Axioms for Measuring without mixing apples and Oranges

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Axioms for measuring utility without mixing apples and oranges*

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Abstract

A mixture set is path-connected via a suitable collection of paths, the most common example being a convex set. Yet in many economic settings, there are pairs of prospects that are not connected by a path of mixtures. Consider the thought experiment of von Neumann and Morgenstern involving a glass of milk, a glass of tea and a cup of coffee: we are often asked to choose between convex combinations of milk and tea, yet the same cannot be said of tea and coffee. We introduce partial mixture sets (which need not be path-connected) and provide a formal extension of the well-known axiomatisation of cardinal, linear utility by Herstein and Milnor. We show that partial mixture sets encompass a variety of settings in the literature and present a novel application to cardinal, nonlinear utility on a stochastic process.

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1 Introduction

In [12], Herstein and Milnor (henceforth HM) introduce mixture sets as a generalisation of the familiar convex sets of prospects. The generalisation is substantial in that, provided an appropriate system of paths connect every pair of prospects in the set, any manifold of locations, prizes, lotteries or actions forms an example of the mixture set. Similar to von Neumann and Morgenstern [23], HM take paths to be continuous relative to the preference ordering that the decision maker supplies.

In this paper we extend the main theorem of HM. The latter provides a set of requirements or axioms that are necessary and sufficient for a real-valued function on a mixture set that measures the decision maker's preference ratings. Like HM, the form of utility we derive is cardinal (unique up to a single, real-valued affine transformation) and suitably linear. For brevity we will refer to such a representation as a CLU.

Our main contribution is to omit the requirement that every pair of prospects is connected via a path of mixtures and identify the minimal strengthening of the HM axioms on preferences that is required to obtain a CLU. Our second contribution is to show that the present framework allows us to model preferences that would otherwise have a nonlinear representation.

Section 2 we introduce partial mixture sets. Partial mixture sets are specified via a partial function (one that is not defined for every pair of prospects. This constitutes a uniform weakening of the axioms for a mixture set of HM. We summarise these axioms with an initial proposition and present a preliminary, abstract analysis of the structure of a partial mixture set.

The remainder of the paper proceeds as follows. In section 3, we introduce the HM axioms on preferences and motivate via examples a strengthened independence axiom and two supplementary axioms: one Archimedean and global and the other relating to measurement and local. We also discuss related axioms and conditions that appear in the literature.

In section 4, we state the main result and, via a sequence of lemmas and propositions, we present the consequences for preferences of combining the axioms. In particular, we show that, for every pair of prospects that is not connected by a path of mixtures, the axioms allow us to construct a concatenation of paths that is monotonic in the preference ordering and that connects that pair. Then, provided these monotone concatenations are suitably unique, we obtain an endogenously derived mixture set on which the HM axioms hold. This yields a constructive proof of our main result.

In section 5, we discuss the main result and our proof in the context of related results in the literature (Fishburn [6], Schmeidler [21], Karni and Safra [14], Karni [13], and Grant et al. [11]) as well as the alternative framework for measuring utility (which also avoids path-connectedness) of Krantz et al. [15]. We demonstrate that various settings in the literature can be written as a partial mixture set: for instance by restricting the set of paths to pairs of prospects that are comonotonic facilitates a connection with the well-known axiomatisation of Schmeidler [21]. To highlight the potential for novel applications, we also describe an experimental setting involving the first and second moments of a Brownian bridge. Through this example, we demonstrate that our focus on a linear representation is not as restrictive as it seems when prospects form a partial mixture set.

Finally, in section 6 we highlight possible extensions of the main theorem.

2 Partial mixture sets

Let X denote a nonempty set of prospects and let I denote the closed unit interval $[0,1] \subseteq \mathbb{R}$. Recall that f is a *path in* X if it is a function from I to X. For instance, when X is a convex set, for every $x, y \in X$, the map $\lambda \mapsto (1-\lambda)x + \lambda y$ defines a (convex) *path from* x *to* y in X.

It is important to note that, like HM, we impose no external topology on X. Instead, paths will be continuous relative to the topology generated by preferences via the axioms we introduce in section 3. For HM, each ordered pair $(x, y) \in X^2$ defines a unique *path of mixtures* ϕ_{xy} in X. Thus, if Φ denotes the resulting set of paths, then HM assume

$$\{(x, y) : \phi_{xy} \in \Phi\} = X^2.$$

As in Kreps [16] and Mongin [17], we may also define the set of paths using a function $\Phi : X^2 \times I \to X$. By allowing Φ to be partially defined, we uniformly weaken the HM axioms for a mixture set.

Definition 1. *Let* $\Phi : X^2 \times I \to X$ *be a partial function.* (X, Φ) *is a* partial mixture set *provided that for every* $x, y \in X$ *such that* $\Phi(x, y, \cdot)$ *is defined,* $\phi_{xy} \overset{\text{def}}{=} \Phi(x, y, \cdot)$ *is a path in* X *and moreover*

$\mathcal{C}_1$ $\phi_{xy}(0) = x$,

$\mathcal{C}_2$ ϕ_{yx} *is defined and* $\phi_{yx}(\lambda) = \phi_{xy}(1-\lambda)$ *for every* $\lambda \in I$*, and*

$\mathcal{C}_3$ *if* $z = \phi_{xy}(\mu)$*, then* ϕ_{xz} *is defined and* $\phi_{xz}(\lambda) = \phi_{xy}(\lambda\mu)$ *for every* $\lambda \in I$.

Where possible, we suppress reference to Φ and adopt X as shorthand for (X, Φ). We also make a small abuse of notation and refer to Φ as the set of paths. Lastly, we will sometimes suppress reference to the endpoints of a path in Φ. For instance, we will use ϕ or $\phi_1, \phi_2, \ldots$ to denote members of Φ.

The following proposition summarises the implications of definition 1.

Proposition 1 (proof on page 24)**.** *For each* $\phi_{xy} \in \Phi$*, the image* $\phi_{xy}(I)$ *is a mixture set and for every* $x', y' \in \phi_{xy}(I)$*, there exists* $\mu, \nu \in I$ *such that*

$$\phi_{x'y'}(\lambda) = \phi_{xy}((1-\lambda)\mu + \lambda\nu) \quad \text{for every } \lambda \in I.$$

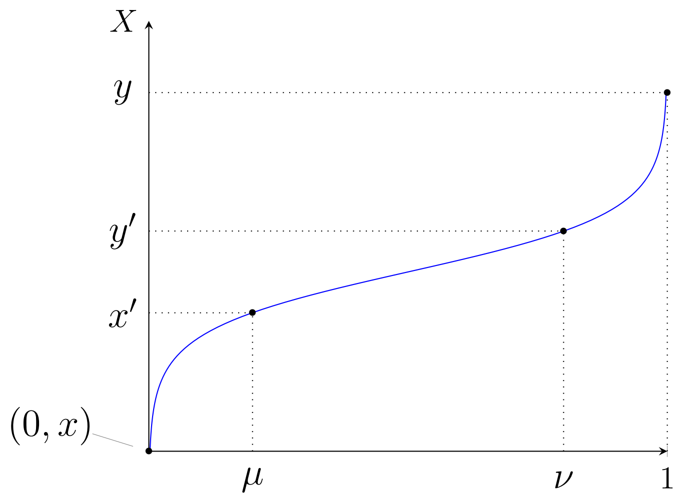

Figure 1: The graph of ϕ_{xy} and $\phi_{x'y'}$ in $I \times X$

Proposition 1 also ensures that the paths themselves are suitable as basic building blocks for the model.

2.1 The structure of a partial mixture set

The fact that Φ is a partial function ensures that the path ϕ_{xy} is well-defined in the sense that it is the unique path from x to y in X that belongs to Φ. (We discuss the extension to nonunique paths in section 6.) Together, conditions $\mathcal{C}_1$ and $\mathcal{C}_2$ ensure that x and y are the appropriate *endpoints* of

ϕ_{xy}. $\mathcal{C}_3$ is the cornerstone of the definition. It ensures that example 1 below cannot be written as a mixture set, even though it is path-connected.

When X is a mixture set, the fact that it has a full set of paths makes it a self-contained block upon which to build the model. In particular, when X is a mixture set, the binary relation $\{(x, y) : \phi_{xy} \in \Phi\}$ is an equivalence relation on X with a single equivalence class. In other words, the latter set is reflexive, symmetric, transitive and also complete. (These properties are unrelated to the corresponding conditions on preferences of section 3.)

By construction, definition 1 allows for incompleteness of $\{(x, y) : \phi_{xy} \in \Phi\}$: there may exist $x, y \in X$ such that $\phi_{xy} \notin \Phi$. Of reflexivity, symmetry and transitivity, only *symmetry* follows directly from definition 1. Indeed, if $\phi_{xy} \in \Phi$, then $\phi_{yx} \in \Phi$ by $\mathcal{C}_2$. By the next proposition, *reflexivity* ($\phi_{xx} \in \Phi$ for every $x \in X$) would follow if every point in X belongs to some path in Φ. In section 3.4, we provide an example of a partial mixture set where we would not expect reflexivity of $\{(x, y) : \phi_{xy} \in \Phi\}$ to hold.

A further consequence of proposition 1 is that, if $\phi_{xy} \in \Phi$, then for every $z \in \phi_{xy}(I)$, both ϕ_{xz} and ϕ_{zy} belong to Φ. *Transitivity* of $\{(x, y) : \phi_{xy} \in \Phi\}$ is the converse of this property: if $\phi_{xz}, \phi_{zy} \in \Phi$, then ϕ_{xy} belongs to Φ and, for some unique $0 < \mu' < 1$, can be written as a *concatenation*

$$\phi_{xy}(\lambda) = \begin{cases} \phi_{xz}(\lambda/\mu') & 0 \leqslant \lambda \leqslant \mu' \\ \phi_{zy}\left((\lambda - \mu')/(1 - \mu')\right) & \mu' \leqslant \lambda \leqslant 1. \end{cases} \tag{1}$$

Equation (1) entails two applications of proposition 1, one for each of the subintervals $[0, \mu']$ and $[\mu', 1]$. In particular, note that on $[\mu', 1]$, the map $\lambda \mapsto (1 - \lambda)\mu' + \lambda 1$ can be inverted to obtain $\lambda \mapsto (\lambda - \mu')/(1 - \mu')$.

We extend this definition of concatenation in our proof of the main result.

3 Axioms on preferences

Preferences are summarised via the binary relation $\precsim$ on X. For each $x, y \in X$, $x \precsim y$ denotes the statement "Val weakly prefers y to x".

Before introducing the axioms on preferences for a cardinal, linear utility representation (CLU) to exist, we provide the formal definition of the latter. Recall that U is a utility representation of preferences on X provided it satisfies the following property: for every $x, y \in X$, $x \precsim y$ if and only if $U(x) \leqslant U(y)$. On a partial mixture set X, the appropriate notion of linearity is the following. The function $U : X \to \mathbb{R}$ is *linear* if, for every $\phi_{xy} \in \Phi$, the composition $U \circ \phi_{xy} : I \to \mathbb{R}$ satisfies

$$(U \circ \phi_{xy})(\lambda) = (1 - \lambda)U(x) + \lambda U(y) \quad \text{for every } \lambda \in I.$$

If a representation U has a certain property (such as linearity), then U is *cardinal* if every other representation V with the same property is related via a single *real-valued affine transformation*: for every $x \in X$, $V(x) = \theta U(x) + \kappa$ for some $\theta > 0$ and $\kappa \in R$.

3.1 The HM axioms

Recall that preferences are *complete* whenever $x, y \in X$ implies $x \precsim y$ or $y \precsim x$; and *transitive* whenever $x \precsim y$ and $y \precsim z$ together imply $x \precsim z$.

Axiom $\mathcal{O}$. $\precsim$ *is* transitive *and* complete *on* X.

As usual, $\prec$ denotes the strict, asymmetric subrelation of $\precsim$ and $\sim$ denotes symmetric subrelation that is known as indifference. An important point to note for what follows is that, when a decision maker's preferences satisfy $\mathcal{O}$, the basic open *order intervals* $\{x' : x \prec x'\}$ and $\{x' : x' \prec y\}$ such that $x, y \in X$ form a subbasis for an *order topology* on X.

Following HM, we will use the following *path continuity* axiom. It is identical to that of HM when X is a mixture set, yet much weaker in general.

Axiom $\mathcal{P}$. *For every $\phi \in \Phi$ and every $z \in X$, the sets $\{\lambda : \phi(\lambda) \precsim z\}$ and $\{\lambda : z \precsim \phi(\lambda)\}$ are closed in I.*

A useful property of $\mathcal{P}$ is that both the minimum and the maximum of $\phi(I)$ in $\precsim$ are well-defined for each $\phi \in \Phi$. This follows from theorem 1 of HM, proposition 1 and the usual extreme value theorem (see Munkres [18]). Since $\mathcal{P}$ does not guarantee the minimum is unique, we adopt the term $\min \phi$ to refer to a representative member of $\min \phi(I)$; $\max \phi$ is defined similarly.

The following independence axiom coincides with that of HM when X is a mixture set. We will show that it is deficient when Φ is partially defined.

Axiom $\mathcal{I}_3$. *If $\phi, \gamma \in \Phi$, $\phi(0) = \gamma(0)$ and $\phi(1) \sim \gamma(1)$, then $\phi(1/2) \sim \gamma(1/2)$.*

3.2 An Archimedean axiom

Axiom $\mathcal{A}$. *If $x \prec y$, then there exists $\phi_0, \ldots, \phi_m \in \Phi$ such that $\min \phi_0 \precsim x$, $y \precsim \max \phi_m$ and, for each $n < m$, $\min \phi_{n+1} \precsim \max \phi_n$.*

$\mathcal{A}$ is an Archimedean condition in the usual mathematical sense and Gilboa and Schmeidler [9, 10] introduce an axiom with similar features for a different model. The following example reveals the difficulties that may arise when X is a partial mixture set and $\mathcal{A}$ fails to hold.

Example 1 (The long line). *Let $\mathbb{A} = \{0, 1, 2, \ldots\}$ denote the well-ordered set of all countable ordinals. Let $\mathcal{O}$ hold on the discrete union $X = \bigsqcup\{X_a : a \in \mathbb{A}\}$, where each X_a is mixture set. Moreover, for every $a \in \mathbb{A}$,*

(i) *$\precsim$ on X_a is order-isomorphic to $\leqslant$ on $\mathbb{R}_+$;*

(ii) $\sup X_a = \min X_{a+1}$, *where* $\sup X_a$ *is the* $\precsim$*-supremum of* X_a *in* X.

As a consequence of the fact that $\mathbb{A}$ in example 1 is uncountable we have

Remark 1 (proof on page 25)**.** *The preferences on* X *of example 1 satisfy both* $\mathcal{P}$ *and* $\mathcal{I}_3$*, but no (real-valued) utility representation exists.*

$\mathcal{A}$ is the weakest axiom that simultaneously rules out example 1 and "connects" every pair of prospects with a finite sequence of paths that are "linked" by indifference sets. The term "linked" is due to Karni [13] where a closely related condition is assumed. (Grant et al. [11, Proposition 6] appeals to a similar condition.) In these articles, the corresponding condition does not play an Archimedean role because X is endowed with an external topology relative to which $\precsim$ is assumed to be continuous.

Perhaps the main drawback of $\mathcal{A}$ is that it contains an existential quantifier.[1] Yet this is a feature of all Archimedean axioms and indeed many standard continuity axioms (see section 3.4 for an example). In our view, many sets that we encounter in our daily lives fail to be path-connected and in such settings it is preferable to explicitly assume $\mathcal{A}$ than to assume the decision maker is working with a mixture set. Moreover, as we show in the setting of section 5.3, it will often be an easy condition to check. Indeed, we propose that experiments be designed with this axiom in mind.

3.3 A stronger independence axiom

$\mathcal{I}_3$ requires that $\phi(0) = \gamma(0)$. This means that it is restricted to pairs of paths that have an endpoint in common. (Pairs of paths that, altogether, have at most three distinct endpoints.) For reasons that we shortly provide, we replace $\mathcal{I}_3$ with the following axiom

[1] I thank an anonymous referee for highlighting this issue.

Axiom $\mathcal{I}_4$. *If $\phi, \gamma \in \Phi$, $\phi(0) \sim \gamma(0)$ and $\phi(1) \sim \gamma(1)$, then $\phi(1/2) \sim \gamma(1/2)$.*

$\mathcal{I}_4$ implies $\mathcal{I}_3$ provided indifference is reflexive. The converse only holds when X is a mixture set, provided indifference is also transitive. $\mathcal{I}_4$ is therefore the natural extension of $\mathcal{I}_3$ to settings where Φ is partially defined.

An early instance of an independence axiom such as $\mathcal{I}_4$ is Samuelson [19]. (Although Samuelson works with a mixture set.) Similar axioms also appear in settings involving partial mixture sets in Fishburn [6] (see also Fishburn [7, ch. 7]) and Karni and Safra [14], Karni [13], and Grant et al. [11].

The following example is based on Karni and Safra, p.324. It shows that $\mathcal{I}_3$ does not yield a linear utility representation.

Example 2 (Preferences satisfying $\mathcal{I}_3$ with no linear representation)**.** *Let ϕ_{xy} and $\phi_{x'y'}$ be a (strictly) $\precsim$-increasing and such that $x \sim x'$ and $y \sim y'$. Any utility representation U satisfies $U(x) = U(x') = r$ and $U(y) = U(y') = s$ for some $r < s \in \mathbb{R}$. If U is linear then ϕ_{xy} and $\phi_{x'y'}$ satisfy*

$$(U \circ \phi_{xy})(1/2) = (U \circ \phi_{x'y'})(1/2) = (r + s)/2.$$

But for $\mathcal{I}_3$ to apply, we need $x = x'$ or $y = y'$. In a mixture set, Φ would contain the path $\phi_{x'y}$, and by applying $\mathcal{I}_3$ twice, yields the desired indifference. Since Φ may not contain any such connecting sequence of paths, $\mathcal{I}_3$ is compatible with $\phi_{xy}(\lambda) \sim \phi_{x'y'}(\lambda^2)$ for each $\lambda \in I$.

We will say that ϕ_{xy} and $\phi_{x'y'}$ are $\precsim$-*synchronised* whenever $x \sim x'$ and $y \sim y'$ together imply $\phi_{xy}(\lambda) \sim \phi_{x'y'}(\lambda)$ for every $\lambda \in I$. The next example shows that further obstacles may arise when paths are not $\precsim$-synchronised.

Example 3 (Overlapping short lines)**.** *Let $X = \bigsqcup \{X_a : a \in \mathbb{A}\}$ be the partial mixture set of example 1 and let both $\mathcal{O}$ and property* (i) *of example 1 hold. However, instead of* (ii)*, for each $a \in \mathbb{A}$ we have* (ii$'$) $\min X_a \sim \min X_{a+1}$ *and for every* $x \in X_a$, $x \sim x'$ *for some* $x' \in X_{a+1}$*;*

(iii$'$) *there exists* $x' \in X_{a+1}$ *such that* $x \prec x'$ *for every* $x \in X_a$.

In contrast with example 1, preferences are not lexicographic. Indeed, (ii$'$) *ensures that the* X_a *overlap and have a common lower bound.* (iii$'$) *ensures that* $\precsim$ *has an uncountable collection of pairwise disjoint open order intervals.*

Remark 2 (proof on page 25)**.** *If* X *and preferences satisfy the properties of example 3, then* $\mathcal{O}$, $\mathcal{P}$, $\mathcal{A}$ *and* $\mathcal{I}_3$ *hold, yet no utility representation exists.*

The proof of the following remark relies on lemma 3 of the next section, yet we include it here for the sake of completeness.

Remark 3 (proof on page 26)**.** $\mathcal{I}_4$ *does not hold for the preferences on* X *of example 3.*

3.4 A measurement axiom

Axiom $\mathcal{M}$**.** *If* $x \prec z \prec y$, *then* $\phi(\lambda) \prec z \prec \phi(\mu)$ *for some* $(\lambda, \mu, \phi) \in I^2 \times \Phi$.

It is worth noting the similarities between $\mathcal{M}$ and a standard continuity axiom [21]: if $x \prec z \prec y$, then for some $0 < \lambda, \mu < 1$, $\phi_{xy}(\lambda) \prec z \prec \phi_{xy}(\mu)$.

To motivate $\mathcal{M}$, we provide an example of preferences that satisfy the preceding axioms, but have no CLU. The example is framed in an unawareness setting and based on Schipper [20]. (Here unawareness means that, at certain levels of awareness, the decision maker fails to include certain prospects in her model).

Example 4. *At awareness level* 0*, Val is unaware of a certain firm, and as such she perceives the only (relevant) prospect to be* x_0 *="status quo". At awareness level* 1*, Val is aware of the firm and understands that a lawsuit* *against the firm is likely. She then perceives the set of prospects to be the image* $\{\phi_1(\lambda) : \lambda \in I\}$ *of some function* ϕ_1*, where* I *is the unit interval. At level* 1*, Val prefers fewer shares to more (lower* λ *is better). At awareness level* 2*, Val is also aware of an innovation. She then perceives the set of prospects to be* $\phi_2(I)$ *and her ranking is reversed, so that more is better.*

Schipper [20] allows Val to step back and reason about her preferences at lower levels of awareness, something that is especially useful in interactive settings. Thus at level 2*,* $X = \{x_0\} \sqcup \phi_1(I) \sqcup \phi_2(I)$ *and we assume that Val's ranking of* X *preserves the underlying ranking at each awareness level. That is,* ϕ_1 *is strictly* $\precsim$*-decreasing and* ϕ_2 *is strictly* $\precsim$*-increasing. In addition, suppose that* $x_0 \sim \phi_1(0) \sim \phi_2(0)$ *and that* $\phi_1(\lambda) \prec \phi_2(\lambda')$ *for every* $\lambda, \lambda' \neq 0$*.*

The issue in example 4 in so far as measurement is concerned is not that x_0 does not belong to a path, it is that x_0 does not belong to the interior of an order interval that is spanned by a path (precisely the content of $\mathcal{M}$).

Remark 4 (proof on page 27)**.** *The preferences on* X *in example 4 have a linear representation, but it is not cardinal.*

4 Derivation of a CLU

In the present section we provide the results that lead to the following

Theorem 1. *Let* X *be a partial mixture set.* $\mathcal{O}$*,* $\mathcal{P}$*,* $\mathcal{A}$*,* $\mathcal{I}_4$ *and* $\mathcal{M}$ *hold if and only if preferences have a cardinal, linear utility representation.*

In section 4.1, we show that the axioms on preferences give rise to complete (as opposed to partial) system of $\precsim$-monotone concatenations (defined next) on X. We then show that the set of $\precsim$-increasing concatenations between any pair $x \prec y$ are unique up to indifference. Finally, in section B of the appendix we show that the system of $\precsim$-monotone concatenations endows the quotient set $X_{/\sim}$ with a mixture set structure and that on this (endogenously generated) mixture set, preferences satisfy the axioms of HM.

4.1 Existence of $\precsim$-monotone concatenations

For a CLU representation to exist on a partial mixture set, the first role of the axioms is to ensure that the image of a path in Φ has similar topological properties to those of I. For instance, in the topology induced by $\mathcal{O}$, $\phi_{xy}(I)$ should be connected. The following extension of Theorem 1 of HM to partial mixture sets shows that this is indeed a consequence of $\mathcal{O}$ and $\mathcal{P}$.

Lemma 1 (proof on page 27)**.** *Let $\mathcal{O}$ and $\mathcal{P}$ hold. If $\phi_{xy} \in \Phi$, $z \in X$ and $x \prec z \prec y$, then there exists $0 < \mu < 1$ such that $z \sim \phi_{xy}(\mu)$.*

Proposition 1 tells us how to write a smaller path in terms of a larger path. Via an generalised version of the concatenations of eq. (1), we construct larger paths from those that belong to Φ. Replacing indifference with equality allows us to link successive paths in the concatenation. In particular,

Definition 2. *$f : I \to X$ is a* Φ-concatenation *if there exists $\phi_0, \ldots, \phi_m$ in Φ such that $f(1) = \phi_m(1)$ and for each n, $\phi_n(1) \sim \phi_{n+1}(0)$ and*

$$f(\nu) = \phi_n((\nu - \mu_n)/(\mu_{n+1} - \mu_n))$$

for every $\nu \in [\mu_n, \mu_{n+1})$ and some $0 = \mu_0 < \cdots < \mu_{m+1} = 1$.

If f is a Φ-concatenation such that $f(0) \sim x$ and $f(1) \sim y$, then we say f is a concatenation from x to y. f inherits continuity from paths via the assumption $\phi_n(1) \sim \phi_{n+1}(0)$ and the fact that the union of finitely many closed sets is closed. $\mathcal{A}$ ensures that such concatenations complete Φ.

Proposition 2 (proof on page 27)**.** *Let $\mathcal{O}$, $\mathcal{P}$ and $\mathcal{A}$ hold. If $x \prec y$, then there exists a Φ-concatenation from x to y.*

In the quest for a CLU, the existence of continuous concatenations that connect each pair of prospects is not enough. Indeed, as the proof of proposition 2 shows, Φ-concatenations may be nonmonotone which we now define.

Definition. *If $x \prec y$, then the concatenation f from x to y is* increasing *provided that, for every $\lambda < \mu$ in I, $f(\lambda) \prec f(\mu)$; if $x \sim y$, then f is increasing if $f(\lambda) \sim f(\mu)$ for every $\lambda, \mu \in I$.* Decreasing *is defined likewise.*

If U is to be is linear, every $\phi_{xy} \in \Phi$ must be monotone. We now show that $\mathcal{I}_3$ (and hence $\mathcal{I}_4$) yield this result and simultaneously improve on lemma 1 to obtain the corresponding versions of theorems 4 and 6 of HM.

Lemma 2 (proof on page 28)**.** *Let $\mathcal{O}$, $\mathcal{P}$ and $\mathcal{I}_3$ hold. If $x \prec z \prec y$ and $\phi_{xy} \in \Phi$, then some unique $0 < \mu < 1$ satisfies $z \sim \phi_{xy}(\mu)$. Moreover, every $\phi \in \Phi$ is monotone.*

Together, lemma 2 and proposition 2 ensure that, for every $x \prec y$, there is an increasing concatenation from x to y.

4.2 Uniqueness of increasing concatenations

By strengthening $\mathcal{I}_3$ to $\mathcal{I}_4$, we improve on lemma 2 by establishing uniqueness (up to indifference) of paths in Φ from x to y.

Lemma 3 (proof on page 29)**.** *Let $\mathcal{O}$, $\mathcal{P}$ and $\mathcal{I}_4$ hold. If $x \prec z \prec y$, then there exists a unique $0 < \mu < 1$ such that $\phi(\mu) \sim z$ for every $\phi \in \Phi$ such that $\phi(0) \sim x$ and $\phi(1) \sim y$.*

Lemma 3 actually suffices for a CLU when X is the discrete union over mixture sets that have an open order interval in common. (For a proof of this fact we refer the reader to the proof of remark 3.)

We now improve on lemma 3 to obtain a similar result for increasing concatenations from x to y. First, we define what it means for a concatenation to be synchronised with Φ and then use lemma 3 to show that each increasing concatenation is indeed synchronised in this way.

Definition. *A concatenation f from x to y is* synchronised with Φ *provided that, for every $\phi \in \Phi$ such that $x \precsim \min\phi$ and $\max\phi \precsim y$,*

$$\phi(\lambda) \sim f((1-\lambda)\mu + \lambda\nu)$$

for every $\lambda \in I$ and for some $\mu, \nu \in I$ that are unique whenever $x \prec y$.

We view the following is an extension of proposition 1 of section 2.1.

Proposition 3 (proof on page 29)**.** *Let $\mathcal{O}$, $\mathcal{P}$, $\mathcal{A}$ and $\mathcal{I}_4$ hold. If $x \prec y$, then every increasing Φ-concatenation from x to y is synchronised with Φ.*

In many ways, proposition 3 is the most important step in the proof of the main theorem. However, it does not imply that increasing concatenations are synchronised with one another. For that, axiom $\mathcal{M}$ is needed.

Together with the other axioms, $\mathcal{M}$ yields the conclusion we have been seeking: that increasing concatenations are unique up to indifference.

Lemma 4 (proof on page 32)**.** *Let $\mathcal{O}$, $\mathcal{P}$, $\mathcal{A}$, $\mathcal{I}_4$ and $\mathcal{M}$ hold. If $x \prec z \prec y$, then there exists a unique $0 < \mu < 1$ such that $f(\mu) \sim z$ for every increasing Φ-concatenation f from x to y.*

Lemma 4 means that it does not matter how we "frame" the paths that form the increasing concatenations, the strength of preference for z relative to x and y is the same. By virtue of its strength, lemma 4 should provide a useful target for experimental testing in a variety of settings.

Building on these results, we prove theorem 1 in section B of the appendix.

5 Discussion and applications

We have shown that if (X, Φ) is a partial mixture set and our axioms hold, then preferences generate an endogenous mixture set $(X_{/\sim}, F)$. The latter mixture set is such that every paths that is absent in Φ is formed in F by concatenating members of Φ: with the indifference relation linking successive members of the concatenation. On $X_{/\sim}$, the axioms of HM hold and we apply the main result of HM to obtain a constructive proof.

It is worth noting that an alternative (nonconstructive) proof is possible if we work with the CLU of $\phi_{xy}(I)$ for each $\phi_{xy} \in \Phi$, assume Φ is well-ordered and, via (transfinite) induction, use positive affine transformations to relocate the image of each $(U \circ \phi_{xy})(I)$. The latter is the strategy of proof in the multilinear utility representation of Fishburn [7] and Fishburn [6, ch.7] as well as Karni and Safra [14, Theorem 2], Karni [13, Theorem 2] Grant et al. [11, Proposition 6]. (In the latter, finite induction suffices because the authors work with a finite collection of subsets that are mixture sets and the union of which exhausts the domain of preferences.)

We argue that our method of proof is itself valuable. It highlights that, a mixture set (albeit endogenous) is implied by cardinality. This is important because, in the absence of preferences, the paths of a partial mixture set impose no external algebraic or topological structure on X whatsoever.

An alternative approach to utility measurement of Krantz et al. [15] assumes that preferences are defined on sets that need not be connected, but elements are assumed to "equally spaced" (as are the integers for instance). The "equally spaced" criterion is reasonable in many real-world applications. Consider for instance the fact that prices on a digital stock market can only be specified certain number of decimal places. Similarly, flour and sugar are typically purchased by the kilogram or pound. Nonetheless, the approach of HM is appealing because people often think in terms of the continuum rather than fractions. Indeed, recall the example of von Neumann and Morgenstern involving mixtures of milk and tea in a glass vs a cup of coffee. It is with probability zero that one might ever accurately measure the amount of milk in glass of tea to a given fraction. By avoiding the need to consider mixtures between tea and coffee (probabilistic or otherwise), partial mixture sets bridge the gap between these two approaches to measurement.

5.1 Three types of partial mixture set

Consider the setting of Fishburn [6], where $X = \times_{\mathbb{A}} X_a$ for some finite set $\mathbb{A}$ and each X_a is player a's set of strategies in a game. (The canonical interpretation is that X is the set that obtains when we exclude correlated strategies from the domain of preferences.) It is easy to see that X is a partial mixture set simply because $\phi_{xy} \in \Phi$ if and only if $x_a \neq y_a$ for at most one $a \in \mathbb{A}$. It is straightforward to show that $\{(x, y) : \phi_{xy} \in \Phi\}$ is reflexive and symmetric. However, take $x, y, z \in X$ such that $\phi_{xy}, \phi_{yz} \in \Phi$. Then although there exists a unique $a \in \mathbb{A}$ such that $x_a \neq y_a$, we may also have $y_a = z_a$ and $y_b \neq z_b$ for some $b \neq a$. Thus $\{(x, y) : \phi_{xy} \in \Phi\}$ is intransitive and does not define an equivalence relation on X. In this setting, the CLU is called a multilinear utility representation since it is linear in each dimension a. Provided our axioms hold, a constructive proof of the existence of a CLU is feasible even in large games, where $\mathbb{A}$ coincides with the unit interval.

A setting where $\{(x, y) : \phi_{xy} \in \Phi\}$ is an equivalence relation is Karni and Safra [14, Theorem 2] and Karni [13, Theorem 2]. For instance, Karni and Safra consider a product $\mathbb{A} = \times_N A_n$ of compact sets where $N = 1, \ldots, n$ and a mixture set $\Delta(N)$. Preferences are then assumed to be continuous in the sense that the upper and lower contour sets are closed in the (externally specified) topology on the product $X = \mathbb{A} \times \Delta(N)$. In this case, $\{a\} \times \Delta(N)$ is a mixture set for each $a \in \mathbb{A}$ and $\phi_{xy} \in \Phi$ if and only if $x, y \in \{a\} \times \Delta(N)$. In effect, $X = \bigsqcup_{\mathbb{A}} \{a\} \times \Delta(N)$. Note that [14] and [13] both derive a CLU.[2]

We have shown that one may form a partial mixture set by either taking the discrete union of mixture sets (the case where $\{(x, y) : \phi_{xy} \in \Phi\}$ is an equivalence relation); or by deleting points in a mixture set (the correlated strategies in Fishburn [6]). But a more subtle type of partial mixture set is obtained by leaving the set of points in (partial) mixture set intact and removing paths from Φ. We consider this latter type of partial mixture in the following subsections.

5.2 Generalised utilitarianism

In the setting of social choice, Grant et al. [11] work with a partial mixture set that resembles Fishburn [6] in the sense that $\{(x, y) : \phi_{xy} \in \Phi\}$ is intransitive on X. Indeed, the authors consider the product $\Delta(N) \times \Delta(C)$ of set of lotteries, where N is a finite set of agents and C is a suitable topological space of prizes. As in Fishburn [6], $\phi_{xy} \in \Phi$ implies that x differs from y in at most one coordinate. The converse however is not true.

In particular, the axioms of Grant et al. [11] do not apply to paths with endpoints $p \times l$ and $p \times l'$ such that $l \neq l'$ and p is nondegenerate, even

[2] Although there is an error in the statement of Theorem 2 of [14], this fact is noted in footnote 5 of [13].

though the set of prospects remains $\Delta(N) \times \Delta(C)$. In Grant et al. [11], the representation is linear on the restricted domain to which axioms apply, yet nonlinear on $\Delta(N) \times \Delta(C)$. We provide an explicit example of this phenomenon in section 5.4.

5.3 Comonotonic prospects

Another setting where prospects are left intact and paths are in effect deleted is the seminal work on (possibly nonadditive) subjective beliefs of Schmeidler [21]. There, each prospect x is a function from the set Ω of states to some convex set of prizes M. The set $X = M^\Omega$ of such prospects is a mixture set provided Φ consists of paths ϕ_{xy} that map $\lambda \mapsto \phi_{xy}(\lambda, \cdot) = (1-\lambda)x(\cdot) + \lambda y(\cdot)$ for every $x, y \in X$, where mixtures are interpreted pointwise on Ω.

The axiomatisation of Schmeidler holds for preferences $\precsim$ on X that satisfy $\mathcal{O}$, $\mathcal{P}$ on Φ, and a restriction of $\mathcal{I}_3$ to pairs of paths with comonotonic endpoints. Schmeidler defines comonotonicity by first inducing a preference relation $\precsim_M$ on M such that $p \precsim_M q$ if and only if the *constant prospects* p^Ω and q^Ω (that are everywhere equal to p and q respectively) satisfy $p^\Omega \precsim q^\Omega$. Then $x, y \in X$ are *comonotonic* if there is no $\omega, \omega' \in \Omega$ such that $x(\omega) \prec_M x(\omega')$ and $y(\omega') \prec_M y(\omega)$. Since it is only possible to know which paths are comonotonic once preferences over constant prospects are known, the set $\Gamma \subsetneq \Phi$ of paths with comonotonic endpoints, defines a partial mixture set (X, Γ) that is endogenous to preferences. The following remark is a consequence of the fact that every member of X is comonotonic with constant prospects.

Remark 5 (proof on page 33)**.** *Let $\mathcal{O}$ hold on $X = M^\Omega$. On the partial mixture set (X, Γ), if $\mathcal{P}$ and $\mathcal{I}_3$ hold, then so do $\mathcal{A}$, $\mathcal{I}_4$ and $\mathcal{M}$.*

Remark 5 implies that if we supplement the axioms of theorem 1 with the monotonicity axiom ($x(\omega) \precsim y(\omega)$ for every $\omega \in \Omega$ implies $x \precsim y$) and the nondegeneracy axiom ($x \prec y$ for some $x, y \in X$), then we obtain a representation that coincides with that of Schmeidler [21] with the following exception. On the partial mixture set (X, Γ) we obtain a CLU, but on (X, Φ), the same utility representation $U : X \to \mathbb{R}$ may have the property that $U \circ \phi : I \to \mathbb{R}$ is nonlinear and discontinuous for some $\phi \in \Phi - \Gamma$. In contrast, in Schmeidler [21], $U \circ \phi$ may be nonlinear, but it is always continuous.

5.4 Application to a Brownian bridge

The first goal of the present section is to demonstrate that, it is possible to exploit the structure of a partial mixture set to represent preferences that would otherwise satisfy the axioms for a properly quadratic utility of Chew, Epstein, and Segal [5, 4]. The second is to develop a novel application. We work with random prospects that consist of points on a Brownian bridge. Recall that the Brownian bridge is a transformation of a Brownian motion with the key property that its value is known at both the initial time 0 and the final time 1. Such processes have been used to model insider information [1] as well as certain bonds and options [2].

Consider a hypothetical experiment where the decision maker, Val, is asked to state her preferences (by ranking pairs of prospects). The set X of prospects is the disjoint union of a set (X_0, Φ_0) of sure prospects such as the real line and the set (X_1, Φ_1) of points and subpaths of a Brownian bridge that we now describe.

For some probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ and index of time $\lambda \in [0, 1]$, let $\left\{B_\lambda \in \mathbb{R}^\Omega : \lambda \in I\right\}$ be the bridge that takes value \$$a$ at time 0 and \$$b$ at time 1. The explicit equation relationship between a Brownian bridge and a

Brownian motion $\{W_\lambda : \lambda \in I\}$ is the following

$$B_\lambda = (1-\lambda)\left(a + \int_0^\lambda \frac{1}{1-r}\,\mathrm{d}W_r\right) + \lambda b \tag{2}$$

Let $\phi(\lambda,\omega) = B_\lambda(\omega)$ for each $(\lambda,\omega) \in I \times \Omega$. Since every $x', y' \in \phi_{xy}(I,\cdot)$ satisfies $x' = \phi_{xy}(\mu,\cdot)$ and $y' = \phi_{xy}(\nu,\cdot)$ for some $\mu,\nu \in I$, take

$$\phi_{x'y'}(\lambda,\cdot) \overset{\text{def}}{=} \phi_{xy}((1-\lambda)\mu + \lambda\nu,\cdot) \quad \text{for each } \lambda \in I.$$

By construction, $\phi_{x'y'}$ satisfies proposition 1, and this ensures that $\phi_{xy}(I,\cdot)$ is a mixture set. In fact, since a time-reversed Brownian bridge is also a Brownian bridge [3, p.9], $\mathcal{C}_2$ may be interpreted literally. Finally, let

$$\mathbf{E} : X \to \mathbb{R} \text{ be the map } x \mapsto \int_\Omega x(\cdot)\,\mathrm{d}\mathbf{P}.$$

The first moment $(\mathbf{E} \circ \phi)(\lambda,\cdot)$ is equal to $(1-\lambda)a + \lambda b$ for each $\lambda \in I$. Let these be Val's certainty equivalents. Then her preferences are represented by $U = \mathbf{E}$. We would therefore "accept" the hypothesis that she is risk neutral because appropriately linear.

Now extend (X,Φ) to be the discrete union of sure prospects (X_0,Φ_0), (X_1,Φ_1) and the set (X_2,Φ_2) of prospects and subpaths of the square ϕ^2_{xy} of a Brownian bridge such that $B_0 = B_1 = 0$. Moreover, suppose that Val's preferences satisfy

$$\phi^2_{xy}(\lambda,\cdot) \sim \lambda(1-\lambda) \text{ for each } \lambda \in I. \tag{3}$$

Clearly, with these certainty equivalents, Val's preferences do not satisfy $\mathcal{I}_4$ on (X,Φ). Indeed, $x^2 \sim y^2 \sim 0 \prec {}^1\!/_4 \sim \phi^2_{xy}({}^1\!/_2)$.

On the other hand, it is clear that the utility representation $U = \mathbf{E}$ is appropriate since it is well-known that the second moment of the present Brownian bridge is indeed $\lambda(1-\lambda)$. The issue is that the partial mixture set

(X, Φ) is misspecified. In the final subsection, we show that by deleting all the $\precsim$-nonmonotone paths and appropriately respecifying the time parameter, we obtain a partial mixture set where $\mathcal{I}_4$ holds. On the resulting partial mixture set, preferences have a cardinal, linear utility representation, but on the original space, preferences are quadratic. Indeed, they satisfy the *strong mixture symmetry* axiom of Chew, Epstein, and Segal [4]:

Axiom. *If* $\phi(0) \sim \phi(1)$*, then* $\phi(\lambda) \sim \phi(1-\lambda)$ *for every* $\lambda \in I$*.*

Although we do not consider the higher moments of a Brownian bridge in the present paper, the methods we describe would certainly apply. Partial mixture sets offer the potential to elicit beliefs using (subjective) moments.

5.5 "Minor surgery" on partial mixture sets

In the derivations that follow, to reduce notational clutter, we write $\phi(\lambda)$ instead of $\phi(\lambda, \cdot)$. The first step is to discard the nonmonotone paths in Φ_2. By (3) and the fact that $\lambda = 1/2$ is the unique argmax of $\lambda(1-\lambda)$, these paths satisfy

$$\phi(\lambda) = \phi^2_{xy}((1-\lambda)\mu + \lambda\nu) \text{ for some } \mu < 1/2 < \nu.$$

The second step is to respecify the remaining members of Φ_2 that violate $\mathcal{I}_4$. Let Φ' be the set of paths in Φ_2 with endpoints $\phi^2_{xy}(\mu)$ and $\phi^2_{xy}(\nu)$ for some $\mu, \nu \leqslant 1/2$. Similarly, let Φ'' consist of those such that $1/2 \leqslant \mu, \nu$.

Let g be the increasing map $\lambda \mapsto 4\lambda(1-\lambda)$ on $[0, 1/2]$. Then, the inverse g^{-1} has image $[0, 1/2]$ and is the increasing map

$$\lambda \mapsto 1/2 - \sqrt{(1-\lambda)/4} \quad \text{on } I.$$

Let γ' be the $\phi^2_{xy} \circ g^{-1}$. Then, for every $x', y' \in \phi^2_{xy}(I)$, there exists $\mu, \nu \in I$

such that $x' = \gamma'(\mu)$ and $y' = \gamma'(\nu)$. For each $\lambda \in I$, let

$$\gamma_{x'y'}(\lambda) \stackrel{\text{def}}{=} \gamma'((1-\lambda)\mu + \lambda\mu).$$

Finally, take Γ' to be the resulting set of paths and note that g defines a bijection between Γ' and Φ'. For Φ'', same procedure yields a function similar to g, and a corresponding path γ''. This yields a set Γ'' of suitably defined with which to replace Φ''. Let Γ be the union of Γ', Γ'', Φ_0 and Φ_1. This brings us to our final result, which is suffices for U to be a CLU for the preferences of the preceding subsection on (X, Γ).

Remark 6. [3] *(X, Γ) is a partial mixture set on which $\mathcal{I}_4$ holds for preferences that are risk-neutral. Moreover, $\gamma'(I)$ and $\gamma''(I)$ are nonconvex.*

6 Extensions

Linear, but not necessarily cardinal, utility When X is a partial mixture set and a CLU exists, our axioms hold. Yet we may wish to weaken the axioms and settle for a representation that is not a CLU. A simple extension of theorem 1 omits $\mathcal{M}$. This yields a (not necessarily cardinal) linear utility representation $U : X \to \mathbb{R}$ such that the image of U is path-connected by $U \circ f$ such that f is a concatenation in Φ.

Nonunique paths in Φ When paths are not uniquely identified by their endpoints, Φ of definition 1 is no longer be a partial function and ϕ_{xy} is ill-defined. However, provided equality is replaced by indifference in $\mathcal{C}_2$ and $\mathcal{C}_3$, the present results may be extended to allow for nonuniqueness. Such an extension is related to, but distinct from, *generalised mixture sets*. In the

[3] See page 34 for proof.

latter, mixtures are everywhere defined by a binary operation and hence Φ is a function on $X^2 \times I$ (see Fishburn and Roberts [8, p.162]).

On nonlinearity In section 5.4 we demonstrated that by removing paths that are nonmonotone in preferences and respecifying those that remain, we may derive a CLU for preferences that otherwise have a proper quadratic representation. In more general settings, preferences on some $\phi \in \Phi$ might yield the certainty equivalent

$$\lambda^3 - {}^3\!/\!_2\,\lambda^2 + {}^2\!/\!_3\,\lambda$$

for each $\lambda \in I$. Then the critical points $1/3$ and $2/3$ of this cubic function determine the partition into intervals on which subpaths of ϕ are monotone. A full exploration of the implications for modelling nonlinearity in this way is beyond the scope of the present article.

A Proofs

PROOF OF PROPOSITION 1 FROM PAGE 5. It suffices to show that for every $x', y' \in \phi(I)$ the path $\phi_{x'y'}$ belongs to Φ. For every such x' and y', there exists $\mu, \nu \in I$ such that $\phi_{xy}(\mu) = x'$ and $\phi_{xy}(\nu) = y'$. By $\mathcal{C}_3$, $\phi_{xx'}, \phi_{xy'} \in \Phi$ and $\phi_{xx'}(\lambda) = \phi_{xy}(\lambda\mu)$ and $\phi_{xy'}(\lambda) = \phi_{xy}(\lambda\nu)$ for every $\lambda \in I$. W.l.o.g., suppose that $\mu \leqslant \nu$.

If $\mu = \nu$, then $x' = y'$. Moreover, by $\mathcal{C}_2$, $\phi_{x'x}(\lambda) = \phi_{xx'}(1-\lambda)$ and by $\mathcal{C}_1$, $x' = \phi_{x'x}(0)$. Then one further application of $\mathcal{C}_3$ yields $\phi_{x'y'}(\lambda) = \phi_{x'x}(\lambda 0) = x'$ for every $\lambda \in I$. In this case, clearly, the proposition holds with $\mu = \nu$ for every $\lambda \in I$.

If $\mu < \nu$, then, for some $\lambda' < 1$, $\mu = \lambda'\nu$. Then $\phi_{xy'}(\lambda') = x'$ and since $\phi_{y'x}(1-\lambda') = \phi_{xy'}(\lambda')$, we see that $x' = \phi_{y'x}(1-\lambda')$. A final application of $\mathcal{C}_3$

yields $\phi_{y'x'}(\lambda) = \phi_{y'x}(\lambda(1-\lambda'))$ for each $\lambda \in I$. Next note that $\lambda' = \mu/\nu$, so that a substitution for λ' and straightforward simplification yields $\phi_{y'x'}(\lambda) = \phi_{y'x}(\lambda(\nu-\mu)/\nu)$ for each $\lambda \in I$. Similarly, we have

$$\begin{aligned} \phi_{y'x}(\lambda(\nu-\mu)/\nu) &= \phi_{xy'}(1-\lambda(\nu-\mu)/\nu) && \text{by } \mathcal{C}_2 \\ &= \phi_{xy}(\nu-\lambda(\nu-\mu)) && \text{by } \mathcal{C}_3. \end{aligned}$$

Let $\lambda \mapsto \kappa = 1-\lambda$. Substituting for λ we have $\phi_{y'x'}(1-\kappa) = \phi_{xy}(\mu+\kappa(\nu-\mu))$. One final application of $\mathcal{C}_2$ to the left-hand-side of the latter equality yields both $\phi_{x'y'}$ and the equation of proposition 1. □

Proof of remark 1 from page 9. In example 1, property (i) ensures that for each a and ϕ, every indifference class of $\precsim$ restricted to X_a or to $\phi(I)$ is a singleton. That is, individually each of these sets is linearly ordered. (ii) ensures that the whole of X is linearly ordered. Finally, the fact that $\mathbb{A}$ is uncountable, means that $(X, \precsim)$ is order-isomorphic to the long line [22]. Since the long line contains an uncountable collection of nonempty, pairwise disjoint open intervals (one for each $a \in \mathbb{A}$), there is no utility representation of preferences.(See the proof of remark 2 for a further discussion of this argument.)

The proof that $\mathcal{I}_3$ holds is immediate because $\precsim$ is linearly ordered.

The proof that $\mathcal{P}$ holds is as follows. Let $z \in X_a$ for some $a \in \mathbb{A}$. If $\max\phi \precsim z$, then by $\mathcal{O}$, $\{\lambda : \phi(\lambda) \precsim z\} = I$. If $\min\phi = \phi(0) \prec z \prec \phi(1) = \max\phi$, then (i) ensures that there is a unique μ such that $\phi(\mu) \sim z$ and for every $\lambda \leqslant \mu$, we have $\phi(\lambda) \precsim z$. Remaining cases are similar and omitted. □

Proof of remark 2 of page 11. The proof that $\mathcal{P}$ holds is very similar to example 1 and therefore omitted.

$\mathcal{I}_3$: If $a \neq b$, then $X_a \cap X_b = \varnothing$ and $\mathcal{I}_3$ holds vacuously for pairs of paths ϕ and γ such that $\phi(I) \subset X_a$ and $\gamma(I) \subset X_b$. If $a = b$, then $\mathcal{I}_3$ holds trivially because whenever the intersection of an indifference set with X_a is nonempty, it is a singleton.

$\mathcal{A}$: Let $x \prec y$ and suppose that $x \in X_a$ and $y \in X_b$ for $a < b$. By (ii′), there exists $x' \in X_b$ such that $x' \sim x$. Now, since X_b is a mixture set, there exists a path ϕ from x' to y in X_b. Then $\mathcal{A}$ holds with $m = 1$.

For the last part of the proposition, by way of contradiction, suppose that preferences have a utility representation $U : X \to \mathbb{R}$. Then the set $U(X)$ satisfies the countable chain condition: every collection of nonempty, pairwise-disjoint, open intervals is countable. Take $x_0 = \min X_0$ and, for each a, take $x_{a+1} \in X_{a+1}$ such that $x \prec x_{a+1}$ for each $x \in X_a$ (these points are well-defined by properties (ii′) and (iii′)). Then collection of $\precsim$-order intervals (x_a, x_{a+1}) such that $a \in \mathbb{A}$ is uncountable and each of its members is nonempty and open. Since $\mathbb{A}$ is well-ordered, for any $a < b$, (x_a, x_{a+1}) and (x_b, x_{b+1}) are pairwise disjoint since $x_{a+1} \prec x_b$. Since this violates the countable chain condition, U cannot be a utility representation. □

Proof of remark 3 from page 11. For each $a \in \mathbb{A}$, since X_a is a mixture set, the fact that the axioms listed in remark 2 HM ensures it has a CLU $U_a : X_a \to \mathbb{R}$. Choose $x_0, y_0 \in X_0$ such that $x_0 \prec y_0$. Then (ii′) ensures that, for each a, there exists $x_a, y_a \in X_a$ such that $x_a \sim x_0$ and $y_a \sim y_0$.

Let $U_0(x_0) = r_0$ and $U_0(y_0) = s_0$. For each a, choose a positive affine transformation $T_a : \mathbb{R} \to \mathbb{R}$ of U_a such that $(T_a \circ U_a)(x_a) = r$ and $(T_a \circ U_a)(y_a) = s$. (If $U_a(x_a) = r_a$ and $U_a(y_a) = s_a$, then let $l_a = (s_0 - r_0)/(s_a - r_a)$ and $k_a = r_0 - lr_a$; then $r \mapsto T_a(r) = k_a + l_a r$ is the required transformation. For each a let $V_a \overset{\text{def}}{=} T_a \circ U_a$. Let $\operatorname{gr} V \overset{\text{def}}{=} \bigcup \operatorname{gr} V_a$.

By way of contradiction, suppose that $\mathcal{I}_4$ holds. Then, for every a, if ϕ_a is the path from x_a to y_a in X_a, then $\phi_a(\lambda) \sim \phi_0(\lambda)$ by lemma 3. By construction, $(V \circ \phi_a)(\lambda) = (V \circ \phi_0)(\lambda)$ for every $a \in \mathbb{A}$. But then, by theorem 7 of

HM (and the discussion on p.296), preferences have a CLU: a contradiction of remark 2. □

PROOF OF REMARK 4 FROM PAGE 12. By proposition 1, $\phi_0(I)$ and $\phi_1(I)$ are both mixture sets. The fact that preferences have a linear representation U follows immediately from two applications of the main theorem of HM.

The reason that U is not cardinal is that for any $0 < \mu < 1$, we may freely define a distinct concatenation f of ϕ and ϕ' such that $f(\mu) = x_0$. This will not do for a cardinal representation, for each distinct pair f and g of such concatenations yields a pair of linear utility representations that are not related via a single positive affine transformation.

Choose f and g such that $f(1/2) = x_0 = g(1/4)$ and $f(1) = g(1) = \phi_1(1)$. This implies that $x_0 \prec g(1/2)$. Let $V := U \circ g \circ f^{-1}$. Then V is a well-defined linear utility by virtue of the fact that f is a bijection. Moreover, by construction, the image $V(X)$ of V is equal to that of U. But since $f^{-1}(x_0) = 1/2$, we have $(g \circ f^{-1})(x_0) = g(1/2)$, so that $V(x_0) = (U \circ g)(1/2)$. Then $U(x_0) = (U \circ g)(1/4) < V(x_0)$.Clearly, there is no positive affine transformation that relates U and V. Thus, U is not cardinal. □

PROOF OF LEMMA 1 FROM PAGE 13. Consider the the set $L = \{\lambda : \phi_{xy}(\lambda) \precsim z\}$. By condition $\mathcal{P}$, L is a closed subset of I. It is nonempty since $\phi_{xy}(0) = x \prec z$. Similarly, the set $U = \{\lambda : z \precsim \phi_{xy}(\lambda)\}$ is closed and nonempty since $z \prec \phi_{xy}(1) = y$. By $\mathcal{O}$, I is the union of L and U. If $L \cap U = \varnothing$, then I is the union of two disjoint, nonempty and closed subsets. Since this would imply that I is disconnected, $L \cap U$ is nonempty. Thus, there exists $\mu \in I$ such that $\phi_{xy}(\mu) \sim z$. The fact that $\mu \neq 0, 1$ follows from $\mathcal{O}$. □

PROOF OF PROPOSITION 2 FROM PAGE 14. Fix $x \prec y$ and let $\phi_0, \ldots, \phi_m$ be a sequence of paths satisfying $\mathcal{A}$. If $m = 0$, then $\min \phi_0 \precsim x \prec y \precsim \max \phi_0$.

By lemma 1, there exist μ and ν such that $\phi_0(\mu) \sim x$ and $\phi_0(\nu) \sim y$. By proposition 1 there exists a path γ such that $\gamma(0) = \phi_0(\mu)$ and $\gamma(1) = \phi_0(\nu)$. Then, for this case, γ is the concatenation we seek.

If $m \geqslant 1$, a similar argument applies and we proceed by induction on the sequence $\phi_0, \ldots, \phi_m$. By the fact that $\mathbb{Z}_+$ is well-ordered, let m be minimal. This ensures that $x \prec \min \phi_1$, $\max \phi_n \prec \min \phi_{n+2}$ and $\max \phi_{m-1} \prec y$. (In other words, each of the paths is necessary.) By the preceding paragraph, there exists γ_0 such that $\gamma_0(0) \sim x$ and $\gamma_0(1) \sim \min \phi_1$. By the preceding argument, there exists minimal μ and maximal ν such that $\mu < \nu$, $\phi_1(\mu) \sim \gamma_0(1)$ and $\phi_1(\nu) \sim \min \phi_2$. By the preceding paragraph, there exists γ_1 such that $\gamma_1(0) \sim \gamma_0(1)$ and $\gamma_1(1) \sim \min \phi_2$. Since m is finite, we obtain a sequence $\gamma_0, \ldots, \gamma_m$ such that $\gamma_n(1) \sim \gamma_{n+1}(0)$ for each n, $x \sim \gamma_0(0)$ and $\gamma_m(1) \sim y$. Now choose and arbitrary sequence $0 = \mu_0 < \cdots < \mu_{m+1} = 1$ and using the present sequence of paths take f to satisfy definition 2. □

Proof of lemma 2 from page 14. Suppose that $x \prec z \prec y$ for some $z \in X$. Then by lemma 1, there exists at least one μ satisfying the required condition. Suppose there exists $\mu' < \mu$ such that $\phi_{xy}(\mu') \sim z$. Then by $\mathcal{O}$, we have $\phi_{xy}(\mu') \sim \phi_{xy}(\mu)$. But since proposition 1 ensures $\phi_{xy}(I)$ is a mixture set and $x \prec y$, theorem 4 of HM implies that $\phi_{xy}(\mu') \prec \phi_{xy}(\mu)$ if and only if $\mu' < \mu$. (This latter theorem applies since all the axioms of HM now hold.)

For the second part, suppose (by way of contradiction) that $z \precsim x \prec y$ and $z \sim \phi_{xy}(\mu)$ for some unique $0 < \mu < 1$. Let $z' = \phi_{xy}(\mu)$. Then $\mathcal{C}_3$ ensures that $\phi_{xz'}(\lambda) = \phi_{xy}(\lambda\mu)$ for every $\lambda \in I$. Indeed, proposition 1 ensures that $\phi_{z'y}(\lambda) = \phi_{xy}((1-\lambda)\mu + \lambda 1)$. $\mathcal{O}$ implies $z' \precsim x \prec y$ and the first part of this proof ensures that $\phi_{z'y}(\nu) \sim x$ for some unique $0 \leqslant \nu < 1$. Let $x' = \phi_{z'y}(\nu)$. Then $x' = \phi_{z'y}(\nu) = \phi_{xy}((1-\nu)\mu + \nu 1)$. Let $\nu^* = (1-\nu)\mu + \nu$. Then $\phi_{xy}(\lambda) \sim x$ for both $\lambda = 0$ and $\lambda = \nu^*$. Clearly $0 < \mu$ implies $0 < \nu^*$:

another contradiction of theorem 4 of HM. □

Proof of lemma 3 from page 14. Let $\phi = \phi_{xy}$. Lemma 2 guarantees the existence of a candidate $0 < \lambda < 1$ such that $\phi(\lambda) \sim z$. Take γ to be any other path in Φ satisfying $\gamma(0) \sim x$ and $\gamma(1) \sim y$. $\mathcal{I}_4$ ensures that $\phi(1/2) \sim \gamma(1/2)$. Condition $\mathcal{C}_3$ then ensures the existence of a subpath $\phi_{0\frac{1}{2}}$ from x to $\phi(1/2)$ such that $\phi_{0\frac{1}{2}}(\lambda) = \phi(\lambda/2)$ for every $\lambda \in I$. A similar path $\gamma_{0\frac{1}{2}}$ exists from x' to $\gamma(1/2)$. An application of $\mathcal{I}_4$ yields $\phi_{0\frac{1}{2}}(1/2) \sim \gamma_{0\frac{1}{2}}(1/2)$. This implies that $\phi(1/4) \sim \gamma(1/4)$. An application of condition $\mathcal{C}_2$ of partial mixture sets and a similar argument applied to the paths $\phi_{\frac{1}{2}1}$ and $\gamma_{\frac{1}{2}1}$ yields $\phi(3/4) \sim \gamma(3/4)$. (Using proposition 1 to translate indifferences on subpaths to indifferences between ϕ and γ.) In this way, the above argument yields $\phi(\rho) \sim \gamma(\rho)$ for every dyadic rational $\rho \in I$. Then, since the dyadic rationals are dense in I, there exists a sequence $\lim_n \rho_n = \lambda$, where recall λ is our candidate for the proof. W.l.o.g., we may take the sequence to be increasing. Then by the proof of lemma 2, $\phi(\rho_n) \precsim \phi(\lambda)$ for each n. $\mathcal{P}$ then ensures that λ belongs to $\{\lambda' : \gamma(\lambda') \precsim \phi(\lambda)\}$. Repeating the argument with the roles of ϕ and γ reversed yields the reverse weak preference, so that $\phi(\lambda) \sim \gamma(\lambda)$, as required. □

Proof of proposition 3 from page 15. Let $\Phi^{\prec} = \{\phi : \phi(0) \prec \phi(1)\}$. For every $\phi \in \Phi^{\prec}$, lemma 2 implies that $\phi(\lambda) \precsim \phi(\mu)$ if and only if $\lambda \leqslant \mu$.

Step 1 (*There exists a minimal, strictly increasing concatenation f from x to y*). By proposition 2, there exists a Φ-concatenation f from x to y. Moreover, since $\mathbb{Z}_+$ is well-ordered, choose f to be composed with the smallest possible number m of paths in Φ. Then f is a concatenatation of paths in $\Phi^{\prec}$, for otherwise, we could exclude a path and obtain a suitable concatenation with even fewer paths. This completes the proof of step 1.

Let $\phi_1, \ldots, \phi_m \in \Phi^{\prec}$ and $0 = \mu_0 < \cdots < \mu_{m+1} = 1$ characterise f. For any $\gamma \in \Phi$, we write $f \leftrightarrow \gamma$ whenever f is synchronised with γ.

STEP 2 (*$f \leftrightarrow \phi_n$ for $n = 0, \ldots, m$*). Take any $n = 0, \ldots, m$. Since $\mu_n < \mu_{n+1}$, $T(\nu) \stackrel{\text{def}}{=} (\nu - \mu_n)/(\mu_{n+1} - \mu_n)$ is well-defined for each $\nu \in [\mu_n, \mu_{n+1})$. Thus, for each $\lambda < 1$, $T^{-1}(\lambda) = \lambda(\mu_{n+1} - \mu_n) + \mu_n = (1-\lambda)\mu_n + \lambda\mu_{n+1}$, so that $\phi_n(\lambda) = (f \circ T^{-1})(\lambda)$ as required. (For $\lambda = 1$, $\phi_n(\lambda) \sim \phi_{n+1}(0) = f(\mu_{n+1})$.)

STEP 3 (*$f \leftrightarrow \gamma$ for every $\gamma \in \Phi^{\prec}$ such that $f(\mu_n) \precsim \gamma(0)$ and $\gamma(1) \precsim f(\mu_{n+1})$*). If $\gamma(0) \sim f(\mu_n)$ and $\gamma(1) \sim f(\mu_{n+1})$, then the fact that $\gamma \leftrightarrow \phi_n$ follows directly by lemma 3. Now suppose that $f(\mu_n) \precsim \gamma(0)$ and $\gamma(1) \precsim f(\mu_{n+1})$, with at least one relation holding strictly. Lemma 2 implies $\phi_n(\mu) \sim \gamma(0)$ and $\gamma(1) \sim \phi_n(\mu')$ some unique $\mu, \mu' \in I$ such that either $0 < \mu$ or $\mu' < 1$.

Since $\gamma \in \Phi^{\prec}$, $\mu < \mu'$ follows by theorem 4 of HM and the fact that $\phi_n \in \Phi^{\prec}$. Write ϕ_n as a concatenation of (at most) three subpaths $\phi', \phi'', \phi''' \in \Phi^{\prec}$ such that in particular $\phi_n(\nu) = \phi''((\nu - \mu)/(\mu' - \mu))$ for each $\mu \leqslant \nu \leqslant \mu'$. Then an inverse transformation T^{-1} similar to the one of step 2 yields $\phi''(\lambda) = \phi_n((1-\lambda)\mu + \lambda\mu')$ for every $\lambda \in I$. Then $\phi''(0) \sim \gamma(0)$ and $\phi''(1) \sim \gamma(1)$, and lemma 3 ensures $\phi''(\lambda) \sim \phi(\lambda)$ for every $\lambda \in I$ and $\phi_n \leftrightarrow \gamma$.

Recall that $\phi_n(\xi) \sim f((1-\xi)\mu_n + \xi\mu_{n+1})$ for every $\xi \in I$. For each $\lambda \in I$, let $\xi = (1-\lambda)\mu + \lambda\mu'$. Then a substitution and straightforward rearrangement yields $\gamma(\lambda) \sim f((1-\lambda)\mu_* + \lambda\mu^*)$ where $\mu_* = (1-\mu)\mu_n + \mu\mu_{n+1}$ and $\mu^* = (1-\mu')\mu_n + \mu'\mu_{n+1}$. Since $\mu_n < \mu_{n+1}$ and $\mu < \mu'$, it is clear that $\mu_* < \mu^*$ as required for $f \leftrightarrow \gamma$.

STEP 4 (*$f \leftrightarrow \gamma$ whenever $\gamma(0) \prec f(\mu_n) \prec \gamma(1)$ for some γ and some n*). W.l.o.g., take γ and n satisfy $f(\mu_{n-1}) \precsim \gamma(0)$ and $\gamma(1) \precsim f(\mu_{n+1})$ with at least one relation holding strictly. (For otherwise, we may take a subpath and combine the present step with step 2.) The difficulty here is that the

μ_n may be incorrectly specified, so that f travels at a different rate on distinct intervals $[\mu_{n-1}, \mu_n)$ and $[\mu_n, \mu_{n+1})$. The proof of remark 4 on page 27 demonstrates the degree of freedom we have in choosing μ_n. We now show that we can always respecify the μ_n and obtain a new concatenation g that is synchronised with γ. Since g and f are composed of the same paths, g satisfies step 2 and step 3 of this proof.

Let $1 \leqslant n \leqslant m$ be the smallest number such that $\gamma(0) \prec f(\mu_n) \prec \gamma(1)$ for some γ. For every $\lambda \in [\mu_{n+1}, 1]$, take g to satisfy $g(\lambda) = f(\lambda)$. Let f_n denote the initial segment f, so that $f_n(\lambda) \stackrel{\text{def}}{=} f(\lambda\mu_n)$ for each $\lambda < 1$ and $f_n(1) = \phi_{n-1}(1)$. That is, f_n is a concatenation of $\phi_0, \ldots, \phi_{n-1}$. On the interval $[0, \mu_{n+1})$, g will be the concatenation of f_n, ϕ_n, but unless f is synchronised with γ to begin with, $g \neq f$.

By lemma 2, there is a unique $0 < \lambda_n < 1$ such that $\gamma(\lambda_n) \sim f(\mu_n) = \phi_{n+1}(0)$. Lemma 2 also ensures that $f_n(\kappa_*) \sim \gamma(0)$ and $\phi_n(\kappa^*) \sim \gamma(1)$ for some unique $0 \leqslant \kappa_* < 1$ and $0 < \kappa^* \leqslant 1$. (Note that $\kappa_* = 0$ if and only if $\gamma(0) \sim x$ and $\kappa^* = 1$ if and only if $\gamma(1) \sim f(\mu_{n+1})$ and, since m is minimal, both do not hold simultaneously.)

We seek g such that $g(\nu_n) \sim \gamma(\lambda_n)$ where

$$\nu_n = (1 - \lambda_n)\mu_* + \lambda_n \mu^* \tag{4}$$

for some unique $\mu_* < \mu^*$ such that $g(\mu_*) \sim \gamma(0)$ and $g(\mu^*) \sim \gamma(1)$. Since g is to be concatenation of f_n on $[0, \mu_n)$, we also require $f_n(\lambda) = g(\lambda\nu_n)$ for every $\lambda < 1$. The latter equality together with the indifferences $g(\mu_*) \sim \gamma(0) \sim f_n(\kappa_*)$ yield the equation $\mu_* = \kappa_*\nu_n$. Similarly, from the concatenation relation between f and ϕ_{n+1} we obtain the equation $\mu^* = (1-\kappa^*)\nu_n + \kappa^*\mu_{n+1}$. Substituting for μ_* and μ^* in (4) and solving for ν_n we find

$$\nu_n = \frac{\lambda_n \kappa^* \mu_{n+1}}{(1 - \lambda_n)(1 - \kappa_*) + \lambda_n \kappa^*}.$$

Now since $\kappa_* < 1$, $0 < \lambda_n < 1$ and $0 < \kappa^*, \mu_{n+1}$, a suitable ν_n exists and uniquely so. By construction, $g \leftrightarrow \gamma$.

By induction, a similar argument applies to every $n < n' \leqslant m$. This completes the proof of this step.

Step 5 (*g is synchronised with every remaining $\gamma \in \Phi$*). For every other $\gamma \in \Phi^{\prec}$, $\gamma(1) \precsim x$ or $y \precsim \gamma(0)$ and by convention $g \leftrightarrow \gamma$. If $\gamma(0) \sim \gamma(1)$, then, since $\gamma(I)$ is a mixture set, theorem 5 of HM ensures that $\gamma(\lambda) \sim \gamma(\mu)$ for every $\lambda, \mu \in I$. In this case $g \leftrightarrow \gamma$ with $\mu = \mu'$. Finally, condition $\mathcal{C}_2$ accounts for every γ such that $\gamma(1) \prec \gamma(0)$. Since these three cases exhaust Φ, our proof is complete. □

Proof of lemma 4 from page 15. Fix $x \prec z \prec y$. By proposition 3, there exists an increasing Φ-concatenation f from x to y. Let $\phi_0, \dots, \phi_m$ and $0 = \mu_0 < \cdots < \mu_{m+1} = 1$ be the sequences that define f. Let g be another increasing Φ-concatenation from x to y and let it be characterised by $\gamma_0, \dots, \gamma_k$ and $0 = \nu_0 < \cdots < \nu_{k+1} = 1$.

Case 1 (for every $n = 1, \dots, m$ and every $j = 1, \dots, k$, $f(\mu_n) \nsim g(\nu_j)$). Note that in this case, we do not need to appeal to $\mathcal{M}$: we can use the paths that make f and g. W.l.o.g., suppose that $f(\mu_1) \prec g(\nu_1)$. By proposition 3, $f(\mu_1) \sim g(\nu')$ for some unique $0 < \nu' < 1$ The fact that $\mu_1 = \nu'$ follows from the argument of the second case with ν' replacing ν_1.

Case 2 (for some $1 \leqslant n \leqslant m$ and $1 \leqslant j \leqslant k$, $f(\mu_n) \sim g(\nu_j)$). W.l.o.g., suppose that $f(\mu_1) \sim g(\nu_1)$, where $\mu_1, \nu_1 < 1$. Together, $\mathcal{M}$ and condition $\mathcal{C}_2$ ensure that, for some $\psi \in \Phi$, $\psi(0) \prec f(\mu_1) \prec \psi(1)$. Lemma 2 then ensures that $\psi(\lambda') \sim f(\mu_1)$ for some unique $\lambda' \in I$. Suppose that $x \precsim \psi(0)$ and $\psi(1) \prec g(\nu_2) \precsim f(\mu_2)$ (otherwise take a subpath of ψ). Since $g(\nu_2) \precsim f(\mu_2)$, proposition 3 ensures the existence of a unique μ'' such that $f(\mu'') \sim g(\nu_2)$.

Let $\phi' \in \Phi$ be the subpath of ϕ_1 such that $\phi'(\xi) = f((1-\xi)\mu_1 + \xi\mu'')$ for every $\xi \in I$. Then there exist unique $\kappa_* < 1$ and $0 < \kappa^*$ such that $\phi_0(\kappa_*) \sim \psi(0)$ and $\phi'(\kappa^*) \sim \psi(1)$. Then, by step 4 of proposition 3 (see page 30),

$$\mu_1 = \frac{\lambda'\kappa^*\mu''}{(1-\lambda')(1-\kappa_*) + \lambda'\kappa^*} \tag{5}$$

Moreover, lemma 3 ensures that κ_* and κ^* also satisfy $\gamma_0(\kappa_*) \sim \psi(0)$ and $\gamma_1(\kappa^*) \sim \psi(1)$.[4] We can therefore write an equation for ν_1 that is similar to (5): the only difference being that ν_2 replaces μ''. Then the only way that $\mu_1 \neq \nu_1$ is if $\mu'' \neq \nu_2$.

If $\nu_2 = 1$, then step 1 on page 29 and the fact that $f(\mu'') \sim g(\nu_2)$ ensure that $\mu'' = 1$, as required. Otherwise, we may repeat the present argument to obtain equations for ν_2 in terms of ν_3 and similarly for μ''. Indeed, by repeating finitely many times we reach $\mu_{m+1} = \nu_{k+1} = 1$, at which point $f(\xi) \sim g(\xi)$ for every $\xi \in I$ and the proof is complete. □

Proof of remark 5. We first show that $\mathcal{I}_4$ holds. Take $\phi, \gamma \in \Phi$ are such that $\phi(0) \sim \gamma(0)$ and $\phi(1) \sim \gamma(1)$. Let $x \stackrel{\text{def}}{=} \phi(0)$. If $x \sim \phi(1)$, then, via lemma 2, $\mathcal{O}$, $\mathcal{P}$ and $\mathcal{I}_3$ imply that ϕ and γ are monotone and $\phi(\lambda) \sim x \sim \gamma(\lambda)$ for every $\lambda \in I$.

Henceforth, w.l.o.g. suppose that $x \prec \phi(1)$. Let x be a constant prospect Since x is comonotonic with $\gamma(1)$, there exists $\gamma' \in \Phi$ such that $\gamma'(0) = x$ and $\gamma'(1) = \gamma(1)$. Then $\mathcal{I}_3$ implies that $\gamma'(1/2) \sim \gamma(1/2)$, $\mathcal{O}$ implies that $\gamma'(1) \sim \phi(1)$ and $\mathcal{I}_3$ then ensures $\gamma'(1/2) \sim \phi(1/)$. A final application of $\mathcal{O}$ yields the conclusion of $\mathcal{I}_4$. The case where all the endpoints of ϕ and γ are nonconstant is similar. In particular, take z to be constant and w.l.o.g. suppose that $\phi(1) \prec z$. Then $\phi_{xz} \in \Gamma$ is increasing and by lemma 2, there exists a unique

[4]Recall that lemma 3 ensures that, for every $\xi \in I$, $\gamma_0(\xi) \sim \phi_0(\xi)$ and $\gamma_1(\xi) \sim \phi'(\xi)$.

μ such that $\phi(1) \sim \phi_{xz}(\mu)$. Indeed, since $\phi(0) = x$, we have $\phi(\lambda) \sim \phi_{xz}(\lambda\mu)$ for every $\lambda \in I$. Similarly, if $y \overset{\text{def}}{=} \gamma(0)$, then since z is a constant prospect, $\phi_{yz} \in \Gamma$ and since $x \sim y$, we have $\phi_{yz}(\lambda\mu) \sim \phi_{xz}(\lambda\mu)$ for every $\lambda \in I$. Then $\mathcal{O}$ implies that $\phi(1) \sim \phi_{yz}(\mu)$ and $\gamma(1) \sim \phi_{yz}(\mu)$. Since $y = \gamma(0)$, we have $\gamma(\lambda) \sim \gamma_{yz}(\lambda\mu)$ for every $\lambda \in I$. This is of course sufficient for $\phi(1/2) \sim \gamma(1/2)$.

For the argument that $\mathcal{A}$ holds, take $x, y, z \in X$ such that $x \prec y$ and let z be a constant prospect. Clearly if $x \prec z \prec y$, then since ϕ_{xz} and ϕ_{zy} belong to Γ, $\mathcal{A}$ holds for x and y. W.l.o.g., suppose that $z \prec x \prec y$. Then clearly $\phi_{zy} \in \Gamma$ ensures that $\mathcal{A}$ holds for x and y.

For the argument that $\mathcal{M}$ holds take $x \prec z \prec y$. If x and y are comonotonic, then $\phi_{xy} \in \Gamma$ and $\mathcal{M}$ holds for x, y, z. Consider the case where x and y are not comonotonic. That is there exists $\omega, \omega' \in \Omega$ such that $x(\omega) \prec_M x(\omega')$ and $y(\omega') \prec_M y(\omega)$. Let $x(\omega) = p$ and $x(\omega') = q$. Then, by the definition of $\prec_M$, $p^\Omega \prec q^\Omega$. We recall that $\mathcal{O}$ implies $z \prec q^\Omega$ or $p^\Omega \prec z$ (see for instance Fishburn [7]). In either case, there exists a constant prospect x' such that $x' \nsim z$ and since both $\phi_{x'y}$ and $\phi_{x'x}$ belong to Γ, $\mathcal{M}$ holds for x, y, z. □

Proof of remark 6 from page 23. The fact that (X, Γ) is partial mixture set follows from by our construction. The argument that $\mathcal{I}_4$ holds is the following. Suppose that $\phi^2_{xy}(\lambda) \sim \lambda(1-\lambda)$ holds for each $\lambda \leqslant 1/2$. Fix an arbitrary $\lambda \leqslant 1/2$. Then note that if $\nu \overset{\text{def}}{=} 4\lambda(1-\lambda)$, then $\phi_{xy}(\lambda) \sim \nu/4$. By construction, $g^{-1}(\nu) = \lambda$, so that a substitution for λ yields $\gamma'(\nu) = (\phi_{xy} \circ g)(\nu) \sim \nu/4$. Finally, $\mathcal{I}_4$ holds because $\gamma'(0) \sim 0$, $\gamma'(1) \sim 1/4$ and $\gamma'(1/2) \sim (0 + 1/4)1/2$. (A similar argument applies to all other $\gamma \in \Gamma_2$.)

We now prove that $\gamma'(I)$ is nonconvex. Since $\gamma'(0) = 0$, we see that that $(1-\nu)\gamma'(0) + \nu\gamma'(1) = \nu\gamma'(1)$. It suffices, therefore, to show that $\nu\gamma'(1)$ is not equal in distribution to $\gamma'(\nu)$ for some $\nu \in I$. In turn, a sufficient condition for the latter is that they differ in expectation.

Recall that for each $\nu \in I$, $g^{-1}(\nu) = \nu/2$. Since $\gamma'(\nu)$ is the square of a Brownian bridge, eq. (2) (with $a = b = 0$) implies

$$(\mathbf{E} \circ \gamma')(\nu) = \mathbf{E}\left\{(1-\nu/2)^2 \cdot \left(\int_0^{\nu/2} \frac{1}{1-r}\, \mathrm{d}W_r\right)^2\right\} = \nu/2 \cdot (1-\nu/2).$$

For instance, if $\nu = 1/2$, then $\nu/2(1-\nu/2) = 3/16$. In contrast,

$$\nu \cdot (\mathbf{E} \circ \gamma')(1) = \nu \cdot \mathbf{E}\left\{(1-1/2)^2 \cdot \left(\int_0^{1/2} \frac{1}{1-r}\, \mathrm{d}W_r\right)^2\right\} = \nu \cdot 1/2 \cdot (1-1/2).$$

So that when $\nu = 1/2$, the latter expectation is equal to $1/8$. $\square$

B Proof of theorem 1

If $\prec = \varnothing$, then by $\mathcal{O}$, $x \sim y$ for every $x, y \in X$. In this case, every utility representation is both linear and cardinal. Conversely, both $\mathcal{A}$ and $\mathcal{M}$ hold vacuously when $\prec = \varnothing$. This ensures that the axioms are necessary and sufficient in this case. Henceforth, suppose that $\prec \neq \varnothing$.

STEP 1 (*Sufficiency of the axioms*). Consider the quotient set $X_{/\sim}$. Each element of $X_{/\sim}$ consists of an indifference class generated by preferences on X. $X_{/\sim}$ is well-defined because $\mathcal{O}$ ensures that the indifference classes partition X. Let $p : X \to X_{/\sim}$ be the natural projection $x \mapsto \{y : y \sim x\}$. Let $F_{/\sim}$ be the set of paths f' in $X_{/\sim}$ such that $f' = p \circ f$ for some increasing concatenation f that is generated by Φ in X. The arguments of the next two paragraphs demonstrate that $(X_{/\sim}, F_{/\sim})$ is a mixture set when $\prec \neq \varnothing$.

If $x \prec y$, then proposition 2 and lemma 4 guarantee the existence and uniqueness (upto indifference) of an increasing concatenation f from x to y. Repeated application of condition $\mathcal{C}_2$ shows that the concatenation g from y

back to x exists and satisfies $g(\lambda) \sim f(1-\lambda)$ for every $\lambda \in I$. For condition $\mathcal{C}_3$, suppose that $z \sim f(\mu)$ for some $z \in X$ and $\mu \in I$. Let $g(\nu) = f(\nu\mu)$ for each $\nu \in I$. If $\mu > 0$, then $g(\nu) = \phi_n((\nu\mu - \mu_n)/(\mu_{n+1} - \mu_n))$ for each n such that $\mu_n < \mu$ and each $\nu \in I$ and we may divide both the numerator and denominator in each ϕ_n to get $g(\nu) = \phi_n((\nu - \mu_n')/(\mu_{n+1}' - \mu_n'))$, where $\mu_n' = \mu_n/\mu$. This shows that g satisfies definition 2. If $\mu = 0$, then $g(\nu) = \phi_0(0)$ for every $\nu \in I$ and by condition $\mathcal{C}_3$, there exists $\phi \in \Phi$ such that $g = \phi$.

If $x \sim y$, then first suppose $x \precsim x'$ for every $x' \in X$. Since $\prec \neq \varnothing$, there exists y' such that $x \prec y'$. In this case, proposition 2 ensures the existence of a concatenation f from x to y'. Then $f(\mu) \sim x$ for $\mu = 0$ and the preceding paragraph completes the proof. Since the case where $x' \precsim x$ for every $x' \in X$ is similar, we proceed to the case where $x' \prec x \prec y'$ for some $x', y' \in X$. In this case, $\mathcal{M}$ and lemma 2 ensure that $x \sim z' = \phi(\mu)$ for some $\phi \in \Phi$ and $0 < \mu < 1$. By proposition 1, $\phi(I)$ is a mixture set and $\phi_{z'z'} \in \Phi$. Finally, since $\phi_{z'z'}(\lambda) \sim x$ for every $\lambda \in I$, $\phi_{z'z'}$ is an increasing concatenation from x to y. Finally, since condition $\mathcal{C}_1$ holds in every case, we have shown that, upto indifference, X a mixture set.

For the axioms, recall from our discussion following definition 2, that, by virtue of $\mathcal{P}$, concatenations inherit continuity from the continuity of members of Φ. Moreover, lemma 4 is clearly sufficient for the property $f(1/2) \sim g(1/2)$ for any pair of increasing concatenations such that $f(0) \sim g(0) \prec f(1) \sim g(1)$. That is to say, increasing concatenations satisfy an independence axiom akin to $\mathcal{I}_4$. We may therefore apply the main theorem of HM to obtain a cardinal and linear utility representation of preferences.

Step 2 (*Necessity of the axioms*). Suppose that U is a CLU of $\precsim$. $\mathcal{O}$, $\mathcal{P}$ and $\mathcal{I}_4$ are well-known to be necessary for a linear utility representation.

Suppose that $\mathcal{A}$ fails to hold. Then for some $x \prec y$, there is some $r' \in \mathbb{R}$

such that $U(x) < r' < U(y)$ and $r' \notin U(X)$. Take $V : X \to \mathbb{R}$ such that $V(z) = U(z)$ for every z such that $U(z) < r'$. For every z such that $r' < U(z)$, let $V(z) = \kappa + \theta U(z)$, where $\theta > 1$ and $\kappa + \theta r' = r'$. This is feasible since $r' - \theta r' < 0$ and we are free to choose $\kappa < 0$. V is a linear utility for $\precsim$, but since it not a positive affine transformation of U, the latter is not cardinal.

Now suppose that $\mathcal{A}$ holds, but $\mathcal{M}$ does not. But this is simply the setting of example 4. With minor modifications, the proof of remark 4 contradicts the assumption that a CLU exists. The remaining possibility is that $\mathcal{A}$ and $\mathcal{M}$ both hold whenever a CLU exists, i.e. that our axioms are necessary.

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