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On Single-peaked Domains and Min-max Rules*

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Abstract

We consider social choice problems where different agents can have different sets of admissible single-peaked preferences. We show every unanimous and strategy-proof social choice function on such domains satisfies Pareto property and tops-onlyness. Further, we characterize all domains on which (i) every unanimous and strategy-proof social choice function is a min-max rule, and (ii) every min-max rule is strategy-proof. As an application of our result, we obtain a characterization of the unanimous and strategy-proof social choice functions on maximal single-peaked domains (Moulin (1980), Weymark (2011)), minimally rich single-peaked domains (Peters et al. (2014)), maximal regular single-crossing domains (Saporiti (2009), Saporiti (2014)), and distance based single-peaked domains.

KEYWORDS: Strategy-proofness, single-peaked preferences, min-max rules, min-max domains, top-connectedness, Pareto property, tops-onlyness.

JEL CLASSIFICATION CODES: D71, D82.

1. Introduction

1.1 Background

We consider a standard social choice problem where an alternative has to be chosen based on privately known preferences of the agents in the society. Such a procedure is known as a *social*

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choice function (SCF). Agents are strategic in the sense that they misreport their preferences whenever it is strictly beneficial for them. An SCF is called *strategy-proof* if no agent can benefit by misreporting her preferences, and is called *unanimous* if whenever all the agents in the society unanimously agree on their best alternative, that alternative is chosen.

Most of the subject matter of social choice theory concerns the study of the unanimous and strategy-proof SCFs for different admissible domains of preferences. In the seminal works by Gibbard (1973) and Satterthwaite (1975), it is shown that if the society has at least three alternatives and there is no particular restriction on the preferences of the agents, then every unanimous and strategy-proof SCF is *dictatorial*, that is, a particular agent in the society determines the outcome regardless of the preferences of the others. The celebrated Gibbard-Satterthwaite theorem hinges crucially on the assumption that the admissible domain of each agent is unrestricted. However, it is well established that in many economic and political applications, there are natural restrictions on such domains. For instance, in the models of locating a firm in a unidimensional spatial market (Hotelling (1929)), setting the rate of carbon dioxide emissions (Black (1948)), setting the level of public expenditure (Romer and Rosenthal (1979)), and so on, preferences admit a natural restriction widely known as *single-peakedness*. Roughly speaking, single-peakedness of a preference implies that there is a prior order over the alternatives such that the preference decreases as one moves away (with respect to the prior order) from her best alternative.

1.2 Motivation and Contribution

The study of single-peaked domains dates back to Black (1948), where it is shown that the pairwise majority rule is strategy-proof on such domains. Later, Moulin (1980) and Weymark (2011) characterize the unanimous and strategy-proof SCFs on these domains.[1,2] However, their characterization rests upon the assumption that the set of admissible preferences of each agent in the society is the *maximal* single-peaked domain, i.e., it contains *all* single-peaked preferences with respect to a given prior order over the alternatives. Note that demanding the existence of

[1] Barberà et al. (1993) and Ching (1997) provide equivalent presentations of this class of SCFs.

[2] A rich literature has developed around the single-peaked restriction by considering various generalizations and extensions (see Barberà et al. (1993), Demange (1982), Schummer and Vohra (2002), Nehring and Puppe (2007a), and Nehring and Puppe (2007b)).

all single-peaked preferences is a strong prerequisite in many practical situations.[3] Furthermore, the assumption that every agent has the same set of admissible preferences is also quite strong. This motivates us to analyze the structure of the unanimous and strategy-proof SCFs on domains where different agents can have different admissible sets of single-peaked preferences. We show that every unanimous and strategy-proof SCF on such domains satisfies *Pareto property* and *tops-onlyness*.[4] Moreover, we show by means of examples that the exact structure of such SCFs will depend heavily on the domain. In order to obtain a tractable structure of the strategy-proof SCFs, we restrict our attention to *top-connected single-peaked domains*. The top-connectedness property with respect to a prior order requires that for every two consecutive (in that prior order) alternatives, there exists a preference that places one at the top and the other at the second-ranked position.[5] In this paper, we provide a characterization of the unanimous and strategy-proof SCFs on top-connected single-peaked domains.

The unanimous and strategy-proof SCFs on the maximal single-peaked domain are known as *min-max rules* (Moulin (1980), Weymark (2011)). Min-max rules are quite popular for their desirable properties like *tops-onlyness*, *Pareto property*, and *anonymity* (for a subclass of min-max rules called *median* rules). Owing to the desirable properties of min-max rules, Barberà et al. (1999) characterize maximal domains on which a *given* min-max rule is strategy-proof. Recently, Arribillaga and Massó (2016) provide necessary and sufficient conditions for the comparability of two min-max rules in terms of their vulnerability to manipulation. Motivated by the importance of the min-max rules, we characterize all domains on which (i) every unanimous and strategy-proof social choice function is a min-max rule, and (ii) every min-max rule is strategy-proof. We call such a domain a *min-max domain*.

Note that min-max domains do *not* require that the admissible preferences of all the agents are the same. Furthermore, it is worth noting that in a social choice problem with m alternatives, the number of preferences of each agent in a min-max domain can range from $2m-2$ to 2^{m-1}, whereas that in the maximal single-peaked domain is exactly 2^{m-1}. Thus, on one hand, our result characterizes the unanimous and strategy-proof SCFs on a large class of single-peaked domains,

[3]See, for instance, the domain restriction considered in models of voting (Tullock (1967), Arrow (1969)), taxation and redistribution (Epple and Romer (1991)), determining the levels of income redistribution (Hamada (1973), Slesnick (1988)), and measuring tax reforms in the presence of horizontal inequity (Hettich (1979)). Recently, Puppe (2016) shows that under mild conditions these domains form subsets of the maximal single-peaked domain.

[4]Chatterji and Sen (2011) provide a sufficient condition on a domain so that every unanimous and strategy-proof SCF on it is tops-only. However, an arbitrary single-peaked domain does not satisfy their condition.

[5]The top-connectedness property is well studied in the literature (see Barberà and Peleg (1990), Aswal et al. (2003), Chatterji and Sen (2011), Chatterji et al. (2014), Chatterji and Zeng (2015), Puppe (2016)) and Roy and Storcken (2016).

and on the other hand, it establishes the full applicability of min-max rules as strategy-proof SCFs.

1.3 Applications

An outstanding example of a top-connected single-peaked domain is a *top-connected regular single-crossing domain*.[6,7] Saporiti (2014) shows that an SCF is unanimous and strategy-proof on a *maximal* single-crossing domain if and only if it is a min-max rule.[8] In contrast, our result shows that an SCF is unanimous and strategy-proof on a *top-connected regular* single-crossing domain if and only if it is a min-max rule. Thus, we extend Saporiti (2014)'s result in two ways: (i) by relaxing the maximality assumption on a single-crossing domain, and (ii) by relaxing the assumption that every agent has the same set of preferences. However, we assume the domains to be regular. Note that in a social choice problem with m alternatives, the number of admissible preferences of each agent in a top-connected regular single-crossing domain can range from $2m-2$ to $m(m-1)/2$, whereas that in the maximal single-crossing domain is exactly $m(m-1)/2$.

Other important examples of top-connected single-peaked domains include *minimally rich single-peaked domains* (Peters et al. (2014)) and *distance based single-peaked domains*. A single-peaked domain is minimally rich if it contains all *left single-peaked* and all *right single-peaked* preferences.[9,10] Further, a single-peaked domain is called distance based if the preferences in it are derived by using some type of distances between the alternatives. It follows from our result that an SCF is unanimous and strategy-proof on these domains if and only if it is a min-max rule.

1.4 Remainder

The rest of the paper is organized as follows. We describe the usual social choice framework in Section 2. In Section 3, we study the structure of the unanimous and strategy-proof SCFs on single-peaked domains, and in Section 4, we characterize such SCFs on top-connected single-peaked

[6] A domain is *regular* if every alternative appears as the top-ranked alternative of some preference in the domain.

[7] Single-crossing domains appear in models of taxation and redistribution (Roberts (1977), Meltzer and Richard (1981)), local public goods and stratification (Westhoff (1977), Epple and Platt (1998), Epple et al. (2001)), coalition formation (Demange (1994), Kung (2006)), selecting constitutional and voting rules (Barberà and Jackson (2004)), and designing policies in the market for higher education (Epple et al. (2006)).

[8] Saporiti (2014) provides a different but equivalent functional form of these SCFs which he calls *augmented representative voter schemes*.

[9] A single-peaked preference is called left (or right) single-peaked if every alternative to the left (or right) of the peak is preferred to every alternative to its right (or left).

[10] Such preferences appear in directional theories of issue voting (Stokes (1963), Rabinowitz (1978), Rabinowitz et al. (1982), Rabinowitz and Macdonald (1989)).

domains. Section 5 characterizes min-max domains. In Section 6, we discuss some applications of our results, and we conclude the paper in the last section. All the omitted proofs are collected in Appendices A - D.

2. Preliminaries

Let $N = \{1, ..., n\}$ be a set of at least two agents, who collectively choose an element from a finite set $X = \{a, a+1, \ldots, b-1, b\}$ of at least three alternatives, where a is an integer. For $x, y \in X$ such that $x \leq y$, we define the intervals $[x, y] = \{z \in X \mid x \leq z \leq y\}$, $[x, y) = [x, y] \setminus \{y\}$, $(x, y] = [x, y] \setminus \{x\}$, and $(x, y) = [x, y] \setminus \{x, y\}$. For notational convenience, whenever it is clear from the context, we do not use braces for singleton sets, i.e., we denote sets $\{i\}$ by i.

A *preference* P over X is a complete, transitive, and antisymmetric binary relation (also called a linear order) defined on X. We denote by $\mathbb{L}(X)$ the set of all preferences over X. An alternative $x \in X$ is called the k^{th} *ranked alternative* in a preference $P \in \mathbb{L}(X)$, denoted by $r_k(P)$, if $|\{a \in X \mid aPx\}| = k - 1$.

Definition 2.1. A preference $P \in \mathbb{L}(X)$ is called *single-peaked* if for all $x, y \in X$, $[x < y \leq r_1(P)$ or $r_1(P) \leq y < x]$ implies yPx.

For an agent i, we denote by $\mathcal{S}_i$ a set of admissible single-peaked preferences. A set $\mathcal{S}_N = \prod_{i \in N} \mathcal{S}_i$ is called a *single-peaked domain*. Note that we do not assume that the set of admissible preferences are the same across all agents. An element $P_N = (P_1, \ldots, P_n) \in \mathcal{S}_N$ is called a *preference profile*. The *top-set* of a preference profile P_N, denoted by $\tau(P_N)$, is defined as $\tau(P_N) = \bigcup_{i \in N} r_1(P_i)$. A set $\mathcal{S}_i$ of admissible preferences of agent i is *regular* if for all $x \in X$, there exists a preference $P \in \mathcal{S}_i$ such that $r_1(P) = x$. Throughout this paper, we assume that the set of admissible preferences of each agent i is regular.

Definition 2.2. A *social choice function* (SCF) f on $\mathcal{S}_N$ is a mapping $f : \mathcal{S}_N \to X$.

Definition 2.3. An SCF $f : \mathcal{S}_N \to X$ is *unanimous* if for all $P_N \in \mathcal{S}_N$ such that $r_1(P_i) = x$ for all $i \in N$ and some $x \in X$, we have $f(P_N) = x$.

Definition 2.4. An SCF $f : \mathcal{S}_N \to X$ satisfies *Pareto property* if for all $P_N \in \mathcal{S}_N$ and all $x, y \in X$, xP_iy for all $i \in N$ implies $f(P_N) \neq y$.

REMARK 2.1. Note that since $\mathcal{S}_i$ is single-peaked for all $i \in N$, an SCF $f : \mathcal{S}_N \to X$ satisfies Pareto property if $f(P_N) \in [\min(\tau(P_N)), \max(\tau(P_N))]$ for all $P_N \in \mathcal{S}_N$.

Definition 2.5. An SCF $f : \mathcal{S}_N \to X$ is *manipulable* if there exists $i \in N$, $P_N \in \mathcal{S}_N$, and $P'_i \in \mathcal{S}_i$ such that $f(P'_i, P_{N\setminus i}) P_i f(P_N)$. An SCF f is *strategy-proof* if it is not manipulable.

Definition 2.6. Two preference profiles $P_N, P'_N \in \mathcal{S}_N$ are called *tops-equivalent* if $r_1(P_i) = r_1(P'_i)$ for all $i \in N$.

Definition 2.7. An SCF $f : \mathcal{S}_N \to X$ is called *tops-only* if for any two tops-equivalent $P_N, P'_N \in \mathcal{S}_N$, $f(P_N) = f(P'_N)$.

Definition 2.8. An SCF $f : \mathcal{S}_N \to X$ is called *uncompromising* if for all $P_N \in \mathcal{S}_N$, all $i \in N$, and all $P'_i \in \mathcal{S}_i$:

(i) if $r_1(P_i) < f(P_N)$ and $r_1(P'_i) \leq f(P_N)$, then $f(P_N) = f(P'_i, P_{N\setminus i})$, and

(ii) if $f(P_N) < r_1(P_i)$ and $f(P_N) \leq r_1(P'_i)$, then $f(P_N) = f(P'_i, P_{N\setminus i})$.

REMARK 2.2. If an SCF satisfies uncompromisingness, then by definition, it is tops-only.

Definition 2.9. Let $\beta = (\beta_S)_{S\subseteq N}$ be a list of 2^n parameters satisfying: (i) $\beta_S \in X$ for all $S \subseteq N$, (ii) $\beta_\emptyset = b$, $\beta_N = a$, and (iii) for any $S \subseteq T$, $\beta_T \leq \beta_S$. Then, an SCF $f^\beta : \mathcal{S}_N \to X$ is called a *min-max rule with respect to* β if

$$f^\beta(P_N) = \min_{S\subseteq N}\{\max_{i\in S}\{r_1(P_i), \beta_S\}\}.$$

REMARK 2.3. Every min-max rule is uncompromising.[11]

3. SCFs on Single-peaked Domains

In this section, we establish two important properties, namely Pareto property and tops-onlyness, of the unanimous and strategy-proof SCFs on arbitrary single-peaked domains.

Theorem 3.1. *Every unanimous and strategy-proof SCF $f : \mathcal{S}_N \to X$ satisfies Pareto property.*

The proof of Theorem 3.1 is relegated to Appendix A.

Theorem 3.2. *Every unanimous and strategy-proof SCF $f : \mathcal{S}_N \to X$ satisfies tops-onlyness.*

The proof of Theorem 3.2 is relegated to Appendix B.

[11] For details, see Weymark (2011).

4. SCFs on Top-connected Single-peaked Domains

In this section, we characterize the unanimous and strategy-proof SCFs on single-peaked domains satisfying top-connectedness. First, we present an example to show that the structure of the unanimous and strategy-proof SCFs on arbitrary single-peaked domains is quite intractable.

Example 4.1. Fix $x, y \in X$ with $y - x \geq 2$. For all $i \in N$, define the set of single-peaked preferences $\mathcal{S}_i^{xy}$ as follows: for all $P \in \mathcal{S}_i^{xy}$, $r_1(P) \in (x, y)$ implies $r_{y-x-1}(P) = x$ and $r_{y-x}(P) = y$. In other words, the set of preferences $\mathcal{S}_i^{xy}$ is such that if the top-alternative of a preference is in the interval (x, y), then all the alternatives in that interval are ranked in the top $y - x - 2$ positions, and x and y are ranked consecutively after those alternatives. Similarly, define $\mathcal{S}_i^{yx}$ as follows: for all $P \in \mathcal{S}_i^{yx}$, $r_1(P) \in (x, y)$ implies $r_{y-x-1}(P) = y$ and $r_{y-x}(P) = x$.

Let $\beta = (\beta_j)_{j \in N}$ be such that $\beta_j \in [x, y]$ for all $j \in N$ with $\min\limits_{j \in N} \beta_j = x$ and $\max\limits_{j \in N} \beta_j = y$. Consider the SCF $f^{xy} : \mathcal{S}_N^{xy} \to X$ as given below:

$$f^{xy}(P_N) = \begin{cases} \text{median}\{r_1(P_2), \ldots, r_1(P_n), \beta_1, \ldots, \beta_n\}, \text{ if } r_1(P_1) \in [x, y) \\ r_1(P_1), \text{ otherwise} \end{cases}$$

Similarly, define $f^{yx} : \mathcal{S}_N^{yx} \to X$ as follows:

$$f^{yx}(P_N) = \begin{cases} \text{median}\{r_1(P_2), \ldots, r_1(P_n), \beta_1, \ldots, \beta_n\}, \text{ if } r_1(P_1) \in (x, y] \\ r_1(P_1), \text{ otherwise} \end{cases}$$

Note that both f^{xy} and f^{yx} are unanimous by definition. We show that f^{xy} is strategy-proof on $\mathcal{S}_N^{xy}$, but manipulable on $\mathcal{S}_N^{yx}$. It follows from similar arguments that f^{yx} is strategy-proof on $\mathcal{S}_N^{yx}$, but manipulable on $\mathcal{S}_N^{xy}$.

Clearly, no agent can manipulate f^{xy} at a profile $P_N \in \mathcal{S}_N^{xy}$ where $r_1(P_1) \notin [x, y)$. Consider a profile $P_N \in \mathcal{S}_N^{xy}$ where $r_1(P_1) \in [x, y)$. Since $f^{xy}(P_N) = \text{median}\{r_1(P_2), \ldots, r_1(P_n), \beta_1, \ldots, \beta_n\}$ and $\mathcal{S}_i^{xy}$ is single-peaked, by the property of a median rule, an agent $i \neq 1$ cannot manipulate at P_N. Now, consider a preference $P_1' \in \mathcal{S}_1^{xy}$. If $r_1(P_1') \in [x, y)$, then $f^{xy}(P_1', P_{N \setminus 1}) = f^{xy}(P_N)$ and hence agent 1 cannot manipulate. On the other hand, if $r_1(P_1') \notin [x, y)$, then $f^{xy}(P_1', P_{N \setminus 1}) = r_1(P_1') \notin [x, y)$. However, by the definition of $\mathcal{S}_1^{xy}$, $u P_1 v$ for all $u \in [x, y)$ and all $v \notin [x, y)$, this means $f^{xy}(P_N) P_1 f^{xy}(P_1', P_{N \setminus 1})$, and hence agent 1 cannot manipulate at P_N.

Now, we show that f^{xy} is manipulable on $\mathcal{S}_N^{yx}$. Consider a profile $P_N \in \mathcal{S}_N^{yx}$ where $r_1(P_1) \in$

(x,y) and $r_1(P_j) = x$ for all $j \neq i$. Then, by the definition of f^{xy}, $f^{xy}(P_N) = x$. Let $P_1' \in \mathcal{S}_1^{yx}$ be such that $r_1(P_1') = y$. Then, $f^{xy}(P_1', P_{N\setminus 1}) = y$. However, since $P_1 \in \mathcal{S}_1^{yx}$, by the definition of $\mathcal{S}_1^{yx}$, yP_1x. This means agent 1 manipulates at P_N via P_1'.

Note that for all $i \in N$, $\mathcal{S}_i^{yx}$ can be obtained from $\mathcal{S}_i^{xy}$ by swapping x and y in the preferences with top-ranked alternative in (x,y). Furthermore, the ranks of x and y are bigger than $y - x - 2$. This shows that, if x and y are far apart, then the structure of a unanimous and strategy-proof SCF will crucially depend on the lower ranked alternatives of the preferences making the presentation of such SCFs intractable. Therefore, we impose a mild restriction called top-connectedness on single-peaked domains, and characterize the unanimous and strategy-proof SCFs on such domains. We begin with a few formal definitions.

Definition 4.1. A set of single-peaked preferences $\mathcal{S}$ is called *top-connected* if for all $x, y \in X$ with $|x - y| = 1$, there exists $P \in \mathcal{S}$ such that $r_1(P) = x$ and $r_2(P) = y$.

Note that the minimum cardinality of a top-connected set of single-peaked preferences with m alternatives is $2m - 2$. Also, since the maximal set of single-peaked preferences is top-connected, the maximum cardinality of such a set is 2^{m-1}. Thus, the class of top-connected set of single-peaked preferences is quite large. In what follows, we provide an example of a top-connected set of single-peaked preferences with five alternatives.

Example 4.2. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$. Then, the set of single-peaked preferences in Table 1 is top-connected.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_5}$
$\mathbf{x_2}$	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_3}$	$\mathbf{x_5}$	$\mathbf{x_4}$
x_3	x_3	x_4	x_1	x_4	x_4	x_2	x_5	x_2	x_5	x_3	x_3
x_4	x_4	x_1	x_4	x_5	x_5	x_5	x_2	x_1	x_2	x_2	x_2
x_5	x_5	x_5	x_5	x_1	x_1	x_1	x_1	x_5	x_1	x_1	x_1

Table 1: A top-connected set of single-peaked preferences

Now, we provide a characterization of the unanimous and strategy-proof SCFs on *top-connected single-peaked domains*.

Theorem 4.1. *Let $\mathcal{S}_i$ be a top-connected set of single-peaked preferences for all $i \in N$. Then, an SCF $f : \mathcal{S}_N \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

The proof of Theorem 4.1 is relegated to Appendix C.

The following corollary is immediate from Theorem 4.1.

Corollary 4.1 (Moulin (1980), Weymark (2011))**.** *Let $\mathcal{S}_i$ be the maximal set of single-peaked preferences for all $i \in N$. Then, an SCF $f : \mathcal{S}_N \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

5. Min-max Domains

In this section, we introduce the notion of min-max domains and provide a characterization of these domains.

Definition 5.1. Let $\mathcal{D}_i \subseteq \mathbb{L}(X)$ for all $i \in N$ and let $\mathcal{D}_N = \prod_{i \in N} \mathcal{D}_i$. Then, $\mathcal{D}_N$ is called a *min-max domain* if

(i) every unanimous and strategy-proof SCF on $\mathcal{D}_N$ is a min-max rule, and

(ii) every min-max rule on $\mathcal{D}_N$ is strategy-proof.

Our next theorem provides a characterization of the min-max domains.

Theorem 5.1. *A domain $\mathcal{D}_N$ is a min-max domain if and only if $\mathcal{D}_i$ is a top-connected set of single-peaked preferences for all $i \in N$.*

The proof of Theorem 5.1 is relegated to Appendix D.

6. Applications

6.1 Regular Single-crossing Domains

In this subsection, we introduce the notion of regular single-crossing domains and provide a characterization of the unanimous and strategy-proof SCFs on these domains.

Definition 6.1. A set of preferences $\mathcal{S}$ is called *single-crossing* if there is a linear order $\lhd$ on $\mathcal{S}$ such that for all $x, y \in X$ and all $P, \hat{P} \in \mathcal{S}$,

$$[x < y, P \lhd \hat{P}, \text{ and } x\hat{P}y] \Rightarrow xPy.$$

Definition 6.2. A single-crossing set of preferences $\mathcal{S}$ is called *maximal* if there is no single-crossing set of preferences $\mathcal{S}'$ such that $\mathcal{S} \subsetneq \mathcal{S}'$.

In what follows, we provide an example of a maximal regular single-crossing set of preferences with five alternatives.

Example 6.1. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$. Then, the set of preferences in Table 2 is maximal regular single-crossing with respect to the linear order given by $P_1 \lhd P_2 \lhd P_3 \lhd P_4 \lhd P_5 \lhd P_6 \lhd P_7 \lhd P_8 \lhd P_9 \lhd P_{10} \lhd P_{11}$. To see this, consider two alternatives, say x_2 and x_4. Then, $x_2 P x_4$ for all $P \in \{P_1, P_2, P_3, P_4, P_5, P_6\}$ and $x_4 P x_2$ for all $P \in \{P_7, P_8, P_9, P_{10}, P_{11}\}$. Therefore, $x_2 \hat{P} x_4$ for some $\hat{P} \in \mathcal{D}$ and $P \lhd \hat{P}$ imply $x_2 P x_4$.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}
$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_5}$
$\mathbf{x_2}$	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_3}$	$\mathbf{x_5}$	$\mathbf{x_4}$
x_3	x_3	x_1	x_4	x_4	x_4	x_2	x_5	x_5	x_3	x_3
x_4	x_4	x_4	x_1	x_5	x_5	x_5	x_2	x_2	x_2	x_2
x_5	x_5	x_5	x_5	x_1	x_1	x_1	x_1	x_1	x_1	x_1

Table 2: A maximal regular single-crossing set of preferences

REMARK 6.1. Note that a maximal regular single-crossing set of preferences is not unique.

The following lemmas establish two crucial properties of a (maximal) regular single-crossing set of preferences.

Lemma 6.1 (Elkind et al. (2014), Puppe (2016)). *Every regular single-crossing set of preferences is single-peaked.*

Lemma 6.2. *Every maximal regular single-crossing set of preferences is top-connected.*

Proof. Let $\mathcal{S}$ be a maximal regular single-crossing set of preferences. Then, by Lemma 6.1, $\mathcal{S}$ is a regular set of single-peaked preferences. Take $x \in X \setminus \{b\}$. We show that there exist $P, P' \in \mathcal{S}$ such that $r_1(P) = r_2(P') = x$ and $r_2(P) = r_1(P') = x + 1$. Without loss of generality, assume for contradiction that for all $P \in \mathcal{S}$ with $r_1(P) = x$, $r_2(P) \neq x + 1$. Because $\mathcal{S}$ is single-peaked, if $x = a$, then $r_2(P) = a + 1$ for all $P \in \mathcal{S}$ with $r_1(P) = a$, which is a contradiction. So, assume

$x \neq a$. Because $\mathcal{S}$ is single-peaked and $x \notin X \setminus \{a, b\}$, for all $P \in \mathcal{S}$ with $r_1(P) = x$, $r_2(P) \neq x + 1$ implies $r_2(P) = x - 1$. Let $\lhd \in \mathbb{L}(\mathcal{S})$ be such that for all $u, v \in X$ and all $P, \hat{P} \in \mathcal{S}$,

$$[u < v, P \lhd \hat{P}, \text{ and } u\hat{P}v] \Rightarrow uPv.$$

Take $\hat{P} \in \mathcal{S}$ with $r_1(\hat{P}) = x$ such that for all $P \in \mathcal{S}$ with $\hat{P} \lhd P$, $r_1(P) \neq x$. Consider the preference $\tilde{P}$ with $r_1(\tilde{P}) = x$ and $r_2(\tilde{P}) = x + 1$ such that for all $u, v \in X \setminus \{x, x + 1\}$, $u\tilde{P}v$ if and only if $u\hat{P}v$. Because $r_1(\tilde{P}) = x$ and $r_2(\tilde{P}) = x + 1$, by our assumption, $\tilde{P} \notin \mathcal{S}$. Therefore, since $\mathcal{S}$ is regular single-crossing, it follows that $\mathcal{S} \cup \tilde{P}$ is also single-crossing with respect to the ordering $\lhd' \in \mathbb{L}(\mathcal{S} \cup \tilde{P})$, where $\lhd'$ is obtained by placing $\tilde{P}$ just after $\hat{P}$ in the ordering $\lhd$, i.e., for all $P, P' \in \mathcal{S}$, $P \lhd' P'$ if and only if $P \lhd P'$, and there is no $P \in \bar{\mathcal{S}}_c$ with $\hat{P} \lhd' P \lhd' \tilde{P}$. However, this contradicts the maximality of $\mathcal{S}$, which completes the proof. ■

The following corollary follows from Theorem 4.1 and Lemma 6.2. It characterizes the unanimous and strategy-proof SCFs on maximal regular single-crossing domains.

Corollary 6.1 (Saporiti (2014))**.** *Let $\mathcal{S}_i$ be a maximal regular single-crossing set of preferences for all $i \in N$. Then, an SCF $f : \mathcal{S}_N \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

The following corollary is obtained from Theorem 4.1 and Lemma 6.1. It characterizes the unanimous and strategy-proof SCFs on top-connected regular single-crossing domains. Note that in a social choice problem with m alternatives, the cardinality of a top-connected regular single-crossing set of preferences can range from $2m - 2$ to $m(m - 1)/2$, whereas that of a maximal regular single-crossing set of preferences is exactly $m(m - 1)/2$.

Corollary 6.2. *Let $\mathcal{S}_i$ be a top-connected regular single-crossing set of preferences for all $i \in N$. Then, an SCF $f : \mathcal{S}_N \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

6.2 Minimally Rich Single-peaked Domains

In this subsection, we present a characterization of the unanimous and strategy-proof SCFs on minimally rich single-peaked domains. The notion of minimally rich single-peaked domains is introduced in Peters et al. (2014). For the sake of completeness, we present below a formal definition of such domains.

Definition 6.3. A single-peaked preference P is called *left single-peaked* (*right single-peaked*) if for all $u < r_1(P) < v$, we have uPv (vPu). Moreover, a set of single-peaked preferences $\mathcal{S}$ is called *minimally rich* if it contains all left and all right single-peaked preferences.

Clearly, a minimally rich set of single-peaked preferences is top-connected. So, we have the following corollary from Theorem 4.1.

Corollary 6.3. *Let $\mathcal{S}_i$ be a minimally rich set of single-peaked preferences for all $i \in N$. Then, an SCF $f : \mathcal{S}_N \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

6.3 Distance based Single-peaked Domains

In this subsection, we introduce the notion of single-peaked domains that are based on distances. Consider the situation where a public facility has to be developed at one of the locations $x_1, \ldots, x_m$. Suppose that there is a street connecting these locations, and for every two locations x_i and x_{i+1}, there are two types of distances, a forward distance from x_i to x_{i+1} and a backward distance from x_{i+1} to x_i. An agent bases her preferences on such distances, i.e., whenever a location is strictly closer than another to her most preferred location, she prefers the former to the latter. Moreover, ties are broken on both sides. We show that under some condition on the distances, such a set of preferences is top-connected single-peaked. Below, we present this notion formally.

Consider the directed line graph $G = \langle X, E \rangle$ on X.[12] A function $d : E \to (0, \infty)$ is called a *distance function* on G. Given a distance function d, define the *distance between two alternatives* x, y as the distance of the path between x and y, i.e., $d(x,y) = d(x, x+1) + \ldots + d(y-1, y)$ if $x < y$ and as $d(x,y) = d(x, x-1) + \ldots + d(y+1, y)$ if $x > y$. A preference P *respects a distance function d* if for all $x, y \in X$, $d(r_1(P), x) < d(r_1(P), y)$ implies xPy. A set of preferences $\mathcal{S}$ is called *single-peaked with respect to a distance function d* if $\mathcal{S} = \{P \in \mathbb{L}(X) \mid P \text{ respects } d\}$.

A distance function satisfies *adjacent symmetry* if $d(x, x+1) = d(x, x-1)$ for all $x \in X \setminus \{a, b\}$. Below, we provide an example of a set of single-peaked preferences with respect to an adjacent symmetric distance function.

Example 6.2. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$. The directed line graph $G = \langle X, E \rangle$ on X and the adjacent symmetric distance function d on E are as given below.

[12] A graph $G = \langle X, E \rangle$ is called a *directed line graph* if $(x,y) \in E \iff |x - y| = 1$.

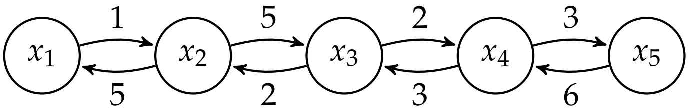

Figure 1: The directed line graph G on X and an adjacent symmetric distance function d on G

Then, the set of preferences in Table 3 is single-peaked with respect to the distance function d.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_5}$
$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_1}$	$\mathbf{x_4}$	$\mathbf{x_2}$	$\mathbf{x_5}$	$\mathbf{x_3}$	$\mathbf{x_4}$
x_3	x_1	x_3	x_2	x_4	x_3	x_5	x_3
x_4	x_4	x_4	x_5	x_5	x_2	x_2	x_2
x_5	x_5	x_5	x_1	x_1	x_1	x_1	x_1

Table 3: A set of single-peaked preferences with respected to the distance function d

Let $G = \langle X, E\rangle$ be the directed line graph on X and let $d : E \to (0, \infty)$ be an adjacent symmetric distance function. Then, it is easy to verify that a set of single-peaked preferences with respect to the distance function d is top-connected. Therefore, we have the following corollary from Theorem 4.1.

Corollary 6.4. *Let $G = \langle X, E\rangle$ be the directed line graph on X and let $d_i : E \to (0, \infty)$ be an adjacent symmetric distance function for all $i \in N$. Suppose that for all $i \in N$, $\mathcal{S}_i$ is a set of single-peaked preferences with respect to the distance function d_i. Then, $f : \mathcal{S}_N \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

7. Concluding Remarks

In this paper, we have studied social choice problems where different agents can have different sets of single-peaked preferences. We have shown that every unanimous and strategy-proof SCFs on such domains satisfy Pareto property and tops-onlyness. We have further shown that if such a domain satisfies a mild restriction called top-connectedness, then every unanimous and strategy-proof SCFs on it is a min-max rule. Outstanding examples of top-connected single-peaked domains are maximal single-peaked domains, minimally rich single-peaked domains, distance based single-peaked domains, and top-connected regular single-crossing domains. Finally, we have introduced the notion of min-max domains, the domains for which the set of unanimous and strategy-proof SCFs coincides with that of min-max rules. We have shown that a domain is a min-max domain if and only if it is a top-connected single-peaked domain.

Appendix A. Proof of Theorem 3.1

Let us introduce a few pieces of notations that we use in the proof of Theorem 3.1. For a profile P_N, we denote by $\min_2(\tau(P_N))$ the second minimum of $\tau(P_N)$. More formally, $\min_2(\tau(P_N)) = x$ if and only if $x \in \tau(P_N)$ and $|\{y \in \tau(P_N) \mid y < x\}| = 1$. Similarly, we denote by $\max_2(\tau(P_N))$ the second maximum of $\tau(P_N)$. Finally, for $k \in \{1, \ldots n\}$, let $\mathcal{S}_N(k) = \{P_N \in \mathcal{S}_N \mid |\tau(P_N)| = k\}$ be the set of preference profiles having k different top-ranked alternatives. Note that $\mathcal{S}_N = \bigcup_{k=1}^{n} \mathcal{S}_N(k)$.

Proof. We prove the theorem by using induction on the number of different top-ranked alternatives at a profile. By Remark 2.1, it is sufficient to prove that $f(P_N) \in [\min(\tau(P_N)), \max(\tau(P_N))]$. By unanimity, $\min(\tau(P_N)) = f(P_N) = \max(\tau(P_N))$ for all $P_N \in \mathcal{S}_N(1)$. Take $k \in \{1, \ldots, n-1\}$. Suppose $f(P_N) \in [\min(\tau(P_N)), \max(\tau(P_N))]$ for all $P_N \in \mathcal{S}_N(k)$. We show that the same holds for all $P_N \in \mathcal{S}_N(k+1)$.

Take $P_N \in \mathcal{S}_N(k+1)$. Without loss of generality, assume for contradiction that $f(P_N) > \max(\tau(P_N))$. Take $i_1 \in N$ such that $r_1(P_{i_1}) = \min(\tau(P_N))$. Consider $P'_{i_1} \in \mathcal{S}_{i_1}$ such that $r_1(P'_{i_1}) = \min_2(\tau(P_N))$. Then, by strategy-proofness, $f(P'_{i_1}, P_{N \setminus i_1}) \notin [\min(\tau(P_N)), f(P_N))$. To see this, note that if $f(P'_{i_1}, P_{N \setminus i_1}) \in [\min(\tau(P_N)), f(P_N))$, then agent i_1 manipulates at P_N via P'_{i_1}. Since $f(P_N) > \max(\tau(P_N))$, this means

$$f(P'_{i_1}, P_{N \setminus i_1}) \notin [\min(\tau(P_N)), \max(\tau(P_N))]. \tag{1}$$

Note that by construction $|\tau(P'_{i_1}, P_{N \setminus i_1})| \leq |\tau(P_N)|$. Clearly, if $|\tau(P'_{i_1}, P_{N \setminus i_1})| = k$, then (1) contradicts our induction hypothesis. Suppose $|\tau(P'_{i_1}, P_{N \setminus i_1})| = k+1$. This means $\min(\tau(P'_{i_1}, P_{N \setminus i_1})) = \min(\tau(P_N))$ and $\max(\tau(P'_{i_1}, P_{N \setminus i_1})) = \max(\tau(P_N))$. If $f(P'_{i_1}, P_{N \setminus i_1}) < \min(\tau(P_N))$ then take $i_2 \in N$ such that $r_1(P_{i_2}) = \max(\tau(P_N))$, and if $f(P'_{i_1}, P_{N \setminus i_1}) > \max(\tau(P_N))$ then take $i_2 \in N$ such that $r_1(P_{i_2}) = \min(\tau(P_N))$. Consider $P'_{i_2} \in \mathcal{S}_{i_2}$ such that $r_1(P'_{i_2}) = \max_2(\tau(P_N))$ if $r_1(P_{i_2}) = \max(\tau(P_N))$, and $r_1(P'_{i_2}) = \min_2(\tau(P_N))$ if $r_1(P_{i_2}) = \min(\tau(P_N))$. Using a similar arguments as for the derivation of (1), it follows that

$$f(P'_{i_1}, P'_{i_2}, P_{N\setminus\{i_1,i_2\}}) \notin [\min(\tau(P_N)), \max(\tau(P_N))]. \tag{2}$$

Continuing in this manner, we construct a sequence of agents $i_1, \dots, i_l$ with $r_1(P_j) \in \{\min(\tau(P_N)), \max($ and $P'_j \in \mathcal{S}_j$ with $r_1(P'_j) \in \{\min_2(\tau(P_N)), \max_2(\tau(P_N))\}$ for all $j = i_1, \dots, i_l$ such that $|\tau(P'_{i_1}, \dots, P'_{i_l}, P_{N\setminus\{i_1,\dots}$ k and $f(P'_{i_1}, \dots, P'_{i_l}, P_{N\setminus\{i_1,\dots,i_l\}}) \notin [\min(\tau(P_N)), \max(\tau(P_N))]$. However, this contradicts our induction hypothesis, which completes the proof of the theorem. ∎

Appendix B. Proof of Theorem 3.2

Proof. Let $P_N \in \mathcal{S}_N$, $i \in N$ and $P'_i \in \mathcal{S}_i$ be such that $r_1(P_i) = r_1(P'_i)$. It is sufficient to prove that $f(P_N) = f(P'_i, P_{N\setminus i})$. Suppose not. Let $r_1(P_i) = r_1(P'_i) = x$, $f(P_N) = y$, and $f(P'_i, P_{N\setminus i}) = y'$. By strategy-proofness, we have yP_iy' and $y'P'_iy$. Since $r_1(P_i) = r_1(P'_i) = x$, by single-peakedness of $\mathcal{S}_i$ and strategy-proofness of f, this means either $y < x < y'$ or $y' < x < y$. Assume without loss of generality that $y < x < y'$. Since $f(P'_i, P_{N\setminus i}) > x$, by Pareto property, we have $\max(\tau(P'_i, P_{N\setminus i})) > x$. Take $i_1 \in N$ such that $r_1(P_{i_1}) = \max(\tau(P'_i, P_{N\setminus i}))$. Consider the preference $P'_{i_1} \in \mathcal{S}_{i_1}$ of agent i_1 where $P'_{i_1} = P'_i$.

Claim 1. $f(P_i, P'_{i_1}, P_{N\setminus\{i,i_1\}}) = y$.
Suppose not. Since $r_1(P_{i_1}) = \max(\tau(P'_i, P_{N\setminus i})) = \max(\tau(P_N))$, we have by Pareto property, $f(P_i, P'_{i_1}, P_{N\setminus\{i,i_1\}}) \leq r_1(P_{i_1})$. Now, if $f(P_i, P'_{i_1}, P_{N\setminus\{i,i_1\}}) < y$, then i_1 manipulates at $(P_i, P'_{i_1}, P_{N\setminus\{i,i_1\}})$ via P_{i_1}. On the other hand, if $f(P_i, P'_{i_1}, P_{N\setminus\{i,i_1\}}) > y$, then i_1 manipulates at P_N via P'_{i_1}. This completes the proof of the claim.

Claim 2. $f(P'_i, P'_{i_1}, P_{N\setminus\{i,i_1\}}) > x$.
Assume for contradiction that $f(P'_i, P'_{i_1}, P_{N\setminus\{i,i_1\}}) \leq x$. If $f(P'_i, P'_{i_1}, P_{N\setminus\{i,i_1\}}) < y$, then by Claim 1, i manipulates at $(P'_i, P'_{i_1}, P_{N\setminus\{i,i_1\}})$ via P_i. On the other hand, if $f(P'_i, P'_{i_1}, P_{N\setminus\{i,i_1\}}) \in (y, x]$, then again by Claim 1, i manipulates at $(P_i, P'_{i_1}, P_{N\setminus\{i,i_1\}})$ via P'_i. Finally, if $f(P'_i, P'_{i_1}, P_{N\setminus\{i,i_1\}}) = y$, then by the facts that $y'P'_{i_1}y$ and $f(P'_i, P_{i_1}, P_{N\setminus\{i,i_1\}}) = y'$, agent i_1 manipulates at $(P'_i, P'_{i_1}, P_{N\setminus\{i,i_1\}})$ via P_{i_1}. This completes the proof of the claim.

Now, we complete the proof of the theorem. Note that since $f(P'_i, P'_{i_1}, P_{N\setminus\{i,i_1\}}) > x$, if $\max(\tau(P'_i, P'_{i_1}, P_{N\setminus\{i,i_1\}})) = x$, then Claim 2 contradicts Pareto property. So, suppose $\max(\tau(P'_i, P'_{i_1}, P_{N\setminus\{i,i_1}$ x. Take $i_2 \in N$ such that $r_1(P_{i_2}) = \max(\tau(P'_i, P'_{i_1}, P_{N\setminus\{i,i_1\}}))$. Consider $P'_{i_2} \in \mathcal{S}_{i_2}$ such that $P'_{i_2} = P'_i$.

Using similar argument as for the proofs of Claim 1 and Claim 2, we have

$$f(P_i, P'_{i_1}, P'_{i_2}, P_{N\setminus\{i,i_1,i_2\}}) = y, \text{ and}$$
$$f(P'_i, P'_{i_1}, P'_{i_2}, P_{N\setminus\{i,i_1,i_2\}}) > x.$$

Continuing in this manner, we can construct a sequence $i_1, \ldots, i_k$ of agents with the properties that (i) for all $j = 1, \ldots, k$, $r_1(P_{i_j}) = \max(\tau(P'_i, P'_{i_1}, \ldots, P'_{i_{j-1}}, P_{N\setminus\{i,i_1,\ldots,i_{j-1}\}}))$ and $P'_{i_j} = P'_i$, and (ii) $\max(\tau(P'_i, P'_{i_1}, \ldots, P'_{i_j}, P_{N\setminus\{i,i_1,\ldots,i_j\}})) = x$ such that

$$f(P'_i, P'_{i_1}, \ldots, P'_{i_j}, P_{N\setminus\{i,i_1,\ldots,i_j\}}) > x.$$

However, since $\max(\tau(P'_i, P'_{i_1}, \ldots, P'_{i_j}, P_{N\setminus\{i,i_1,\ldots,i_j\}})) = x$ and $f(P'_i, P'_{i_1}, \ldots, P'_{i_j}, P_{N\setminus\{i,i_1,\ldots,i_j\}}) > x$, we have a contradiction to the Pareto property of f. This completes the proof of the theorem. ■

Appendix C. Proof of Theorem 4.1

Proof. (If part) Note that a min-max rule is unanimous by definition (on any domain). We show that such a rule is strategy-proof on $\mathcal{S}_N$. For all $i \in N$, let $\bar{\mathcal{S}}_i$ be the maximal set of single-peaked preferences. By Weymark (2011), a min-max rule is strategy-proof on $\bar{\mathcal{S}}_N$. Since $\mathcal{S}_i \subseteq \bar{\mathcal{S}}_i$ for all $i \in N$, a min-max rule must be strategy-proof on $\mathcal{S}_N$. This completes the proof of the if part.

(Only-if part) Let $\mathcal{S}_i$ be a top-connected single-peaked set of preferences for all $i \in N$ and let $f : \mathcal{S}_N \to X$ be a unanimous and strategy-proof SCF. We show that f is a min-max rule. First, we establish a few properties of f in the following sequence of lemmas.

By Theorem 3.1 and Theorem 3.2, f must satisfy Pareto property and tops-onlyness. Our next lemma shows that f is uncompromising.

Lemma C.1. *The SCF f is uncompromising.*

Proof. Let $P_N \in \mathcal{S}_N$, $i \in N$, and $P'_i \in \mathcal{S}_i$ be such that $r_1(P_i) < f(P_N)$ and $r_1(P'_i) \leq f(P_N)$. It is sufficient to show $f(P'_i, P_{N\setminus i}) = f(P_N)$. Suppose $r_1(P_i) = x$, $f(P_N) = y$, and $f(P'_i, P_{N\setminus i}) = y'$. By means of strategy-proofness, we assume that $r_1(P'_i) = y'$ and $\min(\tau(P'_i, P_{N\setminus i})) = y'$.[13] Assume for contradiction that $y \neq y'$.

[13] Since $f(P'_i, P_{N\setminus i}) = y'$, if $r_1(P'_i) \neq y'$, then by strategy-proofness, $f(P''_i, P_{N\setminus i}) = y'$ for some $P''_i \in \mathcal{S}_i$ with $r_1(P''_i) = y'$. Similarly, if $r_1(P_j) < y'$ for some $j \in N \setminus i$, then by strategy-proofness, $f(P'_i, P'_j, P_{N\setminus\{i,j\}}) = y'$ for some $P'_j \in \mathcal{S}_j$ with $r_1(P'_j) = y'$.

By strategy-proofness, we must have $y' < x$. This is because, if $y' \in [x, y)$, then agent i manipulates at P_N via P'_i. On the other hand, if $y' > y$, then by means of the fact that $r_1(P'_i) \leq y$, agent i manipulates at $(P'_i, P_{N \setminus i})$ via P_i.

Let $T = \{j \in N \mid r_1(P_j) < x\}$. For $j \in T$, let $P'_j \in \mathcal{S}_j$ be such that $r_1(P'_j) = x$.

Claim 1. $f(P'_T, P_{N \setminus T}) = y$.

If T is empty, then there is nothing to show. Suppose T is non-empty. Take $j \in T$ such that $r_1(P_j) = \min(\tau(P_N))$. Because $f(P_N) = y$ and $r_1(P_j) = \min(\tau(P_N))$, by strategy-proofness and Pareto property, we have $f(P'_j, P_{N \setminus j}) = y$. Next, take $k \in T$ such that $r_1(P_k) = \min(\tau(P'_j, P_{N \setminus j}))$. Using similar logic, we have $f(P'_j, P'_k, P_{N \setminus \{j,k\}}) = y$. Continuing in this manner, we have

$$f(P'_T, P_{N \setminus T}) = y.$$

This completes the proof of Claim 1.

Let $T' = T \cup i$. For all $j \in T'$, let $\tilde{P}_j \in \mathcal{S}$ be such that $r_1(\tilde{P}_j) = x$.

Claim 2. $f(\tilde{P}_{T'}, P_{N \setminus T'}) = x$.

Take $j \in T'$ such that $r_1(P_j) = y'$. Consider the preference $P''_j \in \mathcal{S}_j$ such that $r_1(P''_j) = y' + 1$. We show $f(P'_i, P''_j, P_{N \setminus \{i,j\}}) \in \{y', y' + 1\}$.[14] Suppose not. By tops-onlyness of f, we can assume $r_2(P''_j) = y'$. However, that means agent j manipulates at $(P'_i, P''_j, P_{N \setminus \{i,j\}})$ via P'_j. This shows $f(P'_i, P''_j, P_{N \setminus \{i,j\}}) \in \{y', y' + 1\}$.

Now, take $k \in T' \setminus j$ (if there is any) such that $r_1(P_k) = y'$. Consider the preference $P''_k \in \mathcal{S}_k$ such that $r_1(P''_k) = y' + 1$. Using similar logic as before, we have $f(P'_i, P''_j, P''_k, P_{N \setminus \{i,j,k\}}) \in \{y', y' + 1\}$. Continuing in this manner, we can construct a profile $\bar{P}_N \in \mathcal{S}_N$ where $r_1(\bar{P}_j) = y' + 1$ for all agents j with $r_1(P_j) = y'$ and $\bar{P}_j = P_j$ for all agents j with $r_1(P_j) > y'$ such that

$$f(\bar{P}_N) \in \{y', y' + 1\}.$$

However, since $\min(\tau(\bar{P}_N)) = y' + 1$, by Pareto property,

$$f(\bar{P}_N) = y' + 1.$$

Using similar logic, we can construct a profile $\hat{P}_N \in \mathcal{S}_N$ where $r_1(\hat{P}_j) = y' + 2$ for all agents j

[14] If $j = i$, then the profile $(P'_i, P''_j, P_{N \setminus \{i,j\}})$ is same as $(P''_i, P_{N \setminus i})$.

with $r_1(\bar{P}_j) = y' + 1$ and $\hat{P}_j = \bar{P}_j$ for all agents j with $r_1(\bar{P}_j) > y' + 1$, and conclude that

$$f(\hat{P}_N) = y' + 2.$$

Continuing in this manner, we move all the agents j in T' to a preference $\tilde{P}_j \in \mathcal{S}_j$ with $r_1(\tilde{P}_j) = x$ while keeping the preferences of all other agents unchanged and conclude that

$$f(\tilde{P}_{T'}, P_{N \setminus T'}) = x.$$

This completes the proof of Claim 2.

Now, we complete the proof of the lemma. Consider the profiles $(P'_T, P_{N\setminus T})$ and $(\tilde{P}_{T'}, P_{N\setminus T'})$. Note that for an agent j, if $r_1(P_j) > x$, then her preference is the same in both the profiles $(P'_T, P_{N\setminus T})$ and $(\tilde{P}_{T'}, P_{N\setminus T'})$. Moreover, for an agent j, if $r_1(P_j) \leq x$, then her top-ranked alternative is x in both the profiles. This means these two profiles are tops-equivalent. However, since $f(P'_T, P_{N\setminus T}) \neq f(\tilde{P}_{T'}, P_{N\setminus T'})$, Claim 1 and 2 contradict tops-onlyness of f. This completes the proof of the lemma. ■

The following lemma establishes that f is a min-max rule.

Lemma C.2. *The SCF f is a min-max rule.*

Proof. For all $S \subseteq N$, let $(P^a_S, P^b_{N\setminus S}) \in \mathcal{S}_N$ be such that $r_1(P^a_i) = a$ for all $i \in S$ and $r_1(P^b_i) = b$ for all $i \in N \setminus S$. Define $\beta_S = f(P^a_S, P^b_{N\setminus S})$ for all $S \subseteq N$. Clearly, $\beta_S \in X$ for all $S \subseteq N$. By unanimity, $\beta_\varnothing = b$ and $\beta_N = a$. Also, by uncompromisingness, $\beta_S \leq \beta_T$ for all $T \subseteq S$.

Take $P_N \in \mathcal{S}_N$. We show $f(P_N) = \min\limits_{S\subseteq N}\{\max\limits_{i\in S}\{r_1(P_i), \beta_S\}\}$. Suppose $S_1 = \{i \in N \mid r_1(P_i) < f(P_N)\}$, $S_2 = \{i \in N \mid f(P_N) < r_1(P_i)\}$, and $S_3 = \{i \in N \mid r_1(P_i) = f(P_N)\}$. By uncompromisingness, $\beta_{S_1\cup S_3} \leq f(P_N) \leq \beta_{S_1}$. Consider the expression $\min\limits_{S\subseteq N}\{\max\limits_{i\in S}\{r_1(P_i), \beta_S\}\}$. Take $S \subseteq S_1$. Then, by Condition (iii) in Definition 2.9, $\beta_{S_1} \leq \beta_S$. Since $r_1(P_i) < f(P_N)$ for all $i \in S$ and $f(P_N) \leq \beta_{S_1} \leq \beta_S$, we have $\max\limits_{i\in S}\{r_1(P_i), \beta_S\} = \beta_S$. Clearly, for all $S \subseteq N$ such that $S \cap S_2 \neq \varnothing$, we have $\max\limits_{i\in S}\{r_1(P_i), \beta_S\} > f(P_N)$. Consider $S \subseteq N$ such that $S \cap S_2 = \varnothing$ and $S \cap S_3 \neq \varnothing$. Then, $S \subseteq S_1 \cup S_3$, and hence $\beta_{S_1\cup S_3} \leq \beta_S$. Therefore, $\max\limits_{i\in S}\{r_1(P_i), \beta_S\} = \max\{f(P_N), \beta_S\} \geq \max\{f(P_N), \beta_{S_1\cup S_3}\}$. Since $\beta_{S_1\cup S_3} \leq f(P_N)$, we have $\max\{f(P_N), \beta_{S_1\cup S_3}\} = f(P_N)$. Combining all these, we have $\min\limits_{S\subseteq N}\{\max\limits_{i\in S}\{r_1(P_i), \beta_S\}\} = \min\{f(P_N), \beta_{S_1}\}$. Because $f(P_N) \leq \beta_{S_1}$, we have $\min\{f(P_N), \beta_{S_1}\} = f(P_N)$. This completes the proof of the lemma. ■

The proof of the only-if part of Theorem 4.1 follows from Lemmas C.1 - C.2. ■

Appendix D. Proof of Theorem 5.1

Proof. The proof of the if part follows from Theorem 4.1. We proceed to prove the only-if part. Let $\mathcal{D}_N$ be a min-max domain. We show that $\mathcal{D}_i$ is top-connected single-peaked for all $i \in N$. We show this in two steps: in Step 1 we show that $\mathcal{D}_i$ is single-peaked for all $i \in N$, and in Step 2, we show that $\mathcal{D}_i$ is top-connected for all $i \in N$.

Step 1. Suppose that $\mathcal{D}_i$ is not single-peaked for some $i \in N$. Then, there is $Q \in \mathcal{D}_i$ and $x, y \in X$ such that $x < y < r_1(Q)$ and xQy. Consider the min-max rule f^β with respect to $(\beta_S)_{S \subseteq N}$ such that $\beta_S = x$ for all $\emptyset \subsetneq S \subsetneq N$. Take $P_N \in \mathcal{D}_N$ such that $P_i = Q$ and $r_1(P_j) = y$ for all $j \in N \setminus i$. By the definition of f^β, $f^\beta(P_N) = y$. Now, take $P'_i \in \mathcal{D}_i$ with $r_1(P'_i) = x$. Again, by the definition of f^β, $f^\beta(P'_i, P_{N \setminus i}) = x$. This means agent i manipulates at P_N via P'_i, which is a contradiction to the assumption that $\mathcal{D}_N$ is a min-max domain. This completes Step 1.

Step 2. In this step, we show that $\mathcal{D}_i$ satisfies top-connectedness for all $i \in N$. Assume for contradiction that $\mathcal{D}_i$ is not top-connected for some $i \in N$. Note that since $\mathcal{D}_i$ is single-peaked, for all $P \in \mathcal{D}_i$, $r_1(P) = a$ (or b) implies $r_2(P) = a + 1$ (or $b - 1$). Because $\mathcal{D}_i$ is single-peaked, for all $P \in \mathcal{D}_i$ and all $x \in X \setminus \{a, b\}$, $r_1(P) = x$ implies $r_2(P) \in \{x - 1, x + 1\}$. Since $\mathcal{D}_i$ violates top-connectedness, assume without loss of generality that there exists $x \in X \setminus \{a, b\}$ such that for all $P \in \mathcal{D}_i$, $r_1(P) = x$ implies $r_2(P) = x - 1$. Consider the following SCF:[15]

$$f(P_N) = \begin{cases} x \text{ if } r_1(P_i) = x \text{ and } xP_j(x-1) \text{ for all } j \in N \setminus i, \\ x - 1 \text{ if } r_1(P_i) = x \text{ and } (x-1)P_j x \text{ for some } j \in N \setminus i, \\ r_1(P_i) \text{ otherwise.} \end{cases}$$

It is left to the reader to verify that f is unanimous and strategy-proof. We show that f is not uncompromising, which in turn means that f is not a min-max rule. Let $P_N \in \mathcal{D}_N$ be such that $r_1(P_i) = x$ and $r_1(P_j) = x - 1$ for some $j \neq i$, and let $P'_i \in \mathcal{D}_i$ be such that $r_1(P'_i) = x + 1$. Then, by the definition of f, $f(P_N) = x - 1$ and $f(P'_i, P_{N \setminus i}) = x + 1$. Therefore, because $f(P_N) = x - 1$ and $x - 1 \leq r_1(P_i) \leq r_1(P'_i)$, the fact that $f(P'_i, P_{N \setminus i}) = x + 1$ is a violation of uncompromisingness. This completes Step 2 and the proof of the only-if part. ■

[15]Here $\mathcal{D}_i$ satisfies the *unique seconds* property defined in Aswal et al. (2003) and the SCF f considered here is similar to the one used in the proof of Theorem 5.1 in Aswal et al. (2003).

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