The effects of government spending under anticipation: the noncausal VAR approach

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Abstract

Fiscal foresight, economic agents receiving information about future fiscal policy, affects the consistency of results about the causal effects of government spending. This study explores the propagation of government spending shocks using a noncausal VAR model that allows for anticipation of exogenous fiscal policy changes. Overcoming the issue of insufficient information, the government spending shock is extracted from an anticipated error term by using institutional information about the conduct of fiscal policy. In addition, the approach nests the conventional causal structural VAR as a special case. In the U.S. economy, the identified spending shock comoves with defence expenditures. The shock increases consumption, employment and output one and a half years prior to its materialisation in government spending. After the shock arrives, real wages respond positively while investment turns negative. The estimated fiscal multiplier is close to unity.

JEL classification: C18, C32, E32, E62
Keywords: Government spending, Fiscal foresight, Nonfundamentalness, Noncausal VAR, Structural VAR

1 Introduction

What is the impact of government spending on the economy? Despite a large body of literature, disagreement prevails how an increase in government spending incorporates into consumption, investment and output. To a great extent, the lack of consensus is due to uncertainty about valid methods to identify exogenous fiscal policy changes and

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to measure the reactions of economic agents. In particular, an identification strategy has to take into consideration that a considerable part of policy measures is implemented with a delay by the nature of political process. The economic agents thus foresee and internalise the policies before they are observed as changes in government spending. If agents' expectations are not included to the empirical model, such fiscal foresight causes an econometric obstacle of retrieving a government spending shock to study the reactions.

As the key tool to study the macroeconomic effects of fiscal policy, the vector autoregressive (VAR) model identifies the spending shock by imposing exclusion, sign or medium-run restrictions, typically based on fiscal rules of the government (Blanchard and Perotti, 2002; Galí et al., 2007; Mountford and Uhlig, 2009; Ben Zeev and Pappa, 2017). However, shown by Ramey (2011b), a large part of exogenous changes in the U.S. government spending is related to the defence expenditures and is anticipated by war dates as well as by professional forecasts. Consequently, the VAR model is at risk to identify a shock that is anticipated by the economic agents, and the true propagation of a fiscal policy shock cannot be revealed. In general, the problem relates to nonfundamentalness as the economic agents possess more information than an econometrician and, therefore, it is not possible to infer the structural shocks from the causal autoregressive representation of the VAR model.[1] As shown by Leeper et al. (2013), the fiscal foresight in a rational expectations model inherently leads to the nonfundamentalness problem through a noninvertible moving average (MA) representation for typical observables included to a VAR model. Ignoring the anticipation of the shock may thus seriously distort the conclusions drawn from the model.

The literature has tackled the fiscal foresight by including proxies for expectations of economic agents based on narrative records, stock market data or professional forecasts in an empirical model. The government spending shock is then recursively identified as innovation to the proxy or by local projections following Jordà (2005).[2] Following the narrative approach, Ramey (2011b) constructed a proxy variable from administrative and news sources about the expected exogenous changes in military spending over time, also used in various subsequent studies to control for information. Using stock market data, Fisher and Peters (2010) recovered the spending shock from the excess returns of U.S. military contractors and Caggiano et al. (2015) from the revisions in the professional forecasters. However, the validity of all these approaches is subject to how well the additional variable catches the information held by the public. Another possibility is to use Blaschke matrices to find the corresponding fundamental representation (see Mertens and Ravn 2010). These theoretical dynamic restrictions may, though, excessively limit the set of possible propagation mechanisms.

I contribute to the fiscal policy literature by estimating the impact of government spending with a noncausal model that implicitly controls for the expectations of the public, while being flexible about the underlying economic process. I deviate from the conven-

[1] The literature about misspecifying the VAR model due to nonfundamentalness traces back to Hansen and Sargent (1980), followed by Hansen and Sargent (1991) and Lippi and Reichlin (1994), and more recently discussed in Fernández-Villaverde et al. (2007), Lütkepohl (2014), Forni and Gambetti (2014) and Beaudry and Portier (2014).

[2] For recent survey, see Leeper et al. (2013) and Ramey (2016).

tional VAR analysis by augmenting the specification with the lead terms of observables, which corresponds to the noncausal VAR model of Lanne and Saikkonen (2013). Without assuming the included variables to be informative enough, the future terms resolve the nonfundamentalness problem as the predictable error term of the model may now contain anticipated economic shocks. The impulse responses to the shocks are then derived from the two-sided MA representation of the model which depends both on the past and future errors.

In contrast to its causal counterpart, the noncausal model can be used to recover a shock that may already be internalised by the economic agents. I parsimoniously identify an anticipated spending shock using typical exclusion restrictions imposed on a fiscal rule according to which the government responds to the recent shocks of the economy with a lag. Under fiscal foresight, the model is able to show the impulse responses of forward-looking variables to an anticipated government spending shock. By contrast, when anticipation does not matter, the model reduces to a causal VAR model with the exclusion restrictions following Blanchard and Perotti (2002). However, to distinguish between causal and noncausal specifications, the estimation requires non-Gaussianity as the models are observationally equivalent by their first and second moments only. To that end, I assume multivariate t-distributed errors, under which the Gaussian structural shocks share a volatility term and normality is nested as limiting case. Consequently, the noncausal model can be estimated by a computationally efficient Gibbs sampler following Lanne and Luoto (2016).

How government spending influences the economy is important for validating the consistency of macroeconomic models as well as for designing fiscal policy. In the neoclassical theory, government spending may either crowd out private consumption and investment or stimulate the economy. The latter occurs when the economy involves nominal rigidities and households are non-Ricardian, eventually leading to a fiscal multiplier larger than one. Under flexible prices instead, spending causes negative consumption and real wage responses as crowding out dominates. In the empirical literature, the exclusion restrictions based on the predetermined fiscal policy tend to produce positive consumption and real wage responses, whereas the studies employing a proxy for expectations document a decline of these variables (Ramey, 2016). In a straightforward manner, the noncausal model is able to give insight to what extent these differences stem from the anticipation of fiscal shocks.

Using the U.S. postwar data, I document non-negligible anticipation of macroeconomic variables in the face of a spending shock. Investment mildly rises during the anticipation phase before turning negative, and the shock increases consumption, employment and real wage. These reactions imply a fiscal multiplier close to one, although it is estimated with high uncertainty. The identified shock is also closely related to defence spending, commonly held as a source of exogeneity in the fiscal policy literature. Finally, I compare my results on different, previously used identification strategies. First, the shock I identify coincides over time with the one obtained by the short-run restrictions from the causal VAR model. However, the impulse responses of the causal VAR incorrectly ignore the anticipation phase and underestimate the size of fiscal multipliers. Second, the results are

insensitive to the inclusion of a proxy variable. Moreover, when examined in the noncausal model, identification relying on the narrative proxy of Ramey (2011b) similar results.

This paper proceeds as follows. The next section presents the methodology based on the noncausal VAR to identify government spending shocks and illustrates the approach in a model of fiscal foresight. Section 3 explores the effects of government spending shocks in the US economy. The last section concludes.

2 Theory

By the institutional structure of the government, introducing a new policy involves a lag between legislation and implementation. When the economic agents see the forthcoming policies, they are likely to hold richer information for decision-making than an econometrician observes. As a result, a structural VAR model is unable to extract exogenous policy changes from fiscal variables only. In this section, I propose an approach to recover a government spending shock when allowing for the misalignment between the information sets of the economic agents and the econometrician. First, I show how impulse responses to the anticipated spending shock can be reproduced by means of noncausality. Second, I illustrate the proposed approach analytically in a model of fiscal foresight. Finally, I review the estimation of the model.

2.1 Identification of government spending shocks under anticipation

Let $y_t = (g_t, y'_{2,t})'$ be an n-dimensional vector of observables with the log of quarterly real government spending $g_t = \log G_t$ and $n-1$ remaining variables of interest collected in vector $y_{2,t}$. Assume the mutually uncorrelated structural shocks in u_t propagate to y_t through the MA representation

$$y_t = \sum_{k=0}^{\infty} B_k u_{t-k} = B(L)u_t, \tag{1}$$

where $u_t = (u_{g,t}, u'_{2,t})'$ consists of the government spending shock $u_{g,t}$ and $(n-1)$ other structural shocks of the economy $u_{2,t}$, L is the usual lag operator and $B(L) = \sum_{k=0}^{\infty} B_k L^k$ an $(n \times n)$ matrix polynomial convergent in the powers of L.

Conventionally, the identification of the government spending shock and the derivation of the impulse responses of y_t are based on the causal VAR(p) model

$$\begin{gathered} A(L)y_t = \varepsilon_t, \\ A(L) = I_n - A_1 L - \ldots - A_p L^p, \ \varepsilon_t \sim (0, \Gamma), \end{gathered} \tag{2}$$

with an MA representation

$$y_t = A(L)^{-1}\varepsilon_t = \sum_{k=0}^{\infty} C_k \varepsilon_{t-k} = C(L)\varepsilon_t, \tag{3}$$

which coincides with (1) as long as its one-step ahead forecast error is a linear combination of structural shocks, i.e., $y_t - \mathrm{E}[y_t|y_{t-1}, y_{t-2}, \ldots] = A(L)y_t = \varepsilon_t = B_0 u_t$. Partition this structural VAR model (2) as

$$\begin{bmatrix} \underset{1\times1}{A_{11}(L)} & \underset{1\times(n-1)}{A_{12}(L)} \\ \underset{(n-1)\times1}{A_{21}(L)} & \underset{(n-1)\times(n-1)}{A_{22}(L)} \end{bmatrix} \begin{bmatrix} \underset{1\times1}{g_t} \\ \underset{(n-1)\times1}{y_{2,t}} \end{bmatrix} = \begin{bmatrix} \underset{1\times1}{\varepsilon_{1,t}} \\ \underset{(n-1)\times1}{\varepsilon_{2,t}} \end{bmatrix} = \begin{bmatrix} \underset{1\times1}{b_{11}} & \underset{1\times(n-1)}{b_{12}} \\ \underset{(n-1)\times1}{b_{21}} & \underset{(n-1)\times(n-1)}{b_{22}} \end{bmatrix} \begin{bmatrix} \underset{1\times1}{u_{g,t}} \\ \underset{(n-1)\times1}{u_{2,t}} \end{bmatrix}. \quad (4)$$

Tracing back to Blanchard and Perotti (2002) (BP, henceforth), the spending shock is identified from the system of equations (4) by imposing exclusion restrictions $b_{12} = \mathbf{0}_{1\times(n-1)}$. Accordingly, based on the predetermined nature of economic policy, it takes at least a quarter for the government to learn about the state of the economy and to implement any measures in response. The spending shock $u_{g,t}$ is thus the only exogenous change that both drives the current spending g_t and is unrelated to the past states of the economy.[3]

However, changes in government spending captured by the identification of BP potentially fails to represent the most recent, unexpected shock of the economy. In particular, the identified shock in the U.S. data is related to defence spending and shown to be predictable by information held by the public (Ramey, 2011b). Consequently, such fiscal foresight prevents the VAR model (2) from producing a forecast error unanticipated both to the economic agents and the econometrician. The error is instead a linear combination of past errors from which a static impact matrix B alone cannot recover the structural shocks (Lippi and Reichlin, 1994). Consequently, the measured effects of government spending may be starkly distorted as the MA representation (3) cannot reveal the underlying impulse responses to the shocks u_t (Ramey, 2011b; Leeper et al., 2013).

The above nonfundamentalness problem boils down to the noninvertibility of the MA representation as economic agents react based on broader information than the history of y_t contains.[4] The invertibility of the MA representation (1) could be attained by enriching y_t with variables reflecting the information set of economic agents (Ramey, 2011b; Fisher and Peters, 2010; Caggiano et al., 2015). Alternatively, imposing dynamic structure on the nonfundamental error term, a Blaschke matrix would recover the spending shock (Mertens and Ravn, 2010).[5] However, while the former approach is subject to the ability of the additional variables to establish invertibility and to identify relevant sources of exogeneity, the latter approach may implicitly impose restrictive structure on the economic process.

As an alternative to the above approaches, consider a representation

$$\begin{bmatrix} g_t \\ y_{2,t} \end{bmatrix} = \sum_{k=0}^{\infty} M_k \epsilon_{t-k} + \begin{bmatrix} 0 \\ f_{2,t} \end{bmatrix}, \quad (5)$$

[3]Originally, Blanchard and Perotti (2002) identify both spending and tax shocks using a non-recursive system that combines the exclusion restrictions $b_{12} = \mathbf{0}_{1\times(n-1)}$ with information about elasticities. Nevertheless, ignoring the tax shock and identifying the spending shock through recursive restrictions produces similar results (Ramey, 2011b, 2016).

[4]In other words, y_t is noninvertible in the past as there exist roots inside the unit circle for $|B(z)|$.

[5]In detail, Mertens and Ravn (2010) derive from a rational expectations model a Blaschke matrix that maps the nonfundamental error term of the VAR model to the anticipated spending shock.

where $M(L) = \sum_{k=0}^{\infty} M_k L^k$, $M_0 = I_n$, is a convergent $(n \times n)$ MA polynomial invertible in L and $\epsilon_t \sim (0, \bar{\Gamma})$ is an independent and identically distributed (iid) error term. In turn, the $(n-1)$-dimensional vector $f_{2,t}$ depends directly on the future values of y_t:

$$f_{2,t} = \Phi_{21,1} g_{t+1} + \Phi_{22,1} y_{2,t+1} + \ldots + \Phi_{21,s} g_{t+s} + \Phi_{22,s} y_{2,t+s}, \tag{6}$$

where $\Phi_{21,k}$ and $\Phi_{22,k}$ for $k = 1, \ldots, s$ are $((n-1) \times 1)$ and $((n-1) \times (n-1))$ matrices, respectively. In particular, the representation (5) differs from the invertible MA representation (3) due to the inclusion of $f_{2,t}$ through which the forward-looking variables in $y_{2,t}$ depend on the future error terms. The forward-looking variables thus react to the future errors that realise in spending with a delay. Moreover, the conditional expectation $\mathrm{E}[\epsilon_{t+j} | y_t, y_{t-1}, \ldots]$, $j > 0$ is nonzero such that errors can be predicted by the past observed variables, in contrast to the forecast error ε_t of the causal VAR.

Specifically, the representation (5) tackles the noninvertibility of the MA representation (1) by allowing y_t to depend on future structural shocks. That is, when the observables induce noninvertibility of (1) to the past, $f_{2,t}$ ensures that the dynamics can be correctly captured with respect to an anticipated error term. Hence, the future terms control for the effects of omitted factors and expectations dismissed by the invertible MA polynomial $M(L)$. On the contrary, when the underlying MA representation (1) is invertible and the causal VAR model is valid, $f_{2,t}$ is approximately zero as the lead terms become superfluous, and the equation (5) reduces to the causal MA representation (3) with $\epsilon_t = \varepsilon_t$.

Now, from the representation (5), identify a government spending shock $\bar{u}_{g,t}$ that is allowed to be anticipated by the variables in $y_{2,t}$. Assume the error term ϵ_t is a static rotation of the anticipated shocks $\bar{u}_t$, containing current or lagged values of the underlying, unanticipated structural shocks u_t. The uncorrelated structural shocks $\bar{u}$ with unit variance are mapped into the error term as

$$\epsilon_t = \bar{B} \bar{u}_t, \tag{7}$$

and $\bar{B}$ satisfies $\mathrm{E}[\epsilon_t \epsilon_t'] = \bar{\Gamma} = \bar{B}\bar{B}'$. Let the first row of $\bar{B}$ be $[\bar{b}_{11} \ \bar{b}_{12}]$ with scalar $\bar{b}_{11}$ and a row vector $\bar{b}_{12}$ of dimension $n-1$. Noting that $M_0 = I_n$, by (5), the impact effect of the current structural shocks on government spending g_t is equal to

$$\epsilon_{1,t} = \bar{b}_{11} \bar{u}_{g,t} + \bar{b}_{12} \bar{u}_{2,t}. \tag{8}$$

Imposing $\bar{b}_{12} = \mathbf{0}_{1 \times (n-1)}$, government follows a fiscal rule where spending is predetermined within one quarter except for exogenous changes due to $\bar{u}_{g,t}$. In other words, the fiscal policy responds contemporaneously only to its own shock in addition to the past variation.

By the above scheme, the spending shock is identified by the strategy of BP but relaxed to be anticipated through term $f_{2,t}$, leaving g_t unchanged prior to t. For $f_{2,t} = \mathbf{0}_{(n-1) \times 1}$, the identification reduces to the original scheme (4) and recovers an unanticipated spending shock analogous to BP. Under noninvertibility instead, the past observables are incapable of recovering the fundamental shock which can then be obtained with the help of $f_{2,t}$. It should, however, be noted that the representations (1) and (5) are not necessarily equivalent in a way that the former could directly be rewritten in terms of $\bar{u}_t$ and f_t

as the latter. A direct mapping may instead only exist if specific structure is imposed on the underlying economic model. The approach is rather a parsimonious departure from invertibility to approximate the true process and to flexibly identify an anticipated government spending shock. The representation (5) additionally covers a wider range of underlying economic dynamics and forms of anticipation than a causal VAR model (2) alone.

The model (5) can be estimated with a noncausal VAR(r,s) model of Lanne and Saikkonen (2013)

$$\Pi(L)\Phi(L^{-1})y_t = \epsilon_t, \tag{9}$$

where

$$\Phi(L^{-1}) = I - \Phi_1 L^{-1} - \ldots - \Phi_s^{-s},$$
$$\Phi_i = \begin{bmatrix} 0 & \mathbf{0}_{1\times(n-1)} \\ \Phi_{21,i} & \Phi_{22,i} \end{bmatrix}, \; i = 1, \ldots, s.$$

and $\Pi(L) = I - \Pi_1 L - \ldots - \Pi_r L^r$. To see this, write the representation (14) equivalently as

$$\Phi(L^{-1})y_t = M(L)\epsilon_t,$$

where $M(L)$ can be inverted to the left-hand side and its inverse be approximated up to a truncation error with the causal polynomial $\Pi(L)$.

The impulse responses to the identified shock $\bar{u}_{g,t}$ are derived from the two-sided MA representation of the model,

$$y_t = \Phi(L^{-1})^{-1}\Pi(L)^{-1}\epsilon_t = \sum_{k=-\infty}^{\infty} \Psi_k \bar{B}\bar{u}_{t-k} \tag{10}$$

through which y_t generally depends both on the past and future shocks. Hence, the impulse responses to a government spending shock are derived from

$$\frac{\partial y_{t+k}}{\partial \bar{u}_{g,t}} = \Psi_k \bar{b}_1, \; k = \ldots, -1, 0, 1, \ldots \tag{11}$$

where $\bar{b}_1$ is the first column of matrix $\bar{B}$, derived from $\bar{\Gamma} = \bar{B}\bar{B}'$ after imposing the exclusion restrictions.[6] By the stability of the matrix polynomials $\Pi(L)$ and $\Phi(L^{-1})$, the coefficients Ψ_k decay to zero as $k \to \pm\infty$.[7] Despite the infinite number of lead terms, the two-sided representation approximates the true model (1) but shows the responses of the most recent shocks at the negative lags with coefficients close to zero beyond the anticipation horizon of the economic agents.

[6] In practice, $\bar{B}$ is derived as lower triangular matrix from the Cholesky decomposition of $\bar{\Gamma}$.

[7] The stability is ensured by $\det \Pi(z) \neq 0$ and $\det \Phi(z) \neq 0$ for $|z| \leq 1$.

2.2 Analytical example: a model of fiscal foresight

Next, I illustrate in an example of fiscal foresight from Leeper et al. (2013) how noninvertibility can be resolved by the above noncausal approach. In this particular setting, a mapping from the theoretical noninvertible model to the noncausal VAR exists.

In the model, a representative household maximises the sum of expected future utility, $\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \log C_t$, by deciding upon consumption under perfect depreciation of capital K_t, exogenous productivity A_t and a proportional tax τ_t on production, $\tau_t Y_t = \tau_t A_t K_{t-1}^{\alpha}$ which is redistributed by the government through lump-sum transfers T_t. By optimising subject to the resulting budget constraint $C_t + K_t + T_t \leq (1-\tau_t) A_t K_{t-1}^{\alpha}$, log-linearising and assuming that $a_t = \log A_t - \log A$ is uncorrelated, the solution for capital becomes

$$k_t = \alpha k_{t-1} + a_t + (1-\theta)\frac{\tau}{1-\tau}\sum_{i=0}^{\infty}\theta^i \mathrm{E}_t \hat{\tau}_{t+i+1}, \tag{12}$$

where $\theta = \alpha\beta(1-\tau) < 1$ and k_t, a_t and $\hat{\tau}_t$ are log-deviations from the steady state.

Consider now agents observing a perfect signal on tax rate q periods forward, i.e., $\hat{\tau}_t = u_{\tau,t-q}$. Furthermore, let $a_t = u_{A,t}$ and assume $u_{A,t}$ and $u_{\tau,t}$ are uncorrelated. Substituting these to the solution yields

$$k_t = \alpha k_{t-1} + u_{A,t} - \kappa(u_{\tau,t-q+2} + \theta u_{\tau,t-q+1} + \ldots + \theta^{q-1} u_{\tau,t}), \tag{13}$$

where $\kappa = (1-\theta)\tau/(1-\tau)$. Under foresight, $q > 0$, present capital is influenced in a way that the most recent news informative about the most distant tax rates is discounted the heaviest by anticipation rate θ. By this inverse discounting, the history of observables is likely insufficient to recover the most recent shocks as they have the least weight on the current dynamics.[8] In particular, the MA representation of the observables $y_t = (\hat{\tau}_t, k_t)'$,

$$\begin{bmatrix} \hat{\tau}_t \\ k_t \end{bmatrix} = \begin{bmatrix} L^q & 0 \\ -\kappa\frac{L^{q-1}+\theta L^{q-2}+\ldots+\theta^{q-1}}{1-\alpha L} & \frac{1}{1-\alpha L} \end{bmatrix} \begin{bmatrix} u_{\tau,t} \\ u_{A,t} \end{bmatrix} = B(L)u_t, \tag{14}$$

is noninvertible in the past since $|B(z)| = z^q = 0$ for $z = 0$, inducing absence of a causal VAR representation.[9]

Nonetheless, y_t is noncausal of the form (5). For $q = 2$, rewrite k_t as

$$\begin{aligned} k_t =& -\kappa\frac{L+\theta}{1-\alpha L}u_{\tau,t} + \frac{1}{1-\alpha L}u_{A,t} \\ =& -\kappa\theta\tau_{t+2} - \kappa(1+\theta\alpha)\tau_{t+1} - \kappa\alpha\frac{1+\alpha\theta}{1-\alpha L}u_{\tau,t-2} + \frac{1}{1-\alpha L}u_{A,t}, \end{aligned}$$

where $u_{\tau,t}$ and $u_{\tau,t-1}$ as signals about future tax rates are substituted out using $u_{\tau,t} = \hat{\tau}_{t+2}$. This leads to the representation of the form (5) for y_t,

$$\begin{bmatrix} 1 & 0 \\ \kappa((1+\theta\alpha)L^{-1} + \theta L^{-2}) & 1 \end{bmatrix} \begin{bmatrix} \hat{\tau}_t \\ k_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{1-\alpha L} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\kappa\alpha(1+\theta a) & 1 \end{bmatrix} \begin{bmatrix} u_{\tau,t-2} \\ u_{A,t} \end{bmatrix},$$

[8] See Leeper et al. (2013) for deeper evaluation.

[9] Leeper et al. (2013) additionally show that the nonfundamental representation produced by a causal VAR can severely misinterpret the effects of tax shocks. Ramey (2009) demonstrates with Monte Carlo evidence that noninvertibility has similar consequences on the inference about government spending shocks.

and multiplying from the left by the inverse of the MA polynomial on the right-hand side,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1-\alpha L \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \kappa((1+\theta\alpha)L^{-1}+\theta L^{-2}) & 1 \end{bmatrix} \begin{bmatrix} \hat{\tau}_t \\ k_t \end{bmatrix} = \epsilon_t \tag{15}$$

which is the noncausal VAR(1,2) model (9) with

$$\epsilon_t = \bar{B}\bar{u}_t,\ \bar{B} = \begin{bmatrix} 1 & 0 \\ -\alpha\kappa(1+\theta a) & 1 \end{bmatrix},\ \bar{u}_t = \begin{bmatrix} u_{\tau,t-2} \\ u_{A,t} \end{bmatrix}.$$

The error term ϵ_t in (15) contains now shocks $u_{\tau,t-2}$ and $u_{A,t}$, the former being anticipated by the economic agents. Moreover, $(\hat{\tau}_t, k_t)'$ has a two-sided MA representation (10) and, as a result, their impulse responses with respect to a tax shock $u_{\tau,t-2}$ read as

$$\frac{\partial y_{t+k}}{\partial u_{\tau,t-2}} = \Psi_k \bar{b}_1,\ k = \ldots, -1, 0, 1, \ldots\ . \tag{16}$$

The impulse responses from the noncausal VAR model can thus be derived in a conventional manner but they additionally are located at the negative lags due to the different time-indexing of the shock. In particular, this time-shifting occurs as noninvertibility prevents obtaining the shock as unanticipated using the current and past values of y_t only. Noncausality facilitates then the recovery of an anticipated shock corresponding to the lagged underlying shock. Figure 1 depicts the impulse responses from the theoretical and noncausal models in the upper and lower plots, respectively, and confirms the equivalence of the MA representations (1) and (10) in this set-up. Evidently, the impulse responses coincide, but through the two-sided MA representation of the noncausal VAR, the timing of the tax shock differs, the anticipation effects hitting at negative lags on capital. The policy shock thus influences capital already at negative k of Ψ_k but those responses are zero at leads beyond $k = -2$.

2.3 Estimation

Next, I outline the estimation of the noncausal VAR(r,s) model (9). I rely on Bayesian methods to tackle the large parameter space arising due to the additional lead terms. I therefore closely follow Lanne and Luoto (2016) who derive a Metropolis-within-Gibbs sampler for the model.[10]

To identify a noncausal VAR(r,s) from a causal VAR($r+s$) model, it is necessary deviate from Gaussianity as the models are observationally equivalent in terms of first and second moments only and cannot be distinguished under normality. In what follows, the error term ϵ_t is assumed to be multivariate t-distributed, implying unique identification of the model parameters through its likelihood function.[11] The noncausal VAR is then equivalently written as

$$\omega_t^{1/2}\Pi(L)\Phi(L^{-1})y_t = \eta_t, \tag{17}$$

[10] Lanne and Saikkonen (2013) propose maximum likelihood estimation of the model.

[11] For details on identifiability, see Lanne and Saikkonen (2013).

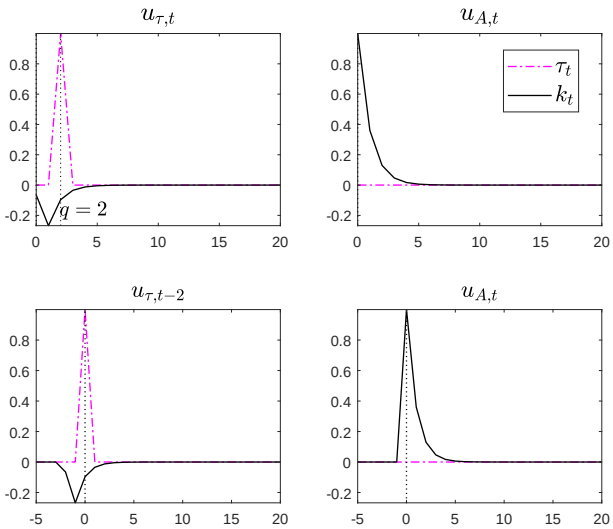


Figure 1: Impulse response functions of the fiscal foresight model to a technology and an anticipated tax shock

The upper panel corresponds to the theoretical impulse responses obtained from the solution. The lower graphs plot the impulse responses obtained from the noncausal VAR(1,2) model.

where $\lambda\omega_t$ is χ^2_λ-distributed and $\eta_t \sim \mathrm{N}(0, \Sigma)$ such that $\bar{\Gamma} = \mathrm{E}[\epsilon_t \epsilon_t'] = \frac{\lambda}{\lambda-2}\Sigma$. Hence, the error distribution is Gaussian conditional on scalar volatility term $\omega_t^{-1/2}$ that controls for leptokurtosis of the time series, i.e., ω_t catches exogenous, common volatility in observables. For small λ, the distribution has fatter tails than under normality. On the other hand, the distribution is closer to Gaussianity for large values of λ. As under normality, η_t is a linear combination of the Gaussian structural shocks which are recovered through rotation matrix $\bar{B}$.[12]

For estimation, the model has a conditional likelihood function shown in Appendix A.1 and a computationally feasible posterior distribution under prior distributions standard in the Bayesian VAR literature. Let vectors π and ϕ_r collect the parameters of the lag and lead terms, respectively. Exploiting (17) and the multiplicative structure of (9), the conditional posterior distribution of parameters is (see Appendix A.1 for details)

$$\begin{aligned}
\phi_r | y, \pi, \Sigma, \omega \sim & \mathrm{N}(\bar{\phi}_r, \bar{V}_{\phi_r}) I(\phi) \\
\pi | y, \phi_r, \Sigma, \omega \sim & \mathrm{N}(\bar{\pi}, \bar{V}_\pi) I(\pi) \\
\Sigma | y, \pi, \phi_r, \omega \sim & \mathrm{iW}(\bar{S}, \bar{\nu}),
\end{aligned}$$

where matrices $\bar{\phi}_r$, $\bar{V}_{\phi_r}$, $\bar{\pi}$, $\bar{V}_\pi$, $\bar{S}$, $\bar{\nu}$ are functions of data and hyperparameters and $I(\cdot)$ is indicator function equal to 1 when the polynomial to which π or ϕ is mapped is stable.

[12]The distributional assumption implies that η_t may contain both anticipated and unanticipated structural shocks that share the same volatility term. Avoiding this potential caveat would require a less parsimonious empirical strategy such as considering an alternative non-Gaussian distribution.

Finally, $\omega = (\omega_{r+1}, \ldots, \omega_{T-s})$ and λ are jointly drawn from

$$\left(\lambda + \epsilon_t(\vartheta)'\Sigma^{-1}\epsilon_t(\vartheta)\right)\omega_t|y, \pi, \phi_r, \Sigma, \lambda \sim \chi^2(\lambda + n), t = r+1, \ldots, T-s,$$

and by a Metropolis-within-Gibbs step from a kernel for λ conditional on y and ω.

3 The impact of government spending in the U.S. economy

This section studies what are the implications of an exogenous change in government spending in the U.S. economy. According to the economic theory, the effects of a government spending shock hinge on the interaction of wealth, intertemporal and distortionary effects (Ramey, 2011a). If the wealth effect of labour supply dominates, the Ricardian households decrease both consumption and investment in response to the increased spending, and crowding-out effects imply a fiscal multiplier smaller than one. In contrast, when the economy involves non-Ricardian and Keynesian elements, a spending shock is instead followed by increasing marginal product of labour and consequently by rising wages, leading to a positive consumption response and stimulative effects of fiscal policy.[13] Finally, distortionary taxation to finance spending dampens the positive effects on consumption, employment and output.

However, validating the effects of government spending poses the econometric challenge due to the fiscal foresight. The noncausal VAR used for the analysis overcomes this issue as the government spending shock may be anticipated. In particular, the approach retains the VAR methodology with the conventional exclusion restrictions, without assuming additional variables to account for the foresight.

3.1 Data and estimation

I use the following U.S. quarterly macroeconomic data. My measure of government spending is government consumption expenditures and gross investment. Output is measured by Gross Domestic Product (GDP), consumption by the sum of consumption of nondurables and services, and investment consists of fixed private investment and consumption of durables. These national accounts variables, taken from the National Income and Product Accounts (NIPA) Tables of Bureau of Economic Analysis, are transformed into real values by the GDP deflator, into per-capita terms by civilian noninstitutional population and expressed in logs. Employment and wages are log per-capita hours and the log real hourly compensation, respectively, from the nonfarm business sector. I derive the average tax rate as all federal receipts divided by the nominal GDP. These seven variables, from which I subtract their quadratic trend, compound the baseline specification and span the quarters from 1945Q1 to 2013Q4. Additionally, I consider annualised inflation, computed as a log difference of GDP deflator, and two interest rates, the 3-month T-bill rate and the 10-Year Treasury Constant Maturity Rate, the latter being available from the second quarter of 1953 onwards.

[13] See Galí et al. (2007) and Ramey (2011a) for more in-depth discussion on propagation mechanisms of government spending.

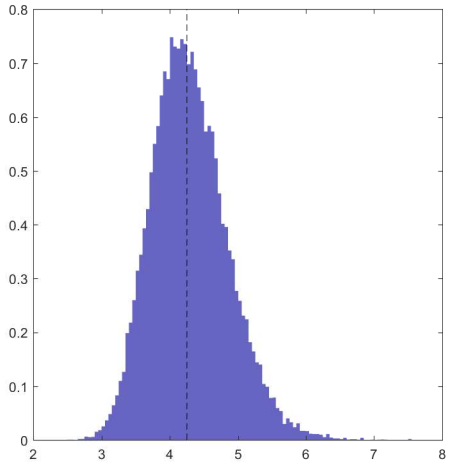


Figure 2: Posterior distribution of degrees-of-freedom parameter λ
Posterior draws of λ from the baseline noncausal VAR. The dashed vertical line is the posterior mean.

The noncausal VAR(r,s) models I estimate include the above variables with the number of lags and leads set to four. Allowing for $r = 4$ lags, on the one hand, allows observables to have an invertible MA polynomial $M(L)$ in (5) general enough to fully catch the variation of structural shocks in the absence of nonfundamentalness. On the other hand, $s = 4$ leads imply rich structure for the noncausal part $f_{2,t}$ if observables are nonfundamental as a result of anticipation. Together, the resulting two-sided MA representation is expected to be general enough to recover the underlying, possibly noninvertible MA representation (1).

I estimate the model with Bayesian methods due to the large parameter space and set a Minnesota-Litterman-type prior distribution as also used by Lanne and Luoto (2016), explained in Appendix A.1. Specifically, I control for tightness of the prior distribution separately for the lag and lead coefficients. By adjusting these overall tightness parameters, the prior about the lag coefficients is less informative, whereas the lead coefficients are shrunk more strongly towards zero. Hence, a priori, the lag terms are more important to determine the dynamics of variables. I proceed by drawing 50,000 times from the posterior distribution. For each draw, I derive the MA representation (10) and impose the exclusion restrictions using Cholesky decomposition $\bar{\Gamma} = \bar{B}\bar{B}'$. As for any Gibbs sampler, the algorithm to obtain posterior draws performs well when the distribution is unimodal. Multimodality, though, easily arises in the estimation of the noncausal VAR, as observed by Lanne and Luoto (2016). Nonetheless, the less loose prior distribution for ϕ_r by the greater overall tightness and the restrictions imposed in (9) are powerful in attaining a unimodal posterior distribution.

For identifying a unique VAR(r,s) specification, it is necessary to assume non-Gaussianity of the error term. However, the assumed multivariate t-distribution nests Gaus-

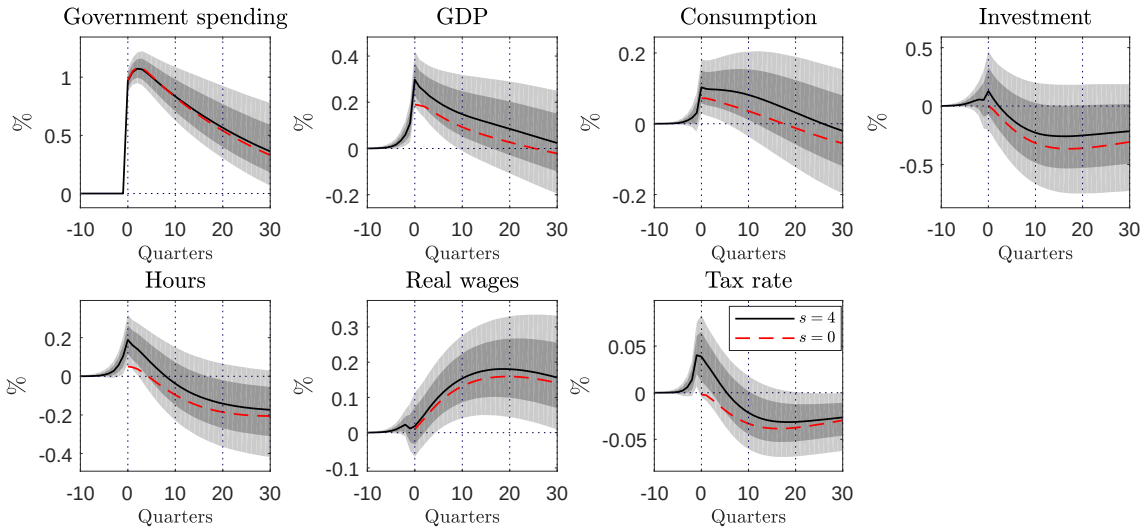


Figure 3: Impulse responses to the government spending shock from the baseline model
Solid lines and dashed lines are the median impulse responses from the noncausal VAR(4,4) and causal VAR(4) models, respectively, to a one standard deviation government spending shock. The dark and light grey shaded regions are the 68 and 90 per cent, respectively, credible sets of the estimated impulse responses from the noncausal model.

sianity for large degrees-of-freedom parameter λ. Low estimates of λ thus immediately suggests the validity of the distributional assumption compared to Gaussianity, implied by excess kurtosis in the error distribution. In Figure 2, I plot the histogram of the posterior draws of λ from the baseline VAR model. The histogram clearly indicates that a large probability mass is located at low degrees of freedom. Moreover, the data strongly dominate the assumed prior mean 10 of λ with a posterior mean of 4.2: the probability of λ being greater than 6 is extremely low. Therefore, the data lend support for the multivariate t-distribution, which facilitates the identification of the noncausal model.

3.2 Impulse responses to a government spending shock

In Figure 3, the solid lines depict the estimated impulse responses to a one standard deviation government spending shock to the seven variables included in the baseline noncausal VAR(4,4) model. Therein, I also report the posterior medians of the estimates together with the 68 and 90 percent credible sets.[14] Because of noncausality, the responses are located both at the negative and positive lags, the former being estimated close to zero beyond lead 10.

According to the noncausal model, government spending increases by one per cent over its trend and output peaks at 0.3 per cent in response to a spending shock $\bar{u}_{g,t}$. The shock materialises in spending from time 0 onwards, implied by the zero restrictions on

[14]Both the posterior median and the credible sets are found by computing the periodwise quantiles from the impulse responses implied by the draws of θ.

the lead terms, while the other variables anticipate this increase from quarter -6 onwards. Except for investment and wages, this anticipation is measured to be statistically significant at the 90 percent level. GDP reacts approximately one and a half years ahead of the future spending increase. Simultaneously, consumption starts to increase and peaks at the realisation of the shock. On the other hand, investment turns negative soon after the shock arrives, remaining below its trend for the subsequent quarters. Hours worked move fast and positively at the anticipation lags and stay over the trend for the following 10 quarters after starting to decrease at the materialisation of the shock. The real wage exhibits a hump-shaped increase, which occurs simultaneously with the decline of hours. Last, the initially, during the anticipation phase increasing average tax rate implies that the induced spending is at least partly tax-funded.

I continue by augmenting the above baseline VAR model with inflation, the short-term rate and the term spread, where the latter is defined as the difference between the 10-year and 3-month rates. As the estimates regarding the seven variable above are indistinguishable from the above results (see Appendix A.2), in panel (a) of Figure 4, I report for the sake of space solely the responses of the three additional variables from this ten-variable VAR(4,4) model.[15] A one percent, exogenous increase in government spending has a small, negative impact on inflation that decreases by 0.15 percentage points. The 3-month rate remains mildly negative during the propagation of the shock, and these two reactions together imply a 0.1 percentage point increase in the ex-post real interest rate. Originating from a smaller change observed in the 10-year rate, the spending shock leads to a higher slope of the term structure, although the effect is insignificant at the most lags.

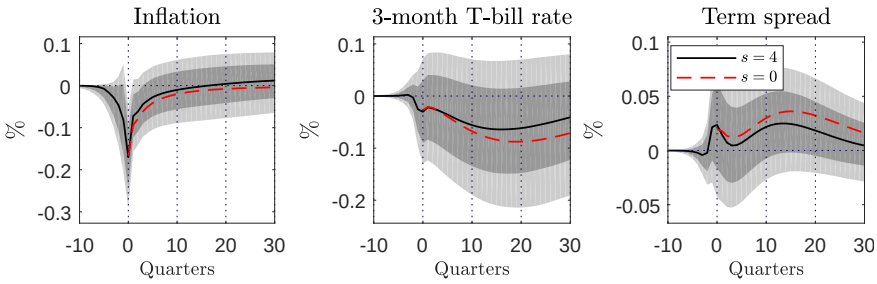
Above, the spending shock caused a positive reaction of consumption and initially increasing but eventually below-trend-level declining investment. For a more in-depth analysis, I replace consumption and investment with their subcomponents in the baseline VAR. Panel (b) of Figure 4 plots the responses of disaggregated consumption and investment components from the re-estimated noncausal VAR(4,4).[16] Both services and nondurable consumption respond significantly and positively to a spending shock, which translates to the previously documented increase of the aggregate consumption. Investment moves mainly due to the reactions of residential and nonresidential investment as durable consumption shows no significant reaction.

Figure 5 plots the four-quarter moving average of the identified shock from the baseline noncausal model together with federal spending divided into three components, consumption expenditures and gross investment on national defence, and non-defence federal spending. Accordingly, the spending shock is closely related to the U.S. defence expenditures, although the link has become somewhat weaker in the past three decades.[17] The spending shock series leads consumption and investment components of defence expendi-

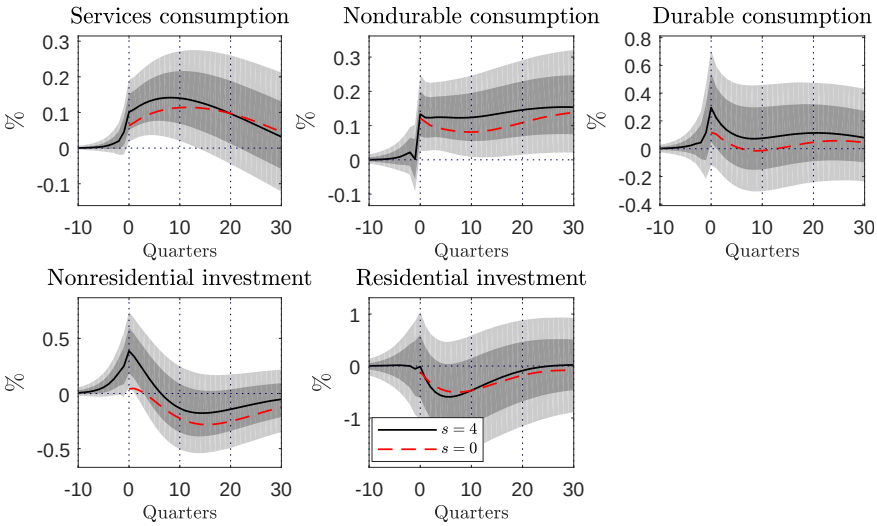
[15] In this specification, data are available only from 1952Q2 onwards. However, the results remain nearly indistinguishable from those of Figure 4 when the 10-year rate is omitted and the model is estimated on data starting from 1945Q1.

[16] Appendix A.2 shows the responses of the other variables included to the model and they coincide with the results from the baseline model.

[17] The relationship becomes the more evident the longer moving average is taken.



(a) Inflation and interest rates



(b) Consumption and investment components

Figure 4: Impulse responses to the government spending shock from the VAR models with additional variables

Solid lines and dashed lines are the median impulse responses from the noncausal VAR(4,4) and causal VAR(4) models, respectively, to a one-standard deviation government spending shock. The dark and light grey shaded regions are the 68 and 90 per cent, respectively, credible sets of the estimated impulse responses from the noncausal model. The impulse responses in panel (a) are from the 10-variable VAR including the baseline and the plotted variables. The impulse respones in panel (b) are computed from the 10-variable VAR including the baseline variables but consumption and investment replaced by the variables shown.

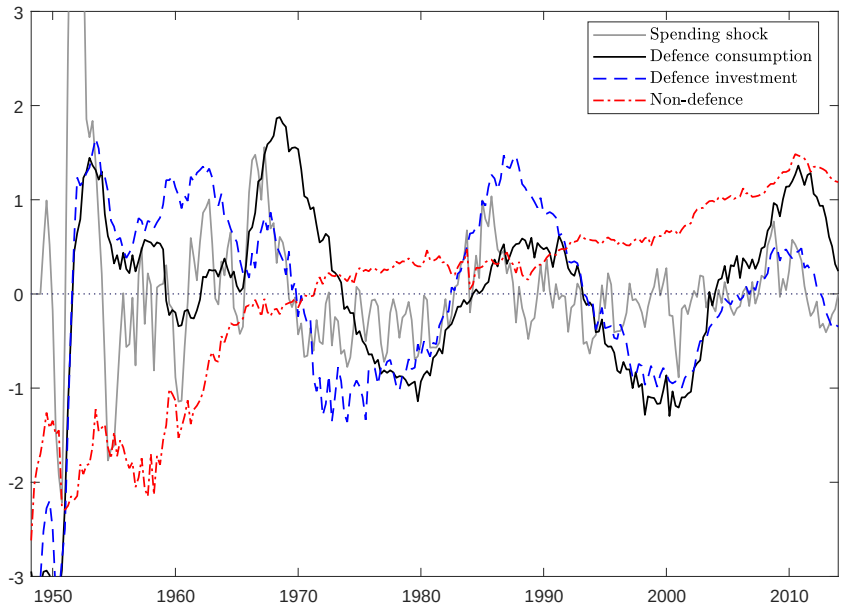


Figure 5: Spending shock, defence and non-defence expenditures

The solid, light grey line depicts the posterior median of the 4-quarter moving average of the spending shock identified from the baseline noncausal VAR(4,4). Dot-dashed and dashed lines are the log real national defense consumption expenditures and gross investment, respectively. Dotted line refers to the log of real federal non-defence consumption expenditures and gross investment. All variables are demeaned and standardised.

tures by one year and is unrelated to the non-defence spending. By these observations, the shock can be characterised by two insights. First, the great persistence of the shock observed in the impulse responses is likely to stem from the nature of defence spending, from decade-lasting military build-ups and wars that the United States was engaged in. Second, the shock is unrelated to the non-defence component of federal spending. It induces instead variation belonging to a particular class of events, the U.S. military expenditures, which likely are orthogonal to the present state of the economy and which have been used to identify exogenous events in spending in the existing literature.[18]

By the noncausal model, it was possible to identify a spending shock that may be predictable to the economic agents, without ruling out causality a priori. To assess the importance of the lead terms, Figures 3 and 4 included in dashed lines the impulse responses from the causal VAR(4) models to an unanticipated spending shock following the identification of Blanchard and Perotti (2002). Setting $s = 0$ and disregarding noncausality, the estimates remain close to their noncausal counterparts at the positive lags with the exception of hours and tax rate that respond only mildly. However, by construction, the causal VAR ignores the responses at the negative lags, despite the fact that the shock may well be anticipated. Moreover, as shown in Figure A.2 in Appendix A.2, the shock identified from the causal VAR virtually coincides over time with the one from the noncausal model. In other words, the causal VAR recovers a shock that aligns with the shock from the noncausal model, but only the latter is able to show the reactions at the anticipatory, negative lags. A causal VAR model under the exclusion restrictions is thus at high risk to catch defence-spending-related events that the economic agents are able to forecast.

In view of the economic theory, I interpret the results as follows. The increase of both output and employment in response to the anticipated spending shock suggests the dominance of the wealth over substitution effects of the households. Moreover, as the shock induces profound increases both in real wage and consumption, there is evidence, to some extent, on the existence of non-Ricardian and Keynesian mechanisms. On the other hand, the eventual decrease of investment suggests crowding out of private business. The path of investment is mostly due to the non-household components as durable consumption or residential investment show no significant reactions. Somewhat surprisingly, the shock has deflationary effects on the price level in contrast with the neoclassical theory and positively reacting consumption.[19]

3.3 The size of fiscal multiplier

Does government spending stimulate the economy? A fiscal multiplier larger than one indicates that an increase in government spending boosts private economy in a way that the benefits dominate the crowding-out and distortionary effects of public consumption and taxation. The spending literature calculates the multiplier by two alternative ways, either as a peak output response relative to the initial government spending impact effect

[18] U.S. defence spending and military events are regarded as sources of exogenous variation, amongst others, by Ramey and Shapiro (1998), Fisher and Peters (2010), Ramey (2011b) and Ben Zeev and Pappa (2017).

[19] The deflationary effects have also been observed by Mountford and Uhlig (2009).

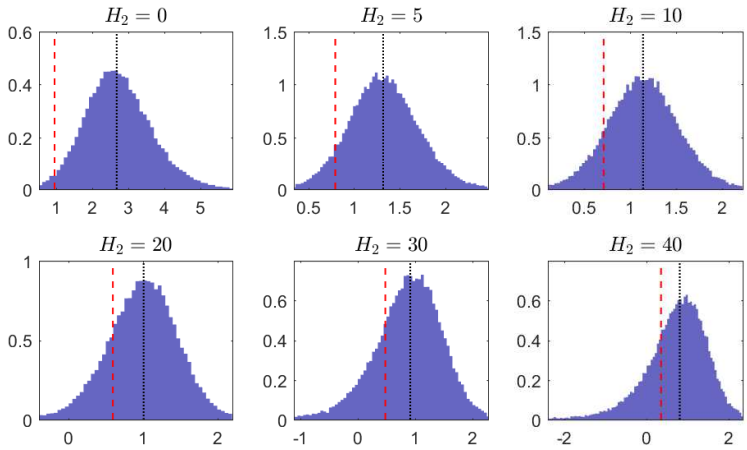


Figure 6: Histogram of posterior draws for fiscal multipliers on different horizons H_2
The multipliers are computed using (18) with $H_1 = -10$ and based on the baseline noncausal VAR(4,4) model. Quantities on the y-axis are normalised such that histograms integrate to 1. Dotted vertical lines are the medians of the multipliers. The red dashed lines are the median fiscal multipliers from the corresponding causal VAR(4) model.

or as a ratio of present value integral of the output response to the integral government spending response. I follow the latter technique, also suggested by Mountford and Uhlig (2009) and Ramey (2016), as the former method tends to overestimate the size of fiscal multiplier. In addition, the latter takes more flexibly timing, persistence and anticipation of the shock into account. The fiscal multiplier is defined as

$$\frac{\sum_{k=H_1}^{H_2}(1+r)^{-(k-H_1)}\frac{\partial \log GDP_{t+k}}{\partial \bar{u}_{g,t}}}{\sum_{k=H_1}^{H_2}(1+r)^{-(k-H_1)}\frac{\partial \log G_{t+k}}{\partial \bar{u}_{g,t}}}\frac{GDP}{G}, \tag{18}$$

where H_1 is set to -10, to the time point of initial reactions according to Figure 4, H_2 is the length of horizon after the shock has realised and r is the long-run real interest rate computed as a sample mean of the difference between the T-bill rate and inflation. Last, GDP/G is the sample mean of the ratio of GDP to government spending and converts the percentage deviations to monetary units in real terms.

Figure 6 reports the posterior distribution of the fiscal multiplier computed by (18) and their medians in dotted lines for various horizons. Under shorter horizons, $H_2 \in \{0, 5, 10\}$, in the upper plots, the median multiplier is above one, being the largest when only the impact effect of government spending is included, $H_2 = 0$. Eventually, the size converges towards one with longer horizons, as seen in the lower graphs. The impulse responses of Figure 3 reveal the mechanism behind this pattern. Government spending showing persistent increase simultaneously with the fast return of GDP to its trend level, the size of multiplier decreases once more inputs are added to the denominator of (18).

Overall, there is great uncertainty whether government spending can be stimulative, seen as large dispersions in the posterior distributions of Figure 6. Even for the shorter

horizons, a significant portion of the probability mass is concentrated on the region below one, and the long-run multiplier with $H_2 = 40$ reaches with high probability both positive and negative values. It is also noteworthy to mention that the multiplier computed here is not purely deficit-based as government spending is accompanied by a distortionary increase in tax rate.[20] In light of this evidence, the overall impact of the identified exogenous government spending on the private economy remains imprecise.[21]

Finally, I draw in dashed lines the posterior medians of fiscal multipliers computed from the causal VAR(4) model with the baseline variables. At all horizons, the multipliers are quantified to be smaller than their corresponding estimates from the noncausal model. Strikingly, despite the positive response of consumption, the causal VAR model with the BP identification has a tendency to produce small multipliers (see also, Ramey, 2016). This well-known controversy can be explained from the causal responses shown in Figure 3. As the causal structural VAR model disregards the anticipation effect in GDP, the nominator of (18) is necessarily smaller relative to the denominator, which results in a smaller multiplier.

3.4 Relation to the proxies and and narrative measures

In the government spending literature, the fiscal foresight and nonfundamentalness have been approached by using a proxy variable either to enrich the information set of a VAR model or to derive the responses to a shock using local projections (Jordà, 2005). By the exclusion restrictions imposed on the error term of the noncausal model (9), the anticipated spending shock can instead be recovered independent of the nonfundamentalness issue exploiting the predetermined nature of government policy. Importantly, including a proxy to the noncausal VAR can then shed light on how informative the variable is about the identified shock.

Concluded from Figure 5, the shock identified in the noncausal VAR reflects defence expenditures, and I thus consider two prominent proxies used in the literature, the narrative defence news and the excess returns of military contractors. First, Ramey's narrative news (Ramey, 2011b) measures information held by the public about the expected discounted value of government spending changes due to foreign policy events. On the other hand, the Fisher-Peters excess returns (Fisher and Peters, 2010) aims to gauge the market expectations about future spending by the returns of top three U.S. military contractors.[22] According to Fisher and Peters (2010), the difference between these series stems from the

[20] Owyang et al. (2013) and Ramey (2016) argue that a trend in the GDP-to-spending ratio leads to a bias in the multiplier estimates computed using (18). I reproduced, for robustness, impulse responses and fiscal multipliers by transforming the national accounts variables with the Gordon-Krenn transformation as the authors suggest. The results both remain qualitatively and are of the same magnitude as those produced in this section.

[21] One possible explanation for this uncertainty may be the time-dependence in the effectiveness of fiscal policy, as examined by Auerbach and Gorodnichenko (2012), Owyang et al. (2013) and Caggiano et al. (2015).

[22] The defence news variable and Fisher-Peters excess returns are available as supplementary data for Ramey (2016) in Valerie Ramey's webpage. The narrative news series, extended by Owyang et al. (2013), spans the whole post-war period until 2013Q4 whereas the last observation for Fisher-Peters data is 2007Q4.

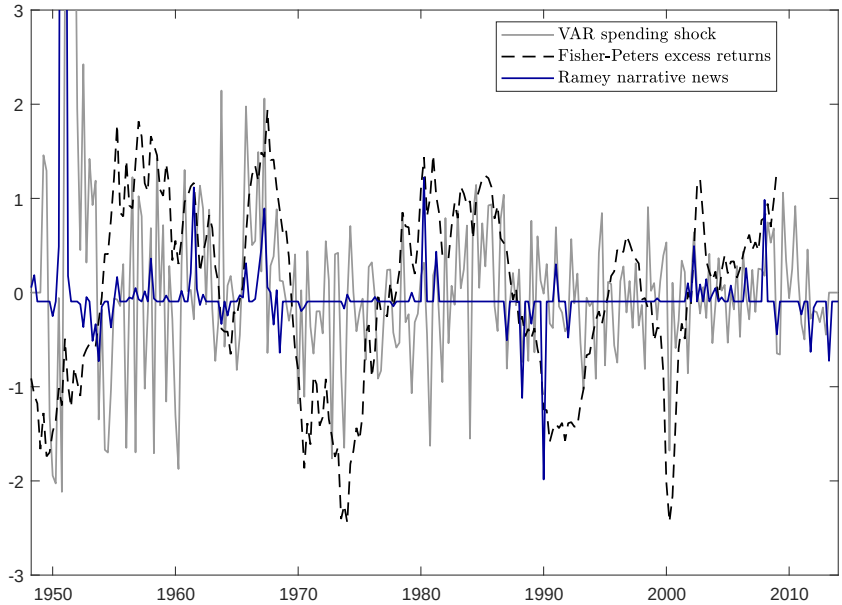


Figure 7: The identified spending shock, Fisher-Peters excess returns and Ramey's narrative news

Grey solid line depicts the median of identified anticipated spending shocks from the baseline VAR(4,4) model, the blue solid line the Ramey (2011b) narrative news of anticipated goverment spending, measured as a share of future government spending of GDP and the black dashed line the excess returns of military contractors constructed by Fisher and Peters (2010).

fact that market expectations about military spending evolve in a more nuanced way than the immediate changes seen in the Ramey's news series. However, Ramey (2016) argues the excess returns series has low instrumental relevance for government spending.

In Figure 7, I plot the proxy variables along with the identified government spending shock estimated from the baseline noncausal VAR(4,4) model. The shock spikes during military events, similar to the the narrative defence news, most notably during the Korean and Vietnam wars at the beginning of 1950s and 1970s, respectively. The Fisher-Peters excess returns comoves with the shock during the 1960s and 1970s as well as during Ronald Reagan's presidency. However, neither of the series is a direct empirical representative of the identified shock.

I continue by adding the proxy variables to the baseline noncausal VAR and allow them to anticipate the shock identified by the exclusion restriction. As expected, a shock-related proxy would respond positively to the future spending increase. Figure 8 graphs the median impulse responses of government spending, GDP and the two respective proxy variables to a one standard deviation shock identified by the exclusion restrictions in both models where the eighth variable is either the narrative news or the Fisher-Peters excess returns.[23] Accordingly, the exclusion or inclusion of either of the proxies does not alter

[23]The excess returns being available only until 2007Q4, the second model spans a shorter time period. However, the results from the baseline model or from the specification with Ramey's narrative news do

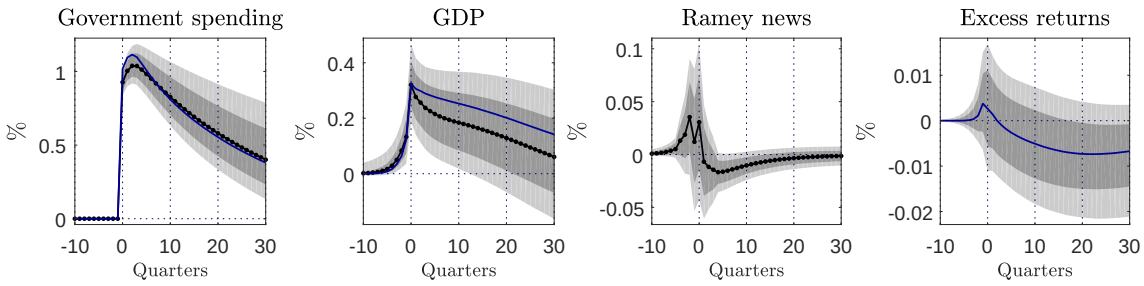


Figure 8: Impulse responses from the VAR models with a proxy variable

Black marked solid lines and blue solid lines are the posterior median impulse responses to the anticipated spending shock from the noncausal VAR(4,4) models augmented with Ramey's narrative news and the Fisher-Peters excess returns, respectively. The dark and light grey shaded regions are the 68 and 90 percent posterior credible sets, respectively, shown in the responses of spending, GDP and Ramey's news from the former and in the response of excess returns from the latter VAR model.

the estimates about the responses of government spending and GDP. Moreover, the shock induces no significant reactions in the proxies, Ramey's news variable responding only slightly positively before the realisation moment.

The relevance of the spending shock can also be analysed by means of its relative contribution to the overall movements in a variable. Formally, the ith variable in the noncausal VAR has an MA representation

$$y_{i,t} = e_i' \sum_{k=-\infty}^{\infty} \Psi_k \left(\bar{b}_1 \bar{u}_{1,t-k} + \bar{b}_2 \bar{u}_{2,t-k} \right), \tag{19}$$

where $e_i = (0, ..., 1, ..., 0)'$ with 1 in its ith element and an $(n \times (n-1))$ matrix $\bar{b}_2$ consists of columns of $\bar{B}$ corresponding to the $n-1$ remaining shocks contained in $\bar{u}_{2,t-k}$. Now, define the fraction of variance of $y_{i,t}$ due to the government spending shock $\bar{u}_{g,t}$ over horizon $[H_1, H_2]$ as

$$\rho(y_{i,t}; H_1, H_2) = \frac{\sum_{k=H_1}^{H_2} e_i' \Psi_k \bar{b}_1 \bar{b}_1' \Psi_k' e_i}{\sum_{k=H_1}^{H_2} e_i' \Psi_k \Sigma \Psi_k' e_i}. \tag{20}$$

Under causality, the fraction (20) reduces to the forecast error variance decomposition of a VAR model over horizon H_2. Once $s > 0$, $\rho(y_{i,t}; H_1, H_2)$ generally gives the fraction of the unconditional variance of variable $y_{i,t}$ explained by the spending shock. In panel (a) of Table 1, I report these fractions in the baseline model and in the models with a proxy variable. The identified shock explains now over 50 per cent of the detrended variation in government spending and approximately 10 per cent in output. In contrast, the shock contributes minimally to the movements in Ramey's news and the excess returns variables. Hence, the proxies are unrelated to the shock identified by the exclusion restriction, which translates to the negligible reactions seen in Figure 8.

Does the use of identification based on proxies lead to different conclusions about the effects of government spending? The mild responses of the proxies to the spending

not alter when using data up to 2007Q4.

Panel (a)	Identification by the exclusion restrictions							
	Baseline VAR		Baseline + Ramey			Baseline + F-P		
Horizon (H_2)	g_t	GDP_t	g_t	GDP_t	Ramey	g_t	GDP_t	F-P
0	100.00	10.34	100.00	10.42	1.81	100.00	9.30	0.80
5	95.35	10.77	94.97	11.63	2.46	92.45	13.47	0.86
10	87.79	9.60	88.22	11.34	2.73	82.48	15.94	1.32
20	72.92	7.67	76.23	10.14	2.90	67.66	18.65	2.40
30	60.73	6.70	66.54	8.77	2.96	59.57	19.46	3.46
40	51.77	6.63	58.69	8.13	3.01	55.16	19.47	4.18
Panel (b)	Identification by the proxy variables							
			Baseline + Ramey			Baseline + F-P		
Horizon (H_2)			g_t	GDP_t	Ramey	g_t	GDP_t	F-P
0			0.43	0.30	93.63	0.94	3.76	85.11
5			0.60	0.32	88.28	1.53	3.03	92.67
10			0.55	0.34	86.96	2.76	3.62	92.80
20			0.50	0.36	85.96	5.70	4.86	88.92
30			0.49	0.37	85.46	8.10	5.91	84.26
40			0.50	0.38	85.09	9.67	6.72	80.77

Table 1: Fractions of variance contributed by the spending shock

The shares shown in percentages are computed using (20) with $H_1 = -10$ as posterior medians. Baseline + Ramey refers to the VAR(4,4) model with the seven baseline variables and the Ramey news, Baseline + F-P to the VAR(4,4) model with the baseline variables and the Fisher-Peters excess returns. In panel (b), the identification is proceeded by the approach maximising the contribution of the shock to the eighth variable, explained in text.

shock and their small contributions suggest that the changes in the proxy variables about defence expenditures consist of events different from those captured by the identified shock. Nonetheless, these events – provided they are exogenous – may induce effects similar to the structurally identified shock. I therefore derive impulse responses to a shock identified by the variation of a proxy. In both of the augmented eight-variable models above, I proceed by finding a shock that explains the most of the overall movements of the proxy, i.e., it maximises the fraction of variance (20) among all possible linear mappings from structural shocks to the reduced-form error term.[24] Given the proxy is mainly driven by the exogenous variation that translates to changes in spending, the identification strategy is valid to recover the causal effects.

Panel (b) of Table 1 reports the shares contributed by these two alternatively identified shocks to the unconditional variance of government spending, GDP and the respective proxy. A single shock is able to explain the major part of the variance of the proxy. However, the shocks barely influence the overall detrended variation of spending and GDP. In Figure 9, I report the results from all identification strategies. First, the blue solid and black marked solid lines draw the impulse responses to the shock identified by the standard exclusion restriction from the augmented models. They are identical to those

[24] Appendix A.3 shows in detail that this identification can be achieved through an eigenvalue problem, similar to the MaxVar approach (Uhlig, 2004; Francis et al., 2014) which rotates the error of a VAR model to find the shock that maximises its amount to the forecast error variance.

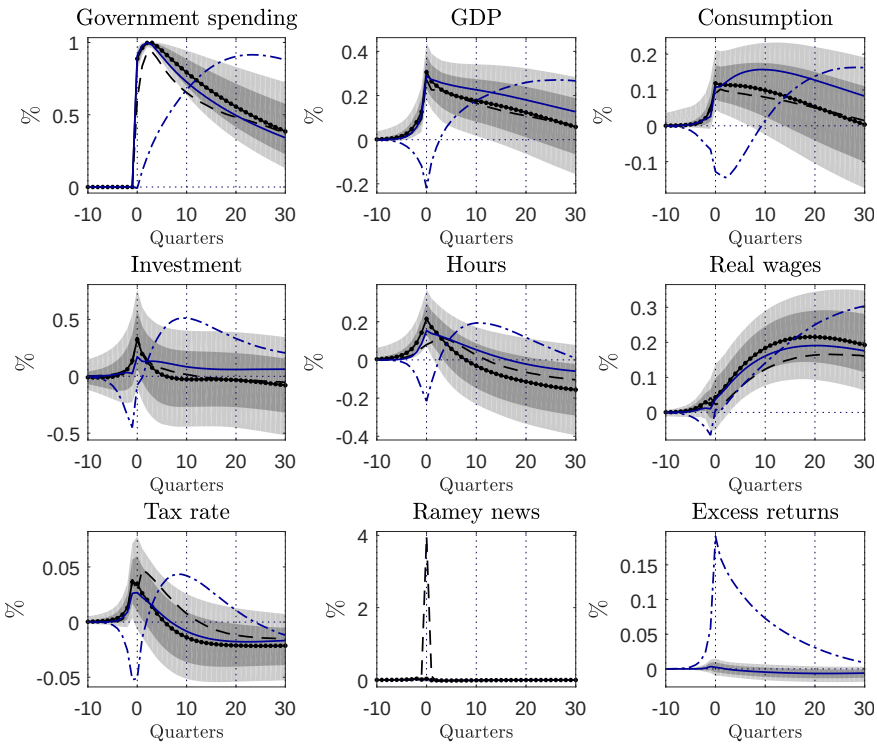


Figure 9: Impulse responses from the VAR Models including a proxy variable and with the alternative identification strategy

Black marked solid lines and blue solid lines are the posterior median impulse responses to the shock identified by the exclusion restrictions from the noncausal VAR(4,4) models augmented with Ramey's narrative news and the Fisher-Peters excess returns, respectively. The black dashed and blue dot-dashed lines are the responses to the alternatively identified Ramey and Fisher-Peters shocks, respectively. The dark and light grey shaded regions are the 68 and 90 percent posterior credible sets, respectively, shown for the baseline variables and the narrative news from the model with Ramey's news and for the excess returns from the second VAR model.

in Figure 8 but are rescaled by normalising the maximum impact on spending to one per cent. Second, the black dashed and blue dashed-dotted lines depict the responses to the Ramey and Fisher-Peters shocks, derived from the alternative identification strategy with respect to the two proxies.

In response to the Ramey shock, the narrative news variable jumps by four percentage points, as expected from the identification strategy that maximises the contribution of the shock to the variable. The jump is followed by a gradual increase of spending which peaks after a year. Broadly, the responses to the Ramey shock are also similar to those from the standard identification and within the credible bands, despite the shocks are empirically unrelated. Conversely, when considering the reactions to the Fisher-Peters shock, these conclusions significantly change. The dissimilarity is partly due to different timing, seen as an immediate jump in the excess returns at date 0 while government spending sets to a long-lasting growth path.

Unlike the previously employed empirical strategies, the noncausal VAR approach can

assess the connection between the structurally identified shock and the empirical proxies, as the error term need not be unanticipated. According to this investigation, the proxies based on the narrative news and excess returns are unlikely to measure the same variation that is captured by the structurally imposed exclusion restrictions. Therefore, the use of these proxies does not directly alleviate the predictability problem of the shock of Blanchard and Perotti (2002). The different results rather stem from factors unrelated to nonfundamentalness such as from the validity of identifying restrictions.

4 Conclusions

This paper addressed the question of the implications of a government spending shock in the face of anticipation. Any empirical strategy attempting to quantify these effects confronts an econometric issue with the timing of the shock as the economic agents are likely to have a larger information set than the model assumes. I resolved the issue of deficient information with the noncausal VAR that is able to incorporate expectations but simultaneously retains the advantages of the VAR methodology by imposing few assumptions on the underlying economy. The analysis of fiscal policy could then be proceeded with a standard identification strategy to recover an anticipated government spending shock.

The noncausal VAR methodology deviates from the forecast error interpretation of the residual but – despite anticipation – facilitates conducting conventional structural analysis. In a simple model of fiscal foresight, I analytically showed that the noncausal model is able to solve the noninvertibility problem. Even though a similar mapping may not exist in a more general setting, the lead terms of the model flexibly are expected to capture anticipation dismissed under invertibility. Importantly, the approach does not rule out the causal case a priori as invertibility of the underlying MA representation is nested in the framework.

In the U.S. postwar economy, the estimated spending shock induced an increase in the forward-looking variables during the anticipatory phase. Spending also turned out to be followed by rising consumption, worked hours and wages, whereas investment was found to decrease as soon as the spending shock materialises. Together, these movements implied a fiscal multiplier close to unity. Importantly, the anticipatory forces are important to take into consideration as they have effect on the measured fiscal multiplier and on the overall impact of forward-looking variables. I also revisited two prominent alternative strategies based on the proxy variables that attempt to circumvent the nonfundamentalness problem. Notably, a proxy to catch the expectations is unlikely to measure the same variation identified by the exclusion restrictions.

Finally, I consider the following areas useful for further research. First, the noncausal approach is readily available for the study of government spending shocks in other economies, as research can be employed using conventional macroeconomic data only, without engaging in costly and demanding data collection of proxy variables. Second, the examination of tax policy with the noncausal model, after imposing adequate structure, can be viewed as a useful extension. Finally, the estimation of the model was based on a simple deviation from Gaussianity which, though, assumed cross-dependent volatility for

the structural shocks. Furthermore, detrended variables were used because the implications of stochastic trends to the model are yet unknown. Using alternative distributions and allowing for nonstationarity could strengthen the robustness of the approach.

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A Appendix

A.1 Estimation of the noncausal VAR(r,s)

I refer to Lanne and Luoto (2016) in the derivation of the following Gibbs sampler algorithm. I additionally consider zero restrictions on the elements of Φ_i, $i = 1, \ldots, s$. Let Π and Φ be matrices stacking Π_i' for $i = 1, \ldots, r$ and Φ_i' for $i = 1, \ldots, s$, respectively. Furthermore, write $\pi = \text{vec}(\Pi)$ and $\phi = \text{vec}(\Phi)$, $\vartheta = (\pi', \phi')'$ and $\theta = (\pi', \phi', \text{vech}(\Sigma)', \lambda)'$. To impose s^* zero restrictions on matrix Φ to satisfy (9), introduce an $((n^2s - s^*) \times 1)$ vector ϕ_r containing the unrestricted parameters of Φ and an $(n^2s \times (n^2s - s^*))$ deterministic matrix R_ϕ which maps the unrestricted parameters to the matrix Φ as $\phi = R_\phi \phi_r$.

The approximate conditional joint density of $y = (y_1, \ldots, y_T)$ on $\omega = (\omega_{r+1}, \ldots, \omega_{T-s})$ is

$$p(y|\omega, \theta) \approx \prod_{r+1}^{T-s} p(\epsilon_t(\vartheta)|\omega_t, \Sigma)$$

with

$$p(\epsilon_t|\omega_t, \Sigma) = \frac{\omega_t^{n/2}}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}\omega_t \epsilon_t(\vartheta)'\Sigma^{-1}\epsilon_t(\vartheta)\right),$$

$$\epsilon_t(\vartheta) = v_t(\phi) - \sum_{j=1}^{r} \Pi_j(\pi) v_{t-j}(\phi),$$

and

$$v_t(\phi) = y_t - \Phi_1(\phi)y_{t+1} - \ldots - \Phi_s(\phi)y_{t+s}.$$

The prior distributions are set as follows: $\pi \sim \text{N}(\underline{\pi}, \underline{V}_\pi)I(\pi)$, $\phi_r \sim \text{N}(\underline{\phi}_r, \underline{V}_{\phi_r})I(\phi)$, $\Sigma \sim \text{iW}(\underline{S}, \underline{\nu})$ and $\lambda \sim \text{Exp}(\underline{\lambda})$, where $I(\cdot)$ is indicator function equal to 1 when the polynomial to which π or ϕ is mapped is stable and iW denotes the inverse Wishart distribution. Furthermore, define the following matrices. First, stack $y_t^* = \omega_t^{1/2}\Pi(L)y_t$ to a $(T-r-s)n \times 1$ vector $\mathbf{y}^*$, and $X_t^* = \omega_t^{1/2}\Pi(L)X_t$ to a $(T-r-s)n \times sn^2$ matrix $\mathbf{X}^*$, where $X_t = I_n \otimes [y_{t+1}' \cdots y_{t+s}']'$. Define similarly matrices $\mathbf{Y}$ and $\mathbf{U}$ by stacking $v_t^* = \omega_t^{1/2} v_t'(\phi)$ and $U_t^* = \omega_t^{1/2}[v_{t-1}'(\phi) \cdots v_{t-r}'(\phi)]'$, respectively, for $t = r+1, \ldots, T-s$.

Following Lanne and Luoto (2016), the full conditional posterior distribution of ϕ_r can be derived as

$$\phi_r|y, \pi, \Sigma, \omega \sim \text{N}(\bar{\phi}_r, \bar{V}_{\phi_r})I(\phi), \;\; \phi = R_\phi \phi_r$$

$$\bar{V}_{\phi_r}^{-1} = \underline{V}_{\phi_r}^{-1} + R_\phi' \mathbf{X}^{*\prime} \mathbf{\Omega} \mathbf{X}^* R_\phi, \;\; \bar{\phi} = \bar{V}_{\phi_r}\left(\underline{V}_{\phi_r}^{-1}\underline{\phi}_r + R_\phi' \mathbf{X}^{*\prime} \mathbf{\Omega} \mathbf{Y}^*\right)$$

and $\mathbf{\Omega} = I_{T-r-s} \otimes \Sigma^{-1}$. The conditional distribution of π reads as

$$\pi|y, \phi, \Sigma, \omega \sim \text{N}(\bar{\pi}, \bar{V}_\pi)I(\pi),$$

$$\bar{V}_\pi^{-1} = \underline{V}_\pi^{-1} + \Sigma^{-1} \otimes \mathbf{U}'\mathbf{U}, \;\; \bar{\pi} = \bar{V}_\pi\left(\underline{V}_\pi^{-1}\underline{\phi} + \text{vec}\left(\mathbf{U}'\mathbf{Y}\Sigma^{-1}\right)\right)$$

Defining further $\bar{S} = \underline{S} + \mathbf{E}'\mathbf{E}$, $\mathbf{E} = \mathbf{Y} - \mathbf{U}\Pi$ and $\bar{\nu} = \underline{\nu} + T - s - r$, the conditional posterior distribution for Σ is

$$\Sigma | y, \pi, \phi, \omega \sim \mathrm{iW}(\bar{S}, \bar{\nu}).$$

The remaining paremeters $\omega = (\omega_{r+1}, \ldots, \omega_{T-r-s})$ and λ are jointly drawn from

$$\left(\lambda + \epsilon_t(\vartheta)'\Sigma^{-1}\epsilon_t(\vartheta)\right)\omega_t | y, \pi, \phi, \Sigma, \lambda \sim \chi^2(\lambda + n), t = r+1, \ldots, T-s$$

and with Metropolis-within-Gibbs step from kernel

$$p(\lambda | y, \omega) \propto \left(2^{\lambda/2}\Gamma(\lambda/2)\right)^{-(T-r-s)} \lambda^{\lambda(T-r-s)/2} \left(\prod_{t=r+1}^{T-s} \omega_t^{(\lambda-2)/2}\right) \exp\left[-\left(\frac{1}{\underline{\lambda}} + \frac{1}{2}\sum_{t=r+1}^{T-s} \omega_t\right)\lambda\right]. \tag{A.1}$$

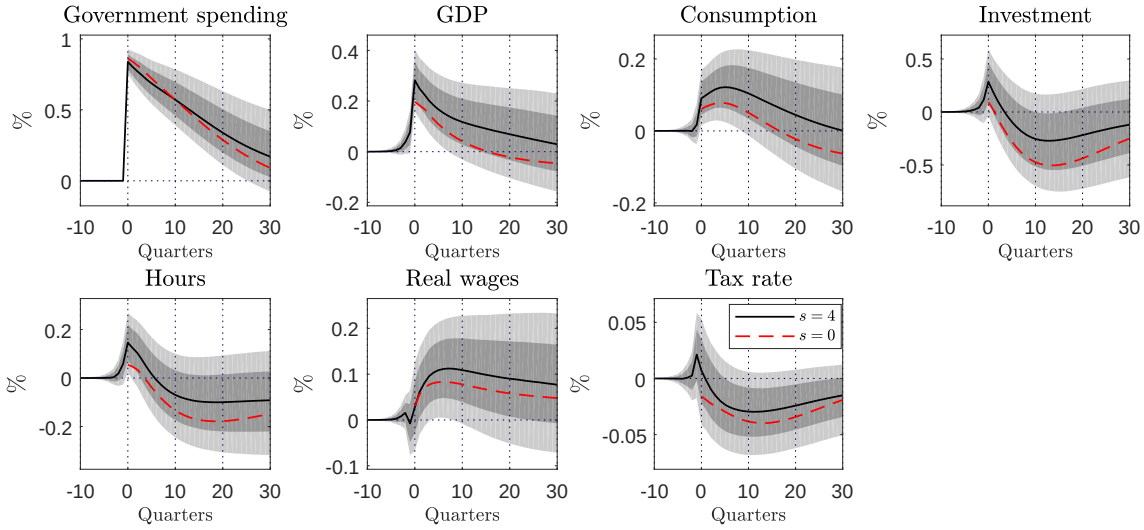
In the last step, I use the univariate normal distribution with mean equal to the mode and variance equal to the inverse of the second hessian of the above kernel as a candidate distribution. The standard Metropolis-Hastings acceptance probability is computed using (A.1).

In the empirical analysis, I use the following Minnesota-Litterman type prior distribution. I set the means of π and ϕ_r, $\underline{\pi}$ and $\underline{\phi}_r$, to 0, and the coefficients are assumed, a priori, independent by having zeros on the off-diagonals of covariance matrices $\underline{V}_\pi$ and $\underline{V}_{\phi_r}$. On the other hand, $\sigma^2_{\pi,ijl}$ and $\sigma^2_{\phi_r,ijl}$, the diagonal elements of $\underline{V}_\pi$ and $\underline{V}_{\phi_r}$ corresponding to the lth lag or lead of variable j in equation i are given by

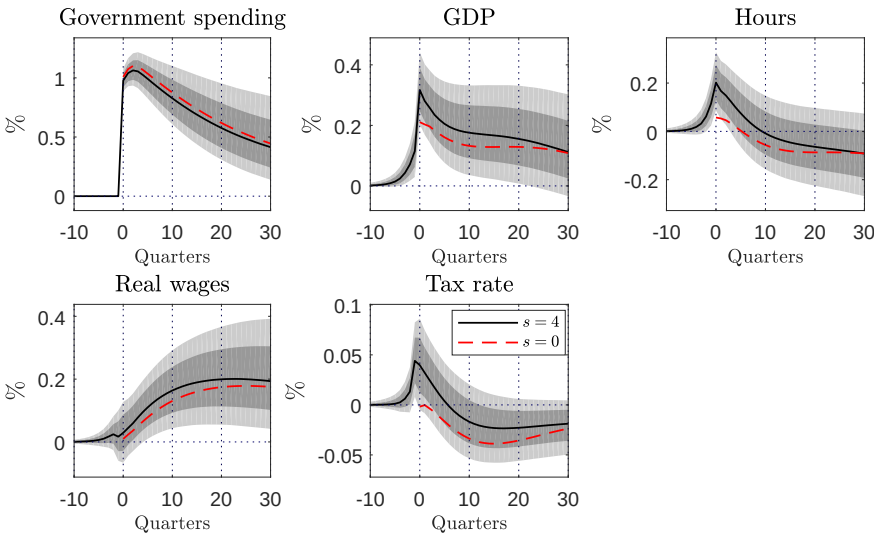
$$\sigma_{\pi,iil} = \frac{\gamma_{1,\pi}}{l^{\gamma_3}},\ \sigma_{\pi,ijl} = \gamma_2 \frac{\gamma_{1,\pi}}{l^{\gamma_3}} \frac{\sigma_i}{\sigma_j},\ i = 1, \ldots, n,\ j = 1, \ldots, n,\ l = 1, \ldots, r,$$
$$\sigma_{\phi_r,iil} = \frac{\gamma_{1,\phi}}{l^{\gamma_3}},\ \sigma_{\phi_r,ijl} = \gamma_2 \frac{\gamma_{1,\phi}}{l^{\gamma_3}} \frac{\sigma_i}{\sigma_j},\ i = 2, \ldots, n,\ j = 1, \ldots, n,\ l = 1, \ldots, s,$$

where σ_i is estimated as the residual standard error from a univariate autoregression with $r + s$ lags on the ith variable, $\gamma_{1,\pi}$ and $\gamma_{1,\phi}$ control for overall tightness, γ_2 for relative tightness and γ_3 is a decay parameter for more distant lags and leads. For these hyperparameters, I use values $\gamma_{1,\pi} = 0.2$, $\gamma_2 = 0.5$ and $\gamma_3 = 1$, standard in the Bayesian VAR literature. Additionally, I set $\gamma_{1,\phi} = 0.15$, which shrinks the lead coefficients moderately but somewhat more towards zero. Last, I use the following values for the remaining hyperparameters: $\underline{S} = (\underline{\nu} - n - 1)\mathrm{diag}(\sigma_1^2, \ldots, \sigma_n^2)$ with degrees-of-freedom parameter $\underline{\nu} = n + 2$ and $\underline{\lambda} = 10$.

A.2 Further empirical results



(a) VAR(4,4) with inflation and interest rates



(b) VAR(4,4) with consumption and investment components

Figure A.1: Impulse responses of the remaining variables to the government spending shock from the VAR models with additional variables

The graphs show the impulse responses of the remaining variables of the model not shown in Figure 4. Solid lines and dashed lines are the median impulse responses from the noncausal VAR(4,4) and causal VAR(4) models, respectively, to a one-standard deviation government spending shock. The dark and light grey shaded regions are the 68 and 90 per cent, respectively, credible sets of the estimated impulse responses from the noncausal model. The impulse responses in panel (a) are from the 10-variable VAR including the baseline and the plotted variables. The impulse respones in panel (b) are computed from the 10-variable VAR including the baseline variables but consumption and investment replaced by the variables shown.

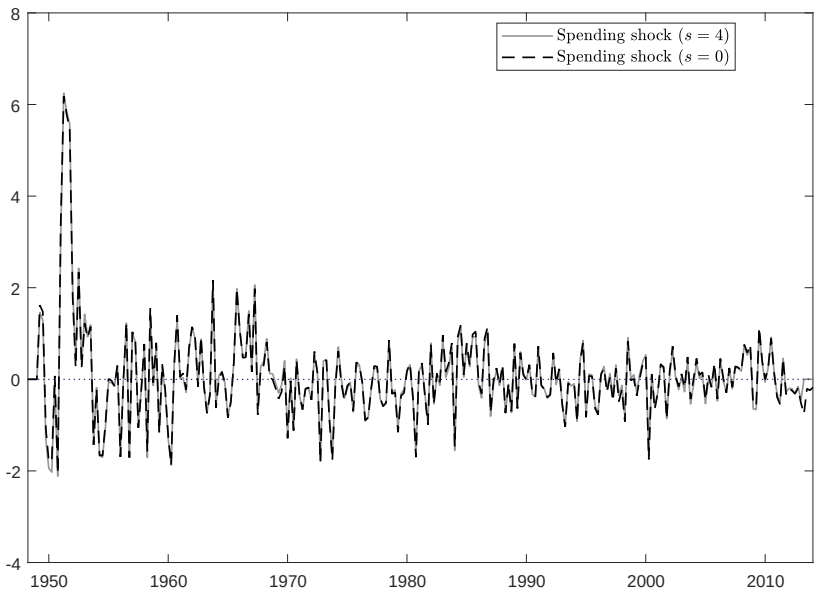


Figure A.2: Spending shocks of the noncausal and causal VAR models

Solid grey lines and the black dashed lines show the posterior medians of the identified spending shock from the noncausal VAR(4,4) and causal VAR(4) models, respectively.

A.3 Identification scheme with a proxy variable

To identify a shock contributing the most to a proxy, the starting point is the two-sided MA representation of y_t

$$y_t = \sum_{k=-\infty}^{\infty} \Psi_k \bar{B} \bar{u}_{t-k},$$

where matrix $\bar{B}$ rotates the structural shocks $\bar{u}_t$ to the reduced-form errors ϵ_t as

$$\epsilon_t = \bar{B}\bar{u}_t.$$

$\bar{B}$ can now be found from

$$\bar{B}\bar{B}' = \bar{\Gamma}$$

as $\mathrm{E}[\epsilon_t \epsilon_t'] = \bar{\Gamma} = \frac{\lambda}{\lambda-2}\Sigma = \mathrm{E}[\bar{B}\bar{u}_t\bar{u}_t'\bar{B}'] = \bar{B}\bar{B}'$. On the other hand, by Cholesky decomposition, $\bar{\Gamma} = \bar{A}\bar{A}'$, or, by introducing an orthogonal matrix W, $\bar{\Gamma} = \bar{A}WW'\bar{A}'$. Consequently, rotation of W yields $\bar{B} = \bar{A}W$. As the interest is in one shock only, it suffices to find the first column of W, w_1 such that $\gamma_1 = \bar{A}w_1$ is the first column of $\bar{B}$.

The MA representation of the ith variable in y_t is then

$$y_{i,t} = \sum_{k=-\infty}^{\infty} e_i' \Psi_k \bar{A} W \bar{u}_{t-k}$$

with variance

$$\mathrm{Var}(y_{i,t}) = \sum_{k=-\infty}^{\infty} e_i' \Psi_k \bar{A} \bar{A}' \Psi_k' e_i$$

$y_{i,t}$ can further be decomposed to the contributions by the n structural shocks

$$y_{i,t} = \sum_{k=-\infty}^{\infty} e_i' \Psi_k \bar{A} [w_1 \bar{u}_{1,t-k} + \ldots + w_n \bar{u}_{n,t-k}].$$

such that the contribution of the first shock to the variable reads as

$$y_{i,t}^1 = \sum_{k=-\infty}^{\infty} e_i' \Psi_k \bar{A} w_1 \bar{u}_{1,t-k} = \sum_{k=-\infty}^{\infty} e_i' \Psi_k \bar{A} w_1 \bar{u}_{1,t-k}.$$

As the aim is to find a shock with the greatest contribution to the ith variable, w_1 is found by maximising

$$\frac{\mathrm{Var}(y_{i,t}^1)}{\mathrm{Var}(y_{i,t})} = \frac{\sum_{k=-\infty}^{\infty} e_i' \Psi_k \bar{A} w_1 w_1' \bar{A}' \Psi_k' e_i}{\sum_{k=-\infty}^{\infty} e_i' \Psi_k \bar{A} \bar{A}' \Psi_k' e_i}$$

subject to the orthogonality of W, $w_1' w_1 = 1$. By rewriting

$$\begin{aligned} e_i' \Psi_k \bar{A} w_1 w_1' \bar{A}' \Psi_k' e_i =& \mathrm{tr}\left(e_i' \Psi_k \bar{A} w_1 w_1' \bar{A}' \Psi_k' e_i \right) \\ =& \mathrm{tr}\left(w_1' \bar{A}' \Psi_k' e_i e_i' \Psi_k \bar{A} w_1 \right) \\ =& \mathrm{tr}\left(w_1' \bar{A}' \Psi_k' E_{ii} \Psi_k \bar{A} w_1 \right) \\ =& \mathrm{tr}\left(w_1' S_k w_1 \right), \end{aligned}$$

the nominator of the objective function is

$$\sum_{k=-H}^{H} w_1' S_k w_1 = w_1' \bar{S} w_1$$

for large H. As the denominator is independent of w_1, the problem can be solved by setting up the Lagrangian

$$\mathcal{L} = w_1' \bar{S} w_1 - \mu (w_1' w_1 - 1).$$

The first-order condition is

$$\bar{S} w_1 = \mu w_1,$$

and since $w_1' \mu w_1 = \mu$, the eigenvector corresponding to the maximal eigenvalue of the positive definite matrix $\bar{S}$ is the optimum.