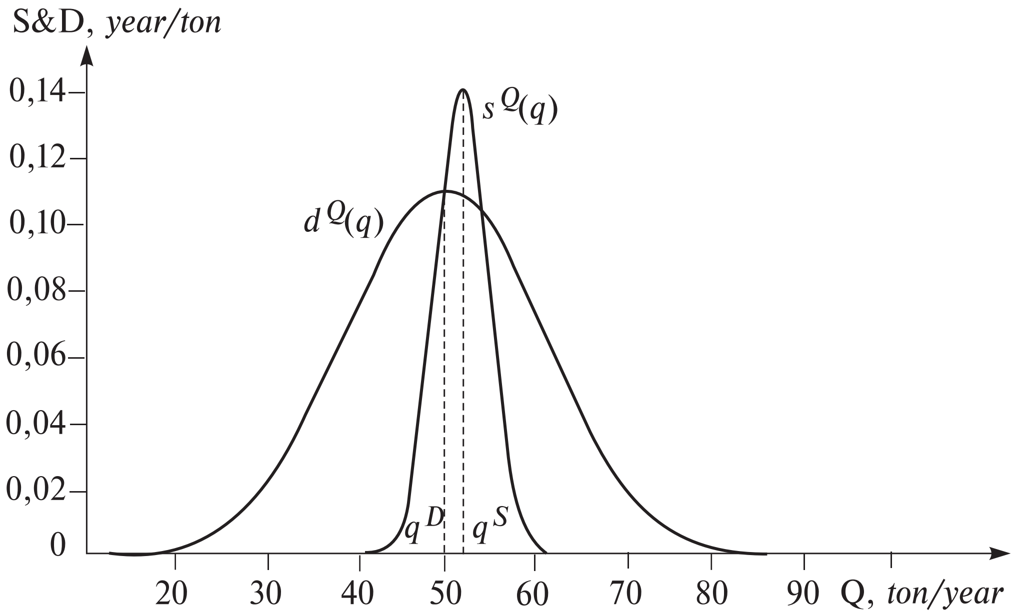

Fig. 4. Graphical representation in the conventional rectangular three-dimensional coordinate system [P, Q, S&D] of the one-dimensional D&S Q-functions, $d^Q(q)$ of the buyer and $s^Q(q)$ of the seller of our model market of grain (see Figs. 1, 2), as the two-dimensional curves with maxima at quantities q^D, and natural widths Γ^{DQ} and Γ^{SQ} respectively. $\Gamma^{DQ} = 26{,}4$ *ton/year*, $\Gamma^{SQ} = 6{,}8$ *ton/year*.

7. Excited States of Quantum Economies

So far, we have confined ourselves to the consideration of only the ground or normal state of quantum economy. However, we know that apart from the normal state there are also an infinite number of excited states with energy E_n, a greater energy compared to that of the normal state (see the stationary economic Schrödinger equations (4)–(6)). The fact that such excited states may exist holds great significance. At present, all we know about excited states is in that their wave functions may have one or more nodes, thus their probability distributions have two or more maximums. That is why the above definition of demand and supply functions hardly makes sense for excited states, or it should somehow be generalized. Intuitively it is also evident that excited states of economy are states of less order and greater chaos in economy, therefore, of its lower efficiency. Since the total demand and supply functions $D(p)$ and $S(p)$ are far from each other in the price-quantity

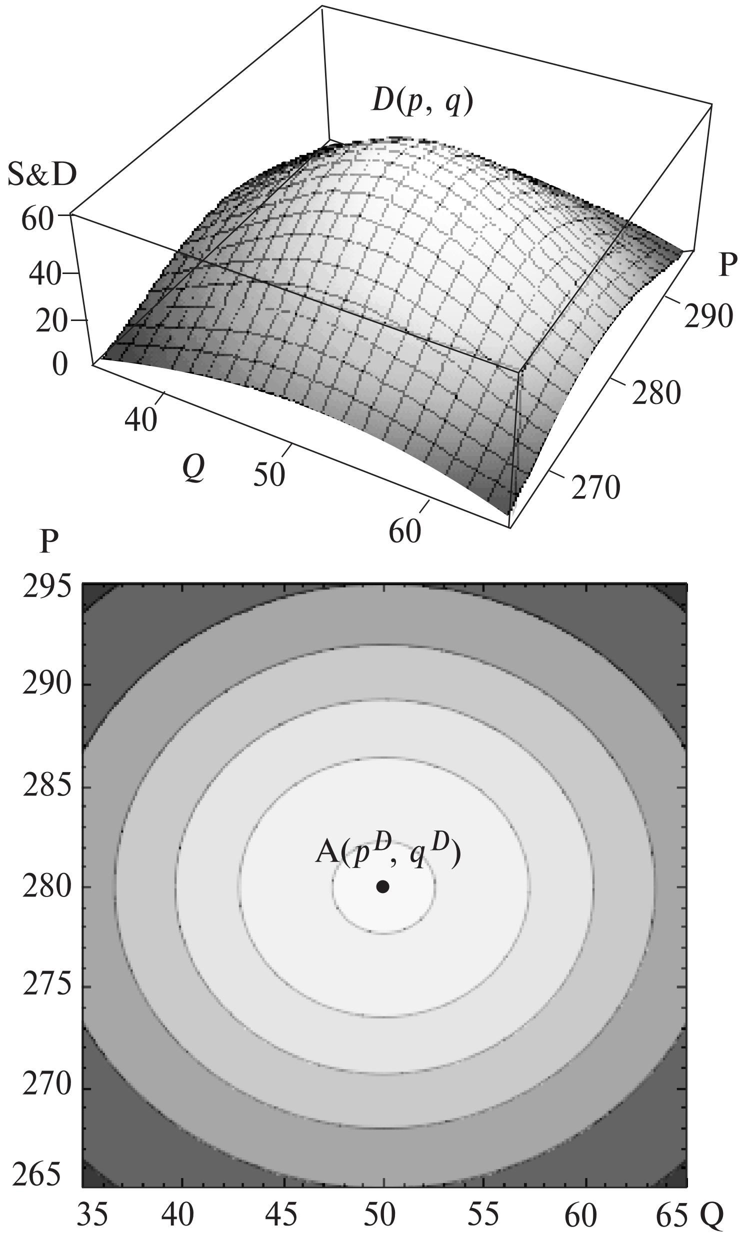

Fig. 5. Graphical representation in the conventional rectangular three-dimensional coordinate system [P, Q, S&D] of the two-dimensional demand function of the buyer as the three-dimensional surface $D(p,q)$ with maximum having the projection on the (P, Q)-plane at the point A (p^D, q^D).

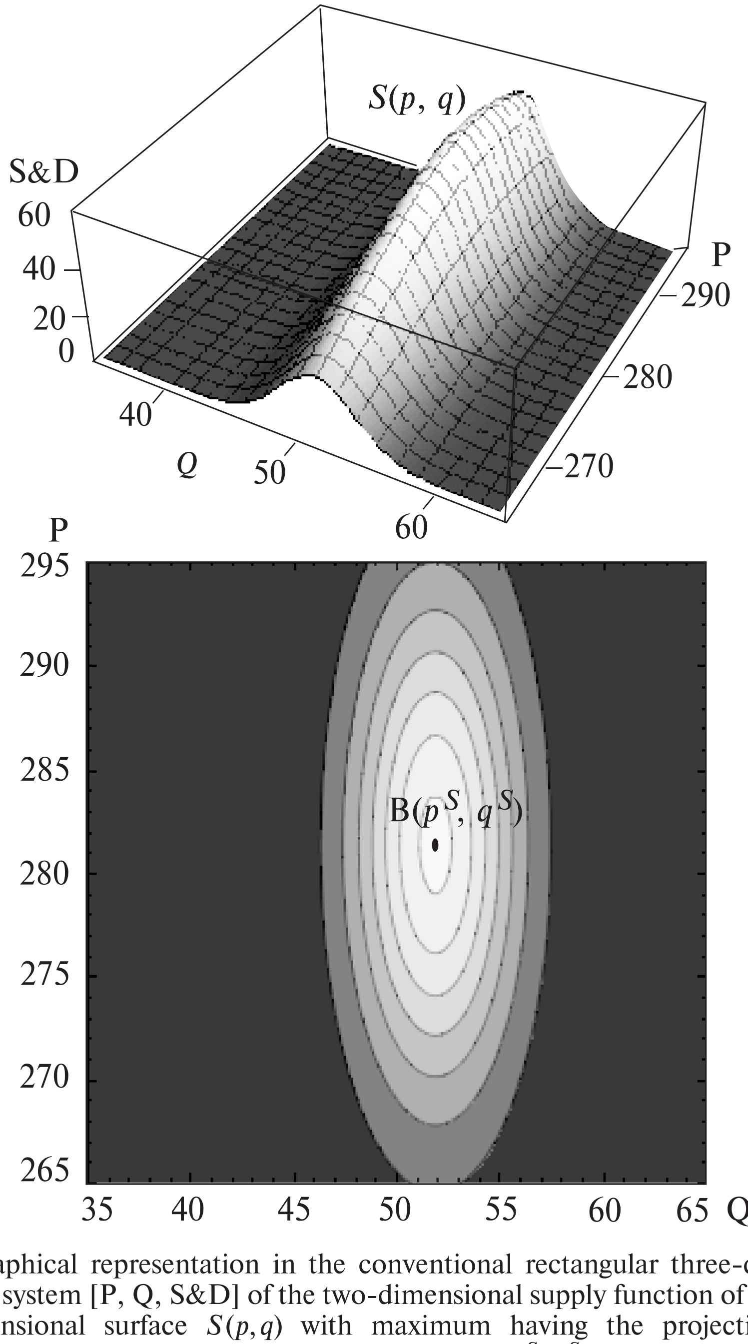

Fig. 6. Graphical representation in the conventional rectangular three-dimensional coordinate system [P, Q, S&D] of the two-dimensional supply function of seller as the three-dimensional surface $S(p,q)$ with maximum having the projection on the (P, Q)-plane at the point B $(p^S; q^S)$.

space they do not overlap enough. The difference $(p^S - p^D)$ between the maximum prices of the total demand (p^S) and supply (p^D) functions $D(p)$ and $S(p)$ correspondingly in the excited state is much more than this difference in the normal state of the economy. Therefore, total sales transactions probabilities are small and as a consequence the total trade volume *MTV* is small. Hence, total production volume (or/and GDP) of the economy is small too.

So, it is because of quantum economy that governments are not advised to make drastic changes in their economic policy, in order to reduce the likelihood of transferring the economy from a normal state to an excited, evidently less efficient state. This conclusion of quantum economy is consistent with the viewpoint of classical economic theory that intervention by the government into the economy is harmful as it reduces the economy's efficiency. This is a known fact. But it is quite another matter when the economy is already in an excited state. In this situation, according to quantum economy it is necessary for the government to cause strong and abrupt time-dependent perturbation $U(t)$, causing the economy to gradually return from the excited state to the normal state. The market itself cannot create this occurrence with its internal forces or/and strengths such as whith constant V potentials. In other words, we may conclude from quantum economy that internal constant forces or/and strengths are in principle not enough to make the economy transfer from an excited state to a normal one, or in actual terms, recover from recession and begin to improve. In this case, the role of government and its interference in economy should be scant, hereby benefitting the whole economy. The problem lies only in the "right" diagnosis of economy, the "right" time and the "right" measures in choosing the suitable time dependent perturbation $U(t)$.

8. Quantum Economic Cycles

So, over the course of the latest development of excited states discussed above, we have obtained the following interesting result concerning the nature of economic cycles. It turned out, that the main cause of an economic crisis is the "non-right" time-dependent perturbations of the economic environment, initially induced by government action. We are not interested here in other institutional and environmental factors which can be treated in this respect in the same way. In other words, this is the occurrence of a transition of the economy from the effective ground state with a large trade volume *MTV* (see Chapter VIII) to the non-effective excited state with a lower trade volume *MTV* under the influence of the government strong time-dependent perturbations.

Furthermore, due to the subsequent drastic “right” measures taken by governments at the “right” time, the economy can overcome recession and transfer from the non-effective excited state to the effective normal state.

Thus, within the framework of the quantum economy the main features of the nature of economic cycles are connected with the transitions (or oscillations) between ground and excited stationary states of the economy. These are caused mainly by time-dependent interventions or measures by the government and other external environment institutions such as overseas markets. This simple quantum two-state model is described by a linear combination of these two states, or in other words, by means of the following time-dependent wave function (see (6)):

$$\Psi(t) = a_{gr}(t) \cdot \psi^0_{gr}(t) + a_{ex}(t) \cdot \psi^0_{ex}(t), \tag{83}$$

$$|a_{gr}(t)|^2 + |a_{ex}(t)|^2 = 1. \tag{84}$$

Here $\psi^0_{gr}(t)$ and $\psi^0_{ex}(t)$ are the wave functions of the ground state and the relevant excited stationary state of the economy, without time-dependent interventions or perturbations of government and other external institutions ($\partial U(t)/\partial t = 0$). Conversely, the respective time-independent interactions of economy agents with the government and institutions can be quite large. In conclusion, if the time-dependent interactions $U(t)$ are relatively weak, we can use the perturbation theory for calculations of co-efficients $a_n(t)$ as one does in quantum mechanics [2].

9. Interrelation Quantum Economy with Other Economic Theories

Concluding, we call attention to the fact that all interactions on markets are strong and comparable in most cases. Therefore, according to the institutional and environmental principle, adequate quantitative economic theory must unambiguously consider influences on the behavior of the market agents both by internal factors (interaction between the market agents) and external factors (influence of other markets and economies, institutes, state, natural factors etc.). For this reason, such theory must represent the synthesis of classical economic theory, institutional economics, evolutionary economics and so forth that quantum economy really does. Moreover, the obtained equations of motion for economies can be calculated by computers using modified programs that calculate the electronic structure of atoms and molecules, or in other words, the methods of atom theories and quantum chemistry of molecules and solids.

10. Conclusions

We understand that in this chapter, we have obtained only a preliminary or initial approximation for more exact quantum economic models that may appear after more thorough and continuous investigations in this field. As it often happens after discovering new approaches and theories, more questions occur than before. First of all, there are questions pertaining to applicability limits regarding the physical modeling of complicated economic systems, as well as to the economic essence of parameters and potentials of models, etc. We hope to clarify all these questions in the future.

As we already mentioned in the previous chapter, we suppose that both the time-dependent and stationary Schrödinger equations derived in this chapter can be regarded, to some extent, as the formal mathematical representations of the market-based trade maximization principle in the price-quantity space. We can also say that the Hamiltonians derived above are, in turn, the mathematical quantum representation of the invisible market hand concept in the price-quantity space.

To summarize, let us return to this chapter's main point and give the following definition of quantum economy. In contrast to probability economics, the phenomenological version of probabilistic economic theory, quantum economy, can be defined as an ab initio version of probabilistic economic theory. Quantum economy is the study of macroscopic market economic phenomena. It looks at the whole of the economic system, using microscopic equations of motion, or the economic Schrödinger equations, at the level of individual market agents.

References

1. Anatoly Kondratenko. *Physical Modeling of Economic Systems. Classical and Quantum Economies*. Novosibirsk: Nauka, 2005. Electronic copy available at: http://ssrn.com/abstract=1304630.
2. Ludwig von Mises. *Human Action: A Treatise on Economics*. Yale University, 1949.
3. J. Doyne Farmer, Martin Shubik, and Eric Smith. *Is Economics the Next Physical Science?* Physics Today 58:37–42 (2005).
4. Jean-Philippe Bouchaud. *Economics Needs a Scientific Revolution*. Nature. October 2008. Vol. 455/30.
5. Anatoly Kondratenko. *Probability Economics: Supply and Deman*d, *Price and Force in the Price — Quantity Space*. Electronic copy available at: http://ssrn.com/abstract=2337462. See also Chapter VII.
6. Anatoly Kondratenko. *Trade Maximization Principle: Market Processes, Supply and Demand Laws, and Equilibrium States*. Electronic copy available at: http://ssrn.com/abstract=2431218. See also Chapter VIII.
7. L.D. Landau, E.M. Lifshitz. *Theoretical Physics, Vol. 3. Quantum Mechanics, Nonrelativistic Theory*. Moscow, Fizmatlit, 2002.
8. Anatoly Kondratenko. *Probability Economics: Supply and Demand in the Price Space*. Electronic copy available at: http://ssrn.com/abstract=2250343. See also Chapter IV.
9. Anatoly Kondratenko. *Probability Economics: Market Price in the Price Space*. Electronic copy available at: http://ssrn.com/abstract=2263708. See also Chapter V.
10. Anatoly Kondratenko. *Probability Economics: Market Force in the Price Space*. Electronic copy available at: http://ssrn.com/abstract=2270306. See also Chapter VI.

AFTERWORD

1. Is Physics-Based Economics Feasible?

A strong consensus has been formed in the economic community on the feasibility of constructing an economic theory by analogy with physics: it is impossible. Our main authority on economic theory, Ludwig von Mises, goes even further. He believes that in general, it is completely impossible to apply mathematics in economic theory, except for the most basic operations, just because that is the very essence of economic phenomena [1]. However, we are trying to do namely this, that is to build a quantitative economic theory, probabilistic economic theory, by analogy with physics. In defense and justification of our point of view we would like to make the following important comments.

First, we believe that the main obstacle to the development of economic theory and empirical economic studies is the lack of quantitative calculation methods for economic parameters and dynamics of economic systems, based on some more or less common first principles. Economic laws are usually formulated intuitively and qualitatively, without the use of reliable formulas and equations. However, these laws form the base of the most important decisions that affect billions of people. On the part of physicists, it looks as if they built nuclear power plants on the basis of a common understanding of the mechanisms of nuclear processes, but not knowing the famous Einstein formula of $E = mc^2$. Therefore, the development of reliable quantitative methods of economic calculations is, in our view, the most important problem of modern economic theory.

Second, there have been a number of attempts to build this type of quantitative theory by various serious authors. The first was Adam Smith with his famous concept of the invisible hand of the market, where the influence of Isaac Newton's classical mechanics can be easily recognized. "Analogies to physics played an important role in the development of economic theory through the 19th century, and some of the founders of neoclassical economic theory, including Irving Fisher, were originally trained as physicists..." [2]. As we now well understand, such a theory must take into account the principal concept of uncertainty and probability. Therefore it must be constructed by analogy with quantum mechanics, rather than classical. But quantum mechanics is such complicated and sophisticated concept that it is impossible to judge it or use it without proper training and experience. Moreover, quantum mechanics was created to describe phenomena in the microscopic world, and it copes perfectly with this task. It cannot be applied directly to phenomena in the macroscopic world. This notion is perfectly understood by all professionals in physics. Nevertheless, it is not forbidden by anyone or anything to try to simulate the behavior of some macroscopic systems, for example, the dynamics of economic systems, using formal Schrödinger equations. This can be done, we are sure, if this challenge is dealt with intelligently and caution is exercised in the interpretation of the obtained results. In particular, one must clearly understand the applicability limits of this approach. Thus, let us emphasize, that quantum economy is naturally modeled by us on quantum mechanics. It is clear that physical economics is, at the present, only in its infancy. However, we appreciate the great potential of the physical economics for economic studies. Specifically, we believe that physical economics gives us the possibility to gain a more penetrating insight into our market economic world.

Third, it is pertinent here to note that there is the urgent need to develop computer programs to solve economic Schrödinger equations . Among other things, this will lead to the ability to produce more or less reasonable or probable economic forecasts, both local and global markets. This will help predict global economic developments as a whole, something that is extremely important, particularly in terms of preventing the development of crisis in the markets and in the economy as a whole. It is appropriate to draw a parallel with weather or climate forecasts. Of course, weather forecasts are known to be somewhat inaccurate, but no one can deny the huge benefits that they bring, despite the fact that the precision of their predictions leaves many people unsatisfied. The same can be said about future economic predictions. The costs of developing methods for the calculations and computer programs will be quite significant, both financially and in terms of time. They can be approximately estimated by comparing them with corresponding costs made in the process of developing similar systems for quantum chemistry. However, we believe that the process will go much more quickly, as we can just borrow a great deal of knowledge and experience from quantum chemistry and other related sciences. It is clear that the countries that first create these types of systems will experience huge advantages in their economic development. In conclusion, one must note the obvious need, to experimentally or empirically test the physical economics with further clarification and development as is the case with any new theory.

Below, we are going to discuss one of the most important concepts in economic theory, the uncertainty and probability concept, as well as the character of economic laws in terms of an analogy with what Richard Feynman did for physics and biology in his famous lectures [3]. We will briefly introduce the complex ideas of this topic as will be understood by economists, as more depth will be beyond the scope of this paper. We will only focus on the structure of the Feynman approach, which we are striving to follow closely in order to make all of our texts easier for readers to understand. We only pose the problem here to stress an analogy with physics, and postpone a more comprehensive discussion of these important questions for the future.

2. The Uncertainty and Probability Principle in Economic Theory

Uncertainty and probability are perpetual facts of life for ordinary people, as well as major corporations, states, and the economic world as a whole. So an economic theory, if it aims to play a serious role in the economic life of a community, should have in its foundation the probabilistic view of the economic world. Obviously, no one knows exactly what will happen tomorrow, much less next year. Because the living environment of people and corporations is perpetually changing, all plans and strategies are probabilistic in nature and require permanent adjustment. As is taught in business schools, “plans are nothing; planning is everything”, a famous phrase often attributed to Dwight D. Eisenhower. The need to survive in a competitive environment affects all economic participants, including market agents. Here, the same patterns are followed as in nature: “It is not the strongest of the species that survives, nor the most intelligent that survives. It is the one that is most adaptable to change”. This is a famous phrase, which is a rather good summary of the popular interpretation of Charles Darwin’s theory of evolution. We see this feature of humankind all the time in practice; it is not a secret that all market agents admit and take into account in all of their plans and behavioral strategies elements of uncertainty and probability that, in turn, are reflected in probability economics. The functions of supply and demand for even the “strongest” or the “smartest” players on the market are represented as probability distributions in the price-quantity space, not just at a single point in it, as one might expect. Even monopolists and monopsonists usually have to take into account elements of uncertainty and probability in their plans. Their supply and demand functions in real life are not very narrow in the price and quantity subspaces, as they, too, depend on the mood of consumers, wholesalers, middlemen and, of course, from the state, regulating their prices.

Thus, probability economic theory is based on the relatively new economic concept of uncertainty and probability. As to the formal content, this concept is nearly identical to the similar concept of physics in the microscopic world, which we have borrowed but given an entirely different meaning and interpretation from that found in physics. On the one hand, in physics there is no doubt that quantum mechanics gives excellent results for the calculations of physical systems, existing in the so-called quantum world, such as for atoms or molecules. On the other hand, in physics there is still no clear consensus on the unambiguous interpretation of the laws of quantum mechanics and its numerical results, which, figuratively speaking, have driven many generations of physicists crazy. To illustrate this fact, one cites the well-known quotation attributed to Richard Feynman, “If you think you understand quantum mechanics you don’t understand quantum mechanics”. The most popular and practical approach to this problem is to avoid any interpretation at all. This approach is summed up in another famous phrase, “Shut up and calculate” (also often attributed to Richard Feynman, but see also [4]).

But in human activity, just as in the market economic world we see a distinctly different picture. Some of the most fundamental activities of both our daily lives and economic lives of corporations are so inextricably entwined with uncertainty and probability that they seem to have become innate and inalienable characteristics, like the pursuit of happiness in one’s private life and profit in business. “The most that can be attained with regard to reality is probability” [1]. Therefore, it is hardly surprising that the assumptions or premises of applying a probabilistic approach to economic problems are rather well substantiated; we all clearly see that that is the way our human world works. One can say it is just the irony of fate that a probabilistic approach (particularly in quantum mechanics) was first developed in physics to deal with physical phenomena on microscopic levels that cannot be perceived by natural means such as by sight, smell, etc., rather than in economics. There, applying a probabilistic approach now appears to be very natural.

To conclude this section, we refer to a Richard Feynman quote from one of his lectures [3] where he formulates the well-known concept regarding the world's most important scientific knowledge, ranging from physics to biology.

"If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generations of creatures, what statement would contain the most information in the fewest words? I believe it is the atomic hypothesis (or the atomic *fact*, or whatever you wish to call it) that *all things are made of atoms — little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another*. In that one sentence, you will see, there is an enormous amount of information about the world, if just a little imagination and thinking are applied... *Everything is made of atoms*. That is the key hypothesis. The most important hypothesis in all of biology, for example, is that *everything that animals do, atoms do. In other words, there is nothing that living things do that cannot be understood from the point of view that they are made of atoms acting according to the laws of physics*. This was not known from the beginning: it took some experimenting and theorizing to suggest this hypothesis, but now it is accepted, and it is the most useful theory for producing new ideas in the field of biology".

If we remind ourselves that the main concept of physics in the microscopic world is the concept of uncertainty and probability, and replace in these claims the word atom with the word agent, and the word thing or animal with the word market, we need not invent new verbal designs to express our probabilistic view of economies and markets: we can just use the claims of Feynman as a matrix for presenting our probabilistic approach to economies and markets. We did precisely the latter in Chapter I, where we formulated the five general principles of physical economics. Very briefly, the main message contained in them is stated here as follows. All markets consist of agents, buyers and sellers that, according to economic laws, are in constant motion in the price-quantity space in continuous search for good bargains between buyers and sellers. Buyers and sellers are strongly attracted to each other at all distances. But buyers compete amongst themselves for the goods of sellers as if they are repelled from each other, increasing market prices, according to the empirical law of demand. And sellers, in turn, competing for the cash of buyers as if they are attracted to one another, decreasing market prices, once again according to the empirical law of supply. Thus, according to the probabilistic picture, all market phenomena can be understood and explained with a sufficiently high degree of accuracy due to this very simple empirical fact: that all markets consist of agents interacting among themselves in accordance with (the known and yet unknown) economic laws, the most important concept of which is that of uncertainty and probability.

3. Character of Economic Laws

Now let us talk a little bit about the character of these economic laws. We will begin with the fact that the character of many of these laws, on the one hand, is very similar to the character of some physical laws. Take for instance the concept of uncertainty and probability, which is valid both in the world of market economy and in the physical microscopic world of nature. On the other hand, many laws of physics are fairly universal and absolute (at least as yet), like the law of conservation of energy, for example. The law cannot be violated, inter alia, and any attempt to create a "perpetual motion machine" will not bring fruitful results. The character of economic laws is not, in principle, such an absolute. In neither the real economy nor in economic theory do such absolute laws exist. In principle, in each case and at any given time economic laws can be violated by anyone. But this cannot happen for long on a large scale. The largest such experiment, used to build socialism in the Soviet Union, ended in failure after seventy years, yet not one generation of people of the former Soviet Union will pay for this experiment.

Another example of ordinary market life, which demonstrates the ability of anyone to violate economic law, is as follows. One of the most general laws of economic theory and economic life can be roughly stated in the following manner. Usually, buyers seek to buy as cheaply as possible, and sellers want to sell goods at the highest price possible. In practice, this translates into the fact that buyers' prices are usually higher than those of the sellers until the conclusion of the transaction, or into the fact that traders buy goods at one price and sell at a higher price to make a profit. In principle, though, a trader with any motive can sell a good at a price lower than he or she bought it, and can do it until they ruin themselves.

We believe such breaches are always present in the market, but that they exist as deviations from the norm. They appear as small fluctuations, unable to develop into a strong trend or to have a strong effect on the economy. So we can simply ignore them in constructing the theoretical models that we have done in all our works. Thus, we can say that in contrast to physical laws, economic laws have only soft imperatives, or are indicative and normative in character. In addition, it requires many years and significant effort efforts to study economic theory and to understand economic laws, as well as to master their use in practice. However, we know that many people making important economic decisions at the level of, say, states and governments, are not usually well educated in economics. Very often, they are policy specialists and masters in all other areas of human activity, so many have committed arbitrary and violent acts against the economy at the expense of future generations and leaders. Formally, decisions of this kind remove markets — and hence economic systems — from the equilibrium state, which we believe is the most effective and therefore the most desirable state of the economy and society as a whole.

4. The Physical Method of Economics

Note that we try to follow analogies drawn from physics as much as possible, because in modern physics many sophisticated theories have been developed. The diversity of various methods and models created is so great that it is easier to borrow ideas from there and adapt them to our own needs rather than reinvent the wheel every time. We believe that as soon as a new science develops enough that the need for quantitative methods emerges, we should consider the possibility of using physical methods of modeling and calculation. As these methods enter a new science, they bring it closer and closer to becoming an established science. The best way to express this idea seems to be to paraphrase Immanuel Kant's famous saying about the role of mathematics in the natural sciences, that "in any special doctrine of nature there can be only as much *proper* science as there is *physics* therein" (we replaced the word mathematics with the word physics). Let us say it in simpler terms in the following manner. At the present time the level of any proper science is determined mainly by the level of penetration of physical concepts, approaches, methods, experiences, cultures etc. into this science. All of this is now true for economics, particularly for probability economics and quantum economy, where uncertainty and probability encompass the same basic ideas as in physics, in which a wide variety of different probabilistic methods of describing microscopic systems has been developed. We think that the concept of uncertainty and probability is no less important in economics, so it makes sense to borrow a lot of experience from physics and rework it imaginatively for use in economics.

But here it is necessary to exercise some caution. We can borrow from physics concepts, but not the laws themselves and their interpretations. We can borrow formal models and techniques from physics, but not the meanings of concepts, formulae and notions. We can borrow forms of equations of motion and their solutions in the most general terms, but not the values of parameters in these equations, etc., since physics and economics deal with rather different systems. For example, many laws of physics, including the famous energy conservation law, are a consequence of various symmetries in nature which cannot exist with any certainty in economic systems. Therefore, a simple mechanical transfer of formulas and models from physics to economics on a "one to one" basis cannot give us something new. We cannot, in principle, simply open economic laws and penetrate deep into economic phenomena. Economic laws and mechanisms of economic processes can be learned only from practical experience and trying to understand them. Today, we can say they are relatively well-known and described in some detail by means of the most common words, in particular, in [1], but only in the most general form at the level of concepts on a qualitative level. We should try to describe these now well-known economic facts, phenomena, and processes through formal concepts and methods developed in physics by the use of adequate mathematical language on a quantitative level. This is what we put forward as the main task of our theories and models. We believe this to be necessary to transfer economic theory to a higher quantitative level. Let us stress that it is too early to say that economic laws are established and well-known. Economics must still walk a long road before it reaches this point, just as physics has done. This process is best described through the well-known stock phrase, "It is through the interplay of observation, prediction and comparison that the laws of nature are slowly clarified". We first need to develop methods of fairly accurate calculations of important characteristics of economic systems, i.e., create quantitative methods in economics, and then carry out a huge number of such calculations and make verifiable predictions on this basis. Only once we check these calculations can we truly write about the laws of economics. This is the long new journey ahead in economics, upon which we should be guided by the catchphrase of Albert Einstein that "the most incomprehensible thing about the world is that it is at all comprehensible".

This is one of the new roads possible in the development of economic theory, or avenues of economic inquiry. Its advantage lies in the fact that it is possible to intensively develop quantitative methods of calculations and to increase the reliability of economic forecasts. We understand this process as follows. First, one should elaborate computer programs and perform very approximate model calculations in order to improve and adjust the form of equations of motion and the parameters of the models, interaction potentials and various constants in potentials and equations. Second, we should carry out a series of calculations for different models and eventually improve all aspects of the models and theories. In the past, this is how the quantum mechanics of atoms, molecules and solid states resulted in the establishment of the special sub-disciplines in physics as quantum chemistry and solid state physics.

According to the point of view of Ludwig von Mises [1], there are two methods of economic inquires, first, the logical method, to which he was undoubtedly devoted, and second, the mathematical method, which has been discarded by him from fundamental consideration. Without going into details concerning this point of view, and without rejecting the mathematical method, in principle we consider it possible to add to these two methods yet a third method, namely the physical method of economic studies. We recognize this term to mean, first of all, the method of agent-based physical modeling developed in this book, which oriented to the description of market agents' actions and the motion of market as a whole. But there can be entirely other approaches, since in physics, as we know, a number of varying theoretical approaches can be applied to the description of different phenomena. As an example, let us point to the already numerous theoretical studies of financial markets within the framework of econophysics.

All this makes it possible to enlarge the framework and to consider the physical method for the study of the economies and markets

References

1. Ludwig von Mises. *Human Action: A Treatise on Economics*. Yale University, 1949.
2. J. Doyne Farmer, Martin Shubik, and Eric Smith. *Is Economics the Next Physical Science?* Physics Today 58:37–42 (2005).
3. Richard P. Feynman, Robert B. Leighton and Matthew Sands. *The Feynman Lectures on Physics*. Addison — Wesley, 1964.
4. N. David Mermin. *Could Feynman Have Said This*. Physics Today 57 (5), 2004.

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