An element-set labelling a Cartesian product by measurable binary relations which leads to postulates of the theory of experience and chance as a theory of co events

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An element-set labelling a Cartesian product by measurable binary relations which leads to postulates of the theory of experience and chance as a theory of co$\sim$events

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Abstract. *We introduce the set-theoretic language for the element-set labelling a Cartesian product by measurable binary relations intended for the labelling, or for the naming of parts and details of the construction that we are going to propose in the theory of experience and chance, or the theory of co$\sim$events that serve as mathematical models of events as dual pairs.*

Keywords. *Eventology, theory of experience and chance, theory of co$\sim$events, measurable binary relation, event, co$\sim$event, experience, chance.*

Before you start working in a set-theoretic space whose objects of interest are simultaneously the *elements* of space, the *sets of elements*, and the *sets of subsets of elements*, it is necessary to stock up some system of «coordinates» suitable for a labelling the space itself, and its parts.

1 A labelling set and a set of its labelling subsets

Here, in my opinion, a slightly peculiar but effective system of set-theoretic coordinates «elements—sets» is quite suitable, based on some labelling set $\mathfrak{X}$ and some set $\mathfrak{S}^{\mathfrak{X}} \subseteq \mathcal{P}(\mathfrak{X})$ of its labelling subsets, and also on the *M-complement*[1] $\mathfrak{X}^{\copyright}$ of the labelling set $\mathfrak{X}$, and on the one-to-one corresponding to the set $\mathfrak{S}^{\mathfrak{X}}$ the set of its labelling subsets $\mathfrak{S}^{\mathfrak{X}^{\copyright}} = \left\{X^{c\copyright} \colon X \in \mathfrak{S}^{\mathfrak{X}}\right\} \subseteq \mathcal{P}\left(\mathfrak{X}^{\copyright}\right)$.

Warning 1 *(relative subsets and relative empty subsets).* Since in the theory of experience and chance one has to deal simultaneously with subsets of sets of different levels, we will need unusual, but convenient notation, directly indicating what subsets of which set is spoken. For example, if we are talking about subsets $x \subseteq \Omega$, $X \subseteq \mathfrak{X}$, or $\mathcal{O} \subseteq \mathcal{P}(\mathfrak{X})$, then denotations of subsets x, X, or $\mathcal{O}$, when appropriate, we will write more fully: $x /\!\!/ \Omega$, $X /\!\!/ \mathfrak{X}$, or $\mathcal{O} /\!\!/ \mathcal{P}(\mathfrak{X})$, directly specifying in which sets these subsets contain. Especially we will have to deal with empty subsets: $\varnothing /\!\!/ \Omega$, $\emptyset /\!\!/ \mathfrak{X}$, or $\emptyset /\!\!/ \mathcal{P}(\mathfrak{X})$, for which we introduce more compact notation: $\varnothing^{\Omega} = \varnothing /\!\!/ \Omega$, $\emptyset^{\mathfrak{X}} = \emptyset /\!\!/ \mathfrak{X}$, or $\emptyset^{\mathcal{P}(\mathfrak{X})} = \emptyset /\!\!/ \mathcal{P}(\mathfrak{X})$, we will talk about them as *relatively empty subsets*, and call Ω*-empty,* $\mathfrak{X}$*-empty, or* $\mathcal{P}(\mathfrak{X})$*-empty subsets* correspondingly.

Consider the *measurable space* $(\Omega, \mathcal{A})$ composed of some set Ω and a sigma-algebra $\mathcal{A}$ of its subsets and we emphasize that: *elements* $\omega \in \Omega$; *measurable subsets* $x \subseteq \Omega$; *some set* $\mathfrak{X} = \{x \colon x \in \mathcal{A}\} \subseteq \mathcal{A}$, composed from measurable subsets $x \in \mathfrak{X}$; and *some set* $\mathfrak{S}^{\mathfrak{X}} \subseteq \mathcal{P}(\mathfrak{X})$ *of subsets* $X \subseteq \mathfrak{X}$, consisting from measurable subsets $x \in X \subseteq \mathfrak{X}$; *until they have no meaningful interpretation* and form only a basis $\mathbf{\Lambda}$ peculiar element-set ***labels*** $\lambda \in \mathbf{\Lambda}$ (*tags, dockets, tickets*, or *names*), intended for a element-set labelling, or a nominating the parts and details of the construction that we are going to propose in the theory of experience and chance [5, 4] as a mathematical model of an event as a dual pair.

Predefinition 1 *(Basic element-set labels). Basic element-set labels* $\lambda \in \mathbf{\Lambda}$ are called as elements, sets and sets of subsets of the measurable space $(\Omega, \mathcal{A})$, and also results of *terraced set-theoretic*

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[1]The set $\mathfrak{X}^{\copyright} = \{x^c \colon x \in \mathfrak{X}\}$ is called a *complement by Minkowski (M-complement)* of the set $\mathfrak{X}$.

operations over them, equipped with their own titles, a list of which can be found in the Appendix on page 122.

We'll fill up the stock of $\mathbf{\Lambda}$ tags with one more label, Cartesian product

$$\mathfrak{X} \times \mathbf{2}^{\mathfrak{X}} = \left\{(x, X) : x \in \mathfrak{X}, X \in 2^{\mathfrak{X}}\right\}, \tag{1}$$

which defines a binary relation

$$\mathcal{R}_{\mathfrak{X},\mathbf{2}^{\mathfrak{X}}} = \left\{(x, X)\colon x \in X, x \in \mathfrak{X}, X \in \mathbf{2}^{\mathfrak{X}}\right\} \subseteq \mathfrak{X} \times \mathbf{2}^{\mathfrak{X}} \tag{2}$$

as a *membership relation* $x \in X$ between elements $x \in \mathfrak{X}$ and subsets $X \in 2^{\mathfrak{X}}$; and also a complementary binary relation

$$\mathcal{R}^c_{\mathfrak{X},\mathbf{2}^{\mathfrak{X}}} = \left\{(x, X)\colon x \notin X, x \in \mathfrak{X}, X \in \mathbf{2}^{\mathfrak{X}}\right\} \subseteq \mathfrak{X} \times \mathbf{2}^{\mathfrak{X}} \tag{3}$$

as a *non-membership relation* $x \notin X$ between elements $x \in \mathfrak{X}$ and subsets $X \in 2^{\mathfrak{X}}$; so that

$$\mathcal{R}_{\mathfrak{X},\mathbf{2}^{\mathfrak{X}}} + \mathcal{R}^c_{\mathfrak{X},\mathbf{2}^{\mathfrak{X}}} = \mathfrak{X} \times \mathbf{2}^{\mathfrak{X}}. \tag{4}$$

Finally, we add to the stock $\mathbf{\Lambda}$ so called *terraced*[2] *label*

$$\Big(\mathsf{Ter}_{X/\!\!/\mathfrak{X}},\quad \mathsf{ter}(X/\!\!/\mathfrak{X})\Big) = \left(\bigcup_{x\in X} x,\quad \bigcap_{x\in X} x \bigcap_{x\in\mathfrak{X}-X} (\Omega - x)\right) \subseteq \Omega \times \Omega, \tag{5}$$

numbered by labels-subsets $X \in \mathbf{2}^{\mathfrak{X}}$ and while defined simply as a pair of indicated measurable subsets of Ω.

To have a full stock we'll stock up in the literal sense «complementary» element-set labels, constructed from: 1) the complements $x^c = \Omega - x$ to measurable subsets $x \subseteq \Omega$, 2) the *M-complementary set* $\mathfrak{X}^{(c)} = \{x^c\colon x \in \mathfrak{X}\} \subseteq \mathcal{A}$ composed from these complements, and 3) the sets $\mathbf{2}^{\mathfrak{X}^{(c)}} = \left\{X^{c(c)}\colon X \in \mathbf{2}^{\mathfrak{X}}\right\} \subseteq \mathcal{P}\left(\mathfrak{X}^{(c)}\right)$ of subsets $X^{c(c)} = (X^c)^{(c)} = (\mathfrak{X} - X)^{(c)} \subseteq \mathfrak{X}^{(c)}$, i.e., such that $X^{c(c)} = \{x^c\colon x \in X^c\} \in \mathbf{2}^{\mathfrak{X}^{(c)}}$.

There we also place a label similar to (11), the Cartesian product

$$\mathfrak{X}^{(c)} \times \mathbf{2}^{\mathfrak{X}^{(c)}} = \left\{(x^c, X^{c(c)}) : x^c \in \mathfrak{X}^{(c)}, X^{c(c)} \in 2^{\mathfrak{X}^{(c)}}\right\}, \tag{6}$$

which defines analogous to (13) a complementary binary relation

$$\mathcal{R}^c_{\mathfrak{X}^{(c)},\mathbf{2}^{\mathfrak{X}^{(c)}}} = \left\{(x^c, X^{c(c)})\colon x^c \in X^{c(c)}, x^c \in \mathfrak{X}^{(c)}, X^{c(c)} \in \mathbf{2}^{\mathfrak{X}^{(c)}}\right\} \subseteq \mathfrak{X}^{(c)} \times \mathbf{2}^{\mathfrak{X}^{(c)}} \tag{7}$$

as a *membership relation* $x^c \in X^{c(c)}$ between elements $x^c \in \mathfrak{X}^{(c)}$ and subsets $X^{c(c)} \in 2^{\mathfrak{X}^{(c)}}$; and also a complementary binary relation

$$\mathcal{R}_{\mathfrak{X}^{(c)},\mathbf{2}^{\mathfrak{X}^{(c)}}} = \left\{(x^c, X^{c(c)})\colon x^c \notin X^{c(c)}, x^c \in \mathfrak{X}^{(c)}, X^{c(c)} \in \mathbf{2}^{\mathfrak{X}^{(c)}}\right\} \subseteq \mathfrak{X}^{(c)} \times \mathbf{2}^{\mathfrak{X}^{(c)}} \tag{8}$$

as a *non-membership relation* $x^c \notin X^{c(c)}$ between elements $x^c \in \mathfrak{X}^{(c)}$ and subsets $X^{c(c)} \in 2^{\mathfrak{X}^{(c)}}$; so that

$$\mathcal{R}^c_{\mathfrak{X}^{(c)},\mathbf{2}^{\mathfrak{X}^{(c)}}} + \mathcal{R}_{\mathfrak{X}^{(c)},\mathbf{2}^{\mathfrak{X}^{(c)}}} = \mathfrak{X}^{(c)} \times \mathbf{2}^{\mathfrak{X}^{(c)}}. \tag{9}$$

[2]Those who are familiar with the beginnings of the eventological theory [3, 2007] should keep their attention to the amazing inevitability of the «splitting» of the previously unified concept of the *terrace event* into two dual halves, the right of which is the *terraced ket-event* which is defined as a terrace event of the first kind $\mathsf{ter}(X/\!\!/\mathfrak{X}) = \bigcap_{x\in X} x \bigcap_{x\in\mathfrak{X}-X} (\Omega - x) \subseteq \Omega$ from the eventological part of the Kolmogorov *probability theory*, and the left one is a *terraced bra-event*, a new concept from the *theory of believabilities*, dual to the *probability theory*, which is defined as terraced event of the 5th kind $\mathsf{Ter}_{X/\!\!/\mathfrak{X}} = \bigcup_{x\in X} x \subseteq \Omega$ from the eventological classification.

Finally, do not forget the similar to (15) *terrace label*

$$\left(\mathsf{Ter}_{X^{c(c)} /\!\!/ \mathfrak{X}^{(c)}}, \quad \mathsf{ter}\left(X^{c(c)} /\!\!/ \mathfrak{X}^{(c)}\right)\right) = \left(\bigcup_{x^c \in X^{c(c)}} x^c, \quad \bigcap_{x^c \in X^{c(c)}} x^c \bigcap_{x^c \in \mathfrak{X}^{(c)} - X^{c(c)}} (\Omega - x^c)\right) \subseteq \Omega \times \Omega, \tag{10}$$

numbered by labels-subsets $X^{c(c)} \in \mathbf{2}^{\mathfrak{X}^{(c)}}$.

Warning 2 *(membership relations and paradoxes of naive set theory).* Some mathematical relations such as «member of» and «subset of», generally speaking, should not be understood as binary relations because its domains and codomains cannot be sets in usual systems of axiomatic set theory. For example, if you try to model the general concept of membership as a binary relation «$\in$», then then for this you will have to define the domain and the codomain, which can be a class of all sets. But such a class is not a set in the naive set theory, and the assumption that the relation «$\in$» is defined on all sets leads to a contradiction from the well-known Russell paradox. At the same time, in the overwhelming majority of mathematical contexts, links to the relation «member of» and «subset of» are absolutely harmless, because they are tacitly limited to some set which is clear from the context. The removal of this problem consists in choosing each time a sufficiently large set A, which contains all objects of interest, and work with the restriction «$\in_A$» instead of «$\in$». Similarly, the relation «$\subseteq$» must also be limited to the relation «$\subseteq_A$» to have some domain A and the codomain $\mathcal{P}(A)$, set of all subsets of A. Therefore, the chain of three membership relations

$$\omega \ \in \ x \ \in \ X \ \in \ \mathbf{2}^{\mathfrak{X}} \ \subseteq \ \mathcal{P}(\mathfrak{X}) \tag{11}$$

will always be understood by me as

$$\omega \ \in_{\Omega} \ x \ \in_{\mathfrak{X}} \ X \ \in_{\mathbf{2}^{\mathfrak{X}}} \ \mathbf{2}^{\mathfrak{X}} \ \subseteq_{\mathcal{P}(\mathfrak{X})} \ \mathcal{P}(\mathfrak{X}), \tag{12}$$

the chain of *limited by default* membership relations.

The stock $\mathbf{\Lambda}$ of element-set labels $\lambda \in \mathbf{\Lambda}$ is intended to construct such a system of element-set «coordinates», which, relying on a duality «element–set», will allow us to divide each concept of the *theory of experience and chance (TEC)* into two dual parts and present it in the form of a conveniently written *dual pair*, i.e., pairs composed of two dual parts. In the bra-ket notation, the dual parts of pairs labelled with the labels $\lambda, \lambda' \in \mathbf{\Lambda}$, are denoted by $\langle\lambda|$ and $|\lambda'\rangle$ correspondingly, the entire dual pair is denoted by $\langle\lambda|\lambda'\rangle$ and is defined as the Cartesian product $\langle\lambda|\lambda'\rangle = \langle\lambda| \times |\lambda'\rangle$ of their dual parts, placing the corresponding concept of the *theory of experience and chance* in the system of element-set «coordinates».

2 Binary relations and quotient-sets

Definition 1 *(Cartesian product of measurable spaces).* Let $\langle\Omega, \mathcal{A}| = (\langle\Omega|, \langle\mathcal{A}|)$ be the measurable bra-space, and $|\Omega, \mathcal{A}\rangle = (|\Omega\rangle, |\mathcal{A}\rangle)$ be the measurable ket-space. Let's denote

$$\langle\Omega|\Omega\rangle = \langle\Omega| \times |\Omega\rangle = \left\{ \langle\omega^*|\omega\rangle \ : \ \langle\omega^*| \in \langle\Omega|, |\omega\rangle \in |\Omega\rangle \right\}$$

the Cartesian product of sets $\langle\Omega|$ and $|\Omega\rangle$ called a *bra-ket-set*. The Cartesian product of the sigma-algebra $\langle\mathcal{A}| \times |\mathcal{A}\rangle$ is a family of subsets of $\langle\Omega| \times |\Omega\rangle$. In general, this family is not closed with respect to countable unions, and hence is not a sigma-algebra. We shall introduce the notation

$$\langle\mathcal{A}|\mathcal{A}\rangle = \sigma(\langle\mathcal{A}| \times |\mathcal{A}\rangle)$$

for the minimal sigma-algebra $\sigma(\langle\mathcal{A}| \times |\mathcal{A}\rangle)$ containing $\langle\mathcal{A}| \times |\mathcal{A}\rangle$, and we shall call it the *bra-ket sigma-algebra*. Then the pair of the bra-ket-set and the bra-ket sigma-algebra, i.e., the measurable bra-ket-space

$$\langle\Omega, \mathcal{A}|\Omega, \mathcal{A}\rangle = (\langle\Omega|\Omega\rangle, \langle\mathcal{A}|\mathcal{A}\rangle),$$

is called the *(Cartesian) product of measurable spaces* $\langle\Omega, \mathcal{A}|$ and $|\Omega, \mathcal{A}\rangle$.

Definition 2 *(cross-sections of a measurable binary relation).* Let $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ be some *measurable binary relation* on $\langle\Omega|\Omega\rangle$. Let's denote

$$\mathcal{R}|_{|\omega\rangle} = \{\langle\omega^*| \in \langle\Omega| \mid \langle\omega^*|\omega\rangle \in \mathcal{R}\} \subseteq \langle\Omega|$$

the *cross-section of* $\mathcal{R}$ *by the ket-point* $|\omega\rangle \in |\Omega\rangle$ which by definition serves as a measurable subset of the bra-set $\langle\Omega|$, i.e., $\mathcal{R}|_{|\omega\rangle} \in \langle\mathcal{A}|$, and

$$\mathcal{R}|_{\langle\omega^*|} = \{|\omega\rangle \in |\Omega\rangle \mid \langle\omega^*|\omega\rangle \in \mathcal{R}\} \subseteq |\Omega\rangle$$

the *cross-section of* $\mathcal{R}$ *by the bra-point* $\langle\omega^*| \in \langle\Omega|$ which by definition serves as a measurable subset of the ket-set $|\Omega\rangle$, i.e., $\mathcal{R}|_{\langle\omega^*|} \in |\mathcal{A}\rangle$.

Complementary to the $\mathcal{R}$ *a binary relation* has a standard denotation $\mathcal{R}^c = \langle\Omega|\Omega\rangle - \mathcal{R}$, and its cross-sections have the form

$$\mathcal{R}^c|_{|\omega\rangle} = \{\langle\omega^*| \in \langle\Omega| \mid \langle\omega^*|\omega\rangle \in \mathcal{R}^c\} \subseteq \langle\Omega|\,,$$

the *cross-section of* $\mathcal{R}^c$ *by the ket-point* $|\omega\rangle \in |\Omega\rangle$, which by definition serves as a measurable subsetof the bra-set $\langle\Omega|$, i.e., $\mathcal{R}^c|_{|\omega\rangle} \in \langle\mathcal{A}|$, and

$$\mathcal{R}^c|_{\langle\omega^*|} = \{|\omega\rangle \in |\Omega\rangle \mid \langle\omega^*|\omega\rangle \in \mathcal{R}^c\} \subseteq |\Omega\rangle\,,$$

the *cross-section of* $\mathcal{R}^c$ *by the bra-point* $\langle\omega^*| \in \langle\Omega|$, which by definition serves as a measurable subset of the ket-set $|\Omega\rangle$, i.e. $\mathcal{R}^c|_{\langle\omega^*|} \in |\mathcal{A}\rangle$.

Property 1 *(duality of cross-sections). The following membership relations are equivalence:*

$$\langle\omega^*| \in \mathcal{R}|_{|\omega\rangle} \iff |\omega\rangle \in \mathcal{R}|_{\langle\omega^*|}. \tag{13}$$

Proof is obvious, since the left and the right membership relations in (13) are equivalence to the same membership relation $\langle\omega^*|\omega\rangle \in \mathcal{R}$, which follows from Definition 2.

Note 1 *(cross-sections of complementary binary relations).* From the definition of cross-sections of complementary binary relations $\mathcal{R}$ and $\mathcal{R}^c$ it follows that cross-sections by $\langle\omega^*|$ and by $|\omega\rangle$ are mutually complementary in the ket-set $|\Omega\rangle$ and in the bra-set $\langle\Omega|$ correspondingly: $\mathcal{R}|_{\langle\omega^*|} + \mathcal{R}^c|_{\langle\omega^*|} = |\Omega\rangle$ and $\mathcal{R}|_{|\omega\rangle} + \mathcal{R}^c|_{|\omega\rangle} = \langle\Omega|$.

2.1 Bra-relation and ket-relation of equivalence, generated by a binary relation

Definition 3 *(two equivalence relations, generated by a binary relation).* Each binary relation $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ deines two equivalence relations: the *bra-relation of equivalence* on $\langle\Omega|$:

$$\langle\mathcal{R}| = \left\{\langle\omega^*, \omega^{*\prime}| : \mathcal{R}|_{\langle\omega^*|} = \mathcal{R}|_{\langle\omega^{*\prime}|}\right\} \subseteq \langle\Omega| \times \langle\Omega|\,, \tag{14}$$

in other words, for $\langle\omega^*, \omega^{*\prime}| \in \langle\Omega| \times \langle\Omega|$

$$\langle\omega^*| \sim_{\langle\mathcal{R}|} \langle\omega^*| \iff \mathcal{R}|_{\langle\omega^*|} = \mathcal{R}|_{\langle\omega^*|}; \tag{15}$$

and the *ket-relation of equivalence* on $|\Omega\rangle$:

$$|\mathcal{R}\rangle = \left\{|\omega, \omega'\rangle : \mathcal{R}|_{|\omega\rangle} = \mathcal{R}|_{|\omega'\rangle}\right\} \subseteq |\Omega\rangle \times |\Omega\rangle\,, \tag{16}$$

in other words, for $|\omega, \omega'\rangle \in |\Omega\rangle \times |\Omega\rangle$

$$|\omega\rangle \sim_{|\mathcal{R}\rangle} |\omega'\rangle \iff \mathcal{R}|_{|\omega\rangle} = \mathcal{R}|_{|\omega'\rangle}; \tag{17}$$

where the following bra-ket-denotations for pairs of bra-points and ket-points are used correspondingly:

$$\begin{gathered} \langle\omega^*, \omega^{*\prime}| = (\langle\omega^*|\,, \langle\omega^{*\prime}|) \in \langle\Omega| \times \langle\Omega| = \langle\Omega, \Omega|\,, \\ |\omega, \omega'\rangle = (|\omega\rangle\,, |\omega'\rangle) \in |\Omega\rangle \times |\Omega\rangle = |\Omega, \Omega\rangle\,. \end{gathered} \tag{18}$$

Definition 4 *($\langle\mathcal{R}|$-equivalent classes).* Let $\langle\mathcal{R}| \subseteq \langle\Omega, \Omega|$ be the bra-relation of equivalence on $\langle\Omega|$. Then for any bra-point $\langle\omega^*| \in \langle\Omega|$ the $\langle\mathcal{R}|$-*equivalent class* of bra-points that $\langle\mathcal{R}|$-equivalent to $\langle\omega^*|$:

$$[\langle\omega^*|]_{\langle\mathcal{R}|} = \left\{\, \langle\omega^{*\prime}| :\ \langle\omega^{*\prime}| \sim_{\langle\mathcal{R}|} \langle\omega^*|\,\right\} = \left\{\langle\omega^{*\prime}| : \mathcal{R}_{\langle\omega^{*\prime}|} = \mathcal{R}_{\langle\omega^*|}\right\} \subseteq \langle\Omega| \tag{19}$$

is defined.

Definition 5 *($|\mathcal{R}\rangle$-equivalent classes).* Let $|\mathcal{R}\rangle \subseteq |\Omega,\Omega\rangle$ be the ket-relation of equivalence on $|\Omega\rangle$. Then for any ket-point $|\omega\rangle \in |\Omega\rangle$ the $|\mathcal{R}\rangle$-*equivalent class* ket-points that $|\mathcal{R}\rangle$-equivalent to $|\omega\rangle$

$$[|\omega\rangle]_{|\mathcal{R}\rangle} = \left\{ |\omega'\rangle : \ |\omega'\rangle \sim_{|\mathcal{R}\rangle} |\omega\rangle \right\} = \left\{ |\omega'\rangle : \mathcal{R}_{|\omega'\rangle} = \mathcal{R}_{|\omega\rangle} \right\} \subseteq |\Omega\rangle \tag{20}$$

is defined.

Definition 6 *(bra-quotient-set).* The *bra-quotient-set of the bra-set $\langle\Omega|$ by equivalence relation $\langle\mathcal{R}|$* is the set $\langle\Omega|/\langle\mathcal{R}|$ composed from $\langle\mathcal{R}|$-equivalent classes:

$$\langle\Omega|/\langle\mathcal{R}| := \left\{ [\langle\omega^*|]_{\langle\mathcal{R}|} : \ \langle\omega^*| \in \langle\Omega| \right\}. \tag{21}$$

Definition 7 *(ket-quotient-set).* The *ket-quotient-set of the ket-set $|\Omega\rangle$ by equivalence relation $|\mathcal{R}\rangle$* is the set $|\Omega\rangle/|\mathcal{R}\rangle$ composed from $|\mathcal{R}\rangle$-equivalent classes:

$$|\Omega\rangle/|\mathcal{R}\rangle := \left\{ [|\omega\rangle]_{|\mathcal{R}\rangle} : \ |\omega\rangle \in |\Omega\rangle \right\}. \tag{22}$$

It is worth emphasizing that a bra-quotient-set and a ket-quotient-set are sets of sets. Each of their elements is itself a set, i.e. $\langle\Omega|/\langle\mathcal{R}| \subseteq \mathcal{P}(\langle\Omega|)$ and $|\Omega\rangle/|\mathcal{R}\rangle \subseteq \mathcal{P}(|\Omega\rangle)$, where $\mathcal{P}(\langle\Omega|)$ and $\mathcal{P}(|\Omega\rangle)$ is the set of all subsets of the set $\langle\Omega|$ and of the set $|\Omega\rangle$ correspondingly.

Definition 8 *(bra-ket-quotient-set).* The *bra-ket-quotient-set of Cartesian product $\langle\Omega|\Omega\rangle$ by equivalence relation $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$* is the partition of Cartesian product $\langle\Omega|\Omega\rangle$ which defined as Cartesian product of quotient-set by Minkowski (Cartesian M-product):

$$\langle\Omega|\Omega\rangle/\mathcal{R} = \langle\Omega|/\langle\mathcal{R}| \ (\times) \ |\Omega\rangle/|\mathcal{R}\rangle = \left\{ [\langle\omega^*|]_{\langle\mathcal{R}|} \times [|\omega\rangle]_{|\mathcal{R}\rangle} : \ \langle\omega^*|\omega\rangle \in \langle\Omega|\Omega\rangle \right\} \subseteq \mathcal{P}(\langle\Omega|\Omega\rangle), \tag{23}$$

composed from Cartesian products of $\langle\mathcal{R}|$-equivalent and $|\mathcal{R}\rangle$-equivalent classes correspondingly.

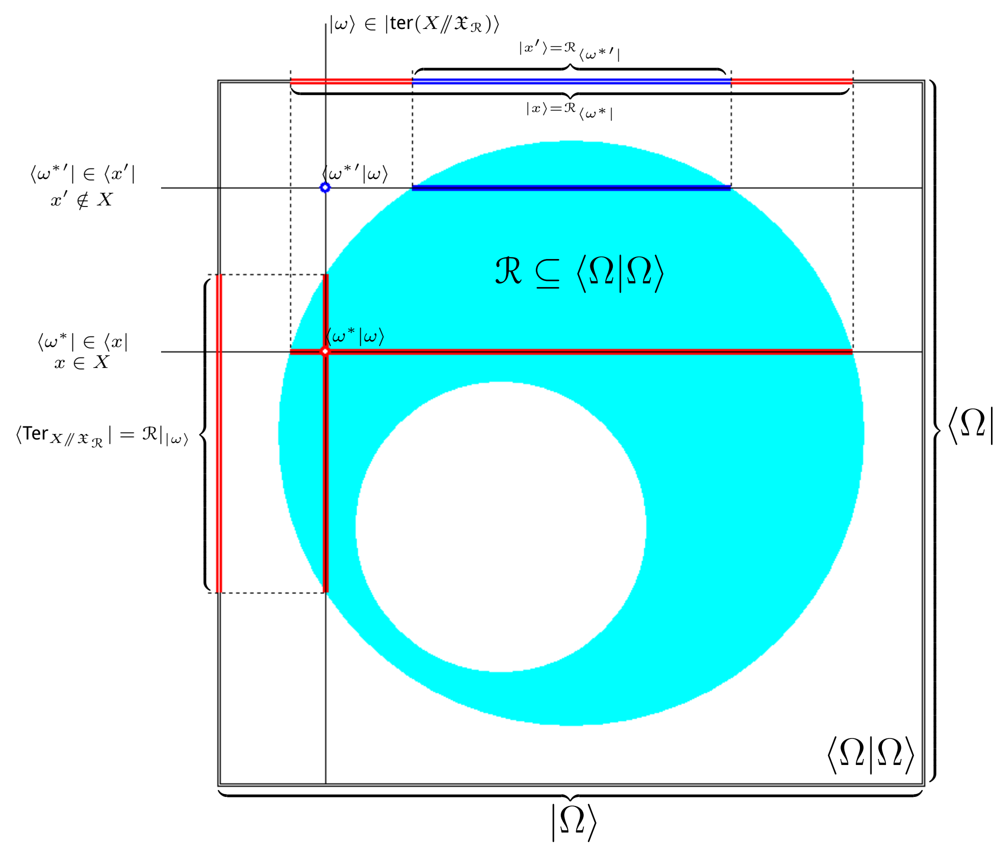

Figure 1: Venn diagram of the binary relation $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ on Cartesian product $\langle\Omega|\Omega\rangle$ with the $\mathcal{R}$-labelling $\langle\mathfrak{X}_{\mathcal{R}}|\mathfrak{2}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ and of three cross-sections of the binary relation: $\mathcal{R}|_{|\omega\rangle}, \mathcal{R}|_{\langle\omega^*|}, \mathcal{R}|_{\langle\omega^{*\prime}|}$ by the ket-point $|\omega\rangle \in |\Omega\rangle$ and the bra-points $\langle\omega^*|, \langle\omega^{*\prime}| \in \langle\Omega|$, where the following membership relations: $\langle\omega^*|\omega\rangle \in \mathcal{R} \Leftrightarrow x \in X$ and $\langle\omega^{*\prime}|\omega\rangle \notin \mathcal{R} \Leftrightarrow x' \notin X$ are equivalence.

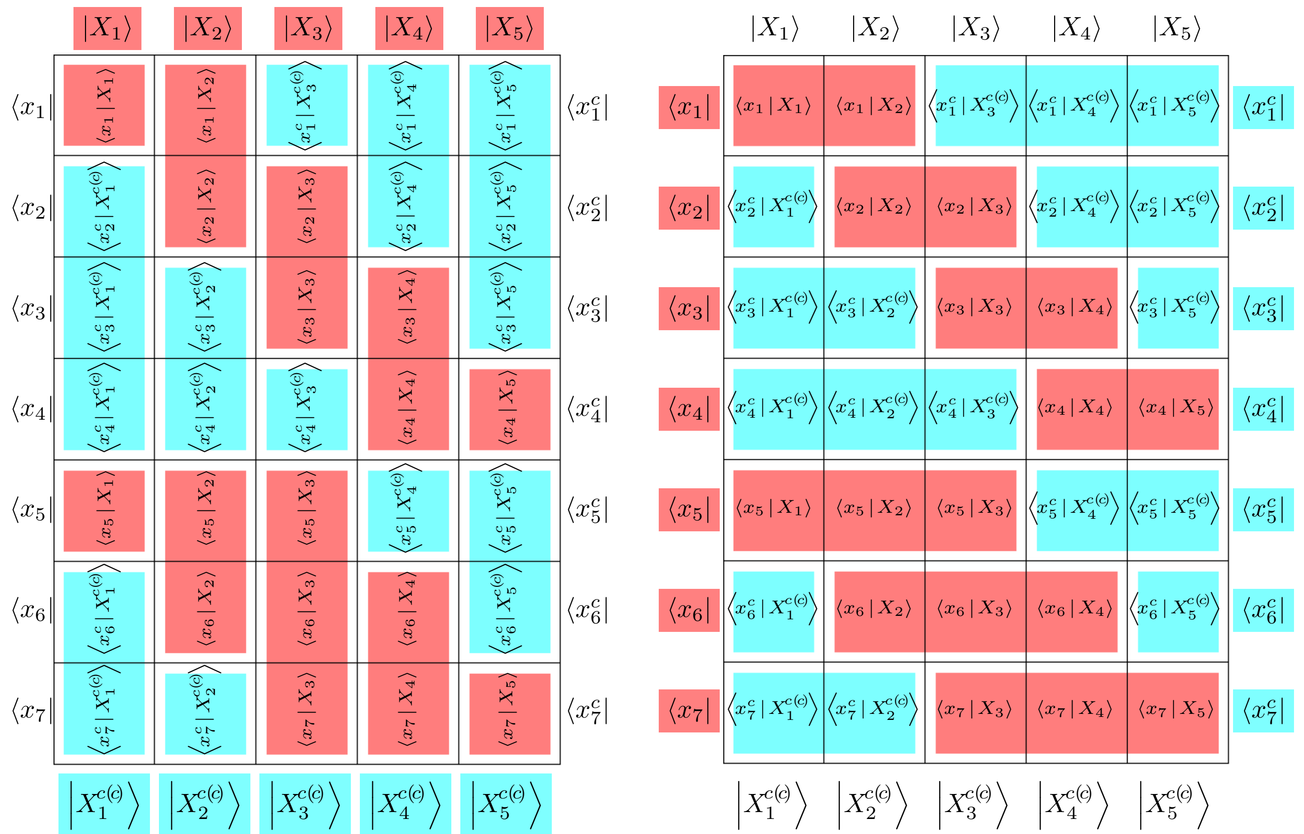

Figure 2: Bra-ket labelling the Cartesian product $\langle\Omega|\Omega\rangle$ by the binary relations $\mathcal{R}$ (red) and the complementary binary relations $\mathcal{R}^c$ (aqua); Venn diagram 7x5; terraced bra-events (red) and complementary terraced bra-events (aqua) — left, ket-events (red) and complementary ket-events (aqua) — right. Due to lack of table-space for terraced ket-events forced abbreviations: $|X\rangle = |\text{ter}(X/\!\!/\mathfrak{X})\rangle$ and $|X^{c(c)}\rangle = |\text{ter}(X^{c(c)}/\!\!/\mathfrak{X}^{(c)})\rangle$ are used here.

3 Element-set labelling by a binary relation

Element-set labelling of the Cartesian product $\langle\Omega|\Omega\rangle$ which will be discussed, is based on the curious fundamental property of measurable binary relations $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$, which, it turns out, allows each of them to define its own *element-set* $\mathcal{R}$*-labelling* of Cartesian product $\langle\Omega|\Omega\rangle$.

To see this, we will build two labellings of Cartesian product $\langle\Omega|\Omega\rangle$ by the binary relation $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ and by its complement $\mathcal{R}^c \subseteq \langle\Omega|\Omega\rangle$. Both mutually complementary labellings are the labellings of the same Cartesian product $\langle\Omega|\Omega\rangle$ with the help of elements and subsets of some labelling set and some set of its labelling subsets, which justifies their name: *element-set labellings*. The labelling set, and the set of its labelling subsets both are completely defined by these binary relations, forming the $\mathcal{R}$-stock of labels $\mathbf{\Lambda}_\mathcal{R}$ that consists from $\mathcal{R}$-labels $\lambda_\mathcal{R} \in \mathbf{\Lambda}_\mathcal{R}$; and the $\mathcal{R}^c$-stock of labels $\mathbf{\Lambda}_\mathcal{R}$, that consists from $\mathcal{R}^c$-labels $\lambda_{\mathcal{R}^c} \in \mathbf{\Lambda}_{\mathcal{R}^c}$.

So, we consider:

- ⋆ the *element-set* $\mathcal{R}$*-labelling* by binary relation $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ of Cartesian product $\langle\Omega|\Omega\rangle$ and three quotient-sets $\langle\Omega|/\mathcal{R}$, $|\Omega\rangle/\mathcal{R}$ and $\langle\Omega|\Omega\rangle/\mathcal{R}$ by this relation with the help of elements and subsets of the labelling set $\mathfrak{X}_\mathcal{R}$ and the set $\mathbf{2}^{\mathfrak{X}_\mathcal{R}} \subseteq \mathcal{P}(\mathfrak{X}_\mathcal{R})$ of its labelling subsets $X \subseteq \mathfrak{X}_\mathcal{R}$, where both sets $\mathfrak{X}_\mathcal{R}$ and $\mathbf{2}^{\mathfrak{X}_\mathcal{R}}$ are defined by relation $\mathcal{R}$ (see definitions below), and
- ⋆ the *element-set* $\mathcal{R}^c$*-labelling* by complementary binary relation $\mathcal{R}^c = \langle\Omega|\Omega\rangle - \mathcal{R}$ of the same Cartesian product $\langle\Omega|\Omega\rangle$ and three quotient-sets $\langle\Omega|/\mathcal{R}^c$, $|\Omega\rangle/\mathcal{R}^c$ and $\langle\Omega|\Omega\rangle/\mathcal{R}^c$ by this relation with the help of elements and subsets of the labelling set $\mathfrak{X}_\mathcal{R}^{(c)} = \{x^c : x \in \mathfrak{X}_\mathcal{R}\}$ and the set $\mathbf{2}^{\mathfrak{X}_\mathcal{R}^{(c)}} = \left\{X^{c(c)} : X \in \mathbf{2}^{\mathfrak{X}_\mathcal{R}}\right\} \subseteq \mathcal{P}\left(\mathfrak{X}_\mathcal{R}^{(c)}\right)$ of its labelling subsets $X^{c(c)} \subseteq \mathfrak{X}_\mathcal{R}^{(c)}$, where both sets $\mathfrak{X}_\mathcal{R}^{(c)}$ and $\mathbf{2}^{\mathfrak{X}_\mathcal{R}^{(c)}}$ are defined by relation $\mathcal{R}^c$ (see definitions below).

The following are definitions and properties that relate to the *element-set* $\mathcal{R}$ *labelling* by binary relation

$\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$, because *element-set $\mathcal{R}^c$-labelling* serves as its routine «complementary reflection», which is easy to construct by looking at the $\mathcal{R}$-labelling (see Figire 2).

3.1 Element-set labelling

The element-set $\mathcal{R}$-labelling of Cartesian product $\langle\Omega|\Omega\rangle$, which uses basic element-set labels (78) from the measurable space $(\Omega, \mathcal{A})$, begins with the construction of an element-set labelling of the measurable ket space $|\Omega, \mathcal{A}\rangle = (|\Omega\rangle, |\mathcal{A}\rangle)$ and its parts with the help of isomorphic images of parts of the measurable space $(\Omega, \mathcal{A})^3$. After defining a number of initial concepts within the ket space, it becomes possible to define the $\mathcal{R}$-labelling set $\mathfrak{X}_\mathcal{R} \subseteq \mathcal{A}$ and the set of labelling subsets $\mathfrak{Z}^{\mathfrak{X}_\mathcal{R}} \subseteq \mathcal{P}(\mathfrak{X}_\mathcal{R})$, in order to complete the construction of the element-set labelling of the ket-space $|\Omega, \mathcal{A}\rangle$, the bra-space $\langle\Omega, \mathcal{A}|$, and, finally, the bra-ket-space $\langle\Omega, \mathcal{A}|\Omega, \mathcal{A}\rangle$.

3.1.1 Element-set labelling of the ket-space

Definition 9 *(ket-points).* The *ket-points* $|\omega\rangle \in |\Omega\rangle$ are labelled by point labels $\omega \in \Omega$.
Definition 10 *(ket-set).* The *ket-set* $|\Omega\rangle$ is labelled by the label Ω and defined as a set of all labelled ket-points:

$$|\Omega\rangle = \{|\omega\rangle : \omega \in \Omega\}. \tag{24}$$

Definition 11 *(ket-subsets).* The *ket-subsets* $|x\rangle \subseteq |\Omega\rangle$ are labelled by labels $x \subseteq \Omega$ and defined as subsets, copmosed from corresponding labelled ket-points:

$$|x\rangle = \{|\omega\rangle : \omega \in x\} \subseteq |\Omega\rangle\,. \tag{25}$$

Definition 12 *(ket-sigma-algebra).* The *ket-sigma-algebra* $|\mathcal{A}\rangle$ is labelled by the label $\mathcal{A}$ and defined as a set of measurable ket-subsets $|x\rangle \subseteq |\Omega\rangle$ labelled by measurable labels $x \in \mathcal{A}$:

$$|\mathcal{A}\rangle = \{|x\rangle : x \in \mathcal{A}\}. \tag{26}$$

Definition 13 *(measurable ket-space).* The *measurable ket-space* $|\Omega, \mathcal{A}\rangle$ is considered to be a labelled by the label of measurable space $(\Omega, \mathcal{A})$ and defined as the pair $|\Omega, \mathcal{A}\rangle = (|\Omega\rangle, |\mathcal{A}\rangle)$.

3.1.2 Two basic labelling sets

Definition 14 *(basic $\mathcal{R}$-labelling set $\mathfrak{X}_\mathcal{R}$).* The *basic $\mathcal{R}$-labelling set $\mathfrak{X}_\mathcal{R} \subseteq \mathcal{A}$ of measurable subset of* Ω is defined by the binary relation $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ as the set of labels

$$\mathfrak{X}_\mathcal{R} = \left\{x \in \mathcal{A}\colon\ |x\rangle = \mathcal{R}|_{\langle\omega^*|}, \langle\omega^*| \in \langle\Omega|\right\} \subseteq \mathcal{A}, \tag{27}$$

ccomposed from measurable subsets $x \subseteq \Omega$ labelling кет-subsets $|x\rangle \subseteq |\Omega\rangle$ that serve by values of the cross-sections: $|x\rangle = \mathcal{R}|_{\langle\omega^*|} \subseteq |\Omega\rangle$ of binary relation $\mathcal{R}$ by bra-points $\langle\omega^*| \in \langle\Omega|$.

Definition 15 *(basic set $\mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}$ of $\mathcal{R}$-labelling subsets).* The *basic set $\mathfrak{Z}^{\mathfrak{X}_\mathcal{R}} \subseteq \mathcal{P}(\mathfrak{X}_\mathcal{R})$ of $\mathcal{R}$-labelling subsets of measurable subsets of* Ω is defined by the binary relation $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ as the set of set-labels

$$\mathfrak{Z}^{\mathfrak{X}_\mathcal{R}} = \left\{X \subseteq \mathfrak{X}_\mathcal{R} : \varnothing^\Omega \notin X,\ \mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R}) \neq \varnothing^\Omega\right\} \subseteq \mathcal{P}(\mathfrak{X}_\mathcal{R}), \tag{28}$$

composed only from labelling subsets $X \subseteq \mathfrak{X}_\mathcal{R}$ that do not contain the Ω-empty label: $\varnothing^\Omega \notin X$, and number the Ω-nonempty terraced labels: $\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R}) \neq \varnothing^\Omega$.

[3] I will not emphasize here that the basis of this isomorphism is the *Minkowski principle (M-principle)* of constructing an operation on sets by means of an isomorphism of operations on elements of these sets, as, for example, in the definition of the operation of *addition of sets by Minkowski* [1, 2], or when defining the *M-complement of the set* (see Footnote[2] on page 130).

3.1.3 Terraced ket-subsets and its properties

Definition 16 *(terraced ket-subsets)*. The *terraced ket-subsets* $|\text{ter}(X /\!\!/ \mathfrak{X}_\mathfrak{R})\rangle \subseteq |\Omega\rangle$ labelled by terraced labels

$$\text{ter}(X /\!\!/ \mathfrak{X}_\mathfrak{R}) = \bigcap_{x \in X} x \bigcap_{x \in \mathfrak{X}_\mathfrak{R} - X} x^c \subseteq \Omega, \tag{29}$$

aree defined in $|\Omega\rangle$ for $X \subseteq \mathfrak{X}_\mathfrak{R}$ by isomorphic formulas

$$\begin{aligned} |\text{ter}(X /\!\!/ \mathfrak{X}_\mathfrak{R})\rangle &= \left\{ |\omega\rangle \in |\Omega\rangle : \omega \in \text{ter}(X /\!\!/ \mathfrak{X}_\mathfrak{R}) \right\} \\ &= \bigcap_{x \in X} |x\rangle \bigcap_{x \in \mathfrak{X}_\mathfrak{R} - X} |x\rangle^c \subseteq |\Omega\rangle \,, \end{aligned} \tag{30}$$

where $x^c = \Omega - x$ and $|x\rangle^c = |\Omega\rangle - |x\rangle$ is the set theoretic complements till Ω and till $|\Omega\rangle$ correspondingly.

Property 2 *(terrcaed formulas of ket-inversing)*. *The terraced ket-subsets $|\text{ter}(X /\!\!/ \mathfrak{X}_\mathfrak{R})\rangle \subseteq |\Omega\rangle$ are linked in $|\Omega\rangle$ with the ket-subsets $|x\rangle \subseteq |\Omega\rangle$ for $x \in \mathfrak{X}_\mathfrak{R}$ and $X \subseteq \mathfrak{X}_\mathfrak{R}$ by terraced formulas of ket-inversing (see [3]):*

$$\begin{aligned} |\text{ter}(X /\!\!/ \mathfrak{X}_\mathfrak{R})\rangle &= \bigcap_{x \in X} |x\rangle \bigcap_{x \in \mathfrak{X}_\mathfrak{R} - X} |x\rangle^c \subseteq |\Omega\rangle \,, \\ |x\rangle &= \sum_{x \in X} |\text{ter}(X /\!\!/ \mathfrak{X}_\mathfrak{R})\rangle \subseteq |\Omega\rangle \,, \end{aligned} \tag{31}$$

where $|x\rangle^c = |\Omega\rangle - |x\rangle$ is the set theoretic complement till $|\Omega\rangle$.

Proof. The first formula of ket-inversing in (31) follows from Definition 16. The second formula follows from the first one since

$$\begin{aligned} \sum_{x \in X} |\text{ter}(X /\!\!/ \mathfrak{X}_\mathfrak{R})\rangle &= \sum_{x \in X} \left(\bigcap_{z \in X} |z\rangle \bigcap_{z \in \mathfrak{X}_\mathfrak{R} - X} |z\rangle^c \right) \\ &= |x\rangle \cap \left(\sum_{X \subseteq \mathfrak{X}_\mathfrak{R} - \{x\}} \left(\bigcap_{z \in X} |z\rangle \bigcap_{z \in (\mathfrak{X}_\mathfrak{R} - \{x\}) - X} |z\rangle^c \right) \right) \\ &= |x\rangle \cap \left(\sum_{X \subseteq \mathfrak{X}_\mathfrak{R} - \{x\}} |\text{ter}(X /\!\!/ \mathfrak{X}_\mathfrak{R} - \{x\})\rangle \right) \\ &= |x\rangle \cap |\Omega\rangle = |x\rangle \,. \end{aligned} \tag{32}$$

Property 3 *(partition of ket-subset and all ket-set)*. *The terraced ket-subsets $|\text{ter}(X /\!\!/ \mathfrak{X}_\mathfrak{R})\rangle \subseteq |\Omega\rangle$ form a partition of ket-subsets $|x\rangle \subseteq |\Omega\rangle$ for each $x \in \mathfrak{X}_\mathfrak{R}$ by formulas:*

$$|x\rangle = \sum_{x \in X \in 2^{\mathfrak{X}_\mathfrak{R}}} |\text{ter}(X /\!\!/ \mathfrak{X}_\mathfrak{R})\rangle \;\subseteq\; |\Omega\rangle \,, \tag{33}$$

in particular,

$$|\Omega\rangle = \sum_{X \in 2^{\mathfrak{X}_\mathfrak{R}}} |\text{ter}(X /\!\!/ \mathfrak{X}_\mathfrak{R})\rangle \,, \tag{34}$$

terraced ket-subsets $|\text{ter}(X /\!\!/ \mathfrak{X}_\mathfrak{R})\rangle$ for $X \in 2^{\mathfrak{X}_\mathfrak{R}}$ form a partition of all the ket-set $|\Omega\rangle$.

Proof is entirely relied on the isomorphism between the measurable ket-space $|\Omega, \mathcal{A}\rangle$ and measurable space of labels $(\Omega, \mathcal{A})$, in which partitions:

$$\begin{aligned} x &= \sum_{x \in X \subseteq \mathfrak{X}} \text{ter}(X /\!\!/ \mathfrak{X}) \;\subseteq\; \Omega, \\ \Omega &= \sum_{X \subseteq \mathfrak{X}} \text{ter}(X /\!\!/ \mathfrak{X}), \end{aligned} \tag{35}$$

isomorphic to (33) and (34) hold (see [3]) for $\mathfrak{X} \subseteq \mathcal{A}$.

3.1.4 Dual element-set labelling bra-space

The element-set labelling of measurable bra-space $\langle\Omega,\mathcal{A}| = (\langle\Omega|\,,\langle\mathcal{A}|)$ generated by the measurable binary relation$\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$, is also constructed using basic element-set labels (78) from the measurable space $(\Omega,\mathcal{A})$. However, unlike the isomorphic labelling of the ket-space $|\Omega,\mathcal{A}\rangle$, this labelling is not isomorphic to $(\Omega,\mathcal{A})$, but is its mapping, which is appropriate to call $\mathcal{R}$-dual isomorphism, and talk about *dual element-set labelling of a bra-space*.

`Definition 17` *(bra-points)*. The ***bra-points*** $\langle\omega^*|$ are labelled by point labels $\omega^* \in \Omega$.

`Definition 18` *(bra-set)*. The ***bra-set*** $\langle\Omega|$ is labelled by the label Ω and defined as the set of all labelled bra-points:

$$\langle\Omega| = \{\langle\omega^*| : \omega^* \in \Omega\}. \tag{36}$$

`Definition 19` *(bra-subset)*. The ***bra-subsets*** $\langle x| \subseteq \langle\Omega|$ are labelled by labels $x \subseteq \Omega$ and defined as the subsets composed from labelled bra-points $\langle\omega^*| \in \langle\Omega|$ the cross-sections $\mathcal{R}|_{\langle\omega^*|}$ by which coincide with the ket-subsets $|x\rangle \subseteq |\Omega\rangle$ lebelled by the same label $x \subseteq \Omega$:

$$\langle x| = \{\langle\omega^*| : \mathcal{R}|_{\langle\omega^*|} = |x\rangle\} \subseteq \langle\Omega|\,. \tag{37}$$

`Property 4` *(bra-subsets as $\langle\mathcal{R}|$-equivalent classes). Each **bra-subset** $\langle x| \in \langle\mathcal{A}|\,, x \in \mathfrak{X}_{\mathcal{R}}$, is the $\langle\mathcal{R}|$-equivalent class $[\langle\omega^*|]_{\langle\mathcal{R}|} \in \langle\Omega|/\langle\mathcal{R}|$ for some $\langle\omega^*| \in \langle\Omega|$ and each $\langle\mathcal{R}|$-equivalent class $[\langle\omega^*|]_{\langle\mathcal{R}|} \in \langle\Omega|/\langle\mathcal{R}|$ is the bra-subset $\langle x|$ for some $x \in \mathfrak{X}_{\mathcal{R}}$. In other words, the following two assertions are equivalent:*

$$\langle x| = [\langle\omega^*|]_{\langle\mathcal{R}|} \iff \langle\omega^*| \in \langle x|\,. \tag{38}$$

`Proof.` By definitions 18 and 4 the left equality in (38)

$$\{\langle\omega^*| : \mathcal{R}_{\langle\omega^*|} = |x\rangle\} = \langle x| = [\langle\omega^*|]_{\langle\mathcal{R}|} = \{\langle\omega^{*\prime}| : \mathcal{R}|_{\langle\omega^{*\prime}|} = \mathcal{R}|_{\langle\omega^*|}\} \tag{39}$$

means that $\mathcal{R}_{\langle\omega^*|} = |x\rangle$, i.e. $\langle\omega^*| \in \langle x|$. Conversely, if the membership relation on the right-hand side of (38) holds, then by the same definitions

$$\langle x| = \{\langle\omega^*| : \mathcal{R}_{\langle\omega^*|} = |x\rangle\} = \{\langle\omega^{*\prime}| : \mathcal{R}|_{\langle\omega^{*\prime}|} = \mathcal{R}|_{\langle\omega^*|}\} = [\langle\omega^*|]_{\langle\mathcal{R}|}. \tag{40}$$

since $\mathcal{R}|_{\langle\omega^{*\prime}|} = \mathcal{R}|_{\langle\omega^*|} = |x\rangle$ for $\langle\omega^{*\prime}|\,,\langle\omega^*| \in \langle x|$.

`Property 5` *(partition of a bra-set by bra-subsets). From Property 4 it immediately follows a partition of the bra-set $\langle\Omega|$ by bra-subsets:*

$$\langle\Omega| = \sum_{x\in\mathfrak{X}_{\mathcal{R}}} \langle x| \tag{41}$$

since the $\langle\mathcal{R}|$-equivalent classes form a partition of the bra-set $\langle\Omega|$.

`Definition 20` *(terraced bra-subsets). The **terraced bra-subsets*** $\langle \mathsf{Ter}_{X/\!\!/\mathfrak{X}_{\mathcal{R}}}| \subseteq \langle\Omega|$ are labelled by terraced labels

$$\mathsf{Ter}_{X/\!\!/\mathfrak{X}_{\mathcal{R}}} = \bigcup_{x\in X} x \subseteq \Omega \tag{42}$$

and defined for each set-label $X \subseteq \mathfrak{X}_{\mathcal{R}}$ as the isomorphic terraced operations

$$\langle \mathsf{Ter}_{X/\!\!/\mathfrak{X}_{\mathcal{R}}}| = \bigcup_{x\in X} \langle x| \;\subseteq\; \langle\Omega| \tag{43}$$

over the bra-subsets $\langle x| \subseteq \langle\Omega|$.

`Definition 21` *(bra-sigma-algebra). The **bra-sigma-algebra*** $\langle\mathcal{A}|$ are labelled by labels $\mathcal{A}$ and defined as the minimal sigma-algebra which cantains the sets of measurable bra-subsets $\langle x| \subseteq \langle\Omega|$ labelled by measurable labels $x \in \mathcal{A}$:

$$\langle\mathcal{A}| = \sigma\big(\{\langle x| : x \in \mathcal{A}\}\big). \tag{44}$$

`Definition 22` *(measurable bra-space). The measurable bra-space* $\langle\Omega, \mathcal{A}|$ is labelled by the label of measurable space $(\Omega, \mathcal{A})$ and defined as the pair $\langle\Omega, \mathcal{A}| = (\langle\Omega|, \langle\mathcal{A}|)$.

`Property 6` *(partition of terraced bra-subset and all bra-set). The bra-subsets* $\langle x| \subseteq \langle\Omega|$ *form a partition of terraced bra-subset* $\langle \mathsf{Ter}_{X /\!/ \mathfrak{X}_\mathcal{R}}| \subseteq \langle\Omega|$ *for each* $X \subseteq \mathfrak{X}_\mathcal{R}$ *by formulas:*

$$\langle \mathsf{Ter}_{X /\!/ \mathfrak{X}_\mathcal{R}}| = \sum_{x \in X} \langle x| \ \subseteq \ \langle\Omega|\,, \tag{45}$$

in particular,

$$\langle \mathsf{Ter}_{\mathfrak{X}_\mathcal{R} /\!/ \mathfrak{X}_\mathcal{R}}| = \sum_{x \in \mathfrak{X}_\mathcal{R}} \langle x| \ = \ \langle\Omega|\,, \tag{46}$$

the bra-subsets $\langle x|$ *for* $x \in \mathfrak{X}_\mathcal{R}$ *form a partition of all the bra-set* $\langle\Omega|$ *(see Property 5).*

`Proof.` Since the bra-subsets $\langle x| \subseteq \langle\Omega|$ are defined (38) by classes of equivalent cross-sections of the binary relation $\mathcal{R}$ they do not intersect: $\langle x| \cap \langle y| = \varnothing^{\langle\Omega|}$ for $x \neq y$ and (45) follows from (43).

`Property 7` *(terraced formulas of bra-inversing). The terraced bra-subsets* $\langle \mathsf{Ter}_{X /\!/ \mathfrak{X}_\mathcal{R}}| \subseteq \langle\Omega|$ *are linked in* $\langle\Omega|$ *with the bra-subsets* $\langle x| \subseteq \langle\Omega|$ *for* $x \in \mathfrak{X}_\mathcal{R}$ *and* $X \subseteq \mathfrak{X}_\mathcal{R}$ *terraced formulas of bra-inversing (see [3]):*

$$\begin{aligned} \langle \mathsf{Ter}_{X /\!/ \mathfrak{X}_\mathcal{R}}| &= \sum_{x \in X} \langle x| \ \subseteq \ \langle\Omega|\,, \\ \langle x| &= \bigcap_{x \in X} \langle \mathsf{Ter}_{X /\!/ \mathfrak{X}_\mathcal{R}}| \bigcap_{x \in \mathfrak{X}_\mathcal{R} - X} \left(\langle\Omega| - \langle \mathsf{Ter}_{X /\!/ \mathfrak{X}_\mathcal{R}}|\right) \subseteq \langle\Omega|\,. \end{aligned} \tag{47}$$

`Proof`. The first formula of bra-inversing in (47) follows from Property 6, the formula (45). The second one follows from the first one By virtue of Lemma 1 since the bra-subsets $\langle x| \subseteq \langle\Omega|$ for $x \in \mathfrak{X}_\mathcal{R}$ satisfy conditions of this lemma forming a partition of $\langle\Omega|$ (see the formula (46)).

`Lemma 1` *(dual formulas of terraced inversing). Let* $\mathfrak{X} \subseteq \mathcal{A}$ *be a set of measurable subsets of* Ω *forming a partition of* Ω*:*

$$\Omega = \sum_{x \in \mathfrak{X}} x, \tag{48}$$

and the X*-partial sums of these subsets have the notation for* $X \subseteq \mathfrak{X}$*:*

$$\mathsf{Ter}_{X /\!/ \mathfrak{X}_\mathcal{R}} = \sum_{x \in X} x \subseteq \Omega. \tag{49}$$

Then for $x \in \mathfrak{X}$ *the dual formulas of terraced inversing are valid:*

$$x = \bigcap_{x \in X} \mathsf{Ter}_{X /\!/ \mathfrak{X}} \bigcap_{x \in \mathfrak{X} - X} \big(\mathsf{Ter}_{X /\!/ \mathfrak{X}}\big)^c \subseteq \Omega. \tag{50}$$

`Proof.` First we note that

$$\big(\mathsf{Ter}_{X /\!/ \mathfrak{X}}\big)^c = \Omega - \mathsf{Ter}_{X /\!/ \mathfrak{X}_\mathcal{R}} = \sum_{x \in X^c} x = \mathsf{Ter}_{X^c /\!/ \mathfrak{X}_\mathcal{R}}, \tag{51}$$

where $X^c = \mathfrak{X} - X$ is the set theoretic complement till $\mathfrak{X}$. Going to complements till Ω in both parts of the equation (50) we get the equivalent formula:

$$x^c = \bigcup_{x \in X} \mathsf{Ter}_{X^c /\!/ \mathfrak{X}} \bigcup_{x \in X^c} \mathsf{Ter}_{X /\!/ \mathfrak{X}} \subseteq \Omega, \tag{52}$$

in which all partial sums $\mathsf{Ter}_{X^c /\!\!/ \mathfrak{X}_\mathcal{R}}$ and $\mathsf{Ter}_{X /\!\!/ \mathfrak{X}_\mathcal{R}}$, included in the union on the right, do not contain $x \in \mathfrak{X}$. In fact, the sum $\mathsf{Ter}_{X^c /\!\!/ \mathfrak{X}_\mathcal{R}} = \sum_{z \in X^c} z$ for $x \in X$ does not contain the term x. And the sum $\mathsf{Ter}_{X /\!\!/ \mathfrak{X}_\mathcal{R}} = \sum_{z \in X} z$ for $x \in X^c$ also does not contain the term x. Hence we get what is required:

$$\bigcup_{x \in X} \mathsf{Ter}_{X^c /\!\!/ \mathfrak{X}} \bigcup_{x \in X^c} \mathsf{Ter}_{X /\!\!/ \mathfrak{X}} = \Omega - x = x^c. \tag{53}$$

3.1.5 Terraced ket-subsets as equivalent classes and other properties of the element-set labelling by a binary relation

`Property 8`. *The following assertions are equivalent for $x \in \mathfrak{X}_\mathcal{R}$:*

$$|\omega\rangle \in |x\rangle \iff \langle x| \subseteq \mathcal{R}|_{|\omega\rangle}, \tag{54}$$
$$|\omega\rangle \notin |x\rangle \iff \langle x| \cap \mathcal{R}|_{|\omega\rangle} = \varnothing^{\langle\Omega|}. \tag{55}$$

`Proof.` The first equivalence (54). Note that by Definition 19 $|\omega\rangle \in |x\rangle$ then and only then when $|\omega\rangle \in \mathcal{R}|_{\langle\omega^*|}$ for all $\langle\omega^*| \in \langle x|$. By virtue of duality of cross-sections (see Property 1) $|\omega\rangle \in \mathcal{R}|_{\langle\omega^*|}$ for all $\langle\omega^*| \in \langle x|$ then and only then when $\langle\omega^*| \in \mathcal{R}|_{|\omega\rangle}$ for all $\langle\omega^*| \in \langle x|$. But this means that $\langle x| \subseteq \mathcal{R}|_{|\omega\rangle}$. The second equivalence (55). Note that by Definition 19 $|\omega\rangle \notin |x\rangle$ then and only then when $|\omega\rangle \notin \mathcal{R}|_{\langle\omega^*|}$ for all $\langle\omega^*| \in \langle x|$. By virtue of duality of cross-sections (see Property 1) $|\omega\rangle \notin \mathcal{R}|_{\langle\omega^*|}$ for all $\langle\omega^*| \in \langle x|$ then and only then when $\langle\omega^*| \notin \mathcal{R}|_{|\omega\rangle}$ for all $\langle\omega^*| \in \langle x|$. But this means that $\langle x| \cap \mathcal{R}|_{|\omega\rangle} = \varnothing^{\langle\Omega|}$.

`Property 9`. *The following assertions are equivalent for $x \in \mathfrak{X}$ and $X \in \mathbf{2}^{\mathfrak{X}_\mathcal{R}}$:*

$$\langle\omega^*| \in \langle x| \iff \mathcal{R}|_{\langle\omega^*|} = |x\rangle, \tag{56}$$
$$|\omega\rangle \in |\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle \iff \mathcal{R}|_{|\omega\rangle} = \langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}_\mathcal{R}}|. \tag{57}$$

`Proof.` The first equivalence (56) is valid by Definition 19. The second equivalence (57). Note that by Definition 16 $|\omega\rangle \in |\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle$ then and only then when $|\omega\rangle \in |x\rangle$ for all $x \in X$ and $|\omega\rangle \notin |x\rangle$ for all $x \notin X$. By Property 8 this means that $\langle x| \subseteq \mathcal{R}|_{|\omega\rangle}$ for all $x \in X$ and $\langle x| \cap \mathcal{R}|_{|\omega\rangle} = \varnothing^{\langle\Omega|}$ for all $x \notin X$. But this is equivalent to the equality $\mathcal{R}|_{|\omega\rangle} = \sum_{x \in X} \langle x|$ due to the fact that the bra-subsets $\langle x| \subseteq \langle\Omega|$ as $\langle\mathcal{R}|$-equivalent classes pairwise disjoint in $\langle\Omega|$. Applying Definition 20 we obtain the required equality: $\mathcal{R}|_{|\omega\rangle} = \langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}_\mathcal{R}}|$.

`Property 10` *(terraced ket-subsets as $|\mathcal{R}\rangle$-equivalent classes). Each terraced ket-subset $|\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle \in |\mathcal{A}\rangle, X \in \mathbf{2}^{\mathfrak{X}_\mathcal{R}}$ is the $|\mathcal{R}\rangle$-equivalent class $[|\omega\rangle]_{|\mathcal{R}\rangle} \in |\Omega\rangle / |\mathcal{R}\rangle$ for some $|\omega\rangle \in |\Omega\rangle$ and each $|\mathcal{R}\rangle$-equivalent class $[|\omega\rangle]_{|\mathcal{R}\rangle} \in |\Omega\rangle / |\mathcal{R}\rangle$ is the terraced ket-subset $|\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle$ for some $X \in \mathbf{2}^{\mathfrak{X}_\mathcal{R}}$. In other words, the following two assertions are equivalent:*

$$|\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle = [|\omega\rangle]_{|\mathcal{R}\rangle} \iff |\omega\rangle \in |\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle. \tag{58}$$

`Proof.` The second equivalence (57) in Property 9 defines a terraced ket-subset as a subset of ket-points:

$$|\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle = \left\{ |\omega\rangle : \mathcal{R}|_{|\omega\rangle} = \langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}_\mathcal{R}}| \right\} \subseteq |\Omega\rangle, \tag{59}$$

which coincides with the $|\mathcal{R}\rangle$-equivalent class $[|\omega\rangle]_{|\mathcal{R}\rangle}$ for any ket-point $|\omega\rangle \in |\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle$ since

$$\left\{ |\omega\rangle : \mathcal{R}|_{|\omega\rangle} = \langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}_\mathcal{R}}| \right\} = \left\{ |\omega'\rangle : \mathcal{R}|_{|\omega'\rangle} = \mathcal{R}|_{|\omega\rangle} = \langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}_\mathcal{R}}| \right\} = [|\omega\rangle]_{|\mathcal{R}\rangle}. \tag{60}$$

`Property 11` *(three partitions of binary relation in the own element-set labelling). For the binary relation $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ on Cartesian product $\langle\Omega|\Omega\rangle$ three partitions are valid: «by rows» $x \in \mathfrak{X}_\mathcal{R}, x \neq \varnothing^\Omega$, «by* *columns»* $X \in \mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}$ *and «by elements», which in its element-set* $\mathcal{R}$*-labelling have the form:*

$$\mathcal{R} = \sum_{\substack{x \in \mathfrak{X}_\mathcal{R} \\ x \neq \varnothing^\Omega}} \langle x | x \rangle \tag{61}$$

— partition «by rows» x,

$$\mathcal{R} = \sum_{X \in \mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}} \langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}_\mathcal{R}} | \mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R}) \rangle \tag{62}$$

— partition «by columns» X,

$$\mathcal{R} = \sum_{X \in \mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}} \sum_{x \in X \in \mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}} \langle x | \mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R}) \rangle \tag{63}$$

— partition «by elements»: «by columns» X and «by rows» x,

$$\mathcal{R} = \sum_{x \in \mathfrak{X}_\mathcal{R}} \sum_{x \in X \in \mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}} \langle x | \mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R}) \rangle \tag{64}$$

— partition «elements»: «by rows» x and «by columns» X.

Property 12 *(formulas of dual link between bra-subsets and ket-subsets). For each* $X \in \mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}$ the terraced bra-subset $\langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}_\mathcal{R}} | \subseteq \langle \Omega |$ *serves as general value of all equivalent cross-sections* $\mathcal{R}|_{|\omega\rangle}$ *by ket-points* $|\omega\rangle \in |\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle$ *from the terraced ket-subset* $|\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle$*, i.e. for* $X \in \mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}$ *the formula of dual link is valid*:

$$|\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle = \{ |\omega\rangle : \mathcal{R}|_{|\omega\rangle} = \langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}_\mathcal{R}} | \} \subseteq |\Omega\rangle . \tag{65}$$

For each $x \in \mathfrak{X}_\mathcal{R}$ *the ket-subset* $|x\rangle \subseteq |\Omega\rangle$ *serves as general value of all equivalent cross-sections* $\mathcal{R}|_{\langle \omega^* |}$ *by bra-points* $\langle \omega^* | \in \langle x |$ *from the bra-subset* $\langle x | \subseteq \langle \Omega |$*, i.e., for* $x \in \mathfrak{X}_\mathcal{R}$ *the formula od dual link is valid*:

$$\langle x | = \{ \langle \omega^* | : \mathcal{R}|_{\langle \omega^* |} = |x\rangle \} \subseteq \langle \Omega | . \tag{66}$$

Proof. The first formula (65) follows from (57) in Property 9 (see also (59)). The second formula (66) is correct by Definition 19.

Property 13 *(labelling the cross-sections of binary relation).*

$$\langle \omega^* | \in \langle x | \iff \mathcal{R}|_{\langle \omega^* |} = \sum_{x \in X} |\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle , \tag{67}$$

$$|\omega\rangle \in |\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle \iff \mathcal{R}|_{|\omega\rangle} = \sum_{x \in X} \langle x | . \tag{68}$$

Proof. A labelling the cross-sections of $\mathcal{R}$ by bra-points $\langle \omega^* | \in \langle x |$ is the first formula (56) in Property 9 taking into account the partition of a ket-subset by terraced ket-subsets (33). A labelling the cross-sections of $\mathcal{R}$ by ket-points $|\omega\rangle \in |\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle$ is the second formula (57) in Property 9.

Definition 23 *(element-set labelling a quotient-set by binary relation).* They say that

$$\langle \Omega | / \mathcal{R} = \langle \mathfrak{X}_\mathcal{R} | = \{ \langle x | : x \in \mathfrak{X}_\mathcal{R} \}, \tag{69}$$

a labelling the bra-quotient-set $\langle \Omega | / \mathcal{R}$ by binary relation $\mathcal{R} \subseteq \langle \Omega | \Omega \rangle$, in which labels $x \in \mathfrak{X}_\mathcal{R}$ of the labelling set $\mathfrak{X}_\mathcal{R}$ label all *bra-subsets* $\langle x | \in \langle \Omega | / \mathcal{R}$ of this quotient-set;

$$|\Omega\rangle / \mathcal{R} = \left| \mathfrak{Z}^{\mathfrak{X}_\mathcal{R}} \right\rangle = \left\{ |\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle : X \in \mathfrak{Z}^{\mathfrak{X}_\mathcal{R}} \right\} , \tag{70}$$

a labelling the ket-quotient-set $|\Omega\rangle / \mathcal{R}$ by binary relation $\mathcal{R} \subseteq \langle \Omega | \Omega \rangle$, in which set-labels $X \in \mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}$ from the set of labelling sets $\mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}$ label the *terraced ket-subsets* $|\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle \in |\Omega\rangle / \mathcal{R}$ of this quotient-set;

$$\langle \Omega | \Omega \rangle / \mathcal{R} = \left\langle \mathfrak{X}_\mathcal{R} \middle| \mathfrak{Z}^{\mathfrak{X}_\mathcal{R}} \right\rangle = \left\{ \langle x | \mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R}) \rangle : x \in \mathfrak{X}_\mathcal{R}, X \in \mathfrak{Z}^{\mathfrak{X}_\mathcal{R}} \right\} , \tag{71}$$

a labelling the bra-ket-quotient-set $\langle\Omega|\Omega\rangle/\mathcal{R}$ by binary relation $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$, in which pairs (x, X), where $x \in \mathfrak{X}_\mathcal{R}$ is an element of the labelling set $\mathfrak{X}_\mathcal{R}$, and $X \in \mathbf{2}^{\mathfrak{X}_\mathcal{R}}$ is the element from the set $\mathbf{2}^{\mathfrak{X}_\mathcal{R}}$ of labelling subsets, label all the *bra-ket-subsets* $\langle x|\mathsf{ter}(X/\!\!/\mathfrak{X}_\mathcal{R})\rangle \in \langle\Omega|\Omega\rangle/\mathcal{R}$ of this quotient-set.

`Predefinition 2` *(ℛ-labelling by basic element-set labels).* $\mathcal{R}$-labelled by *basic element-set labels* $\lambda \in \mathbf{\Lambda}$ parts of bra-space and ket-space are supplied with general notation $\langle\lambda_\mathcal{R}|$ and $|\lambda_\mathcal{R}\rangle$, a list of which can be found in Appendix on page 122.

3.1.6 Measurable binary relation as a membership relation

`Theorem 1` *(measurable binary relation as a membership relation). Any measurable binary relation $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ on Cartesian product $\langle\Omega|\Omega\rangle$ is equivalent to the membership relation*

$$\mathcal{R}_{\langle\mathfrak{X}_\mathcal{R}|\mathbf{2}^{\mathfrak{X}_\mathcal{R}}\rangle} = \Big\{ \langle x|\mathsf{ter}(X/\!\!/\mathfrak{X}_\mathcal{R})\rangle : x \in X \Big\} \subseteq \Big\langle \mathfrak{X}_\mathcal{R} \Big| \mathbf{2}^{\mathfrak{X}_\mathcal{R}} \Big\rangle \tag{72}$$

in the element-set $\mathcal{R}$-labelling $\Big\langle \mathfrak{X}_\mathcal{R} \Big| \mathbf{2}^{\mathfrak{X}_\mathcal{R}} \Big\rangle$ of the quotient-set $\langle\Omega|\Omega\rangle/\mathcal{R}$. In other words,

$$\mathcal{R} = \Big\{ \langle\omega^*|\omega\rangle \in \langle\Omega|\Omega\rangle : \ \langle\omega^*|\omega\rangle \in \langle x|\mathsf{ter}(X/\!\!/\mathfrak{X}_\mathcal{R})\rangle \in \mathcal{R}_{\langle\mathfrak{X}_\mathcal{R}|\mathbf{2}^{\mathfrak{X}_\mathcal{R}}\rangle} \Big\} \subseteq \langle\Omega|\Omega\rangle\,. \tag{73}$$

`Proof` is based on the equivalence

$$\langle x|\mathsf{ter}(X/\!\!/\mathfrak{X}_\mathcal{R})\rangle \subseteq \mathcal{R} \quad \Longleftrightarrow \quad x \in X, \tag{74}$$

from which it follows that the membership relation $\langle\omega^*|\omega\rangle \in \mathcal{R}$ is equivalent to the fulfillment of two membership relations: $\langle\omega^*|\omega\rangle \in \langle x|\mathsf{ter}(X/\!\!/\mathfrak{X}_\mathcal{R})\rangle$ and $x \in X$ which proves the theorem.

4 Appendix

`List 1` *(basic element-set labels). The basic element-set labels* $\lambda \in \mathbf{\Lambda}$, or simply *basic labels*, are defined as the following elements, sets, and sets of subsets of the measurable space $(\Omega, \mathcal{A})$, and also as *terraced set theoretic operations*[4] over them, are equipped with their own names[5]:

$$\lambda = \begin{cases} \Omega & \textit{label of a set}, \\ \mathcal{A} & \textit{label of a sigma-algebra}, \\ (\Omega, \mathcal{A}) & \textit{label of a measurable space}, \\ \mathfrak{X} \subseteq \mathcal{A} & \textit{set-label}, \\ \mathfrak{S}^{\mathfrak{X}} \subseteq \mathcal{P}(\mathfrak{X}) & \textit{set of set-labels}, \\ \omega, \omega^* \in \Omega, & \textit{point label}, \\ x \in \mathfrak{X},\ x \subseteq \Omega & \textit{label}, \\ x^c \in \mathfrak{X}^{(c)},\ x^c \subseteq \Omega & \textit{label}, \\ \varnothing^{\Omega} \subseteq \Omega & \Omega\textit{-empty label}, \\ X \subseteq \mathfrak{X} & \textit{set-label}, \\ X^{c(c)} \subseteq \mathfrak{X}^{(c)} & \textit{set-label}, \\ \emptyset^{\mathfrak{X}} \subseteq \mathfrak{X} & \Omega\textit{-empty set-label}, \\ \mathsf{ter}(X /\!\!/ \mathfrak{X}) = \bigcap_{x \in X} x \bigcap_{x \in \mathfrak{X} - X} x^c \subseteq \Omega & \textit{terraced label}, \\ \mathsf{ter}\left(X^{c(c)} /\!\!/ \mathfrak{X}^{(c)}\right) = \mathsf{ter}(X /\!\!/ \mathfrak{X}) \subseteq \Omega & \textit{terraced label}, \\ \mathsf{Ter}_{X /\!\!/ \mathfrak{X}} = \bigcup_{x \in X} x \subseteq \Omega & \textit{terraced label}, \\ \mathsf{Ter}_{X^{c(c)} /\!\!/ \mathfrak{X}^{(c)}} = \bigcup_{x \in \mathfrak{X} - X} x^c \subseteq \Omega & \textit{terraced label}. \end{cases} \tag{75}$$

taking into account the definition of terraced sets and the validity of terraced equalities in Ω (see [3]):

$$\begin{aligned} \mathsf{ter}\left(X^{c(c)} /\!\!/ \mathfrak{X}^{(c)}\right) &= \bigcap_{x^c \in X^{c(c)}} x^c \bigcap_{x^c \in \mathfrak{X}^{(c)} - X^{c(c)}} x = \bigcap_{x \in X} x \bigcap_{x \in \mathfrak{X} - X} x^c = \mathsf{ter}(X /\!\!/ \mathfrak{X}) \subseteq \Omega, \\ \mathsf{Ter}_{X^{c(c)} /\!\!/ \mathfrak{X}^{(c)}} &= \bigcup_{x^c \in X^{c(c)}} x^c = \bigcup_{x \in \mathfrak{X} - X} x^c \subseteq \Omega. \end{aligned} \tag{76}$$

`List 2` *($\mathcal{R}$-labelling by basic element-set labels).* $\mathcal{R}$-labelled by *basic element-set labels* $\lambda \in \mathbf{\Lambda}$ parts of the bra-space and the ket-space, is equipped by the following denotations:

$$\langle \lambda_{\mathcal{R}} | = \begin{cases} \langle \Omega | & \textit{bra-set}, \\ \langle \mathcal{A} | & \textit{bra-sigma-algebra}, \\ \langle \Omega, \mathcal{A} | & \textit{measurable bra-space}, \\ \langle \mathfrak{X}_{\mathcal{R}} | & \textit{bra-quotient-set}, \\ \langle \omega^* | \in \langle \Omega | & \textit{bra-point}, \\ \langle x | \subseteq \langle \Omega |, x \in \mathfrak{X}_{\mathcal{R}} & \textit{bra-subset}, \\ \langle x^c | \subseteq \langle \Omega |, x \in \mathfrak{X}_{\mathcal{R}} & \textit{complementary bra-subset}, \\ \langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}_{\mathcal{R}}} | = \sum_{x \in X} \langle x | \subseteq \langle \Omega |, X \in \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} & \textit{terraced bra-subset}, \\ \langle \mathsf{Ter}_{X^{c(c)} /\!\!/ \mathfrak{X}^{(c)}_{\mathcal{R}}} | = \langle \Omega | - \langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}_{\mathcal{R}}} | \subseteq \langle \Omega |, X \in \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} & \textit{complementary terraced bra-subset}; \end{cases} \tag{77}$$

[4]In [3, 2007] the *terraced set theoretic operation* of the 1-st type is defined as the subset $\mathsf{ter}(X /\!\!/ \mathfrak{X}) = \bigcap_{x \in X} x \bigcap_{x \in \mathfrak{X} - X} (\Omega - x) \subseteq \Omega$, and the *terraced set theoretic operation* of the 5-th type as the subset $\mathsf{Ter}_{X /\!\!/ \mathfrak{X}} = \bigcup_{x \in X} x \subseteq \Omega$.

[5]where $X^{c(c)} = (\mathfrak{X} - X)^{(c)} = \{x^c : x \in \mathfrak{X} - X\}$ is an M-complementary subset $X^c = \mathfrak{X} - X$.

$$|\lambda_{\mathcal{R}}\rangle = \begin{cases} |\Omega\rangle & \textit{ket-set}, \\ |\mathcal{A}\rangle & \textit{ket-sigma-algebra}, \\ |\Omega, \mathcal{A}\rangle & \textit{measurable ket-space}, \\ |\mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}}\rangle & \textit{ket-quotient-set}, \\ |\omega\rangle \in |\Omega\rangle\,, & \textit{ket-point}, \\ |x\rangle \subseteq |\Omega\rangle\,, x \in \mathfrak{X}_{\mathcal{R}} & \textit{ket-subset}, \\ |x^c\rangle \subseteq |\Omega\rangle\,, x \in \mathfrak{X}_{\mathcal{R}} & \textit{complementary ket-subset}, \\ |\mathsf{ter}(X /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle = \bigcap\limits_{x \in X} |x\rangle \bigcap\limits_{x \in \mathfrak{X}_{\mathcal{R}} - X} |x\rangle^c \subseteq \Omega, X \in \mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}} & \textit{terraced ket-subset}, \\ |\mathsf{ter}\left(X^{c(c)} /\!\!/ \mathfrak{X}_{\mathcal{R}}^{(c)}\right)\rangle = |\mathsf{ter}(X /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle \subseteq |\Omega\rangle\,, X \in \mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}} & \textit{complementary terraced ket-subset}. \end{cases} \tag{78}$$

`List 3` *($\mathcal{R}$-labelling the bra-ket-set and its parts).* Each measurable binary relation $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ characterizes a *element-set labelling* the bra-ket-set $\langle\Omega|\Omega\rangle$ and its parts which are defined and called by the following way:
1) the *bra-point* $\langle\omega^*| \in \langle\Omega|$ and the *ket-point* $|\omega\rangle \in |\Omega\rangle$, labelled by labels $\omega^* \in \Omega$ and $\omega \in \Omega$;
2) the *quotient-sets*:

$$\begin{aligned} \langle\Omega|/\mathcal{R} &= \langle\mathfrak{X}_{\mathcal{R}}| = \{\langle x| : x \in \mathfrak{X}_{\mathcal{R}}\}, \\ |\Omega\rangle/\mathcal{R} &= \left|\mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}}\right\rangle = \left\{|\mathsf{ter}(X /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle : X \in \mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}}\right\}, \\ \langle\Omega|\Omega\rangle/\mathcal{R} &= \left\langle \mathfrak{X}_{\mathcal{R}} \,\middle|\, \mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}} \right\rangle = \langle\mathfrak{X}_{\mathcal{R}}| \times |\mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}}\rangle = \left\{\langle x|\mathsf{ter}(X /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle : x \in \mathfrak{X}_{\mathcal{R}},\ X \in \mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}}\right\}; \end{aligned} \tag{79}$$

the *complementary quotient-sets*:

$$\begin{aligned} \langle\Omega|/\mathcal{R}^c &= \langle\mathfrak{X}_{\mathcal{R}}^{(c)}| = \{\langle x^c| : x \in \mathfrak{X}_{\mathcal{R}}\}, \\ |\Omega\rangle/\mathcal{R}^c &= \left|\mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}^{(c)}}\right\rangle = \left\{|\mathsf{ter}(X^{c(c)} /\!\!/ \mathfrak{X}_{\mathcal{R}}^{(c)})\rangle : X \in \mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}}\right\}, \\ \langle\Omega|\Omega\rangle/\mathcal{R}^c &= \left\langle \mathfrak{X}_{\mathcal{R}}^{(c)} \,\middle|\, \mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}^{(c)}} \right\rangle = \langle\mathfrak{X}_{\mathcal{R}}^{(c)}| \times |\mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}^{(c)}}\rangle = \left\{\langle x^c|\mathsf{ter}(X^{c(c)} /\!\!/ \mathfrak{X}_{\mathcal{R}}^{(c)})\rangle : x \in \mathfrak{X}_{\mathcal{R}},\ X \in \mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}}\right\}; \end{aligned} \tag{80}$$

3) the *bra-subset*

$$\langle x| = \{\langle\omega^*| : \mathcal{R}|_{\langle\omega^*|} = |x\rangle\} \tag{81}$$

consists from bra-points $\langle\omega^*| \in \langle\Omega|$, the cross-sections $\mathcal{R}$ by which is equivalent to the ket-subset $|x\rangle \subseteq |\Omega\rangle$ that is labelled by the same $\mathcal{R}$-labels $x \in \mathfrak{X}_{\mathcal{R}}$;
4) the *terraced ket-subset*

$$|\mathsf{ter}(X /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle = \{|\omega\rangle : \mathcal{R}|_{|\omega\rangle} = \langle\mathsf{Ter}_{X /\!\!/ \mathfrak{X}_{\mathcal{R}}}|\} \tag{82}$$

consists from ket-points $|\omega\rangle \in |\Omega\rangle$, the cross-sections $\mathcal{R}$ by which are equivalent to the ket-subset $\langle\mathsf{Ter}_{X /\!\!/ \mathfrak{X}_{\mathcal{R}}}| \subseteq \langle\Omega|$ that labelled by the same $\mathcal{R}$-labels $X \in \mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}}$.
5) the *complementary bra-subset*

$$\langle x^c| = \{\langle\omega^*| : \mathcal{R}^c|_{\langle\omega^*|} = |x^c\rangle\} \tag{83}$$

consists from bra-points $\langle\omega^*| \in \langle\Omega|$, the cross-sections $\mathcal{R}^c$ by which are equivalent to the ket-subset $|x^c\rangle \subseteq |\Omega\rangle$ that $\mathcal{R}^c$-labelled by the same labels $x \in \mathfrak{X}_{\mathcal{R}}$;
6) the *complementary terraced ket-subset*

$$|\mathsf{ter}(X^{c(c)} /\!\!/ \mathfrak{X}_{\mathcal{R}}^{(c)})\rangle = \{|\omega\rangle : \mathcal{R}^c|_{|\omega\rangle} = \langle\mathsf{Ter}_{X^{c(c)} /\!\!/ \mathfrak{X}_{\mathcal{R}}^{(c)}}|\} \tag{84}$$

consists from ket-points $|\omega\rangle \in |\Omega\rangle$, the cross-sections $\mathcal{R}^c$ by which are equivalent to the bra-subset $\langle\mathsf{Ter}_{X^{c(c)} /\!\!/ \mathfrak{X}_{\mathcal{R}}^{(c)}}|$ that $\mathcal{R}^c$-labelled by the same label $X \in \mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}}$.
7) the *binary quotient-relation* $\mathcal{R}_{\langle\mathfrak{X}_{\mathcal{R}}|\mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}}\rangle} \subseteq \langle\Omega|\Omega\rangle/\mathcal{R}$ *is the binary relation*

$$\mathcal{R}_{\langle\mathfrak{X}_{\mathcal{R}}|\mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}}\rangle} = \left\{\langle x|\mathsf{ter}(X /\!\!/ \mathfrak{X})\rangle : x \in X;\ x \in \mathfrak{X}_{\mathcal{R}},\ X \in \mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}}\right\} \subseteq \langle\mathfrak{X}_{\mathcal{R}}|\mathbf{\mathfrak{z}}^{\mathfrak{X}_{\mathcal{R}}}\rangle \tag{85}$$

on the quotient-set $\langle \mathfrak{X}_{\mathcal{R}} | \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} \rangle = \langle \Omega | \Omega \rangle / \mathcal{R}$, which is equivalent to the membership relation «$x \in X$» on $\mathfrak{X}_{\mathcal{R}} \times \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}$.

8) the *complementary binary quotient-relation* $\mathcal{R}^c_{\langle \mathfrak{X}_{\mathcal{R}} | \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} \rangle} \subseteq \langle \Omega | \Omega \rangle / \mathcal{R}^c$ *is the binary relation*

$$\mathcal{R}^c_{\langle \mathfrak{X}_{\mathcal{R}} | \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} \rangle} = \left\{ \langle x^c | \, |\mathsf{ter}(X^{c(c)} /\!\!/ \mathfrak{X}^{(c)}_{\mathcal{R}}) \rangle \rangle : x^c \in X^{c(c)};\, x \in \mathfrak{X}_{\mathcal{R}},\, X \in \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}_{\mathcal{R}} \right\} \subseteq \langle \mathfrak{X}_{\mathcal{R}} | \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} \rangle \tag{86}$$

on the quotient-set $\langle \mathfrak{X}_{\mathcal{R}} | \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} \rangle = \langle \Omega | \Omega \rangle / \mathcal{R}^c$, which is equivalent to the membership relation «$x^c \in X^{c(c)}$» on $\mathfrak{X}_{\mathcal{R}} \times \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}$.

`List 4` *`(general denotations of` $\mathcal{R}$`-labelled subsets).`* The measurable binary relation $\mathcal{R} \subseteq \langle \Omega | \Omega \rangle$ generates the $\mathcal{R}$-labelling of subsets from the bra-ket-set $\langle \Omega | \Omega \rangle$, the bra-set $\langle \Omega |$, and the ket-set $\langle \Omega |$. Some of these subsets, and also some *Cartesian products* of these subsets have a common name: *$\mathcal{R}$-labelled subsets*, and corresponding common denotations:

$$\langle \lambda_{\mathcal{R}} | = \begin{cases} \langle x | \,, & x \in \mathfrak{X}_{\mathcal{R}}, \\ \langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}_{\mathcal{R}}} | \,, & X \in \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}, \end{cases}$$

$$| \lambda_{\mathcal{R}} \rangle = \begin{cases} | x \rangle \,, & x \in \mathfrak{X}_{\mathcal{R}}, \\ | \mathsf{ter}(X /\!\!/ \mathfrak{X}_{\mathcal{R}}) \rangle \,, & X \in \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}, \end{cases} \tag{87}$$

$$\langle \lambda_{\mathcal{R}} | \lambda'_{\mathcal{R}} \rangle = \langle x | \mathsf{ter}(X /\!\!/ \mathfrak{X}_{\mathcal{R}}) \rangle \,,\; x \in \mathfrak{X}_{\mathcal{R}},\, X \in \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}$$

They say that the $\mathcal{R}$-labelled *by the same label* $\lambda_{\mathcal{R}} \in \mathbf{\Lambda}$ bra-subset $\langle \lambda_{\mathcal{R}} |$ and the ket-subset $| \lambda_{\mathcal{R}} \rangle$, and also $\mathcal{R}$-labelled by terraced labels $\lambda_{\mathcal{R}} \in \mathbf{\Lambda}$ that numbered *by the same set-label*, the terraced bra-subset $\langle \lambda_{\mathcal{R}} |$ and the terraced ket-subset $| \lambda_{\mathcal{R}} \rangle$ are *bra-ket-dual to each other* and form the *pair of bra-ket-dual subsets* in the form of Cartesian product

$$\langle \lambda_{\mathcal{R}} | \lambda_{\mathcal{R}} \rangle = \langle \lambda_{\mathcal{R}} | \times | \lambda_{\mathcal{R}} \rangle = \begin{cases} \langle x | x \rangle \,, & x \in \mathfrak{X}_{\mathcal{R}}, \\ \langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}_{\mathcal{R}}} | \mathsf{ter}(X /\!\!/ \mathfrak{X}_{\mathcal{R}}) \rangle \,, & X \in \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}. \end{cases} \tag{88}$$

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