The theory of dual co event means

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Abstract. *This work is the third, but not the last, in the cycle begun by the works [23, 22] about the new theory of experience and chance as the theory of co$\sim$events. Here I introduce the concepts of two co$\sim$event means, which serve as dual co$\sim$event characteristics of some co$\sim$event. The very idea of dual co$\sim$event means, has become the development of two concepts: mean-measure set [16] and mean-probable event [20, 24], which were first introduced as two independent characteristics of the set of events, so that then, within the framework of the theory of experience and of chance, the idea can finally get the opportunity to appear as two dual faces of the same co$\sim$event. I must admit that, precisely, this idea, hopelessly long and lonely stood at the sources of an indecently long string of guesses and insights, did not tire of looming, beckoning to the new co$\sim$event description of the dual nature of uncertainty, which I called the theory of experience and chance, or the certainty theory. The constructive final push to the idea of dual co$\sim$event means has become two surprisingly suitable examples, with which I was fortunate to get acquainted in 2015, each of which is based on the statistics of the experienced-random experiment in the form of a co$\sim$event.*

Keywords. *Eventology, theory of experience and chance, event, co$\sim$event, experience, chance, to happen, to experience, to occur, probability, believability, mean-believable (mean-experienced) terraced bra-event, mean-probable (mean-possible) ket-event, mean-believable-probability (mean-experienced-possible) co$\sim$event, experienced-random experiment, dual event means, dual co$\sim$event means, bra-menas, ket-means, Bayesian analysis, approval voting, forest approval voting.*

1 Introduction

This work is the third, but not the last, in the cycle begun by the works [23, 22] about the new theory of experience and chance as the theory of *co$\sim$events*[1]. Here I introduce the concepts of two co$\sim$event means, which serve as dual event characteristics of some co$\sim$event. The very idea of *dual co$\sim$event means*, has become the development of two concepts: *mean-measure set* [16][2] and *mean-probable event* [20, 24], which were first introduced as two independent characteristics of the set of events, so that then, within the framework of the theory of experience and chance, the idea can finally get the opportunity to appear as two dual faces of the same co$\sim$event. I must admit that, precisely, this idea, hopelessly long and lonely stood at the sources of an indecently long string of guesses and insights, did not tire of looming, beckoning to the new co$\sim$event description of the dual nature of uncertainty, which I called the *theory of experience and chance*, or the *certainty theory*. The constructive final push to the idea of dual co$\sim$event means has become two surprisingly suitable examples, with which I was fortunate to get acquainted in 2015, each of which gives a statistics of the results of the experienced-random experiment in the form of a co$\sim$event.

2 Mean characteristics of a set of events in eventology

In [16], as well as in [1, p. 644], you can find the definition of the concept of the *mean-measure set*, which was first introduced by me in 1973, and published in [21, 1975], and [15, 1977]. *The mean-measure set*

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[1]A «co$\sim$event», a derivative of an «event», is a new English word that corresponds to the Russian term «so$\sim$bytie», derived from «sobytie», that signifies co-being, coexistence (about the Russian term «so-bytie» see also [9, p. 25] and [3].)

[2]See my primary sources are [15, 21], and also links to «Vorobyev's expectation» in [14, 11, 8].

is the mean-set characteristic of a *random set* whose values are subsets of a measurable space with a measure. This characteristic plays the same role for a random set as for a *random element* with values from a linear space[3] is the *expectation*, or *mean value*. There are two concepts existing in the eventology [18, 2007]: *mean-measure set of events* and *mean-probable event* [20, 24, 2012] is the result of applying the idea of defining the mean-measure set within two of different measurable spaces with measures: *mean-probable event* is the mean-set characteristic of ***events, as measurable subsets of the space of elementary outcomes***, and *mean-measure set of events* [16] is the mean-set characteristic of the ***measurable subsets of events occurred from a given set of events***.

In this section, the old notation is used, which was usually used within the framework of the probabilistic [10], as well as the eventologic paradigm [18] before postulating the theory of experience and chance as a theory of co∼events [22]:

$$
\begin{array}{rl}
(\Omega, \mathcal{A}, \mathbf{P}) & \text{— the probability space,} \\
\Omega & \text{— the space of elementary outcomes } \omega \in \Omega, \\
\mathcal{A} & \text{— the sigma-algebra of events } x \subseteq \Omega, \\
\mathbf{P} & \text{— the probability measure on } \mathcal{A}, \\
\mathfrak{X} \subset \mathcal{A} & \text{— the (finite) set of events } x \in \mathfrak{X}, \\
(\mathfrak{X}, \mathcal{A}^{\mathfrak{X}}, \mathbf{B}) & \text{— the measurable space with the measure } \mathbf{B}\text{, normalized to unity,} \\
\mathcal{A}^{\mathfrak{X}} & \text{— the sigma-algebra of subsets } X \subseteq \mathfrak{X} \text{ of events } x \in \mathfrak{X}, \\
\mathfrak{Z}^{\mathfrak{X}} \subseteq \mathcal{A}^{\mathfrak{X}} & \text{— a set of some } \mathcal{A}^{\mathfrak{X}}\text{-measurable subsets } X \subseteq \mathfrak{X}, \\
\mathbf{B} & \text{— the measure on } \mathcal{A}^{\mathfrak{X}}\text{, normalized to unity}^{a}.
\end{array}
\tag{1}
$$

[a]For the finite set $\mathfrak{X} \subset \mathcal{A}$ the measure $\mathbf{B}$ may be, in particular, proportional to the power of subsets: $\mathbf{B}(X) = |X|/|\mathfrak{X}|$, $X \in \mathcal{A}^{\mathfrak{X}}$, including, for example, $\mathbf{B}(x) = \mathbf{B}(\{x\}) = 1/|\mathfrak{X}|$, $x \in \mathfrak{X}$.

2.1 Mean-measure set of events

In eventology, each *set of events* $\mathfrak{X} \subset \mathcal{A}$ uniquely relates the concept of a *random element as a random set of events*

$$K_{\mathfrak{X}} : \Omega \to \mathfrak{Z}^{\mathfrak{X}}$$

defined on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ with values

$$K_{\mathfrak{X}}(\omega) = \{x \in \mathfrak{X} : \omega \in x\} \in \mathfrak{Z}^{\mathfrak{X}}$$

from the area $\mathfrak{Z}^{\mathfrak{X}} \subseteq \mathcal{A}^{\mathfrak{X}} \subseteq \mathcal{P}(\mathfrak{X})$, that is contained in the sigma-algebra of a measurable space $(\mathfrak{X}, \mathcal{A}^{\mathfrak{X}}, \mathbf{B})$ with the measure $\mathbf{B}$ normalized to unity. The value $K_{\mathfrak{X}}(\omega)$ of the random set of events $K_{\mathfrak{X}}$ on the $\omega \in \Omega$ is interpreted as a subset $X(\omega) = \{x \in \mathfrak{X} \colon \omega \in x\} \in \mathfrak{Z}^{\mathfrak{X}}$ consisting only of those events $x \in \mathfrak{X}$ that happens when the elementary outcome $\omega \in \Omega$ happens.

The random set of events $K_{\mathfrak{X}}$ is defined by the family $\left\{p(X /\!\!/ \mathfrak{X}), X \in \mathfrak{Z}^{\mathfrak{X}}\right\}$ of probabilities

$$p(X /\!\!/ \mathfrak{X}) = \mathbf{P}(\{\omega \colon K_{\mathfrak{X}}(\omega) = X\}) = \mathbf{P}(\mathsf{ter}(X /\!\!/ \mathfrak{X}))$$

of terraced events

$$\mathsf{ter}(X /\!\!/ \mathfrak{X}) = \bigcap_{x \in X} x \bigcap_{x \in \mathfrak{X} - X} x^c \subseteq \Omega,$$

that form a partition, generated by $\mathfrak{X}$, of the space of elementary outcomes Ω:

$$\Omega = \sum_{X \in \mathfrak{Z}^{\mathfrak{X}}} \mathsf{ter}(X /\!\!/ \mathfrak{X}).$$

For the *random set of events* $K_{\mathfrak{X}}$ literally the same as in the general case for a *random set* of arbitrary points [16, 1] the following concept is defined.

`Definition 1` *(mean-measure set of events).* Let

$$\mathcal{E}_\alpha K_{\mathfrak{X}} = \{x \in \mathfrak{X} : \mathbf{P}(x) > \alpha\} \subseteq \{x \in \mathfrak{X} : \mathbf{P}(x) \geqslant \alpha\} = \mathcal{E}^\alpha K_{\mathfrak{X}} \in \mathcal{A}^{\mathfrak{X}},$$

[3]For a random variable, a vector, a matrix, a function, etc.

then the *mean-measure set of events* for the random set of events $K_{\mathfrak{X}}$ is any set of events $\mathcal{E}K_{\mathfrak{X}} \in \mathcal{A}^{\mathfrak{X}}$ that satisfies two inclusion relations:

$$\mathcal{E}_\alpha K_{\mathfrak{X}} \subset \mathcal{E}K_{\mathfrak{X}} \subseteq \mathcal{E}^\alpha K_{\mathfrak{X}}$$

for some level $\alpha \in [0,1]$ such that the approximate equality[4] $\mathbf{B}(\mathcal{E}K_{\mathfrak{X}}) \approx \mathbf{E_P}(\mathbf{B}(K_{\mathfrak{X}}))$ is performed with the smallest error, which will be briefly denoted below: $\mathbf{B}(\mathcal{E}K_{\mathfrak{X}}) \underset{\min}{\approx} \mathbf{E_P}(\mathbf{B}(K_{\mathfrak{X}}))$.

In other words, one of the two equalities holds:

$$\mathcal{E}K_{\mathfrak{X}} = \begin{cases} \mathcal{E}_\alpha K_{\mathfrak{X}}, & \mathbf{E_P}(\mathbf{B}(K_{\mathfrak{X}})) - \mathbf{B}(\mathcal{E}_\alpha K_{\mathfrak{X}}) < \mathbf{B}(\mathcal{E}^\alpha K_{\mathfrak{X}}) - \mathbf{E_P}(\mathbf{B}(K_{\mathfrak{X}})), \\ \mathcal{E}^\alpha K_{\mathfrak{X}}, & \mathbf{E_P}(\mathbf{B}(K_{\mathfrak{X}})) - \mathbf{B}(\mathcal{E}_\alpha K_{\mathfrak{X}}) \geqslant \mathbf{B}(\mathcal{E}^\alpha K_{\mathfrak{X}}) - \mathbf{E_P}(\mathbf{B}(K_{\mathfrak{X}})), \end{cases}$$

or:

$$\mathcal{E}K_{\mathfrak{X}} = \begin{cases} \mathcal{E}_\alpha K_{\mathfrak{X}}, & \mathbf{E_P}(\mathbf{B}(K_{\mathfrak{X}})) - \mathbf{B}(\mathcal{E}_\alpha K_{\mathfrak{X}}) \leqslant \mathbf{B}(\mathcal{E}^\alpha K_{\mathfrak{X}}) - \mathbf{E_P}(\mathbf{B}(K_{\mathfrak{X}})), \\ \mathcal{E}^\alpha K_{\mathfrak{X}}, & \mathbf{E_P}(\mathbf{B}(K_{\mathfrak{X}})) - \mathbf{B}(\mathcal{E}_\alpha K_{\mathfrak{X}}) > \mathbf{B}(\mathcal{E}^\alpha K_{\mathfrak{X}}) - \mathbf{E_P}(\mathbf{B}(K_{\mathfrak{X}})). \end{cases}$$

Lemma 1 *(extremal properties of mean-measure set of events).* The *mean-measure set of events* $\mathcal{E}K_{\mathfrak{X}}$ *minimizes the mean distance* $\mathbf{E_P}\rho(K_{\mathfrak{X}}, X) = \mathbf{E_P}(\mathbf{B}(K_{\mathfrak{X}} \,\Delta\, X))$, *mean measure* $\mathbf{B}$ *of symmetrical difference*[5]*:*

$$\mathbf{E_P}\rho(K_{\mathfrak{X}}, \mathcal{E}K_{\mathfrak{X}}) = \min_{\substack{X \in \mathcal{A}^{\mathfrak{X}} \\ \mathbf{B}(X) \underset{\min}{\approx} \mathbf{E_P}(\mathbf{B}(K_{\mathfrak{X}}))}} \mathbf{E_P}\rho(K_{\mathfrak{X}}, X),$$

between the random element $K_{\mathfrak{X}}$ *and those subsets of events* $X \in \mathcal{A}^{\mathfrak{X}}$ *for which the approximate equality of their measure* $\mathbf{B}(X)$ *and the mean measure* $\mathbf{E_P}(\mathbf{B}(K_{\mathfrak{X}}))$ *is performed with the least error.*

Proof of the lemma does not differ from a proof of analogous statements about extremal properties of *Vorobyev's mean* for random finite sets [16], [21] or for random closed sets [14], [11], [8], and others.

2.2 Mean-probable event

Based on the idea of [16, 1] already used in the definition of the *mean-measure set* (see the previous paragraph), the eventology defines [20, 24] *mean-probable event* playing the role of mean-set characteristic of events $x \in \mathfrak{X}$ ***as subsets of*** Ω. In the same way as a *mean-measure set* [16] plays the role of the mean-set characteristic of the values of the random element $K_{\mathfrak{X}}$ ***as subsets of*** $\mathfrak{X}$.

Definition 2 *(mean-weighted probability of events from a set of events).* Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, and $(\mathfrak{X}, \mathcal{A}^{\mathfrak{X}}, \mathbf{B})$ be a measurable space with the measure $\mathbf{B}$ normalized to unity. For the set of events $\mathfrak{X} \subseteq \mathcal{A}$ a *mean-weighted by the measure* $\mathbf{B}$ *probability* $\mathbf{P}$ *of events from* $\mathfrak{X}$ is defined by formula:

$$\widehat{\mathbf{P}}_{\mathfrak{X}} = \sum_{x \in \mathfrak{X}} \mathbf{P}(x)\mathbf{B}(x). \tag{2}$$

Definition 3 *(mean-probable event).* Let

$$\widehat{x}^{\mathfrak{X}}_\alpha = \sum_{X\,:\,\mathbf{B}(X) > \pi} \mathsf{ter}(X /\!\!/ \mathfrak{X}) \;\subseteq\; \sum_{X\,:\,\mathbf{B}(X) \geqslant \pi} \mathsf{ter}(X /\!\!/ \mathfrak{X}) = \widehat{x}^\alpha_{\mathfrak{X}} \;\in\; \mathcal{A}^{\mathfrak{X}}, \tag{3}$$

then the *mean-probable event* for the set of events $\mathfrak{X} \subset \mathcal{A}$ is any event $\widehat{x}_{\mathfrak{X}} \in \mathcal{A}^{\mathfrak{X}}$ which satisfies two inclusions:

$$\widehat{x}^{\mathfrak{X}}_\alpha \subset \widehat{x}_{\mathfrak{X}} \subseteq \widehat{x}^\alpha_{\mathfrak{X}}, \tag{4}$$

for some level $\beta \in [0,1]$ such that approximate equality $\mathbf{P}\left(\widehat{x}_{\mathfrak{X}}\right) \approx \widehat{\mathbf{P}}_{\mathfrak{X}}$ holds with the least error which will be briefly denoted as $\mathbf{P}\left(\widehat{x}_{\mathfrak{X}}\right) \underset{\min}{\approx} \widehat{\mathbf{P}}_{\mathfrak{X}}$. In other words, the *mean-probable event* happens with a probability that differs from mean-weighted by $\mathbf{B}$ probability $\mathbf{P}_{\mathfrak{X}}$ of events $x \in \mathfrak{X}$ (3) with the smallest error.

[4] Here $\mathbf{E_P}(\mathbf{B}(K_{\mathfrak{X}})) = \sum_{X \in \mathbf{2}^{\mathfrak{X}}} \mathbf{B}(X)p(X /\!\!/ \mathfrak{X})$ is an expectation of r.v. $\mathbf{B}(K_{\mathfrak{X}})$ by the probability measure $\mathbf{P}$ that for the finite set $\mathfrak{X} \subset \mathcal{A}$ is defined by this formula, but may be also calculated by the Robbins theorem [12]: $\mathbf{E_P}(\mathbf{B}(K_{\mathfrak{X}})) = \sum_{x \in \mathfrak{X}} \mathbf{P}(x)\mathbf{B}(x)$.

[5] $K_{\mathfrak{X}} \,\Delta\, X = K_{\mathfrak{X}} \cap X^c + (K_{\mathfrak{X}})^c \cap X = K_{\mathfrak{X}} \cap (\mathfrak{X} - X) + (\mathfrak{X} - K_{\mathfrak{X}}) \cap X$.

In other words, one of the two equalities holds:

$$\widehat{x}_{\mathfrak{X}} = \begin{cases} \widehat{x}^{\mathfrak{X}}_{\alpha}, & \widehat{\mathbf{P}}_{\mathfrak{X}} - \mathbf{P}\left(\widehat{x}^{\mathfrak{X}}_{\alpha}\right) < \mathbf{P}\left(\widehat{x}^{\alpha}_{\mathfrak{X}}\right) - \widehat{\mathbf{P}}_{\mathfrak{X}} \\ \widehat{x}^{\alpha}_{\mathfrak{X}}, & \widehat{\mathbf{P}}_{\mathfrak{X}} - \mathbf{P}\left(\widehat{x}^{\mathfrak{X}}_{\alpha}\right) \geqslant \mathbf{P}\left(\widehat{x}^{\alpha}_{\mathfrak{X}}\right) - \widehat{\mathbf{P}}_{\mathfrak{X}}, \end{cases} \tag{5}$$

or:

$$\widehat{x}_{\mathfrak{X}} = \begin{cases} \widehat{x}^{\mathfrak{X}}_{\alpha}, & \widehat{\mathbf{P}}_{\mathfrak{X}} - \mathbf{P}\left(\widehat{x}^{\mathfrak{X}}_{\alpha}\right) \leqslant \mathbf{P}\left(\widehat{x}^{\alpha}_{\mathfrak{X}}\right) - \widehat{\mathbf{P}}_{\mathfrak{X}} \\ \widehat{x}^{\alpha}_{\mathfrak{X}}, & \widehat{\mathbf{P}}_{\mathfrak{X}} - \mathbf{P}\left(\widehat{x}^{\mathfrak{X}}_{\alpha}\right) > \mathbf{P}\left(\widehat{x}^{\alpha}_{\mathfrak{X}}\right) - \widehat{\mathbf{P}}_{\mathfrak{X}}. \end{cases} \tag{6}$$

`Definition 4` *`(probabilistic distance of an event till a set of events).`* *A probabilistic distance of an event $\lambda \in \mathcal{A}$ till a set of events $\mathfrak{X} \subset \mathcal{A}$* is defined by formaula

$$\rho(\lambda, \mathfrak{X}) = \sum_{x \in \mathfrak{X}} \mathbf{P}(\lambda \,\Delta\, x)\mathbf{B}(x), \tag{7}$$

as a mean-weighted by $\mathbf{B}$ probability of symmetric differences of events $x \in \mathfrak{X}$ and the event $\lambda \in \mathcal{A}$.

`Lemma 2` *`(extremal properties of the mean-probable event).`* The *mean-probable event $\widehat{x}_{\mathfrak{X}}$ for the set of events $\mathfrak{X}$ minimizes the probabilistic distance till $\mathfrak{X}$:*

$$\rho(\widehat{x}_{\mathfrak{X}}, \mathfrak{X}) = \min_{\substack{\lambda \in \mathcal{A}^{\mathfrak{X}} \\ \mathbf{P}(\lambda) \underset{\min}{\approx} \widehat{\mathbf{P}}_{\mathfrak{X}}}} \rho(\lambda, \mathfrak{X}) \tag{8}$$

among such events $\lambda \in \mathcal{A}^{\mathfrak{X}}$ that occur with probability $\mathbf{P}(\lambda)$ which differs from mean-weighted by $\mathbf{B}$ probability $\widehat{\mathbf{P}}_{\mathfrak{X}}$ of events from $\mathfrak{X}$ (3) with the smallest error.

Proof of the lemma does not differ from a proof of analogous statements about extremal properties of *Vorobyev's mean* for random finite sets [16], [21] or for random closed sets [14], [11], [8], and others.

Probabilistic-eventological paradigm	**Paradigm of co∼event $\mathcal{R} \subseteq \langle\Omega\|\Omega\rangle$ (co∼event paradigm)**	
	Beleivability bra-space $\langle\Omega, \mathcal{A}, \mathbf{B}\|$	**Probability ket-space $\|\Omega, \mathcal{A}, \mathbf{B}\rangle$**
$(\Omega, \mathcal{A}, \mathbf{P})$ — probability space		$\|\Omega, \mathcal{A}, \mathbf{P}\rangle$ — probability ket-space
Ω — space of elementary outcomes $\omega \in \Omega$	$\langle\Omega\|$ — space of elementary bra-incomes $\langle\omega\| \in \langle\Omega\|$	$\|\Omega\rangle$ — space of elementary ket-outcomes $\|\omega\rangle \in \|\Omega\rangle$
$\mathcal{A}$ — sigma-algebra of events $x \subseteq \Omega$	$\langle\mathcal{A}\|$ — sigma-algebra of bra-events $\langle x\| \subseteq \langle\Omega\|$	$\|\mathcal{A}\rangle$ — sigma-algebra of ket-events $\|x\rangle \subseteq \|\Omega\rangle$
$\mathbf{P}$ — probability measure on $\mathcal{A}$		$\mathbf{P}$ — probability measure on $\|\mathcal{A}\rangle$
$(\mathfrak{X}, \mathcal{A}^{\mathfrak{X}}, \mathbf{B})$ — measurable space with the measure $\mathbf{B}$	$\langle\Omega, \mathcal{A}, \mathbf{B}\|$ — believability bra-space	
$\mathfrak{X}$ — set of events $x \in \mathfrak{X} \subseteq \mathcal{A}$	$\mathfrak{X}_{\mathcal{R}}$ — $\mathcal{R}$-labelling set of labels $x \in \mathfrak{X}_{\mathcal{R}} \subseteq \mathcal{A}_{\mathfrak{X}_{\mathcal{R}}} \subseteq \mathcal{A}$, $\langle\mathfrak{X}_{\mathcal{R}}\|$ — set of bra-events $\langle x\| \in \langle\mathfrak{X}_{\mathcal{R}}\| \subseteq \langle\mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}\| \subseteq \langle\mathcal{A}\|$, $\langle\mathfrak{X}_{\mathcal{R}}\|$ — $\mathcal{R}$-labelling of bra-quotient-set $\langle\Omega\|/\mathcal{R}$	
$\mathcal{A}_{\mathfrak{X}}$ — sigma-algebra of events $x \in \mathfrak{X} \subseteq \mathcal{A}_{\mathfrak{X}} \subseteq \mathcal{A}$	$\mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}$ — sigma-algebra of labels $x \in \mathfrak{X}_{\mathcal{R}} \subseteq \mathcal{A}_{\mathfrak{X}_{\mathcal{R}}} \subseteq \mathcal{A}$ $\langle\mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}\|$ — sigma-algebra of bra-events $\langle x\| \in \langle\mathfrak{X}_{\mathcal{R}}\| \subseteq \langle\mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}\| \subseteq \langle\mathcal{A}\|$	
$\mathfrak{S}^{\mathfrak{X}}$ — set of subsets of events $X \subseteq \mathfrak{X}$ such that $X \in \mathfrak{S}^{\mathfrak{X}} \subseteq \mathcal{A}^{\mathfrak{X}} \subseteq \mathcal{A}$		$\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}$ — set of $\mathcal{R}$-labelling set labels $X \subseteq \mathfrak{X}_{\mathcal{R}}$ such that $X \in \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} \subseteq \mathcal{A}^{\mathfrak{X}_{\mathcal{R}}} \subseteq \mathcal{A}$ $\|\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ — set of terraced events $\|\mathrm{ter}(X /\!\!/ \mathfrak{X})\rangle \in \|\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}\rangle \subseteq \|\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}\rangle \subseteq \|\mathcal{A}\rangle$, $\|\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ — $\mathcal{R}$-labelling of ket-quotient-set $\|\Omega\rangle/\mathcal{R}$
$\mathcal{A}^{\mathfrak{X}}$ — sigma-algebra of subsets $X \subseteq \mathfrak{X}$ of events $x \in \mathfrak{X} \subseteq \mathcal{A}$		$\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}$ — sigma-algebra of $\mathcal{R}$-labelling set labels $X \subseteq \mathfrak{X}_{\mathcal{R}}$ such that $X \in \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} \subseteq \mathcal{A}^{\mathfrak{X}_{\mathcal{R}}} \subseteq \mathcal{A}$ $\|\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ — sigma-algebra of terraced ket-events $\|\mathrm{ter}(X /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle \in \|\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}\rangle \subseteq \|\mathcal{A}\rangle$
$\mathbf{B}$ — measure on $\mathcal{A}^{\mathfrak{X}}$ normalized to unity	$\mathbf{B}$ — believability measure on $\langle\mathcal{A}\|$	

Table 1: Probabilistic-eventological and co∼event paradigms.

3 At the border of paradigms

I led you to the frontier where the probabilistic-eventological paradigm borders on co$\sim$event paradigm. We will be delayed for a short time on the border itself to set forth the previous paragraph this time[6] in the language of the *theory of experience and chance* [22], and to more clearly see the origins of this new theory in terms of the *mean-measure set of events* and the *mean-probable event*. It was these two notions of the mean-set characteristics of the set of events originated within the probability theory [10] and the eventology theory [18], and after demonstrating the duality properties to each other, pushed me to construct the *theory of experience and chance* as a *theory of co$\sim$event*.

3.1 Random bra-element

The *set of bra-events* $\langle\mathfrak{X}| \subset \langle\mathcal{A}|$ and the set of terraced bra-events $\langle\mathbf{2}^{\mathfrak{X}}| = \left\{\langle\mathsf{Ter}_{X/\!\!/\mathfrak{X}}| : X \in \mathbf{2}^{\mathfrak{X}}\right\} \subseteq \langle\mathcal{A}_{\mathfrak{X}}| \subseteq \langle\mathcal{A}|$ are uniquely associated with the notion of *random bra-element*

$$\langle K_{\mathfrak{X}}| : |\Omega\rangle \to \langle\mathbf{2}^{\mathfrak{X}}|\,,$$

defined on the probability space $|\Omega, \mathcal{A}, \mathbf{P}\rangle$. On the elementary ket-outcome $|\omega\rangle \in |\mathsf{ter}(X/\!\!/\mathfrak{X})\rangle\,, X \in \mathbf{2}^{\mathfrak{X}}$ this random bra-element takes the value

$$\langle K_{\mathfrak{X}}|\,(|\omega\rangle) = \langle\mathsf{Ter}_{X/\!\!/\mathfrak{X}}| \in \langle\mathbf{2}^{\mathfrak{X}}|$$

from a bra-area $\langle\mathbf{2}^{\mathfrak{X}}|$ that is contained in the sigma-algebra $\langle\mathcal{A}_{\mathfrak{X}}| \subseteq \langle\mathcal{A}|$ of believability bra-space $\langle\Omega, \mathcal{A}, \mathbf{B}|$ generated by bra-events $\langle x| \in \langle\mathfrak{X}_{\mathfrak{R}}| \subseteq \langle\mathcal{A}_{\mathfrak{X}}| \subseteq \langle\mathcal{A}|$. Its value $\langle K_{\mathfrak{X}}|\,(|\omega\rangle)$ is interpreted as a terraced bra-event $\langle\mathsf{Ter}_{X/\!\!/\mathfrak{X}}| \in \langle\mathbf{2}^{\mathfrak{X}}|$ that is experienced with believability

$$b(X/\!\!/\mathfrak{X}) = \mathbf{B}(\langle\mathsf{Ter}_{X/\!\!/\mathfrak{X}}|) = \sum_{x \in X} B(\langle x|) = \sum_{x \in X} b_x, \tag{9}$$

when the terraced ket-event $|\mathsf{ter}(X/\!\!/\mathfrak{X})\rangle$ happens, i.e. the elementary outcome $|\omega\rangle \in |\mathsf{ter}(X/\!\!/\mathfrak{X})\rangle$ happens with probability

$$p(X/\!\!/\mathfrak{X}) = \mathbf{P}(|\mathsf{ter}(X/\!\!/\mathfrak{X})\rangle). \tag{10}$$

The *random bra-element* $\langle K_{\mathfrak{X}}|$ is defined by

1) a family $\left\{p(X/\!\!/\mathfrak{X}), X \in \mathbf{2}^{\mathfrak{X}}\right\}$ of probabilities $p(X/\!\!/\mathfrak{X}) = \mathbf{P}(|\mathsf{ter}(X/\!\!/\mathfrak{X})\rangle)$ of terraced ket-events

$$|\mathsf{ter}(X/\!\!/\mathfrak{X})\rangle = \bigcap_{x \in X} |x\rangle \bigcap_{x \in \mathfrak{X}-X} |x\rangle^c \subseteq |\Omega\rangle\,,$$

that form partition, generated by $|\mathfrak{X}\rangle$, of the space of elementary ket-outcomes $|\Omega\rangle$:

$$|\Omega\rangle = \sum_{X \in \mathbf{2}^{\mathfrak{X}}} |\mathsf{ter}(X/\!\!/\mathfrak{X})\rangle\,;$$

2) a family $\langle\mathbf{2}^{\mathfrak{X}}| = \left\{\langle\mathsf{Ter}_{X/\!\!/\mathfrak{X}}| : X \in \mathbf{2}^{\mathfrak{X}}\right\}$ of its values, terraced bra-events

$$\langle\mathsf{Ter}_{X/\!\!/\mathfrak{X}}| = \sum_{x \in X} \langle x| \subseteq \langle\Omega|\,,$$

that is experienced with belieavbility $b(X/\!\!/\mathfrak{X}) = \mathbf{B}(\langle\mathsf{Ter}_{X/\!\!/\mathfrak{X}}|)$ and defined as sums of bra-events $\langle x| \in \langle\mathfrak{X}|$ forming a partition of the space of elementary bra-incomes $\langle\Omega|$:

$$\langle\Omega| = \sum_{x \in \mathfrak{X}} \langle x|\,.$$

[6] I'm afraid I will have to repeat this exposition for the third time already within the framework of the co$\sim$event paradigm (see paragraph 6.1 on page 166) to make the necessary improvements to the means definitions.

3.2 Mean-believable terraced bra-event

For the *random bra-element* $\langle K_{\mathfrak{X}}|$ literally the same as for a *random set* of arbitrary elements [16, 1], and for a *random set of events* [19, 24] I define the *mean-believable terraced bra-event* as the terraced bra-event, which is denoted by

$$\mathcal{E}\langle K_{\mathfrak{X}}| \in \langle \mathcal{A}_{\mathfrak{X}}|\,,$$

is experienced with believability $\mathbf{B}\left(\mathcal{E}\langle K_{\mathfrak{X}}|\right)$ that differs of mean-probable believability

$$\mathbf{E}_{\mathbf{P}}(\mathbf{B}(\langle K_{\mathfrak{X}}|)) = \sum_{X \in \mathbf{2}^{\mathfrak{X}}} \mathbf{B}(\langle \mathsf{Ter}_{X/\!\!/\mathfrak{X}}|)\mathbf{P}(\langle \mathsf{ter}(X/\!\!/\mathfrak{X})|) = \sum_{X \in \mathbf{2}^{\mathfrak{X}}} b(X/\!\!/\mathfrak{X})p(X/\!\!/\mathfrak{X}) \tag{11}$$

of terraced bra-events $\langle \mathsf{Ter}_{X/\!\!/\mathfrak{X}}| \in \langle \mathbf{2}^{\mathfrak{X}}|$ with the least error, and plays the role of its mean-set characteristics ***as bra-subsets*** $\langle \mathsf{Ter}_{X/\!\!/\mathfrak{X}}| \subseteq \langle \Omega|$.

`Definition 5` *(mean-believable terraced bra-event).* Let

$$\mathcal{E}_{\alpha}\langle K_{\mathfrak{X}}| = \{\langle x| : \mathbf{P}(|x\rangle) > \alpha\} \;\subseteq\; \{\langle x| : \mathbf{P}(|x\rangle) \geqslant \alpha\} = \mathcal{E}^{\alpha}\langle K_{\mathfrak{X}}| \;\in\; \langle \mathcal{A}_{\mathfrak{X}}|\,,$$

then the *mean-believable terraced bra-event* of the random bra-element $\langle K_{\mathfrak{X}}|$ is any terraced bra-event $\mathcal{E}\langle K_{\mathfrak{X}}| \in \langle \mathcal{A}_{\mathfrak{X}}|$ that holds two inclusions:

$$\mathcal{E}_{\alpha}\langle K_{\mathfrak{X}}| \;\subset\; \mathcal{E}\langle K_{\mathfrak{X}}| \;\subseteq\; \mathcal{E}^{\alpha}\langle K_{\mathfrak{X}}|\,,$$

for some level $\alpha \in [0, 1]$ such that approximate equality

$$\mathbf{B}(\mathcal{E}\langle K_{\mathfrak{X}}|) \approx \mathbf{E}_{\mathbf{P}}(\mathbf{B}(\langle K_{\mathfrak{X}}|)) \tag{12}$$

is satisfied with the smallest error. This will be briefly denoted below

$$\mathbf{B}(\mathcal{E}\langle K_{\mathfrak{X}}|) \underset{\min}{\approx} \mathbf{E}_{\mathbf{P}}(\mathbf{B}(\langle K_{\mathfrak{X}}|)). \tag{13}$$

As a result, we get the *medium-tied terraced bra-event* that is experienced with believability differing from the mean-probable believability $\mathbf{E}_{\mathbf{P}}(\mathbf{B}(\langle K_{\mathfrak{X}}|))$ of the random bra-element $\langle K_{\mathfrak{X}}|$ with the least error.

In other words, one of the two equalities holds:

$$\mathcal{E}\langle K_{\mathfrak{X}}| = \begin{cases} \mathcal{E}_{\alpha}\langle K_{\mathfrak{X}}|\,, & \mathbf{E}_{\mathbf{P}}(\mathbf{B}(\langle K_{\mathfrak{X}}|)) - \mathbf{B}(\mathcal{E}_{\alpha}\langle K_{\mathfrak{X}}|) < \mathbf{B}(\mathcal{E}^{\alpha}\langle K_{\mathfrak{X}}|) - \mathbf{E}_{\mathbf{P}}(\mathbf{B}(\langle K_{\mathfrak{X}}|)), \\ \mathcal{E}^{\alpha}\langle K_{\mathfrak{X}}|\,, & \mathbf{E}_{\mathbf{P}}(\mathbf{B}(\langle K_{\mathfrak{X}}|)) - \mathbf{B}(\mathcal{E}_{\alpha}\langle K_{\mathfrak{X}}|) \geqslant \mathbf{B}(\mathcal{E}^{\alpha}\langle K_{\mathfrak{X}}|) - \mathbf{E}_{\mathbf{P}}(\mathbf{B}(\langle K_{\mathfrak{X}}|)), \end{cases}$$

or:

$$\mathcal{E}\langle K_{\mathfrak{X}}| = \begin{cases} \mathcal{E}_{\alpha}\langle K_{\mathfrak{X}}|\,, & \mathbf{E}_{\mathbf{P}}(\mathbf{B}(\langle K_{\mathfrak{X}}|)) - \mathbf{B}(\mathcal{E}_{\alpha}\langle K_{\mathfrak{X}}|) \leqslant \mathbf{B}(\mathcal{E}^{\alpha}\langle K_{\mathfrak{X}}|) - \mathbf{E}_{\mathbf{P}}(\mathbf{B}(\langle K_{\mathfrak{X}}|)), \\ \mathcal{E}^{\alpha}\langle K_{\mathfrak{X}}|\,, & \mathbf{E}_{\mathbf{P}}(\mathbf{B}(\langle K_{\mathfrak{X}}|)) - \mathbf{B}(\mathcal{E}_{\alpha}\langle K_{\mathfrak{X}}|) > \mathbf{B}(\mathcal{E}^{\alpha}\langle K_{\mathfrak{X}}|) - \mathbf{E}_{\mathbf{P}}(\mathbf{B}(\langle K_{\mathfrak{X}}|)). \end{cases}$$

`Definition 6` *(believability distance).* The *believability distance* of the terraced bra-event $\langle \mathsf{Ter}_{W/\!\!/\mathfrak{X}}| \in \langle \mathbf{2}^{\mathfrak{X}}|$ till the random bra-element $\langle K_{\mathfrak{X}}|$ is a value

$$\mathbf{E}_{\mathbf{P}}\Big(\mathbf{B}\big(\,\langle K_{\mathfrak{X}}| \,\Delta\, \langle \mathsf{Ter}_{W/\!\!/\mathfrak{X}}|\,\big)\Big) = \sum_{X \in \mathbf{2}^{\mathfrak{X}}} \mathbf{B}(\langle \mathsf{Ter}_{X/\!\!/\mathfrak{X}}| \,\Delta\, \langle \mathsf{Ter}_{W/\!\!/\mathfrak{X}}|)\mathbf{P}(|\mathsf{ter}(X/\!\!/\mathfrak{X})\rangle),$$

mean-probable believability of its symmetrical difference.

`Theorem 1` *(extremal properties of mean-believable terraced bra-event).* The *mean-believable terraced bra-event $\mathcal{E}\langle K_{\mathfrak{X}}|$ of the random bra-element $\langle K_{\mathfrak{X}}|$ minimizes its believability distance (mean-probable believability of symmetrical difference)*

$$\mathbf{E}_{\mathbf{P}}(\mathbf{B}(\langle K_{\mathfrak{X}}| \,\Delta\, \mathcal{E}\langle K_{\mathfrak{X}}|)) = \min_{X \,:\, b(X/\!\!/\mathfrak{X}) \underset{\min}{\approx} \mathbf{E}_{\mathbf{P}}(\mathbf{B}(\langle K_{\mathfrak{X}}|))} \mathbf{E}_{\mathbf{P}}(\mathbf{B}(\langle K_{\mathfrak{X}}| \,\Delta\, \langle \mathsf{Ter}_{X/\!\!/\mathfrak{X}}|))$$

till the random bra-element $\langle K_{\mathfrak{X}}|$ among those terraced bra-events $\langle \mathsf{Ter}_{X/\!\!/\mathfrak{X}}| \in \langle \mathcal{A}_{\mathfrak{X}}|$ for which the approximate equality (13) holds with smallest error.

Proof differs from the proof of Lemma 1 only by denotations.

3.3 Experienced ket-element

The measure is uniquely connected with the concept

The *set of ket-events* $|\mathfrak{X}\rangle \subset |\mathcal{A}\rangle$ is uniquely connected with the notion of *experienced ket-element*

$$|K_{\mathfrak{X}}\rangle : \langle\Omega| \to |\mathfrak{X}\rangle \, ,$$

defined on the believability bra-space $\langle\Omega, \mathcal{A}, \mathbf{P}|$. On the elementary bra-income $\langle\omega| \in \langle x| \, , x \in \mathfrak{X}$ this experienced ket-element takes a value

$$|K_{\mathfrak{X}}\rangle \left(\langle\omega|\right) = |x\rangle \in |\mathfrak{X}\rangle$$

from the ket-area $|\mathfrak{X}\rangle$ that is contained in the sigma-algebra $|\mathcal{A}^{\mathfrak{X}}\rangle \subseteq |\mathcal{A}\rangle$ of the probability space $|\Omega, \mathcal{A}, \mathbf{B}\rangle$ geberated by terraced ket-events $|\mathsf{ter}(X /\!\!/ \mathfrak{X})\rangle \in |\mathcal{A}^{\mathfrak{X}}\rangle \subseteq |\mathcal{A}\rangle$. Its value $|K_{\mathfrak{X}}\rangle \left(\langle\omega|\right)$ is interpreted as the ket-event $|x\rangle \in |\mathfrak{X}\rangle$ that *happens* with probability

$$p_x = \mathbf{P}(|x\rangle) = \sum_{x \in X} \mathbf{P}(|\mathsf{ter}(X /\!\!/ \mathfrak{X})\rangle) = \sum_{x \in X} p(X /\!\!/ \mathfrak{X}), \tag{14}$$

and causes the bra-event $\langle x|$ *to be experienced*, i.e., causes all elementary bra-incomes $\langle\omega| \in \langle x|$ *to be experienced* with believability

$$b_x = \mathbf{B}(\langle x|). \tag{15}$$

The *experienced ket-element* $|K_{\mathfrak{X}}\rangle$ is defined by

1) a family $\{b_x, x \in \mathfrak{X}\}$ of believabilities $b_x = \mathbf{B}(\langle x|)$ of bra-events $\langle x| \subseteq \langle\Omega|$ that form a partition, generated by $\langle\mathfrak{X}|$, of the space of elementary bra-incomes $\langle\Omega|$:

$$\langle\Omega| = \sum_{x \in \mathfrak{X}} \langle x| \, ;$$

2) a family $|\mathfrak{X}\rangle = \{|x\rangle : x \in \mathfrak{X}\}$ of its values, ket-events

$$|x\rangle = \sum_{x \in X} |\mathsf{ter}(X /\!\!/ \mathfrak{X})\rangle \subseteq |\Omega\rangle \, ,$$

that happens with probability $p_x = \mathbf{P}(|x\rangle)$ and is defined as sums of terraced ket-events $|\mathsf{ter}(X /\!\!/ \mathfrak{X})\rangle \in |\Omega\rangle$ forming a partition of space of elementary ket-outcomes $|\Omega\rangle$:

$$|\Omega\rangle = \sum_{x \in \mathfrak{X}} |\mathsf{ter}(X /\!\!/ \mathfrak{X})\rangle \, .$$

3.4 Mean-probable ket-event

On the basis of the same idea [16, 1] and the eventological definition of the *mean-probable event* [20, 24] I define the *mean-probable event*

$$\mathcal{E}|K_{\mathfrak{X}}\rangle \in |\mathcal{A}^{\mathfrak{X}}\rangle \, ,$$

as a ket-event that happens with probability $\mathbf{P}\left(\mathcal{E}|K_{\mathfrak{X}}\rangle\right)$ which differs of mean-believability probability

$$\mathbf{E}_{\mathbf{B}}(\mathbf{P}(|K_{\mathfrak{X}}\rangle)) = \sum_{x \in \mathfrak{X}} \mathbf{P}(|x\rangle)\mathbf{B}(\langle x|) = \sum_{x \in \mathfrak{X}} p_x b_x$$

of ket-events $|x\rangle \in |\mathfrak{X}\rangle$ with the least error, and plays a role of its mean-set characteristic ***as ket-subsets*** $|x\rangle \subseteq |\Omega\rangle$. In the same way as *mean-believability terraced bra-events* $\mathcal{E}\langle K_{\mathfrak{X}}|$ plays the role of the mean-set characteristic of the values of the random bra-element $\langle K_{\mathfrak{X}}|$ ***as bra-subsets*** $\langle\mathsf{Ter}_{X /\!\!/ \mathfrak{X}}| \subseteq \langle\Omega|$.

Definition 7 *(mean-probable ket-event).* Let

$$\mathcal{E}_\alpha|K_{\mathfrak{X}}\rangle = \{|\mathsf{ter}(X /\!\!/ \mathfrak{X})\rangle : \mathbf{B}(\langle\mathsf{Ter}_{X /\!\!/ \mathfrak{X}}|) > \alpha\} \;\; \subseteq \;\; \{|\mathsf{ter}(X /\!\!/ \mathfrak{X})\rangle : \mathbf{B}(\langle\mathsf{Ter}_{X /\!\!/ \mathfrak{X}}|) \geqslant h\} = \mathcal{E}^\alpha|K_{\mathfrak{X}}\rangle \;\; \in \;\; |\mathcal{A}^{\mathfrak{X}}\rangle \, ,$$

then the *mean-probable ket-event* of the experienced ket-element $|K_{\mathfrak{X}}\rangle$ is any ket-events $\mathcal{E}|K_{\mathfrak{X}}\rangle \in |\mathcal{A}^{\mathfrak{X}}\rangle$ that holds two inclusions:

$$\mathcal{E}_\alpha|K_{\mathfrak{X}}\rangle \;\subset\; \mathcal{E}|K_{\mathfrak{X}}\rangle \;\subseteq\; \mathcal{E}^\alpha|K_{\mathfrak{X}}\rangle$$

for some level $\alpha \in [0,1]$ such that the approximate equality

$$\mathbf{P}(\mathcal{E}|K_{\mathfrak{X}}\rangle) \approx \mathbf{E}_{\mathbf{B}}(\mathbf{P}(|K_{\mathfrak{X}}\rangle)) \tag{16}$$

holds with the smallest error. This will be briefly denoted below

$$\mathbf{P}(\mathcal{E}|K_{\mathfrak{X}}\rangle) \underset{\min}{\approx} \mathbf{E}_{\mathbf{B}}(\mathbf{P}(|K_{\mathfrak{X}}\rangle)). \tag{17}$$

As a result, we get the *mean-probable event* that happens with probability that differs from the mean-believability probability $\mathbf{E}_{\mathbf{B}}(\mathbf{P}(|K_{\mathfrak{X}}\rangle))$ of the experienced ket-element $|K_{\mathfrak{X}}\rangle$ with the least error.

In other words, one of the two equalities holds:

$$\mathcal{E}|K_{\mathfrak{X}}\rangle = \begin{cases} \mathcal{E}_\alpha|K_{\mathfrak{X}}\rangle\,, & \mathbf{E}_{\mathbf{B}}(\mathbf{P}(|K_{\mathfrak{X}}\rangle)) - \mathbf{P}(\mathcal{E}_\alpha|K_{\mathfrak{X}}\rangle) < \mathbf{P}(\mathcal{E}^\alpha|K_{\mathfrak{X}}\rangle) - \mathbf{E}_{\mathbf{B}}(\mathbf{P}(|K_{\mathfrak{X}}\rangle)),\\ \mathcal{E}^\alpha|K_{\mathfrak{X}}\rangle & \mathbf{E}_{\mathbf{B}}(\mathbf{P}(|K_{\mathfrak{X}}\rangle)) - \mathbf{P}(\mathcal{E}_\alpha|K_{\mathfrak{X}}\rangle) \geqslant \mathbf{P}(\mathcal{E}^\alpha|K_{\mathfrak{X}}\rangle) - \mathbf{E}_{\mathbf{B}}(\mathbf{P}(|K_{\mathfrak{X}}\rangle)), \end{cases}$$

or:

$$\mathcal{E}|K_{\mathfrak{X}}\rangle = \begin{cases} \mathcal{E}_\alpha|K_{\mathfrak{X}}\rangle\,, & \mathbf{E}_{\mathbf{B}}(\mathbf{P}(|K_{\mathfrak{X}}\rangle)) - \mathbf{P}(\mathcal{E}_\alpha|K_{\mathfrak{X}}\rangle) \leqslant \mathbf{P}(\mathcal{E}^\alpha|K_{\mathfrak{X}}\rangle) - \mathbf{E}_{\mathbf{B}}(\mathbf{P}(|K_{\mathfrak{X}}\rangle)),\\ \mathcal{E}^\alpha|K_{\mathfrak{X}}\rangle & \mathbf{E}_{\mathbf{B}}(\mathbf{P}(|K_{\mathfrak{X}}\rangle)) - \mathbf{P}(\mathcal{E}_\alpha|K_{\mathfrak{X}}\rangle) > \mathbf{P}(\mathcal{E}^\alpha|K_{\mathfrak{X}}\rangle) - \mathbf{E}_{\mathbf{B}}(\mathbf{P}(|K_{\mathfrak{X}}\rangle)). \end{cases}$$

Definition 8 *(probabilistic distance)*. The ***probabilistic distance*** of the ket-event $|w\rangle \in \mathfrak{X}$ till the experienced ket-element $|K_{\mathfrak{X}}\rangle$ is a mean-believability probability of its symmetrical difference:

$$\mathbf{E}_{\mathbf{B}}\Big(\mathbf{P}\big(\,|K_{\mathfrak{X}}\rangle \,\Delta\, |w\rangle\,\big)\Big) = \sum_{x\in\mathfrak{X}} \mathbf{P}(|x\rangle \,\Delta\, |w\rangle)\mathbf{B}(\langle x|).$$

Theorem 2 *(extremal properties of mean-probable ket-events)*. The *mean-probable ket-event $\mathcal{E}|K_{\mathfrak{X}}\rangle$ of the experienced ket-element $|K_{\mathfrak{X}}\rangle$ minimizes its probabilistic distance*

$$\mathbf{E}_{\mathbf{B}}\Big(\mathbf{P}\big(\,|K_{\mathfrak{X}}\rangle \,\Delta\, \mathcal{E}|K_{\mathfrak{X}}\rangle\,\big)\Big) = \min_{x\,:\,p_x \underset{\min}{\approx} \mathbf{E}_{\mathbf{B}}(\mathbf{P}(|K_{\mathfrak{X}}\rangle))} \mathbf{E}_{\mathbf{B}}\Big(\mathbf{P}\big(\,|K_{\mathfrak{X}}\rangle \,\Delta\, |x\rangle\,\big)\Big)$$

till the experienced ket-element $|K_{\mathfrak{X}}\rangle$ among those ket-events $|x\rangle \in |\mathcal{A}^{\mathfrak{X}}\rangle$ for which the approximate equality (17) holds with the least error.

Proof differs of the proof of the lemma 2 only by denotations.

4 Beyond the probabilistic-eventological paradigm

If the new paradigm is an extension of the old one, follows from it, then all the old concepts can be translated into a new language without going beyond the boundary of the old paradigm, but not vice versa. In my opinion, the *border of old and new paradigms* contains those concepts of a new paradigm that can still be defined and interpreted within the old paradigm. However, new concepts emerging beyond its boundaries, *outside the old paradigm*, can no longer be defined and interpreted in the old way. The concepts of the *mean-measure set of events* and the *mean-probable event*, defined in the paragraph 2 within the *probabilistic-eventological paradigm* (see the first column of the Table 1), and also on the boundary of this paradigm in the paragraph 3 literally translated into the language of the theory of experience and chance as a *mean-believability terraced bra-event* and *mean-probable ket-event* (see the second and third columns of the Table 1), — this is what you can first rely on to go beyond it to define there within the framework of the new *co$\sim$event paradigm* such generalizations of these concepts, which can no longer be defined or interpreted within the framework of the old *probabilistic-eventological* one.

In order to achieve the goal and determine the dual co$\sim$event means, I need to develop in this work the theory of ordered co$\sim$event, as well as with co$\sim$event ordered by believability and probability measures. Then I need to introduce the notion of N-tuple ordered co$\sim$event and for the third time refine the definitions of *mean-believable* and *mean-probable* event means for the ordered co∼event, to define, finally, for each co∼event dual co∼event means as *double-tuple ordered co∼event whose certainty coincides with the certainty of the given co∼event*.

In conclusion, I intend to discuss in advance the prospects of a rather unexpected idea of the interpretation of *believability as a conditional probability*, and *probability as conditional believability*, which naturally arises in the definition of dual co∼event means, which, in my opinion, serves by co∼event justification of Bayesian analysis in statistics, and which will be discussed in detail in my next papers on the theory of co∼event.

4.1 Ordered and strictly ordered co∼events

Definition 9 *(ordered co∼event).* The co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ is called an *ordered co∼event* whenever all its cross-sections by bra-points $\langle\omega^*| \in \langle\Omega|$ as subsets $\mathcal{R}|_{\langle\omega^*|} \subseteq |\Omega\rangle$ of the ket-space $|\Omega\rangle$ are ordered by inclusion:

$$\mathcal{R}|_{\langle\omega^*|} \subseteq \mathcal{R}|_{\langle\omega^*|} \subseteq |\Omega\rangle \text{ or } \mathcal{R}|_{\langle\omega^*|} \subseteq \mathcal{R}|_{\langle\omega^*|} \subseteq |\Omega\rangle \tag{18}$$

for any pair of different bra-incomes $\langle\omega^*| \neq \langle\omega^*|\,, \langle\omega^*|\,, \langle\omega^*| \in \langle\Omega|$.

Property 1. If the co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ is *ordered* than its cross-sections by all ket-points $|\omega\rangle \in |\Omega\rangle$ as subsets $\mathcal{R}|_{|\omega\rangle} \subseteq \langle\Omega|$ of the bra-set $\langle\Omega|$ are also ordered by inclusion:

$$\mathcal{R}|_{|\omega\rangle} \subseteq \mathcal{R}|_{|\omega'\rangle} \subseteq \langle\Omega| \text{ or } \mathcal{R}|_{|\omega'\rangle} \subseteq \mathcal{R}|_{|\omega\rangle} \subseteq \langle\Omega| \tag{19}$$

for any pair of different bra-incomes $|\omega\rangle \neq |\omega'\rangle\,, |\omega\rangle\,, |\omega'\rangle \in |\Omega\rangle$.

Proof. By the definition 9 the ordered co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ defines on the bra-set $\langle\Omega|$ the order «$\preceq$» by rule

$$\langle\omega^*| \preceq \langle\omega^{*\prime}| \Longleftrightarrow \mathcal{R}|_{\langle\omega^*|} \subseteq \mathcal{R}|_{\langle\omega^{*\prime}|}$$

for each pair of bra-points $\langle\omega^*|\,, \langle\omega^{*\prime}| \in \langle\Omega|$. Therefore for each ket-point $|\omega\rangle \in |\Omega\rangle$ the cross-section by this ket-point

$$\mathcal{R}|_{|\omega\rangle} = \{\langle\omega^*| :\ \langle\omega^*|\omega\rangle \in \mathcal{R}\}$$

is representable in the form

$$\mathcal{R}|_{|\omega\rangle} = \left\{\langle\omega^*| :\ \langle\omega^*|^{-}_{\langle\Omega|}(|\omega\rangle) \preceq \langle\omega^*|\right\},$$

where $\langle\omega^*|^{-}_{\langle\Omega|}(|\omega\rangle)$ is the smallest bra-point in section $\mathcal{R}|_{|\omega\rangle}$ on the order «$\preceq$» on the $\langle\Omega|$. Hence we obtain the required result: for any two ket-points $|\omega\rangle\,, |\omega'\rangle \in |\Omega\rangle$ and cross-sections $\mathcal{R}|_{|\omega\rangle}, \mathcal{R}|_{|\omega'\rangle}$ by them either $\mathcal{R}|_{|\omega\rangle} \subseteq \mathcal{R}|_{|\omega'\rangle}$ if $\langle\omega^*|^{-}_{\langle\Omega|}(|\omega\rangle) \preceq \langle\omega^*|^{-}_{\langle\Omega|}(|\omega'\rangle)$, or $\mathcal{R}|_{|\omega'\rangle} \subseteq \mathcal{R}|_{|\omega\rangle}$ if $\langle\omega^*|^{-}_{\langle\Omega|}(|\omega'\rangle) \preceq \langle\omega^*|^{-}_{\langle\Omega|}(|\omega\rangle)$ is satisfied.

Lemma 3 *(about the strict order of ket-events and terraced bra-events on the labelling the bra-ket space by an ordered co∼event). Let the ordered co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ generates the labelling $\langle\mathfrak{X}_\mathcal{R}|\mathcal{S}^{\mathfrak{X}_\mathcal{R}}\rangle$, then*

1) ket-events $|x\rangle \subseteq |\Omega\rangle\,, x \in \mathfrak{X}_\mathcal{R}$ are strictly ordered by inclusion:

$$|x\rangle \subset |x'\rangle \subseteq |\Omega\rangle \text{ or } |x'\rangle \subset |x\rangle \subseteq |\Omega\rangle \tag{20}$$

for any pair of different labels $x \neq x'$, $x, x' \in \mathfrak{X}_\mathcal{R}$;

2) terraced bra-events $\langle\mathsf{Ter}_{X/\!\!/\mathfrak{X}_\mathcal{R}}| \subseteq \langle\Omega|\,, X \in \mathcal{S}^{\mathfrak{X}_\mathcal{R}}$, are strictly ordered by inclusion:

$$\langle\mathsf{Ter}_{X/\!\!/\mathfrak{X}_\mathcal{R}}| \subset \langle\mathsf{Ter}_{X'/\!\!/\mathfrak{X}_\mathcal{R}}| \subseteq \langle\Omega| \text{ or } \langle\mathsf{Ter}_{X'/\!\!/\mathfrak{X}_\mathcal{R}}| \subset \langle\mathsf{Ter}_{X/\!\!/\mathfrak{X}_\mathcal{R}}| \subseteq \langle\Omega| \tag{21}$$

for any pair of different set labels $X \neq X'$, $X, X' \in \mathcal{S}^{\mathfrak{X}_\mathcal{R}}$.

Proof. Ket-events $|x\rangle \subseteq |\Omega\rangle\,, x \in \mathfrak{X}_\mathcal{R}$ are classes of equivalent cross-sections $\mathcal{R}|_{\langle\omega^*|}$ by bra-incomes $\langle\omega^*| \in \langle\Omega|$, and terraced bra-events $\langle\mathsf{Ter}_{X/\!\!/\mathfrak{X}_\mathcal{R}}| \subseteq \langle\Omega|\,, X \in \mathcal{S}^{\mathfrak{X}_\mathcal{R}}$ are classes of equivalent cross-sections $\mathcal{R}|_{|\omega\rangle}$ by ket-outcomes $|\omega\rangle \in |\Omega\rangle$. Both can not coincide when $x \neq x'$ and $X \neq X'$ by its definitions [22] and therefore they are strictly ordered by virtue of the ordering of the co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$.

`Definition 10` *(a strictly ordered labelling the bra-ket space).* If both labelling sets $\mathfrak{X}_{\mathcal{R}}$ and $\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}$ of the labelling $\langle \mathfrak{X}_{\mathcal{R}} | \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} \rangle$ generated by co∼event $\mathcal{R} \subseteq \langle \Omega | \Omega \rangle$ are strictly ordered by inclusions «$\subseteq_{\Omega}$» and «$\subseteq_{\mathfrak{X}_{\mathcal{R}}}$» correspondingly, then $\langle \mathfrak{X}_{\mathcal{R}} | \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} \rangle$ is called the *strictly ordered labelling the bra-ket-space* $\langle \Omega | \Omega \rangle$.

`Corollary 1` *(strict orders on the labelling generated by the ordered co∼event).* The ordered co∼event $\mathcal{R} \subseteq \langle \Omega | \Omega \rangle$ generates the strict ordered labelling $\langle \mathfrak{X}_{\mathcal{R}} | \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} \rangle$ of bra-ket-space $\langle \Omega | \Omega \rangle$. In other words, the ordered co∼event $\mathcal{R}$ generates on the first labelling set $\mathfrak{X}_{\mathcal{R}}$ the strict order for inclusion:

$$x \subset x' \text{ or } x' \subset x \tag{22}$$

for any pair of different labels $x \neq x', x, x' \in \mathfrak{X}_{\mathcal{R}}$; and on the second labelling set $\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}$ generates the strict order for inclusion:

$$X \subset X' \text{ or } X' \subset X \tag{23}$$

for any pair of different set labels $X \neq X', X, X' \in \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}$.

`Proof` follows follows from the definitions 9 and 10, the property 1 on page 154, the lemma 3 on page 154 and an additivity of probability $\mathbf{P}$ and believability $\mathbf{B}$.

4.2 N-tuple ordered co∼event

`Property 2` *(ordered co∼event generated a finite[7] labelling). Let $\mathcal{R} \subseteq \langle \Omega | \Omega \rangle$ be the ordered co∼event generated the finite labelling $\langle \mathfrak{X}_{\mathcal{R}} | \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} \rangle$, $N = |\mathfrak{X}_{\mathcal{R}}|_{\neq \varnothing}$ be the number of nonempty labels in $\mathfrak{X}_{\mathcal{R}}$, $\mathfrak{S}^N = \left| \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} \right|_{\neq \emptyset}$ be the number of nonempty labels in $\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}$. Then $N = \mathfrak{S}^N$.*

`Proof.` Let $(\Omega, \mathcal{A})$ be the labelling measurable space, $x_1, \ldots, x_N, \ldots$ be nonempty labels $x_i \in \mathcal{A}$, numbered in descending order for strict inclusion:

$$\Omega \supseteq x_1 \supset x_2 \supset \ldots \supset x_N \supset \varnothing, \tag{24}$$

and $X_1, \ldots, X_N, \ldots$ be nonempty set labels $X_i \subseteq \mathcal{A}$, numbered in ascending order for strict inclusion:

$$\emptyset \subset X_1 \subset X_2 \subset \ldots \subset X_N \subseteq \mathfrak{X}_{\mathcal{R}}. \tag{25}$$

Prove, that there is such $N = 0, 1, 2, \ldots$ that a labelling the ordered co∼event $\mathcal{R}$ has one of the following forms:

$$\langle \mathfrak{X}_{\mathcal{R}} | \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} \rangle = \begin{cases} \left. \begin{array}{l} \langle \{x_1, \ldots, x_N\} | \{X_N, \ldots, X_1\} \rangle \\ \langle \{x_1, \ldots, x_N\} | \{X_N, \ldots, X_1, \emptyset\} \rangle \\ \langle \{x_1, \ldots, x_N, \varnothing\} | \{X_N, \ldots, X_1\} \rangle \\ \langle \{x_1, \ldots, x_N, \varnothing\} | \{X_N, \ldots, X_1, \emptyset\} \rangle \end{array} \right\}, & \mathcal{R} \neq \emptyset \subseteq \langle \Omega | \Omega \rangle, \ N > 0, \\ \\ \langle \{\varnothing\} | \{\emptyset\} \rangle, & \mathcal{R} = \emptyset \subseteq \langle \Omega | \Omega \rangle, \ N = 0. \end{cases} \tag{26}$$

This follows from the corollary 1 on strict orders by inclusion «$\subset_{\Omega}$» and «$\subset_{\mathfrak{X}_{\mathcal{R}}}$» correspondingly on labelling sets $\mathfrak{X}_{\mathcal{R}}$ and $\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}$ generated by the ordered co∼event $\mathcal{R} \subseteq \langle \Omega | \Omega \rangle$. The property 2 follows from (27).

`Definition 11` *(N-tuple ordered co∼event).* 1) The ordered co∼event $\mathcal{R} \subseteq \langle \Omega | \Omega \rangle$ with the finite labelling $\langle \mathfrak{X}_{\mathcal{R}} | \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} \rangle$ is called *N-tuple ordered co∼event* if $N = |\mathfrak{X}_{\mathcal{R}}|_{\neq \varnothing}$; 2) the set

$$\mathfrak{R}_N = \left\{ \mathcal{R} \subseteq \langle \Omega | \Omega \rangle : \mathcal{R} - \text{ordered co}\sim\text{event}, |\mathfrak{X}_{\mathcal{R}}|_{\neq \varnothing} = N \right\} \subseteq \langle \mathcal{A} | \mathcal{A} \rangle \tag{27}$$

for $N = 0, 1, 2, \ldots$ is called the *set of N-tuple ordered co∼events*[8]; 3) the set

$$\mathfrak{R}_{\leqslant N} = \sum_{1 \leqslant n \leqslant N} \mathfrak{R}_n \tag{28}$$

[7] For a defining dual co∼event means a definition of co∼event with *finite labelling* is enough. The *infinite labelling* co∼events has their own useful features and will be considered in the following works.

[8] where $\langle \mathcal{A}_{\mathfrak{X}_{\mathcal{R}}} |$ is the sigma-algebra of bra-events $\langle x | \in \langle \mathfrak{X}_{\mathcal{R}} | \subseteq \langle \mathcal{A}_{\mathfrak{X}_{\mathcal{R}}} | \subseteq \langle \mathcal{A} |$; and $| \mathcal{A}^{\mathfrak{X}_{\mathcal{R}}} \rangle$ is the sigma-algebra of terraced ket-events $|\mathrm{ter}(X /\!\!/ \mathfrak{X}_{\mathcal{R}}) \rangle \in | \mathcal{A}^{\mathfrak{X}_{\mathcal{R}}} \rangle \subseteq | \mathcal{A} \rangle$.

is called the *set of n-tuple nonempty ordered co$\sim$events for all* $n = 1, \ldots, N$.

`Property 3` *(partition of a set of N-tuple nonempty ordered co$\sim$event).* From (27) it follows that for $N = 1, 2, \ldots$

$$\mathfrak{R}_N = \mathfrak{R}_N^{00} + \mathfrak{R}_N^{01} + \mathfrak{R}_N^{10} + \mathfrak{R}_N^{11}, \tag{29}$$

i.e. the set of N-tuple nonempty[9] ordered co$\sim$events is partitioned on 4 subsets that correspond to 4 types of N-tuple nonempty ordered co$\sim$events:

$$\begin{aligned}
\mathfrak{R}_N^{11} &= \Big\{\mathcal{R} \subseteq \langle\Omega|\Omega\rangle : \mathcal{R}\text{— ordered co}\sim\text{event}, |\mathfrak{X}_\mathcal{R}| = N, |\mathbf{\mathcal{Z}}^{\mathfrak{X}_\mathcal{R}}| = N\Big\}, \\
\mathfrak{R}_N^{10} &= \Big\{\mathcal{R} \subseteq \langle\Omega|\Omega\rangle : \mathcal{R}\text{— ordered co}\sim\text{event}, |\mathfrak{X}_\mathcal{R}| = N, |\mathbf{\mathcal{Z}}^{\mathfrak{X}_\mathcal{R}}| = N+1\Big\}, \\
\mathfrak{R}_N^{01} &= \Big\{\mathcal{R} \subseteq \langle\Omega|\Omega\rangle : \mathcal{R}\text{— ordered co}\sim\text{event}, |\mathfrak{X}_\mathcal{R}| = N+1, |\mathbf{\mathcal{Z}}^{\mathfrak{X}_\mathcal{R}}| = N\Big\}, \\
\mathfrak{R}_N^{11} &= \Big\{\mathcal{R} \subseteq \langle\Omega|\Omega\rangle : \mathcal{R}\text{— ordered co}\sim\text{event}, |\mathfrak{X}_\mathcal{R}| = N+1, |\mathbf{\mathcal{Z}}^{\mathfrak{X}_\mathcal{R}}| = N+1\Big\}.
\end{aligned} \tag{30}$$

`Property 4` *(about the connection of labels and set labels of ordered co$\sim$event).* Between labels $x_1, \ldots, x_N$ numbered in descending order for strict inclusion and set labels $X_1, \ldots, X_N$ numbered in ascending order for strict inclusion of N-tuple ordered co$\sim$event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$:

$$\begin{aligned}
&\Omega \supseteq x_1 \supset_\Omega x_2 \supset_\Omega \ldots \supset_\Omega x_N \supset_\Omega \varnothing, \\
&\emptyset \subset_{\mathfrak{X}_\mathcal{R}} X_1 \subset_{\mathfrak{X}_\mathcal{R}} X_2 \subset_{\mathfrak{X}_\mathcal{R}} \ldots \subset_{\mathfrak{X}_\mathcal{R}} X_N \subset_{\mathfrak{X}_\mathcal{R}} \mathfrak{X}_\mathcal{R},
\end{aligned} \tag{31}$$

the following relations hold:

$$\begin{aligned}
X_1 &= \{x_1\}, \\
X_2 &= \{x_1, x_2\}, \\
&\ldots, \\
X_{N-1} &= \{x_1, \ldots, x_{N-1}\}, \\
X_N &= \{x_1, \ldots, x_N\},
\end{aligned} \tag{32}$$

Or, what's the same,

$$X_n = \sum_{i=1}^{n} \{x_i\} \tag{33}$$

for $n = 1, \ldots, N$.

Proof immediately follows from the fact that set labels $X \subseteq \mathfrak{X}_\mathcal{R}$ are defined as sets of labels $x \in \mathfrak{X}_\mathcal{R}$.

`Property 5` *(on the strict order of terraced bra-events and ket-events for ordered co$\sim$event).* Terraced bra-events $\langle \mathsf{Ter}_{X_1 /\!\!/ \mathfrak{X}_\mathcal{R}}|$ and ket-events $|x\rangle, x \in \mathfrak{X}_\mathcal{R}$ are strictly ordered by inclusion in accordance with the strict order for the inclusion of labels and set labels in the labelling the N-tuple ordered co-event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$:

$$\begin{aligned}
&|\Omega\rangle \supseteq |x_1\rangle \supset |x_2\rangle \supset \ldots \supset |x_N\rangle \supset \varnothing, \\
&\emptyset \subseteq \langle \mathsf{Ter}_{X_1 /\!\!/ \mathfrak{X}_\mathcal{R}}| \subset \langle \mathsf{Ter}_{X_2 /\!\!/ \mathfrak{X}_\mathcal{R}}| \subset \ldots \subset \langle \mathsf{Ter}_{X_N /\!\!/ \mathfrak{X}_\mathcal{R}}| \subseteq \langle\Omega|.
\end{aligned} \tag{34}$$

Proof follows from defining relations (see [22]) for ket-events and terraced bra-events:

$$\begin{aligned}
|x\rangle &= \sum_{x \in X} |\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle, \quad x \in \{x_1, \ldots, x_N\} \\
\langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}_\mathcal{R}}| &= \sum_{x \in X} \langle x|, \quad X \subseteq \{x_1, \ldots, x_N\}.
\end{aligned} \tag{35}$$

[9] $\mathcal{R} \neq \emptyset$, i.e. $N \neq 0$.

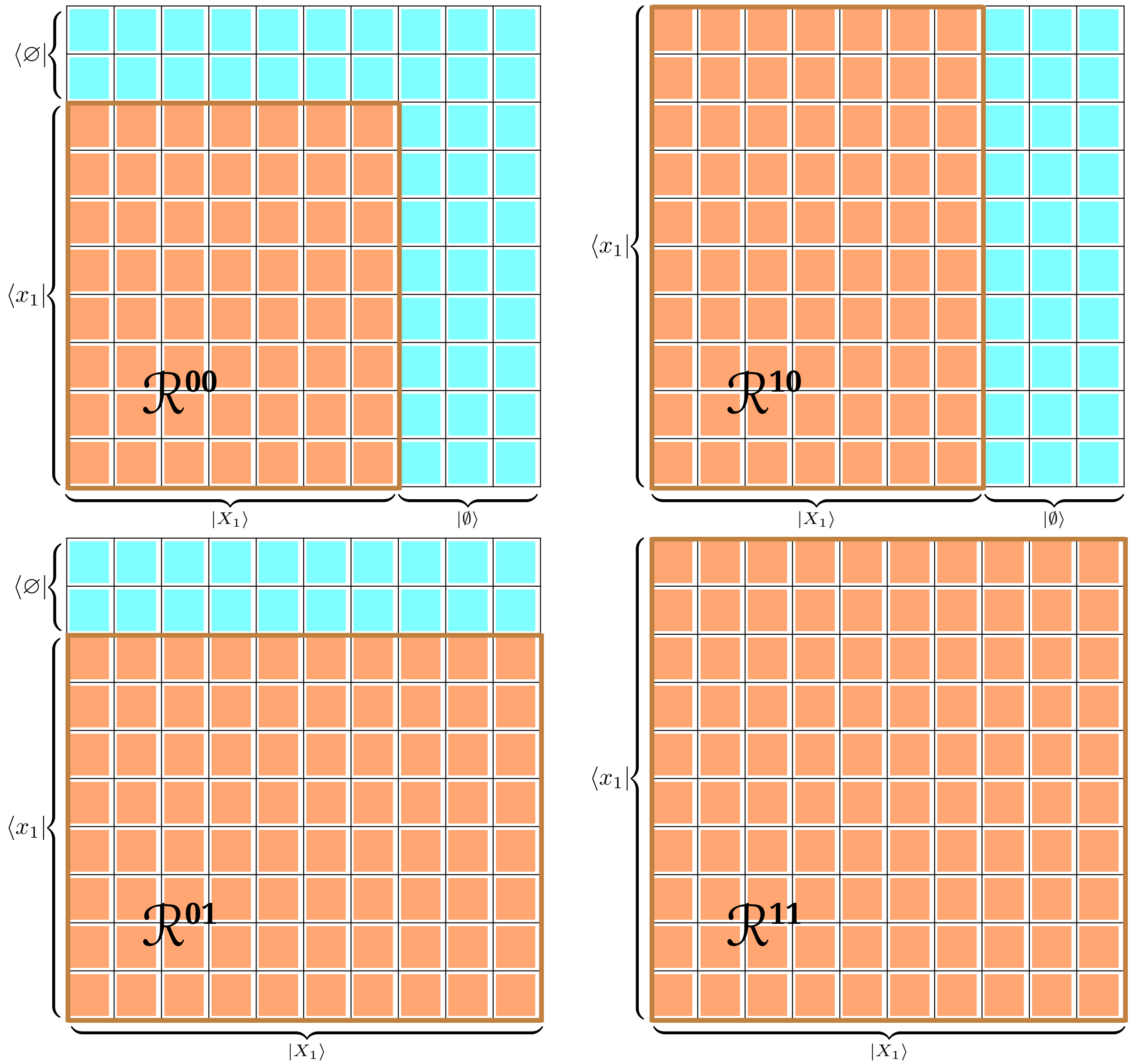

Table 2: Venn diagrams of 4 types of monoplet ordered co∼events $\mathcal{R}^{00}, \mathcal{R}^{01}, \mathcal{R}^{10}, \mathcal{R}^{11} \subseteq \langle\Omega|\Omega\rangle$.

4.3 Monoplet and doublet ordered co∼events

Let us consider in more detail monoplet and doublet ordered co∼events $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ with the finite labelling $\langle\mathfrak{X}_\mathcal{R}|\mathbf{Z}^{\mathfrak{X}_\mathcal{R}}\rangle$, i.e. ordered co∼events that form labelling sets $\mathfrak{R}_1$ or $\mathfrak{R}_2$.

4.3.1 Monoplet ordered co∼event

From (27) it is clear that a labelling the monoplet ordered co∼events may be one of 4 types:

$$\langle\mathfrak{X}_\mathcal{R}|\mathbf{Z}^{\mathfrak{X}_\mathcal{R}}\rangle = \begin{cases} \langle\{x_1\}|\{X_1\}\rangle\,, \\ \langle\{x_1\}|\{X_1,\emptyset\}\rangle\,, \\ \langle\{x_1,\varnothing\}|\{X_1\}\rangle\,, \\ \langle\{x_1,\varnothing\}|\{X_1,\emptyset\}\rangle\,, \end{cases} \tag{36}$$

Table 3: Venn diagrams of 4 types of doublet ordered co∼events $\mathcal{R}^{00}, \mathcal{R}^{01}, \mathcal{R}^{10}, \mathcal{R}^{11} \subseteq \langle\Omega|\Omega\rangle$.

in accordance with which, the set $\mathfrak{R}_1$ is divided into 4 subsets:

$$\begin{aligned}
\mathfrak{R}_1^{11} &= \left\{\mathcal{R} \subseteq \langle\Omega|\Omega\rangle : \mathfrak{X}_\mathcal{R} = \{x_1\}, \mathbf{2}^{\mathfrak{X}_\mathcal{R}} = \{X_1\}\right\}, \\
\mathfrak{R}_1^{10} &= \left\{\mathcal{R} \subseteq \langle\Omega|\Omega\rangle : \mathfrak{X}_\mathcal{R} = \{x_1\}, \mathbf{2}^{\mathfrak{X}_\mathcal{R}} = \{\emptyset, X_1\}\right\}, \\
\mathfrak{R}_1^{01} &= \left\{\mathcal{R} \subseteq \langle\Omega|\Omega\rangle : \mathfrak{X}_\mathcal{R} = \{\varnothing, x_1\}, \mathbf{2}^{\mathfrak{X}_\mathcal{R}} = \{X_1\}\right\}, \\
\mathfrak{R}_1^{00} &= \left\{\mathcal{R} \subseteq \langle\Omega|\Omega\rangle : \mathfrak{X}_\mathcal{R} = \{\varnothing, x_1\}, \mathbf{2}^{\mathfrak{X}_\mathcal{R}} = \{\emptyset, X_1\}\right\},
\end{aligned} \tag{37}$$

consisting from monoplet ordered co∼events of corresponding 4 types (see Table 2).

4.3.2 Doublet ordered co∼events

From (27) it is clear that a labelling the doublet ordered co∼events may be one of 4 types:

$$\langle\mathfrak{X}_\mathcal{R}|\mathbf{2}^{\mathfrak{X}_\mathcal{R}}\rangle = \begin{cases} \langle\{x_1, x_2\}|\{X_2, X_1\}\rangle, \\ \langle\{x_1, x_2\}|\{X_1, X_2, \emptyset\}\rangle, \\ \langle\{x_1, x_2, \varnothing\}|\{X_2, X_1\}\rangle, \\ \langle\{x_1, x_2, \varnothing\}|\{X_2, X_1, \emptyset\}\rangle, \end{cases} \tag{38}$$

in accordance with which, the set $\mathfrak{R}_2$ is divided into 4 subsets:

$$\begin{aligned}
\mathfrak{R}_1^{11} &= \Big\{\mathcal{R} \subseteq \langle\Omega|\Omega\rangle : \mathfrak{X}_{\mathcal{R}} = \{x_1, x_2\}, \mathbf{\mathfrak{S}}^{\mathfrak{X}_{\mathcal{R}}} = \{X_2, X_1\}\Big\},\\
\mathfrak{R}_1^{10} &= \Big\{\mathcal{R} \subseteq \langle\Omega|\Omega\rangle : \mathfrak{X}_{\mathcal{R}} = \{x_1, x_2\}, \mathbf{\mathfrak{S}}^{\mathfrak{X}_{\mathcal{R}}} = \{X_2, X_1, \emptyset\}\Big\},\\
\mathfrak{R}_1^{01} &= \Big\{\mathcal{R} \subseteq \langle\Omega|\Omega\rangle : \mathfrak{X}_{\mathcal{R}} = \{x_1, x_2, \varnothing\}, \mathbf{\mathfrak{S}}^{\mathfrak{X}_{\mathcal{R}}} = \{X_2, X_1\}\Big\},\\
\mathfrak{R}_1^{00} &= \Big\{\mathcal{R} \subseteq \langle\Omega|\Omega\rangle : \mathfrak{X}_{\mathcal{R}} = \{x_1, x_2, \varnothing\}, \mathbf{\mathfrak{S}}^{\mathfrak{X}_{\mathcal{R}}} = \{X_2, X_1, \emptyset\}\Big\},
\end{aligned} \tag{39}$$

consisting from doublet ordered co∼events of corresponding 4 types (see Table 3).

5 Orders and equivalences in a certainty space, controlled by believability and probability

Definition 12 *(believability bra-space, probability ket space and certainty (believability-probability) bra-ket space).* The measurable space $\langle\Omega, \mathcal{A}, \mathbf{B}| = (\langle\Omega|, \langle\mathcal{A}|, \mathbf{B})$ with believability measure $\mathbf{B}$, normalized to unity, is called a *believability bra-space*. The measurable space $|\Omega, \mathcal{A}, \mathbf{P}\rangle = (|\Omega\rangle, |\mathcal{A}\rangle, \mathbf{P})$ with probability measure $\mathbf{P}$, normalized to unity, is called a *probability ket-space*. The Cartesian product of these measurable spaces $\langle\Omega, \mathcal{A}, \mathbf{B}|\Omega, \mathcal{A}, \mathbf{P}\rangle = (\langle\Omega|\Omega\rangle, \langle\mathcal{A}|\mathcal{A}\rangle, \mathbf{\Phi})$ with certainty (believability-probability) measure $\mathbf{\Phi} = \mathbf{B} \times \mathbf{P}$, which is defined as a product of believability $\mathbf{B}$ and probability $\mathbf{P}$, is called a *certainty (believability-probability) bra-ket-space*.

Definition 13 *($\mathbf{B}$-order, strict $\mathbf{B}$-order, and $\mathbf{B}$-equivalence).* The believability measure $\mathbf{B}$ defines on $\mathbf{\mathfrak{S}}^{\mathfrak{X}_{\mathcal{R}}} \subseteq \mathcal{P}(\mathfrak{X}_{\mathcal{R}})$ for each pair $(X, X') \in \mathbf{\mathfrak{S}}^{\mathfrak{X}_{\mathcal{R}}} \times \mathbf{\mathfrak{S}}^{\mathfrak{X}_{\mathcal{R}}}$ a *relation of* $\mathbf{B}$*-order*:

$$X \precsim_{\mathbf{B}} X' \iff b(X /\!\!/ \mathfrak{X}_{\mathcal{R}}) \leqslant b(X' /\!\!/ \mathfrak{X}_{\mathcal{R}}),$$

a *relation of strict* $\mathbf{B}$*-order*:

$$X \prec_{\mathbf{B}} X' \iff b(X /\!\!/ \mathfrak{X}_{\mathcal{R}}) < b(X' /\!\!/ \mathfrak{X}_{\mathcal{R}}),$$

and a *relation of* $\mathbf{B}$*-equivalence*:

$$X \overset{\mathbf{B}}{\sim} X' \iff b(X /\!\!/ \mathfrak{X}_{\mathcal{R}}) = b(X' /\!\!/ \mathfrak{X}_{\mathcal{R}}).$$

where

$$b(X /\!\!/ \mathfrak{X}_{\mathcal{R}}) = \mathbf{B}(\langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}_{\mathcal{R}}}|)$$

is a value of believability $\mathbf{B}$ on the *terrced bra-event* $\langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}_{\mathcal{R}}}| \subseteq \langle\Omega|$ of the believability bra-space $\langle\Omega, \mathcal{A}, \mathbf{B}|$.

Definition 14 *($\mathbf{P}$-order, strict $\mathbf{P}$-order, and $\mathbf{P}$-equivalence).* The probability measure $\mathbf{P}$ defines on $\mathfrak{X}_{\mathcal{R}}$ foe each pair $(x, x') \in \mathfrak{X}_{\mathcal{R}} \times \mathfrak{X}_{\mathcal{R}}$ a *relation of* $\mathbf{P}$*-order*:

$$x \precsim_{\mathbf{P}} x' \iff p_{x /\!\!/ \mathfrak{X}_{\mathcal{R}}} \leqslant p_{x' /\!\!/ \mathfrak{X}_{\mathcal{R}}},$$

a *relation of strict* $\mathbf{P}$*-order*:

$$x \prec_{\mathbf{P}} x' \iff p_{x /\!\!/ \mathfrak{X}_{\mathcal{R}}} < p_{x' /\!\!/ \mathfrak{X}_{\mathcal{R}}},$$

and a *relation of* $\mathbf{P}$*-equivalence*:

$$x \overset{\mathbf{P}}{\sim} x' \iff p_{x /\!\!/ \mathfrak{X}_{\mathcal{R}}} = p_{x' /\!\!/ \mathfrak{X}_{\mathcal{R}}}.$$

where

$$p_{x /\!\!/ \mathfrak{X}_{\mathcal{R}}} = \mathbf{P}(|x\rangle)$$

is a value of probability $\mathbf{P}$ on *ket-event* $|x\rangle \subseteq |\Omega\rangle$ of the probability ket-space $|\Omega, \mathcal{A}, \mathbf{P}\rangle$.

5.1 B-quotient-labelling, generated by co∼event

Definition 15 *(B-quotient-labelling).* Denote $\mathfrak{S}^{\mathfrak{X}_\mathcal{R}}/\overset{\mathbf{B}}{\sim} = \{[X_1], \dots, [X_N], \emptyset\}$ a quotient-set of the finite set $\mathfrak{S}^{\mathfrak{X}_\mathcal{R}}$ by relation of $\mathbf{B}$-equivalence. Then by definition the quotient-set $\mathfrak{S}^{\mathfrak{X}_\mathcal{R}}/\overset{\mathbf{B}}{\sim}$ the set labels $X_n \in \mathfrak{S}^{\mathfrak{X}_\mathcal{R}}$, by which its elements are «labelled» as classes of $\mathbf{B}$-equivalence $[X_n] \in \mathfrak{S}^{\mathfrak{X}_\mathcal{R}}/\overset{\mathbf{B}}{\sim}$, are strict $\mathbf{B}$-ordered, like so:

$$X_1 \succ_\mathbf{B} X_2 \succ_\mathbf{B} \dots \succ_\mathbf{B} X_N \succ_\mathbf{B} \emptyset.$$

We uniquely associate this strictly $\mathbf{B}$-ordered set labels with other strictly $\mathbf{B}$-ordered set labels $X^\mathbf{B}_n$, $n = 1, \dots, N$, which are defined as

$$X^\mathbf{B}_n = \{x^\mathbf{B}_n, \dots, x^\mathbf{B}_N\} \subseteq \mathfrak{X}^\mathbf{B}_\mathcal{R},$$

nonempty strict $\mathbf{B}$-ordered subsets of nonempty labels $x_i \neq \varnothing_\Omega \subseteq \Omega$ from the set

$$\mathfrak{X}^\mathbf{B}_\mathcal{R} = \{x^\mathbf{B}_1, \dots, x^\mathbf{B}_N, \varnothing_\Omega\} \subseteq \mathcal{A},$$

that together with the empty set label $\emptyset_{\mathfrak{X}^\mathbf{B}_\mathcal{R}} \subseteq \mathfrak{X}^\mathbf{B}_\mathcal{R}$ form the set

$$\mathfrak{S}^{\mathfrak{X}_\mathcal{R}}_\mathbf{B} = \{X^\mathbf{B}_1, \dots, X^\mathbf{B}_N, \emptyset_{\mathfrak{X}^\mathbf{B}_\mathcal{R}}\} \subseteq \mathcal{P}(\mathfrak{X}^\mathbf{B}_\mathcal{R}).$$

The $\mathbf{B}$-*quotient-labelling* of the labelling $\langle \mathfrak{X}_\mathcal{R} | \mathfrak{S}^{\mathfrak{X}_\mathcal{R}} \rangle$ is the labelling $\langle \mathfrak{X}^\mathbf{B}_\mathcal{R} | \mathfrak{S}^{\mathfrak{X}_\mathcal{R}}_\mathbf{B} \rangle$ of bra-ket-set $\langle \Omega | \Omega \rangle$ by labelling sets $\mathfrak{X}^\mathbf{B}_\mathcal{R} \subseteq \mathcal{A}$ and $\mathfrak{S}^{\mathfrak{X}_\mathcal{R}}_\mathbf{B} \subseteq \mathcal{P}(\mathfrak{X}^\mathbf{B}_\mathcal{R})$, which define the set of terraced $\mathbf{B}$-quotient-ket-events

$$|\mathfrak{S}^{\mathfrak{X}_\mathcal{R}}_\mathbf{B}\rangle = \left\{ |\mathsf{ter}(X^\mathbf{B} /\!\!/ \mathfrak{X}^\mathbf{B}_\mathcal{R})\rangle, X^\mathbf{B} \in \mathfrak{S}^{\mathfrak{X}_\mathcal{R}}_\mathbf{B} \right\} \tag{40}$$

and the set of $\mathbf{B}$-quotient-bra-events

$$\langle \mathfrak{X}^\mathbf{B}_\mathcal{R} | = \{ \langle x^\mathbf{B} |, x^\mathbf{B} \in \mathfrak{X}^\mathbf{B}_\mathcal{R} \} \tag{41}$$

by the following way. Terraced $\mathbf{B}$-quotient-ket-events in (41) are defined for each $X^\mathbf{B} \in \mathfrak{S}^{\mathfrak{X}_\mathcal{R}}_\mathbf{B}$ through terraced ket-events in the initial labelling from $|\mathfrak{S}^{\mathfrak{X}_\mathcal{R}}\rangle$ by formulas:

$$|\mathsf{ter}(X^\mathbf{B} /\!\!/ \mathfrak{X}^\mathbf{B}_\mathcal{R})\rangle = \sum_{X^\mathbf{B} \overset{\mathbf{B}}{\sim} W \in \mathfrak{S}^{\mathfrak{X}_\mathcal{R}}} |\mathsf{ter}(W /\!\!/ \mathfrak{X}_\mathcal{R})\rangle,$$

and $\mathbf{B}$-quotient-bra-events in (42), and terraced $\mathbf{B}$-quotient-bra-events $\langle \mathsf{Ter}_{X^\mathbf{B}_n /\!\!/ \mathfrak{X}^\mathbf{B}_\mathcal{R}} |$, $n = 1, \dots, N$ satisfy the constraints, which are defined by values of believability measure $\mathbf{B}$ on terraced bra-events in the initial labelling $\langle \mathfrak{X}_\mathcal{R} | \mathfrak{S}^{\mathfrak{X}_\mathcal{R}} \rangle$:

$$b_{x^\mathbf{B}} = \mathbf{B}(\langle x^\mathbf{B} |) = \begin{cases} b(X^\mathbf{B}_n /\!\!/ \mathfrak{X}^\mathbf{B}_\mathcal{R}) - b(X^\mathbf{B}_{n+1} /\!\!/ \mathfrak{X}^\mathbf{B}_\mathcal{R}), & x^\mathbf{B} = x^\mathbf{B}_n, \quad n = 1, \dots, N-1, \\ b(X^\mathbf{B}_N /\!\!/ \mathfrak{X}^\mathbf{B}_\mathcal{R}), & x^\mathbf{B} = x^\mathbf{B}_N, \\ 1 - b(X^\mathbf{B}_1 /\!\!/ \mathfrak{X}^\mathbf{B}_\mathcal{R}), & x^\mathbf{B} = \varnothing_\Omega, \end{cases} \tag{42}$$

where

$$b(X^\mathbf{B}_n /\!\!/ \mathfrak{X}^\mathbf{B}_\mathcal{R}) = \mathbf{B}(\langle \mathsf{Ter}_{X^\mathbf{B}_n /\!\!/ \mathfrak{X}^\mathbf{B}_\mathcal{R}} |) = \mathbf{B}(\langle \mathsf{ter}(W /\!\!/ \mathfrak{X}_\mathcal{R}) |) = b(W /\!\!/ \mathfrak{X}_\mathcal{R}), \quad X^\mathbf{B}_n \overset{\mathbf{B}}{\sim} W, \quad n = 1, \dots, N. \tag{43}$$

Property 6 *(B-quotient-partition of a ket-set and a bra-set).* Terraced $\mathbf{B}$-quotient-ket-events in (41) and $\mathbf{B}$-quotient-bra-events in (42) form a $\mathbf{B}$-quotient-partition of the ket-set $|\Omega\rangle$ and bra-set $\langle \Omega |$ correspondingly:

$$\begin{aligned} |\Omega\rangle &= \sum_{X^\mathbf{B} \in \mathfrak{S}^{\mathfrak{X}_\mathcal{R}}_\mathbf{B}} |\mathsf{ter}(X^\mathbf{B} /\!\!/ \mathfrak{X}^\mathbf{B}_\mathcal{R})\rangle = |\mathsf{ter}(\emptyset_{\mathfrak{X}^\mathbf{B}_\mathcal{R}} /\!\!/ \mathfrak{X}^\mathbf{B}_\mathcal{R})\rangle + \sum_{n=1}^N |\mathsf{ter}(X^\mathbf{B}_n /\!\!/ \mathfrak{X}^\mathbf{B}_\mathcal{R})\rangle, \\ \langle \Omega | &= \sum_{x^\mathbf{B} \in \mathfrak{X}^\mathbf{B}_\mathcal{R}} \langle x^\mathbf{B} | = \langle \varnothing_\Omega | + \sum_{n=1}^N \langle x^\mathbf{B}_n |. \end{aligned} \tag{44}$$

Property 7 *(probabilities of terraced $\mathbf{B}$-quotient-ket-events and $\mathbf{B}$-quotient-ket-events).*

$$p(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) = \mathbf{P}(|\mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}})\rangle) = \begin{cases} p(X^{\mathbf{B}}_n /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) = \displaystyle\sum_{X^{\mathbf{B}} \overset{\mathbf{B}}{\sim} W \in \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}} p(W /\!\!/ \mathfrak{X}_{\mathcal{R}}), & X^{\mathbf{B}} = X^{\mathbf{B}}_n, \quad n = 1, \ldots, N, \\ p(\emptyset_{\mathfrak{X}_{\mathcal{R}}} /\!\!/ \mathfrak{X}_{\mathcal{R}}) & X^{\mathbf{B}} = \emptyset_{\mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}}, \end{cases} \tag{45}$$

$$p_{x^{\mathbf{B}}} = \mathbf{P}(|x^{\mathbf{B}}\rangle) = \sum_{x^{\mathbf{B}} \in X^{\mathbf{B}} \in \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}} \mathbf{P}(|\mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}})\rangle) = \sum_{x^{\mathbf{B}} \in X^{\mathbf{B}} \in \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}} p(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}).$$

5.2 $\mathbf{P}$-quotient-labelling, generated by co∼event

Definition 16 *($\mathbf{P}$-quotient-labelling).* Denote $\mathfrak{X}_{\mathcal{R}} / \overset{\mathbf{P}}{\sim} = \{[x_1], \ldots, [x_N], \emptyset\}$ a quotient-set of the finite set $\mathfrak{X}_{\mathcal{R}}$ by relation of $\mathbf{P}$-equivalence. Then by definition of the quotient-set $\mathfrak{X}_{\mathcal{R}} / \overset{\mathbf{P}}{\sim}$ labels $x_n \in \mathfrak{X}_{\mathcal{R}}$, by which its elements are «labelled» as classes of $\mathbf{P}$-equivalence $[x_n] \in \mathfrak{X}_{\mathcal{R}} / \overset{\mathbf{P}}{\sim}$, are strict $\mathbf{P}$-ordered, like as:

$$x_1 \succ_{\mathbf{P}} x_2 \succ_{\mathbf{P}} \ldots \succ_{\mathbf{P}} x_N \succ_{\mathbf{P}} \varnothing_\Omega.$$

We uniquely associate this nonempty strictly $\mathbf{P}$-ordered labels with other nonempty strictly $\mathbf{P}$-ordered labels $x^{\mathbf{P}}_n$, $n = 1, \ldots, N$, which together with the empty label $\varnothing_\Omega \subseteq \Omega$ from the set

$$\mathfrak{X}^{\mathbf{P}}_{\mathcal{R}} = \{x^{\mathbf{P}}_1, \ldots, x^{\mathbf{P}}_N, \varnothing_\Omega\} \subseteq \mathcal{A},$$

and form from them N nonempty set labels

$$X^{\mathbf{P}}_n = \{x^{\mathbf{P}}_1, \ldots, x^{\mathbf{P}}_n\} \subseteq \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}}$$

for $n = 1, \ldots, N$, which together with the empty set label $\emptyset_{\mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}} \subseteq \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}$ form the set

$$\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{P}} = \{X^{\mathbf{P}}_1, \ldots, X^{\mathbf{P}}_N, \emptyset_{\mathfrak{X}^{\mathbf{P}}_{\mathcal{R}}}\} \subseteq \mathcal{P}(\mathfrak{X}^{\mathbf{P}}_{\mathcal{R}}).$$

The $\mathbf{P}$-*quotient-labelling* of the labelling $\langle \mathfrak{X}_{\mathcal{R}} | \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} \rangle$ is a labelling $\langle \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}} | \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{P}} \rangle$ of bra-ket-set $\langle \Omega | \Omega \rangle$ by labelling sets $\mathfrak{X}^{\mathbf{P}}_{\mathcal{R}} \subseteq \mathcal{A}$ and $\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{P}} \subseteq \mathcal{P}(\mathfrak{X}^{\mathbf{P}}_{\mathcal{R}})$, which define the set of $\mathbf{P}$-bra-events

$$\langle \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}} | = \{\langle x^{\mathbf{P}} | \,, x^{\mathbf{P}} \in \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}}\} \tag{46}$$

and the set of terraced $\mathbf{P}$-ket-events

$$|\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{P}}\rangle = \left\{ |\mathsf{ter}(X^{\mathbf{P}} /\!\!/ \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}})\rangle \,, X^{\mathbf{P}} \in \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{P}} \right\} \tag{47}$$

by the following way. The $\mathbf{P}$-bra-events in (47) are defined for each $x^{\mathbf{P}} \in \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}}$ through bra-events in the initial labelling from $\langle \mathfrak{X}_{\mathcal{R}} |$ by formulas:

$$\langle x^{\mathbf{P}} | = \sum_{x^{\mathbf{P}} \overset{\mathbf{P}}{\sim} w \in \mathfrak{X}_{\mathcal{R}}} \langle w | \,,$$

and terraced $\mathbf{P}$-ket-events in (48) satisfy the constraints, which are defined by values of believability measure $\mathbf{P}$ on ket-events in the initial labelling $\langle \mathfrak{X}_{\mathcal{R}} | \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} \rangle$:

$$p(X^{\mathbf{P}} /\!\!/ \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}}) = \mathbf{P}(|\mathsf{ter}(X^{\mathbf{P}} /\!\!/ \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}})\rangle) = \begin{cases} p_{x^{\mathbf{P}}_n} - p_{x^{\mathbf{P}}_{n+1}}, & X^{\mathbf{P}} = X^{\mathbf{P}}_n, \quad n = 1, \ldots, N-1, \\ p_{x^{\mathbf{P}}_N}, & X^{\mathbf{P}} = X^{\mathbf{P}}_N, \\ 1 - p_{x^{\mathbf{P}}_1}, & X^{\mathbf{P}} = \emptyset_{\mathfrak{X}^{\mathbf{P}}_{\mathcal{R}}}. \end{cases} \tag{48}$$

where

$$p_{x^{\mathbf{P}}} = \mathbf{P}(|x^{\mathbf{P}}\rangle) = \mathbf{P}(|w\rangle) = p_w, \quad x^{\mathbf{P}} \sim w \in \mathfrak{X}_{\mathcal{R}}. \tag{49}$$

Property 8 *($\mathbf{P}$-partition of a ket-set and a bra-set).* Terraced $\mathbf{P}$-ket-events in (48) and $\mathbf{P}$-bra-events in (47) form a partition of the ket-set $|\Omega\rangle$ and the bra-set $\langle\Omega|$ correspondingly:

$$
\begin{aligned}
|\Omega\rangle &= \sum_{X^{\mathbf{P}} \in \mathfrak{S}_{\mathbf{P}}^{\mathfrak{X}_{\mathcal{R}}}} |\mathsf{ter}(X^{\mathbf{P}} /\!\!/ \mathfrak{X}_{\mathcal{R}}^{\mathbf{P}})\rangle = |\mathsf{ter}(\emptyset_{\mathfrak{X}_{\mathcal{R}}^{\mathbf{P}}} /\!\!/ \mathfrak{X}_{\mathcal{R}}^{\mathbf{P}})\rangle + \sum_{n=1}^{N} |\mathsf{ter}(X_n^{\mathbf{P}} /\!\!/ \mathfrak{X}_{\mathcal{R}}^{\mathbf{P}})\rangle\,, \\
\langle\Omega| &= \sum_{x^{\mathbf{P}} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{P}}} \langle x^{\mathbf{P}}| = \langle\varnothing_{\Omega}| + \sum_{n=1}^{N} \langle x_n^{\mathbf{P}}|\,.
\end{aligned}
\tag{50}
$$

Property 9 *(believabilities of terraced $\mathbf{B}$-quotient-bra-events and $\mathbf{B}$-quotient-bra-events).* Believabilities terraced $\mathbf{B}$-quotient-bra-events and $\mathbf{B}$–quotient-bra-events are defined by values of believability measure $\mathbf{B}$ on bra-events in the initial labelling $\langle\mathfrak{X}_{\mathcal{R}}|\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}\rangle$:

$$
\begin{aligned}
b(X^{\mathbf{P}} /\!\!/ \mathfrak{X}_{\mathcal{R}}^{\mathbf{P}}) = \mathbf{B}(\langle\mathsf{ter}(X^{\mathbf{P}} /\!\!/ \mathfrak{X}_{\mathcal{R}}^{\mathbf{P}})|) &= \sum_{x^{\mathbf{P}} \in X^{\mathbf{P}}} \mathbf{B}(\langle x^{\mathbf{P}}|) = \sum_{x^{\mathbf{P}} \in X^{\mathbf{P}}} b_{x^{\mathbf{P}}}, \\
b_{x^{\mathbf{P}}} = \mathbf{B}(\langle x^{\mathbf{P}}|) &= \sum_{x^{\mathbf{P}} \overset{\mathbf{P}}{\sim} w \in \mathfrak{X}_{\mathcal{R}}} \mathbf{B}(\langle w|) = \sum_{x^{\mathbf{P}} \overset{\mathbf{P}}{\sim} w \in \mathfrak{X}_{\mathcal{R}}} b_w.
\end{aligned}
\tag{51}
$$

5.3 Quotient-projections of co$\sim$event

Definition 17 *($\mathbf{B}$-quotient-projection and $\mathbf{P}$-quotient-projection co$\sim$event).* The co$\sim$event $\mathcal{R}_{\mathbf{P}} \subseteq \langle\Omega|\Omega\rangle$ is called a $\mathbf{B}$-*quotient-projection*, and co$\sim$event $\mathcal{R}_{\mathbf{P}} \subseteq \langle\Omega|\Omega\rangle$ is called a $\mathbf{P}$-*quotient-projection of co$\sim$event* $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$, if $\mathcal{R}_{\mathbf{B}}$ generates the $\mathbf{B}$-quotient-labelling $\langle\mathfrak{X}_{\mathcal{R}}^{\mathbf{B}}|\mathfrak{S}_{\mathbf{B}}^{\mathfrak{X}_{\mathcal{R}}}\rangle$, and $\mathcal{R}_{\mathbf{P}}$ generates the $\mathbf{P}$-quotient-labelling $\langle\mathfrak{X}_{\mathcal{R}}^{\mathbf{P}}|\mathfrak{S}_{\mathbf{P}}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ of the bra-ket-space $\langle\Omega|\Omega\rangle$.

Property 10 *(strict order quotient-projections co$\sim$event).* By the definitions $\mathbf{B}$-quotient-labelling and $\mathbf{P}$-quotient-labelling (see Definitions 15 and 16) the $\mathbf{B}$-quotient-projection $\mathcal{R}_{\mathbf{B}} \subseteq \langle\Omega|\Omega\rangle$ and $\mathbf{P}$-quotient-projection $\mathcal{R}_{\mathbf{P}} \subseteq \langle\Omega|\Omega\rangle$ of the arbitrary co$\sim$event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ are always the ordered co$\sim$event which generate the strictly ordered quotient-labelling $\langle\mathfrak{X}_{\mathcal{R}}^{\mathbf{B}}|\mathfrak{S}_{\mathbf{B}}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ and $\langle\mathfrak{X}_{\mathcal{R}}^{\mathbf{P}}|\mathfrak{S}_{\mathbf{P}}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ bra-ket-space $\langle\Omega|\Omega\rangle$.

Property 11 *(certainty quotient-projections co$\sim$event).* The certainty of$\mathbf{B}$-quotient-projection$\mathcal{R}_{\mathbf{B}} \subseteq \langle\Omega|\Omega\rangle$ and $\mathbf{P}$-quotient-projection $\mathcal{R}_{\mathbf{P}} \subseteq \langle\Omega|\Omega\rangle$ of any co$\sim$event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ coincide with the certainty of this co$\sim$event:

$$
\mathbf{\Phi}(\mathcal{R}_{\mathbf{B}}) = \mathbf{\Phi}(\mathcal{R}_{\mathbf{P}}) = \mathbf{\Phi}(\mathcal{R}). \tag{52}
$$

Proof. Let us use the fact (see Property 11 in [23]) that for each co$\sim$event $\mathcal{R}, \mathcal{R}_{\mathbf{B}}, \mathcal{R}_{\mathbf{P}} \subseteq \langle\Omega|\Omega\rangle$ the corresponding partitions are valid, which in their own element-set labellings $\langle\mathfrak{X}_{\mathcal{R}}|\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}\rangle$, $\langle\mathfrak{X}_{\mathcal{R}}^{\mathbf{B}}|\mathfrak{S}_{\mathbf{B}}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ and $\langle\mathfrak{X}_{\mathcal{R}}^{\mathbf{P}}|\mathfrak{S}_{\mathbf{P}}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ are written as follows:

$$
\mathcal{R} = \sum_{x \in \mathfrak{X}_{\mathcal{R}}} \langle x|x\rangle = \sum_{X \in \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}} \langle\mathsf{Ter}_{X /\!\!/ \mathfrak{X}_{\mathcal{R}}}|\mathsf{ter}(X /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle = \sum_{x \in \mathfrak{X}_{\mathcal{R}}} \sum_{x \in X \in \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}} \langle x|\mathsf{ter}(X /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle, \tag{53}
$$

$$
\mathcal{R}_{\mathbf{B}} = \sum_{x^{\mathbf{B}} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{B}}} \langle x^{\mathbf{B}}|x^{\mathbf{B}}\rangle = \sum_{X^{\mathbf{B}} \in \mathfrak{S}_{\mathbf{B}}^{\mathfrak{X}_{\mathcal{R}}}} \langle\mathsf{Ter}_{X^{\mathbf{B}} /\!\!/ \mathfrak{X}_{\mathcal{R}}^{\mathbf{B}}}|\mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}_{\mathcal{R}}^{\mathbf{B}})\rangle = \sum_{x^{\mathbf{B}} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{B}}} \sum_{x^{\mathbf{B}} \in X^{\mathbf{B}} \in \mathfrak{S}_{\mathbf{B}}^{\mathfrak{X}_{\mathcal{R}}}} \langle x^{\mathbf{B}}|\mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}_{\mathcal{R}}^{\mathbf{B}})\rangle, \tag{54}
$$

$$
\mathcal{R}_{\mathbf{P}} = \sum_{x^{\mathbf{P}} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{P}}} \langle x^{\mathbf{P}}|x^{\mathbf{P}}\rangle = \sum_{X^{\mathbf{P}} \in \mathfrak{S}_{\mathbf{P}}^{\mathfrak{X}_{\mathcal{R}}}} \langle\mathsf{Ter}_{X^{\mathbf{P}} /\!\!/ \mathfrak{X}_{\mathcal{R}}^{\mathbf{P}}}|\mathsf{ter}(X^{\mathbf{P}} /\!\!/ \mathfrak{X}_{\mathcal{R}}^{\mathbf{P}})\rangle = \sum_{x^{\mathbf{P}} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{P}}} \sum_{x^{\mathbf{P}} \in X^{\mathbf{P}} \in \mathfrak{S}_{\mathbf{P}}^{\mathfrak{X}_{\mathcal{R}}}} \langle x^{\mathbf{P}}|\mathsf{ter}(X^{\mathbf{P}} /\!\!/ \mathfrak{X}_{\mathcal{R}}^{\mathbf{P}})\rangle. \tag{55}
$$

Hence, since the measure $\mathbf{\Phi}$ is additive, we have

$$\mathbf{\Phi}(\mathcal{R}) = \sum_{x\in\mathfrak{X}_\mathcal{R}} \mathbf{\Phi}(\langle x|x\rangle) = \sum_{X\in\mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}} \mathbf{\Phi}(\langle \mathsf{Ter}_{X/\!\!/\mathfrak{X}_\mathcal{R}}|\mathsf{ter}(X/\!\!/\mathfrak{X}_\mathcal{R})\rangle) = \sum_{x\in\mathfrak{X}_\mathcal{R}} \sum_{x\in X\in\mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}} \mathbf{\Phi}(\langle x|\mathsf{ter}(X/\!\!/\mathfrak{X}_\mathcal{R})\rangle), \tag{56}$$

$$\mathbf{\Phi}(\mathcal{R}_\mathbf{B}) = \sum_{x^\mathbf{B}\in\mathfrak{X}^\mathbf{B}_\mathcal{R}} \mathbf{\Phi}(\langle x^\mathbf{B}|x^\mathbf{B}\rangle) = \sum_{X^\mathbf{B}\in\mathfrak{Z}_\mathbf{B}^{\mathfrak{X}_\mathcal{R}}} \mathbf{\Phi}(\langle \mathsf{Ter}_{X^\mathbf{B}/\!\!/\mathfrak{X}^\mathbf{B}_\mathcal{R}}|\mathsf{ter}(X^\mathbf{B}/\!\!/\mathfrak{X}^\mathbf{B}_\mathcal{R})\rangle) = \sum_{x^\mathbf{B}\in\mathfrak{X}^\mathbf{B}_\mathcal{R}} \sum_{x^\mathbf{B}\in X^\mathbf{B}\in\mathfrak{Z}_\mathbf{B}^{\mathfrak{X}_\mathcal{R}}} \mathbf{\Phi}(\langle x^\mathbf{B}|\mathsf{ter}(X^\mathbf{B}/\!\!/\mathfrak{X}^\mathbf{B}_\mathcal{R})\rangle), \tag{57}$$

$$\mathbf{\Phi}(\mathcal{R}_\mathbf{P}) = \sum_{x^\mathbf{P}\in\mathfrak{X}^\mathbf{P}_\mathcal{R}} \mathbf{\Phi}(\langle x^\mathbf{P}|x^\mathbf{P}\rangle) = \sum_{X^\mathbf{P}\in\mathfrak{Z}_\mathbf{P}^{\mathfrak{X}_\mathcal{R}}} \mathbf{\Phi}(\langle \mathsf{Ter}_{X^\mathbf{P}/\!\!/\mathfrak{X}^\mathbf{P}_\mathcal{R}}|\mathsf{ter}(X^\mathbf{P}/\!\!/\mathfrak{X}^\mathbf{P}_\mathcal{R})\rangle) = \sum_{x^\mathbf{P}\in\mathfrak{X}^\mathbf{P}_\mathcal{R}} \sum_{x^\mathbf{P}\in X^\mathbf{P}\in\mathfrak{Z}_\mathbf{P}^{\mathfrak{X}_\mathcal{R}}} \mathbf{\Phi}(\langle x^\mathbf{P}|\mathsf{ter}(X^\mathbf{P}/\!\!/\mathfrak{X}^\mathbf{P}_\mathcal{R})\rangle). \tag{58}$$

By Axiom 13 of the theory of experience and chance (see [22]) for every $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ to each $\mathcal{R}$-labelled co∼event $\langle\lambda^*_\mathcal{R}|\lambda_\mathcal{R}\rangle \in \langle\mathcal{A}|\mathcal{A}\rangle$ a nonnegative real number is associated with its certainty, which is equal to $\mathbf{\Phi}(\langle\lambda^*_\mathcal{R}|\lambda_\mathcal{R}\rangle) = \mathbf{B}(\langle\lambda^*_\mathcal{R}|)\mathbf{P}(|\lambda_\mathcal{R}\rangle)$. Using the standard notation of the theory of experience and chance:

$$\begin{gathered} b_x = \mathbf{B}(\langle x|), \quad p_x = \mathbf{P}(|x\rangle), \\ b(X/\!\!/\mathfrak{X}_\mathcal{R}) = \mathbf{B}(\langle \mathsf{Ter}_{X/\!\!/\mathfrak{X}_\mathcal{R}}|), \quad p(X/\!\!/\mathfrak{X}_\mathcal{R}) = \mathbf{P}(|\mathsf{ter}(X/\!\!/\mathfrak{X}_\mathcal{R})\rangle), \end{gathered} \tag{59}$$

we get

$$\mathbf{\Phi}(\mathcal{R}) = \sum_{x\in\mathfrak{X}_\mathcal{R}} b_x p_x = \sum_{X\in\mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}} b(X/\!\!/\mathfrak{X}_\mathcal{R})p(X/\!\!/\mathfrak{X}_\mathcal{R}) = \sum_{x\in\mathfrak{X}_\mathcal{R}} \sum_{x\in X\in\mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}} b_x p(X/\!\!/\mathfrak{X}_\mathcal{R}), \tag{60}$$

$$\mathbf{\Phi}(\mathcal{R}_\mathbf{B}) = \sum_{x^\mathbf{B}\in\mathfrak{X}^\mathbf{B}_\mathcal{R}} b_{x^\mathbf{B}} p_{x^\mathbf{B}} = \sum_{X^\mathbf{B}\in\mathfrak{Z}_\mathbf{B}^{\mathfrak{X}_\mathcal{R}}} b(X^\mathbf{B}/\!\!/\mathfrak{X}^\mathbf{B}_\mathcal{R})p(X^\mathbf{B}/\!\!/\mathfrak{X}^\mathbf{B}_\mathcal{R}) = \sum_{x^\mathbf{B}\in\mathfrak{X}^\mathbf{B}_\mathcal{R}} \sum_{x^\mathbf{B}\in X^\mathbf{B}\in\mathfrak{Z}_\mathbf{B}^{\mathfrak{X}_\mathcal{R}}} b_{x^\mathbf{B}} p(X^\mathbf{B}/\!\!/\mathfrak{X}^\mathbf{B}_\mathcal{R}), \tag{61}$$

$$\mathbf{\Phi}(\mathcal{R}_\mathbf{P}) = \sum_{x^\mathbf{P}\in\mathfrak{X}^\mathbf{P}_\mathcal{R}} b_{x^\mathbf{P}} p_{x^\mathbf{P}} = \sum_{X^\mathbf{P}\in\mathfrak{Z}_\mathbf{P}^{\mathfrak{X}_\mathcal{R}}} b(X^\mathbf{P}/\!\!/\mathfrak{X}^\mathbf{P}_\mathcal{R})p(X^\mathbf{P}/\!\!/\mathfrak{X}^\mathbf{P}_\mathcal{R}) = \sum_{x^\mathbf{P}\in\mathfrak{X}^\mathbf{P}_\mathcal{R}} \sum_{x^\mathbf{P}\in X^\mathbf{P}\in\mathfrak{Z}_\mathbf{P}^{\mathfrak{X}_\mathcal{R}}} b_{x^\mathbf{P}} p(X^\mathbf{P}/\!\!/\mathfrak{X}^\mathbf{P}_\mathcal{R}). \tag{62}$$

Taking into account (61), (62) and (63) and applying the set-summation technique [17], we get what is required:

$$\mathbf{\Phi}(\mathcal{R}) = \sum_{x\in\mathfrak{X}_\mathcal{R}} b_x p_x = \sum_{x^\mathbf{P}\in\mathfrak{X}^\mathbf{P}_\mathcal{R}} \sum_{x^\mathbf{P}\overset{\mathbf{P}}{\sim} w\in\mathfrak{X}_\mathcal{R}} b_w p_w = \sum_{x^\mathbf{P}\in\mathfrak{X}^\mathbf{P}_\mathcal{R}} p_{x^\mathbf{P}} \sum_{x^\mathbf{P}\overset{\mathbf{P}}{\sim} w\in\mathfrak{X}_\mathcal{R}} b_w = \sum_{x^\mathbf{P}\in\mathfrak{X}^\mathbf{P}_\mathcal{R}} p_{x^\mathbf{P}} b_{x^\mathbf{P}} = \mathbf{\Phi}(\mathcal{R}_\mathbf{P}), \tag{63}$$

$$\begin{aligned} \mathbf{\Phi}(\mathcal{R}) &= \sum_{X\in\mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}} b(X/\!\!/\mathfrak{X}_\mathcal{R})p(X/\!\!/\mathfrak{X}_\mathcal{R}) = \sum_{X^\mathbf{B}\in\mathfrak{Z}_\mathbf{B}^{\mathfrak{X}_\mathcal{R}}} \sum_{X^\mathbf{B}\overset{\mathbf{B}}{\sim} W\in\mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}} b(W/\!\!/\mathfrak{X}_\mathcal{R})p(W/\!\!/\mathfrak{X}_\mathcal{R}) \\ &= \sum_{X^\mathbf{B}\in\mathfrak{Z}_\mathbf{B}^{\mathfrak{X}_\mathcal{R}}} b(X^\mathbf{B}/\!\!/\mathfrak{X}^\mathbf{B}_\mathcal{R}) \sum_{X^\mathbf{B}\overset{\mathbf{B}}{\sim} W\in\mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}} p(W/\!\!/\mathfrak{X}_\mathcal{R}) = \sum_{X^\mathbf{B}\in\mathfrak{Z}_\mathbf{B}^{\mathfrak{X}_\mathcal{R}}} b(X^\mathbf{B}/\!\!/\mathfrak{X}^\mathbf{B}_\mathcal{R})p(X^\mathbf{B}/\!\!/\mathfrak{X}^\mathbf{B}_\mathcal{R}) = \mathbf{\Phi}(\mathcal{R}_\mathbf{B}). \end{aligned} \tag{64}$$

5.4 Quotient-indicator of the co∼event on the quotient-space

Definition 18 *(quotient-space).* The $\mathbf{P}$-$\mathbf{B}$-quotient-labelling

$$\left\langle \mathfrak{X}^\mathbf{P}_\mathcal{R} \,\middle|\, \mathfrak{Z}_\mathbf{B}^{\mathfrak{X}_\mathcal{R}} \right\rangle = \langle \mathfrak{X}^\mathbf{P}_\mathcal{R}| \times |\mathfrak{Z}_\mathbf{B}^{\mathfrak{X}_\mathcal{R}}\rangle = \left\{ \langle x^\mathbf{P}|\mathsf{ter}(X^\mathbf{B}/\!\!/\mathfrak{X}^\mathbf{B}_\mathcal{R})\rangle : x^\mathbf{P} \in \mathfrak{X}^\mathbf{P}_\mathcal{R}, X^\mathbf{B} \in \mathfrak{Z}_\mathbf{B}^{\mathfrak{X}_\mathcal{R}} \right\} \tag{65}$$

of the space $\langle\Omega|\Omega\rangle$ generated by co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ is called a *quotient-space*[10] of the space $\langle\Omega|\Omega\rangle$ with respect to $\mathcal{R}$, i.e. by relations of $\mathbf{P}$-equivalence «$\overset{\mathbf{P}}{\sim}$» on $\langle\Omega|$ and of $\mathbf{B}$-equivalence «$\overset{\mathbf{B}}{\sim}$» on $|\Omega\rangle$.

Definition 19 *(certainty quotient-distribution of measure on a quotient-space).* For each co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ the *certainty quotient-distribution (quotient-c.d.) of measure*[11] $\mathbf{\Phi}$ on the quotient-space $\langle \mathfrak{X}^\mathbf{P}_\mathcal{R}|\mathfrak{Z}_\mathbf{B}^{\mathfrak{X}_\mathcal{R}}\rangle$ of the space $\langle\Omega|\Omega\rangle$ is a family

$$\left\{ \varphi_{x^\mathbf{P}}(X^\mathbf{B}/\!\!/\mathfrak{X}^\mathbf{B}_\mathcal{R}) \colon \langle x^\mathbf{P}|\mathsf{ter}(X^\mathbf{B}/\!\!/\mathfrak{X}^\mathbf{B}_\mathcal{R})\rangle \in \langle \mathfrak{X}^\mathbf{P}_\mathcal{R}|\mathfrak{Z}_\mathbf{B}^{\mathfrak{X}_\mathcal{R}}\rangle \right\} \tag{66}$$

[10] see illustrations of notions introduced in this paragraph in figurees 3 — 18.

of certainties $\varphi_{x^{\mathbf{P}}}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) = \mathbf{\Phi}(\langle x^{\mathbf{P}} | \mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) \rangle)$ of elementary quotient-incomes-outcomes $\langle x^{\mathbf{P}} | \mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) \rangle \in \langle \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}} | \mathbf{\mathfrak{Z}}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}} \rangle$.

`Theorem 3` *(formula of certainty quotient-distribution of measure).* The *certainty quotient-distribution of the measure $\mathbf{\Phi}$ on the quotient-space $\langle \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}} | \mathbf{\mathfrak{Z}}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}} \rangle$ of the space $\langle \Omega | \Omega \rangle$ is calculated for $\langle x^{\mathbf{P}} | \mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) \rangle \in \langle \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}} | \mathbf{\mathfrak{Z}}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}} \rangle$ by the following formulas:*

$$\varphi_{x^{\mathbf{P}}}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) = \sum_{x^{\mathbf{P}} \overset{\mathbf{P}}{\sim} w \in \mathfrak{X}_{\mathcal{R}}} \sum_{X^{\mathbf{B}} \overset{\mathbf{B}}{\sim} W \in \mathbf{\mathfrak{Z}}^{\mathfrak{X}_{\mathcal{R}}}} \varphi_w(W /\!\!/ \mathfrak{X}_{\mathcal{R}}), \tag{67}$$

where $\langle w | \mathsf{ter}(W /\!\!/ \mathfrak{X}_{\mathcal{R}}) \rangle \in \langle \mathfrak{X}_{\mathcal{R}} | \mathbf{\mathfrak{Z}}^{\mathfrak{X}_{\mathcal{R}}} \rangle$ is an elementary income-outcome, and $\varphi_w(W /\!\!/ \mathfrak{X}_{\mathcal{R}}) = \mathbf{\Phi}(\langle w | \mathsf{ter}(W /\!\!/ \mathfrak{X}_{\mathcal{R}}) \rangle)$ is its certainty.

Proof follows from the partition of elementary quotient-income-outcomes into elementary income-outcomes:

$$\langle x^{\mathbf{P}} | \mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) \rangle = \sum_{x^{\mathbf{P}} \overset{\mathbf{P}}{\sim} w \in \mathfrak{X}_{\mathcal{R}}} \sum_{X^{\mathbf{B}} \overset{\mathbf{B}}{\sim} W \in \mathbf{\mathfrak{Z}}^{\mathfrak{X}_{\mathcal{R}}}} \langle w | \mathsf{ter}(W /\!\!/ \mathfrak{X}_{\mathcal{R}}) \rangle, \tag{68}$$

and from the additivity of the certainty measure $\mathbf{\Phi}$.

`Definition 20` *(certainty quotient-distribution of co∼event).* The *certainty quotient-distribution (quotient-c.d.) of co∼event*[11] $\mathcal{R} \subseteq \langle \Omega | \Omega \rangle$ on the quotient-space $\langle \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}} | \mathbf{\mathfrak{Z}}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}} \rangle$ of the space $\langle \Omega | \Omega \rangle$ is a family

$$\left\{ \varphi^{\mathcal{R}}_{x^{\mathbf{P}}}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) \colon \langle x^{\mathbf{P}} | \mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) \rangle \in \langle \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}} | \mathbf{\mathfrak{Z}}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}} \rangle \right\}, \tag{69}$$

consisting from certainties $\varphi^{\mathcal{R}}_{x^{\mathbf{P}}}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) = \mathbf{\Phi}(\mathcal{R} \cap \langle x^{\mathbf{P}} | \mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) \rangle)$ of intersections of co∼event $\mathcal{R}$ with elementary quotient-incomes-outcomes $\langle x^{\mathbf{P}} | \mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) \rangle \in \langle \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}} | \mathbf{\mathfrak{Z}}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}} \rangle$.

`Theorem 4` *(formula of certainty quotient-distribution of co∼event).* The *certainty quotient-distribution of co∼event $\mathcal{R} \subseteq \langle \Omega | \Omega \rangle$ on the quotient-space $\langle \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}} | \mathbf{\mathfrak{Z}}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}} \rangle$ of the space $\langle \Omega | \Omega \rangle$ is calculated for each elementary quotient-income-outcome $\langle x^{\mathbf{P}} | \mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) \rangle \in \langle \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}} | \mathbf{\mathfrak{Z}}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}} \rangle$ by the formula:*

$$\varphi^{\mathcal{R}}_{x^{\mathbf{P}}}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) = \sum_{x^{\mathbf{P}} \overset{\mathbf{P}}{\sim} w \in \mathfrak{X}_{\mathcal{R}}} \sum_{\substack{w \in W \\ X^{\mathbf{B}} \overset{\mathbf{B}}{\sim} W \in \mathbf{\mathfrak{Z}}^{\mathfrak{X}_{\mathcal{R}}}}} \varphi_w(W /\!\!/ \mathfrak{X}_{\mathcal{R}}), \tag{70}$$

where $\varphi_w(W /\!\!/ \mathfrak{X}_{\mathcal{R}}) = \mathbf{\Phi}(\langle w | \mathsf{ter}(W /\!\!/ \mathfrak{X}_{\mathcal{R}}) \rangle)$ is a certainty of elementary income-outcome $\langle w | \mathsf{ter}(W /\!\!/ \mathfrak{X}_{\mathcal{R}}) \rangle \in \langle \mathfrak{X}_{\mathcal{R}} | \mathbf{\mathfrak{Z}}^{\mathfrak{X}_{\mathcal{R}}} \rangle$.

Proof follows from the obvious partition:

$$\mathcal{R} \cap \langle x^{\mathbf{P}} | \mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) \rangle = \sum_{x^{\mathbf{P}} \overset{\mathbf{P}}{\sim} w \in \mathfrak{X}_{\mathcal{R}}} \sum_{X^{\mathbf{B}} \overset{\mathbf{B}}{\sim} W \in \mathbf{\mathfrak{Z}}^{\mathfrak{X}_{\mathcal{R}}}} \big(\mathcal{R} \cap \langle w | \mathsf{ter}(W /\!\!/ \mathfrak{X}_{\mathcal{R}}) \rangle \big), \tag{71}$$

the additivity of certainty measure $\mathbf{\Phi}$, and the fact that the relation of membership $\langle \omega^* | \omega \rangle \in \mathcal{R}$ is performed whenever $\langle \omega^* | \omega \rangle \in \langle w | \mathsf{ter}(W /\!\!/ \mathfrak{X}_{\mathcal{R}}) \rangle$ and $w \in W$ (see the proof in [22]).

`Definition 21` *(indicator of co∼event on bra-ket-space).* The *indicator of co∼event* $\mathcal{R} \subseteq \langle \Omega | \Omega \rangle$ on the bra-ket-space $\langle \Omega | \Omega \rangle$ is a Boolean[11] function

$$\mathbf{1}_{\mathcal{R}} \colon \langle \Omega | \Omega \rangle \to \{0, 1\}, \tag{72}$$

such that

$$\mathbf{1}_{\mathcal{R}}(\langle \omega^* | \omega \rangle) = \begin{cases} 1, & \langle \omega^* | \omega \rangle \in \mathcal{R}, \\ 0, & \text{otherwise}. \end{cases} \tag{73}$$

[11] Here *Boolean function* is a function that takes values 0 or 1.

Definition 22 *(quotient-indicator co∼event on the quotient-space).* The *quotient-indicator of co∼event*[11] $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ on the quotient-space $\langle \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}} | \mathbf{Ƨ}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}} \rangle$ of the bra-ket space $\langle\Omega|\Omega\rangle$ is a real function

$$\mathbf{1}^{\mathbf{PB}}_{\mathcal{R}} \colon \left\langle \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}} \,\middle|\, \mathbf{Ƨ}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}} \right\rangle \to [0,1], \tag{74}$$

that on each elementary quotient-income-outcome $\langle x^{\mathbf{P}} | \mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}})\rangle \in \langle \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}} | \mathbf{Ƨ}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}} \rangle$ takes a value

$$\mathbf{1}^{\mathbf{PB}}_{\mathcal{R}}(\langle x^{\mathbf{P}} | \mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}})\rangle) = \frac{\varphi^{\mathcal{R}}_{x^{\mathbf{P}}}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}})}{\varphi_{x^{\mathbf{P}}}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}})} \in [0,1], \tag{75}$$

where $\varphi^{\mathcal{R}}_{x^{\mathbf{P}}}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) = \mathbf{\Phi}(\mathcal{R} \cap \langle x^{\mathbf{P}} | \mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}})\rangle)$ and $\varphi_{x^{\mathbf{P}}}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) = \mathbf{\Phi}(\langle x^{\mathbf{P}} | \mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}})\rangle)$ is a value of the certainty quotient-distribution of co∼event $\mathcal{R}$ and a value of the certainty quotient-distribution of $\mathbf{\Phi}$ on the elemenatry quotient-income-outcome $\langle x^{\mathbf{P}} | \mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}})\rangle$ correspondingly.

Property 12 *(quotient-chracteristics of a complement co∼event).* We give without proof the formulas for calculating quotient-characteristics of the complement co∼event $\mathcal{R}^c = \langle\Omega|\Omega\rangle - \mathcal{R}$, which are related to the corresponding quotient-characteristics of the co∼event $\mathcal{R}$ by the obvious relations:

$$\begin{aligned} \varphi^{\mathcal{R}^c}_{x^{\mathbf{P}}}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) &= \mathbf{\Phi}(\mathcal{R}^c \cap \langle x^{\mathbf{P}} | \mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}})\rangle) \\ &= \varphi_{x^{\mathbf{P}}}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) - \varphi^{\mathcal{R}}_{x^{\mathbf{P}}}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}) = \sum_{x^{\mathbf{P}} \overset{\mathbf{P}}{\sim} w \in \mathfrak{X}_{\mathcal{R}}} \; \sum_{\substack{w \notin W \\ X^{\mathbf{B}} \overset{\mathbf{B}}{\sim} W \in \mathbf{Ƨ}^{\mathfrak{X}_{\mathcal{R}}}}} \varphi_w(W /\!\!/ \mathfrak{X}_{\mathcal{R}}) \end{aligned} \tag{76}$$

are certainties of intersections of $\mathcal{R}^c$ with elementary quotient-incomes-outcomes $\langle x^{\mathbf{P}} | \mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}})\rangle \in \langle \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}} | \mathbf{Ƨ}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}} \rangle$, form the *certainty quotient-distribution of the complement co∼event*[11];

$$\mathbf{1}_{\mathcal{R}^c}(\langle \omega^* | \omega \rangle) = 1 - \mathbf{1}_{\mathcal{R}}(\langle \omega^* | \omega \rangle) = \begin{cases} 1, & \langle \omega^* | \omega \rangle \in \mathcal{R}^c, \\ 0, & \text{otherwise}, \end{cases} \tag{77}$$

is an *indicator of the complement co∼event* on bra-ket-space;

$$\mathbf{1}^{\mathbf{PB}}_{\mathcal{R}^c}(\langle x^{\mathbf{P}} | \mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}})\rangle) = 1 - \mathbf{1}^{\mathbf{PB}}_{\mathcal{R}}(\langle x^{\mathbf{P}} | \mathsf{ter}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}})\rangle) = \frac{\varphi^{\mathcal{R}^c}_{x^{\mathbf{P}}}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}})}{\varphi_{x^{\mathbf{P}}}(X^{\mathbf{B}} /\!\!/ \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}})} \in [0,1], \tag{78}$$

is a *quotient-indicator of the complement co∼event*[11] on quotient-space.

6 Dual co∼event means

Let $\langle \Omega, \mathcal{A}, \mathbf{B} | \Omega, \mathcal{A}, \mathbf{P} \rangle = \left(\langle \Omega|\Omega \rangle, \langle \mathcal{A}|\mathcal{A} \rangle, \mathbf{\Phi} \right)$ be the certainty bra-ket-space with the sigma-algebra[12] $\langle \mathcal{A}|\mathcal{A} \rangle$ and the certainty measure $\mathbf{\Phi}$; $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ be the co∼event that is defined as a measurable binary relation $\mathcal{R} \in \langle \mathcal{A}|\mathcal{A} \rangle$ on $\langle\Omega|\Omega\rangle$ (see [22]); and $\mathbf{\Phi}(\mathcal{R})$ be the value of its certainty measure. The co∼event $\mathcal{R}$ is a proposed by the theory of experience and chance [22] mathematical model of dual uncertainty that arises between the observer and the observation in the process of an experienced-random experiment. Therefore, the problem of the mean description of such uncertainty seems quite natural. What is the mean description of the dual uncertainty of the results of the experienced-random experiment? What new and relevant ideas can this mean description offer, if any? I will try to answer these questions, noting at once that in this paper only the mean description of the only one co∼event is considered, and the mean description of the set of co∼events remains outside of the paper and it will be considered in my next papers.

Any co∼event as a measurable binary relation on $\langle\Omega|\Omega\rangle$, i.e. some its measurable subset $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$, generates the element-set labelling $\langle \mathfrak{X}_{\mathcal{R}} | \mathbf{Ƨ}^{\mathfrak{X}_{\mathcal{R}}} \rangle = \langle \mathfrak{X}_{\mathcal{R}} | \times | \mathbf{Ƨ}^{\mathfrak{X}_{\mathcal{R}}} \rangle$ of the bra-ket-space $\langle\Omega|\Omega\rangle$ (see [23]), where

$$\begin{aligned} \langle \mathfrak{X}_{\mathcal{R}} | &= \{ \langle x | \, , x \in \mathfrak{X}_{\mathcal{R}} \} = \langle \Omega | / \mathcal{R}, \\ | \mathbf{Ƨ}^{\mathfrak{X}_{\mathcal{R}}} \rangle &= \{ | \mathsf{ter}(X /\!\!/ \mathfrak{X}_{\mathcal{R}}) \rangle \, , X \in \mathbf{Ƨ}^{\mathfrak{X}_{\mathcal{R}}} \} = | \Omega \rangle / \mathcal{R} \end{aligned} \tag{79}$$

[12] In order not to complicate the definitions of dual co∼event means by excessive technical details, it will always be assumed that the sigma-algebras $\langle \mathcal{A}|$ and $|\mathcal{A}\rangle$ are quite «rich», so that for each pair of nested elements, for example, $\lambda_1 \subseteq \lambda_2 \in \langle \mathcal{A}|$, contain an intermediate third element $\lambda_1 \subseteq \lambda \subseteq \lambda_2$ of arbitrary measure $\mathbf{B}(\lambda)$ such that $\mathbf{B}(\lambda_1) \leqslant \mathbf{B}(\lambda) \leqslant \mathbf{B}(\lambda_2)$.

are the corresponding quotient-sets by co$\sim$event $\mathcal{R}$. Elements of these quotient-sets serve bra-events $\langle x| \subseteq \langle\Omega|, x \in \mathfrak{X}_\mathcal{R}$, and terraced ket-events $|\text{ter}(X/\!\!/\mathfrak{X}_\mathcal{R})\rangle \subseteq |\Omega\rangle, X \in \mathfrak{s}^{\mathfrak{X}_\mathcal{R}}$.

Besides bra-events and terraced ket-events the element-set labelling also defines for each set label $X \in \mathfrak{s}^{\mathfrak{X}_\mathcal{R}} \subseteq \mathcal{P}(\mathfrak{X}_\mathcal{R})$ and for each label $x \in \mathfrak{X}_\mathcal{R} \subseteq \mathcal{A}$ the terraced bra-events and ket-events correspondingly:

$$\langle \text{Ter}_{X/\!\!/\mathfrak{X}_\mathcal{R}}| = \sum_{x \in X \in \mathfrak{s}^{\mathfrak{X}_\mathcal{R}}} \langle x| \subseteq \langle\Omega|, \tag{80}$$

$$|x\rangle = \sum_{x \in X \in \mathfrak{s}^{\mathfrak{X}_\mathcal{R}}} |\text{ter}(X/\!\!/\mathfrak{X}_\mathcal{R})\rangle \subseteq |\Omega\rangle. \tag{81}$$

The mean description of the co$\sim$event $\mathcal{R}$, interesting to me in this work, is the *dual co$\sim$event means* with definitions which are based on concepts of *dual event means*:

- mean terraced bra-event among terraced bra-events (81) and
- mean ket-event among ket-events (82),

that will be considered for the third time in paragraphs 6.1.2 and 6.1.4, as co$\sim$event improvements of *mean-believable terraced bra-event* and *mean-probable ket-event* correspondingly (see also initial definitions in the paragraphs 2.1 and 2.2 on pages 146 and 148) discussed in paragraphs 3.2 and 3.4 on pages 150 and 152.

Difference from the former[13]**.** In defining of dual event means for the co$\sim$event $\mathcal{R}$ I will need not only bra-events $\langle x| \in \langle\mathfrak{X}_\mathcal{R}|$ and terraced bra-events $\langle\text{Ter}_{X/\!\!/\mathfrak{X}_\mathcal{R}}| \in \langle\mathfrak{s}^{\mathfrak{X}_\mathcal{R}}|$, or ket-events $|x\rangle \in |\mathfrak{X}_\mathcal{R}\rangle$ and terraced bra-events $|\text{ter}(X/\!\!/\mathfrak{X}_\mathcal{R})\rangle \in |\mathfrak{s}^{\mathfrak{X}_\mathcal{R}}\rangle$, i.e. elements of the sigma-subalgebra $\langle\mathcal{A}_{\mathfrak{X}_\mathcal{R}}| \subseteq \langle\mathcal{A}|$ and sigma-subalgebra $|\mathcal{A}^{\mathfrak{X}_\mathcal{R}}\rangle \subseteq \langle\mathcal{A}|$, but elements of more wide sigma-algebras $\langle\mathcal{A}|$ and $|\mathcal{A}\rangle$, which are supposed to be *quite rich*[14], in order to contain all possible value of defined concepts. Elements of quite rich sigma-algebras $\langle\mathcal{A}|$ and $\langle\mathcal{A}|$ I will continue to be called *bra-events* and *ket-events* correspondingly. And in order to better distinguish these bra-events and ket-events from bra-events and ket-events from sigma-subalgebras $\langle\mathcal{A}_{\mathfrak{X}_\mathcal{R}}|$ and $|\mathcal{A}^{\mathfrak{X}_\mathcal{R}}\rangle$ I will use for their designation other labels (more often $w \in \mathcal{A}$ for bra-events $\langle w| \in \langle\mathcal{A}|$ and ket-events $|w\rangle \in |\mathcal{A}\rangle$) avoiding previous labels $x \in \mathfrak{X}_\mathcal{R}$ and set labels $X \in \mathfrak{s}^{\mathfrak{X}_\mathcal{R}}$. The fee for extending the range of values of dual event means is not so much their nonuniqueness, which is quite natural, how much is the impossibility of their labelling in the framework of the sigma-algebra generated by the labelling sets $\mathfrak{X}_\mathcal{R}$ and $\mathfrak{s}^{\mathfrak{X}_\mathcal{R}}$. Of course, this is not an acceptable property. However, I will show how to «partially return» the possibility of convenient labelling, defining even broader concepts of dual co$\sim$ event means.

6.1 Event means for an ordered co$\sim$event

Let $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ (see Definition 9 on page 153) be an ordered co$\sim$event on the certainty bra-ket-space $\langle\Omega, \mathcal{A}, \mathbf{B}|\Omega, \mathcal{A}, \mathbf{P}\rangle$, which generates the labelling set of labels $\mathfrak{X}_\mathcal{R} \subseteq \mathcal{A}$ and the labelling set of set labels $\mathfrak{s}^{\mathfrak{X}_\mathcal{R}} \subseteq \mathcal{P}(\mathfrak{X}_\mathcal{R})$ forming the element-set labelling $\langle\mathfrak{X}_\mathcal{R}|\mathfrak{s}^{\mathfrak{X}_\mathcal{R}}\rangle$ of the space.

By properties 2, 4 and 5 on page 155–156 wherein the numbers of nonempty labels in two labelling sets of the ordered co$\sim$event $\mathcal{R}$ coincide: $|\mathfrak{X}_\mathcal{R}|_{\neq\varnothing} = \left|\mathfrak{s}^{\mathfrak{X}_\mathcal{R}}\right|_{\neq\emptyset} = N$, the labelling sets themselves have the form:

$$\langle\mathfrak{X}_\mathcal{R}|\mathfrak{s}^{\mathfrak{X}_\mathcal{R}}\rangle = \begin{cases} \begin{cases} \langle x_1, \ldots, x_N | X_N, \ldots, X_1\rangle \\ \langle x_1, \ldots, x_N | X_N, \ldots, X_1, \emptyset\rangle \\ \langle x_1, \ldots, x_N, \varnothing | X_N, \ldots, X_1\rangle \\ \langle x_1, \ldots, x_N, \varnothing | X_N, \ldots, X_1, \emptyset\rangle \end{cases}, & \mathcal{R} \neq \emptyset \subseteq \langle\Omega|\Omega\rangle, \ N > 0, \\ \langle\varnothing|\emptyset\rangle, & \mathcal{R} = \emptyset \subseteq \langle\Omega|\Omega\rangle, \ N = 0, \end{cases} \tag{82}$$

ket-events and terraced bra-events are strict ordered by inclusion:

$$\begin{gathered} |\Omega\rangle \supseteq |x_1\rangle \supset |x_2\rangle \supset \ldots \supset |x_N\rangle \supset \varnothing_{|\Omega\rangle}, \\ \emptyset_{\langle\Omega|} \subseteq \langle\text{Ter}_{X_1/\!\!/\mathfrak{X}_\mathcal{R}}| \subset \langle\text{Ter}_{X_2/\!\!/\mathfrak{X}_\mathcal{R}}| \subset \ldots \subset \langle\text{Ter}_{X_N/\!\!/\mathfrak{X}_\mathcal{R}}| \subseteq \langle\Omega|. \end{gathered} \tag{83}$$

[13]See definitions of means for a *random set* of arbitrary elements [16, 1] and for a *random set of events* [19, 24]

[14]About the *quite rich* sigma-algebra see Footnote 13 on page 165.

and for $n = 1, \ldots, N$ the formulas are valid:

$$
\begin{aligned}
|x_n\rangle &= \sum_{i=n}^{N} |\mathsf{ter}(X_i /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle \,, \\
\langle \mathsf{Ter}_{X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}}| &= \sum_{i=1}^{n} \langle x_i| \,.
\end{aligned}
\tag{84}
$$

6.1.1 Random bra-element generated by an ordered co∼event

The ordered event $\mathcal{R}$ generating the *set of bra-events* $\langle \mathfrak{X}_{\mathcal{R}}| = \{\langle x| : x \in \mathfrak{X}_{\mathcal{R}}\} \subset \langle \mathcal{A}|$ and the set of terraced bra-events $\langle \mathbf{ƨ}^{\mathfrak{X}_{\mathcal{R}}}| = \left\{\langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}}| : X \in \mathbf{ƨ}^{\mathfrak{X}}\right\} \subseteq \langle \mathcal{A}_{\mathfrak{X}}| \subseteq \langle \mathcal{A}|$ defines on the probability ket-space $|\Omega, \mathcal{A}, \mathbf{P}\rangle$ the *random bra-element*

$$\langle \mathcal{R}| : |\Omega\rangle \to \langle \mathbf{ƨ}^{\mathfrak{X}_{\mathcal{R}}}| \,,$$

which in the case of elementary ket-outcome $|\omega\rangle \in |\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle \,, X_n \in \mathbf{ƨ}^{\mathfrak{X}_{\mathcal{R}}}$ takes a value

$$\langle \mathcal{R}|\omega\rangle = \langle \mathcal{R}| \, (|\omega\rangle) = \langle \mathsf{Ter}_{X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}}| \in \langle \mathbf{ƨ}^{\mathfrak{X}_{\mathcal{R}}}|$$

from the bra-area $\langle \mathbf{ƨ}^{\mathfrak{X}_{\mathcal{R}}}|$ that is contained in the sigma-subalgebra $\langle \mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}| \subseteq \langle \mathcal{A}|$ believability bra-space $\langle \Omega, \mathcal{A}, \mathbf{B}|$ generated by bra-events $\langle x| \in \langle \mathfrak{X}_{\mathcal{R}}| \subseteq \langle \mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}| \subseteq \langle \mathcal{A}|$. Its value $\langle \mathcal{R}|\omega\rangle$ is interpreted as the terraced bra-event $\langle \mathsf{Ter}_{X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}}| \in \langle \mathbf{ƨ}^{\mathfrak{X}_{\mathcal{R}}}|$ which is *experienced* with believability

$$b(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}) = \mathbf{B}(\langle \mathsf{Ter}_{X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}}|) = \sum_{i=1}^{n} B(\langle x_i|) = \sum_{i=1}^{n} b_{x_i} , \tag{85}$$

when the terraced ket-event $|\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle$ *happens,* i.e. elementary ket-income $|\omega\rangle \in |\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle$ happens with probability

$$p(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}) = \mathbf{P}(|\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle). \tag{86}$$

The *random bra-element* $\langle \mathcal{R}|$ is defined by

1) a family $\langle \mathbf{ƨ}^{\mathfrak{X}_{\mathcal{R}}}| = \left\{\langle \mathsf{Ter}_{X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}}| : n = 1, \ldots, N\right\}$ of its values, terraced bra-events

$$\langle \mathsf{Ter}_{X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}}| = \sum_{i=1}^{n} \langle x_i| \subseteq \langle \Omega| \,,$$

which are experienced with believability $b(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}) = \mathbf{B}(\langle \mathsf{Ter}_{X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}}|)$ and are defined as sums of bra-events $\langle x| \in \langle \mathfrak{X}_{\mathcal{R}}|$ forming the partition of the space of elementary bra-incomes $\langle \Omega|$:

$$\langle \Omega| = \sum_{x \in \mathfrak{X}_{\mathcal{R}}} \langle x| \,;$$

2) a family $\left\{p(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}), X_n \in \mathbf{ƨ}^{\mathfrak{X}_{\mathcal{R}}}\right\}$ of probabilities

$$p(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}) = \mathbf{P}\Big(\{\, |\omega\rangle : \langle \mathcal{R}|\omega\rangle = \langle \mathsf{Ter}_{X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}}| \,\}\Big) = \mathbf{P}\Big(\, |\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle \Big)$$

of terraced ket-events

$$|\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle = \bigcap_{x \in X_n} |x\rangle \bigcap_{x \in \mathfrak{X}_{\mathcal{R}} - X_n} |x\rangle^c \subseteq |\Omega\rangle \,,$$

on which the random bra-element takes corresponding values $\langle \mathsf{Ter}_{X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}}| \subseteq \langle \Omega|$ and which forms a partition of the space of elementary ket-incomes $|\Omega\rangle$ generated by the set of ket-events $|\mathfrak{X}_{\mathcal{R}}\rangle$:

$$|\Omega\rangle = \sum_{X_n \in \mathbf{ƨ}^{\mathfrak{X}_{\mathcal{R}}}} |\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle \,.$$

6.1.2 Mean-believable bra-event for an ordered co$\sim$event

Definition 23 *(mean-believable bra-event).* Let for the ordered co$\sim$event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ with the element-set labelling $\langle\mathfrak{X}_\mathcal{R}|\mathfrak{S}^{\mathfrak{X}}_\mathcal{R}\rangle$ (83) the designations are entered:

$$\langle\mathcal{E}_n\mathcal{R}| = \begin{cases} \langle\mathsf{Ter}_{X_n/\!\!/\mathfrak{X}_\mathcal{R}}|\,, & n = 1,\dots,N, \\ \langle\mathsf{Ter}_{X_0/\!\!/\mathfrak{X}_\mathcal{R}}| = \langle\mathsf{Ter}_{\emptyset/\!\!/\mathfrak{X}_\mathcal{R}}| = \varnothing_{\langle\Omega|}, & n = 0, \end{cases} \tag{87}$$

$$b(X_0/\!\!/\mathfrak{X}_\mathcal{R}) = \mathbf{B}(\langle\mathsf{Ter}_{X_0/\!\!/\mathfrak{X}_\mathcal{R}}|) = \mathbf{B}(\langle\mathsf{Ter}_{\emptyset/\!\!/\mathfrak{X}_\mathcal{R}}|) = \mathbf{B}(\varnothing_{\langle\Omega|}) = 0,$$

then the ***mean-believable bra-event*** of its random bra-element $\langle\mathcal{R}|$ is defined as any bra-event $\langle\mathcal{E}\mathcal{R}| \in \langle\mathcal{A}|$ with believability $\mathbf{B}(\langle\mathcal{E}\mathcal{R}|) = \mathbf{\Phi}(\mathcal{R})$ which satisfies the inclusions:

$$\langle\mathcal{E}_n\mathcal{R}| \;\subset\; \langle\mathcal{E}\mathcal{R}| \;\subseteq\; \langle\mathcal{E}_{n+1}\mathcal{R}|\,, \tag{88}$$

for some $n = 0,1,\dots,N-1$ such that $\mathbf{\Phi}(\mathcal{R})$ satisfies the inequalities:

$$b(X_n/\!\!/\mathfrak{X}_\mathcal{R}) \;<\; \mathbf{\Phi}(\mathcal{R}) \;\leqslant\; b(X_{n+1}/\!\!/\mathfrak{X}_\mathcal{R}). \tag{89}$$

Property 13 *(mean-believable bra-event).* From Definition 23 it follows that the ***mean-believable bra-event*** $\langle\mathcal{E}\mathcal{R}| \in \langle\mathcal{A}|$
1) is concluded between bra-events from sigma-algebras $\langle\mathcal{A}_{\mathfrak{X}_\mathcal{R}}|$;
2) is unlabellable, if $\langle\mathcal{E}\mathcal{R}| \in \langle\mathcal{A}| - \langle\mathcal{A}_{\mathfrak{X}_\mathcal{R}}|$;
3) is representable in the form:

$$\langle\mathcal{E}\mathcal{R}| = \langle\mathcal{E}_n\mathcal{R}| + \langle w|\,, \tag{90}$$

for some $n = 0,1,\dots,N-1$ such that $\mathbf{\Phi}(\mathcal{R})$ satisfies the inequalities (91), where

$$\langle w| \;=\; \langle\mathcal{E}\mathcal{R}| - \langle\mathcal{E}_n\mathcal{R}| \subseteq \langle x_{n+1}|\,, \tag{91}$$

$$\mathbf{B}(\langle w|) \;=\; \mathbf{\Phi}(\mathcal{R}) - b(X_n/\!\!/\mathfrak{X}_\mathcal{R}). \tag{92}$$

4) is experienced with believability coinciding with mean-probable believability of the random bra-element $\langle\mathcal{R}|$: $\mathbf{B}(\langle\mathcal{E}\mathcal{R}|) = \mathbf{E}_\mathbf{P}(\,\mathbf{B}(\langle\mathcal{R}|)\,)$;
5) plays the role of the mean-set characteristic of values of the random bra-element $\langle\mathcal{R}|$, terraced bra-events $\langle\mathsf{Ter}_{\mathfrak{X}_\mathcal{R}}|\,, X \in \mathfrak{S}^{\mathfrak{X}_\mathcal{R}}$, ***as bra-subsets*** of the bra-set $\langle\Omega|$.

Proof. I will prove only the third and fourth properties assuming that the rest do not need proof.
3) From defining definitions (89) we have (92):

$$\langle w| = \langle\mathcal{E}\mathcal{R}| - \langle\mathcal{E}_n\mathcal{R}| \subseteq \langle\mathcal{E}_{n+1}\mathcal{R}| - \langle\mathcal{E}_n\mathcal{R}| = \langle\mathsf{Ter}_{X_{n+1}/\!\!/\mathfrak{X}_\mathcal{R}}| - \langle\mathsf{Ter}_{X_n/\!\!/\mathfrak{X}_\mathcal{R}}| = \langle x_{n+1}|\,. \tag{93}$$

And from the fact that by Definition 23 $\mathbf{\Phi}(\langle\mathcal{E}\mathcal{R}|) = \mathbf{\Phi}(\mathcal{R})$ (93) follows.
4) Firstly, the mean-probable believability of the random bra-element $\langle\mathcal{R}|$, i.e. an expectation of believability of the random bra-element $\langle\mathcal{R}|$ by probability measure, has the form:

$$\begin{aligned} \mathbf{E}_\mathbf{P}(\,\mathbf{B}(\langle\mathcal{R}|)\,) &= \sum_{X \in \mathfrak{S}^{\mathfrak{X}_\mathcal{R}}} \mathbf{B}(\langle\mathsf{Ter}_{X/\!\!/\mathfrak{X}_\mathcal{R}}|)\mathbf{P}(|\mathsf{ter}(X/\!\!/\mathfrak{X}_\mathcal{R})\rangle) \\ &= \sum_{X \in \mathfrak{S}^{\mathfrak{X}_\mathcal{R}}} b(X/\!\!/\mathfrak{X}_\mathcal{R})p(X/\!\!/\mathfrak{X}_\mathcal{R}). \end{aligned} \tag{94}$$

Secondly, by the Robbins-Fubini theorem (see in [22]) a certainty of the co$\sim$event $\mathcal{R}$ coincides with (95):

$$\begin{aligned} \mathcal{R} &= \sum_{X \in \mathfrak{S}^{\mathfrak{X}_\mathcal{R}}} \langle\mathsf{Ter}_{X/\!\!/\mathfrak{X}_\mathcal{R}}|\mathsf{ter}(X/\!\!/\mathfrak{X}_\mathcal{R})\rangle\,, \\ \mathbf{\Phi}(\mathcal{R}) &= \sum_{X \in \mathfrak{S}^{\mathfrak{X}_\mathcal{R}}} \mathbf{\Phi}(\langle\mathsf{Ter}_{X/\!\!/\mathfrak{X}_\mathcal{R}}|\mathsf{ter}(X/\!\!/\mathfrak{X}_\mathcal{R})\rangle), \\ &= \sum_{X \in \mathfrak{S}^{\mathfrak{X}_\mathcal{R}}} \mathbf{B}(\langle\mathsf{Ter}_{X/\!\!/\mathfrak{X}_\mathcal{R}}|)\mathbf{P}(|\mathsf{ter}(X/\!\!/\mathfrak{X}_\mathcal{R})\rangle), \\ &= \sum_{X \in \mathfrak{S}^{\mathfrak{X}_\mathcal{R}}} b(X/\!\!/\mathfrak{X}_\mathcal{R})p(X/\!\!/\mathfrak{X}_\mathcal{R}) = \mathbf{E}_\mathbf{P}(\,\mathbf{B}(\langle\mathcal{R}|)\,), \end{aligned} \tag{95}$$

as required.

Definition 24 *(believability distance).* The *believability distance* of an arbitrary bra-event $\langle w| \in \langle\mathcal{A}|$ till the random bra-element $\langle\mathcal{R}|$ is the value

$$\rho\big(\langle\mathcal{R}|, \langle w|\big) = \mathbf{E}_{\mathbf{P}}\Big(\mathbf{B}\big(\langle\mathcal{R}| \;\Delta\; \langle w|\big)\Big) = \sum_{X \in 2^{\mathfrak{X}_{\mathcal{R}}}} \mathbf{B}\Big(\langle \mathsf{Ter}_{X/\!\!/\mathfrak{X}_{\mathcal{R}}}| \;\Delta\; \langle w|\Big)\mathbf{P}\Big(|\mathsf{ter}(X/\!\!/\mathfrak{X}_{\mathcal{R}})\rangle\Big),$$

the mean-probable believability of their symmetrical difference.

Theorem 5 *(extremal properties of a mean-believable bra-event). Mean-believability bra-event $\langle\mathcal{E}\mathcal{R}|$ of the random bra-element $\langle\mathcal{R}|$ minimizes its believability distance*

$$\rho\big(\langle\mathcal{R}|, \langle\mathcal{E}\mathcal{R}|\big) = \min_{\substack{\langle w| \in \langle\mathcal{A}| \\ \mathbf{B}(\langle w|) = \mathbf{E}_{\mathbf{P}}(\mathbf{B}(\langle\mathcal{R}|))}} \rho\big(\langle\mathcal{R}|, \langle w|\big)$$

till the random bra-element $\langle\mathcal{R}|$ among bra-events $\langle w| \in \langle\mathcal{A}|$ a believability of which is equal to the mean-probable believability of the bra-element.

Proof differs from the proof of the lemma 1 only in notation.

6.1.3 Experienced ket-element generated by an ordered co∼event

The *set of ket-events* $|\mathfrak{X}_{\mathcal{R}}\rangle \subset |\mathcal{A}\rangle$ uniquely associated with the notion of the *experienced ket-element*

$$|\mathcal{R}\rangle : \langle\Omega| \to |\mathfrak{X}_{\mathcal{R}}\rangle,$$

defined on the believability bra-space $\langle\Omega, \mathcal{A}, \mathbf{P}|$, which on the elementary bra-income $\langle\omega^*| \in \langle x|, x \in \mathfrak{X}$ takes a value

$$\langle\omega^*|\mathcal{R}\rangle = |\mathcal{R}\rangle(\langle\omega^*|) = |x\rangle \in |\mathfrak{X}_{\mathcal{R}}\rangle$$

from the ket-area $|\mathfrak{X}_{\mathcal{R}}\rangle$ that is contained in the sigma-subalgebra $|\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}\rangle \subseteq |\mathcal{A}\rangle$ of the probability ket-space $|\Omega, \mathcal{A}, \mathbf{B}\rangle$ generated by terraced ket-events $|\mathsf{ter}(X_i/\!\!/\mathfrak{X}_{\mathcal{R}})\rangle \in |\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}\rangle \subseteq |\mathcal{A}\rangle$. Its value $\langle\omega^*|\mathcal{R}\rangle$ is interpreted as the ket-event $|x_n\rangle \in |\mathfrak{X}_{\mathcal{R}}\rangle$ which *happens* with probability

$$p_{x_n} = \mathbf{P}(|x_n\rangle) = \sum_{i=n}^{N} \mathbf{P}(|\mathsf{ter}(X_i/\!\!/\mathfrak{X}_{\mathcal{R}})\rangle) = \sum_{i=n}^{N} p(X_i/\!\!/\mathfrak{X}_{\mathcal{R}}), \tag{96}$$

and forces the bra-event $\langle x_n|$ to be *experienced*, i.e., forces all elementary bra-incomes $\langle\omega^*| \in \langle x_n|$ to be *experienced* with believability

$$b_{x_n} = \mathbf{B}\Big(\{\langle\omega^*| : \langle\omega^*|\mathcal{R}\rangle = |x_n\rangle\}\Big) = \mathbf{B}\big(\langle x_n|\big). \tag{97}$$

The *experienced ket-element* $|\mathcal{R}\rangle$ is defined by

1) a family $|\mathfrak{X}_{\mathcal{R}}\rangle = \{|x\rangle : x \in \mathfrak{X}\}$ of its values, ket-events

$$|x_n\rangle = \sum_{i=n}^{N} |\mathsf{ter}(X_i/\!\!/\mathfrak{X}_{\mathcal{R}})\rangle \subseteq |\Omega\rangle,$$

that happens with probability $p_{x_n} = \mathbf{P}(|x_n\rangle)$ and is defined as sums of terraced ket-events $|\mathsf{ter}(X_i/\!\!/\mathfrak{X}_{\mathcal{R}})\rangle \in |\Omega\rangle$ forming a partition of the space of elementary ket-outcomes $|\Omega\rangle$:

$$|\Omega\rangle = \sum_{x \in \mathfrak{X}_{\mathcal{R}}} |\mathsf{ter}(X/\!\!/\mathfrak{X}_{\mathcal{R}})\rangle.$$

2) a family $\{b_x, x \in \mathfrak{X}\}$ of believabilities

$$b_{x_n} = \mathbf{B}\Big(\{\langle\omega^*| : \langle\omega^*|\mathcal{R}\rangle = |x_n\rangle\}\Big) = \mathbf{B}\big(\langle x_n|\big)$$

of bra-events $\langle x_n| \subseteq \langle\Omega|$, on which the experienced ket-element takes values $|x_n\rangle \subseteq |\Omega\rangle$ and which form a partition, generated by $\langle\mathfrak{X}_{\mathcal{R}}|$, of the space of elementary bra-incomes $\langle\Omega|$:

$$\langle\Omega| = \sum_{x \in \mathfrak{X}_{\mathcal{R}}} \langle x|.$$

6.1.4 Mean-probable ket-event for an ordered co∼event

Definition 25 *(mean-probable ket-event).* Let for the ordered co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ with the element-set labelling $\langle\mathfrak{X}_\mathcal{R}|\mathfrak{Z}^{\mathfrak{X}}_\mathcal{R}\rangle$ (83) the denotations are introduced:

$$|\mathcal{E}_n\mathcal{R}\rangle = \begin{cases} |x_n\rangle\,, & n = 1,\ldots,N, \\ |x_{N+1}\rangle = \varnothing_{|\Omega\rangle}, & n = N+1, \end{cases} \tag{98}$$

$$p_{x_{N+1}} = \mathbf{P}(|x_{N+1}\rangle) = \mathbf{P}(\varnothing_{|\Omega\rangle}) = 0,$$

then the *mean-probable ket-event* of its experienced ket-element $|\mathcal{R}\rangle$ is defined as any ket-event $|\mathcal{E}\mathcal{R}\rangle \in |\mathcal{A}\rangle$ with probability $\mathbf{P}(|\mathcal{E}\mathcal{R}\rangle) = \mathbf{\Phi}(\mathcal{R})$ that satisfies the inclusions:

$$|\mathcal{E}_{n+1}\mathcal{R}\rangle \ \subset\ |\mathcal{E}\mathcal{R}\rangle \ \subseteq\ |\mathcal{E}_n\mathcal{R}\rangle\,, \tag{99}$$

for some $n = 1,\ldots,N$ such that $\mathbf{\Phi}(\mathcal{R})$ satisfies the inequalities:

$$p_{x_{n+1}} \ <\ \mathbf{\Phi}(\mathcal{R}) \ \leqslant\ p_{x_n}. \tag{100}$$

Property 14 *(mean-probable ket-event).* From Definition 25 it follows that the ***mean-probable ket-event*** $|\mathcal{E}\mathcal{R}\rangle \in |\mathcal{A}\rangle$
1) is concluded between ket-events from the sigma-algebra $|\mathcal{A}^{\mathfrak{X}_\mathcal{R}}\rangle$;
2) is unlabellable if $|\mathcal{E}\mathcal{R}\rangle \in |\mathcal{A}\rangle - |\mathcal{A}^{\mathfrak{X}_\mathcal{R}}\rangle$;
3) is presentable in the form

$$|\mathcal{E}\mathcal{R}\rangle = |\mathcal{E}_{n+1}\mathcal{R}\rangle + |w\rangle \tag{101}$$

for some $n = 1,\ldots,N$ such that $\mathbf{\Phi}(\mathcal{R})$ satisfies the inequalities (101), where

$$|w\rangle = |\mathcal{E}\mathcal{R}\rangle - |\mathcal{E}_{n+1}\mathcal{R}\rangle \subseteq \mathsf{ter}(X_n /\!\!/ \mathfrak{X}_\mathcal{R}), \tag{102}$$
$$\mathbf{P}(|w\rangle) = \mathbf{\Phi}(\mathcal{R}) - p_{x_{n+1}}, \tag{103}$$

4) happens with probability coinciding with mean-believable probability of the experienced ket-element $|\mathcal{R}\rangle$: $\mathbf{P}(|\mathcal{E}\mathcal{R}\rangle) = \mathbf{E}_{\mathbf{B}}(\,\mathbf{P}(|\mathcal{R}\rangle)\,)$;
5) plays the role of mean-set characteristic of values of the experienced ket-element $|\mathcal{R}\rangle$, ket-events $|x\rangle\,, x \in \mathfrak{X}_\mathcal{R}$, ***as ket-subsets of*** the ket-set $|\Omega\rangle$.

Proof. I will prove only the third and fourth properties, leaving the reader to reflect on others.
3) From defining inclusions (100) we have (103):

$$|w\rangle = |\mathcal{E}\mathcal{R}\rangle - |\mathcal{E}_{n+1}\mathcal{R}\rangle \subseteq |\mathcal{E}_n\mathcal{R}\rangle - |\mathcal{E}_{n+1}\mathcal{R}\rangle = |x_n\rangle - |x_{n+1}\rangle = \mathsf{ter}(X_n /\!\!/ \mathfrak{X}_\mathcal{R}). \tag{104}$$

And from the fact that by Definition 25 $\mathbf{\Phi}(|\mathcal{E}\mathcal{R}\rangle) = \mathbf{\Phi}(\mathcal{R})$ (104) follows;
4) Firstly, the mean-probable believability of the experienced ket-element $|\mathcal{R}\rangle$, i.e. an expectation of probability of the experienced ket-element $|\mathcal{R}\rangle$ by believability measure, has the form:

$$\begin{aligned} \mathbf{E}_{\mathbf{B}}(\,\mathbf{P}(|\mathcal{R}\rangle)\,) &= \sum_{x\in\mathfrak{X}_\mathcal{R}} \mathbf{B}(\langle x|)\mathbf{P}(|x\rangle) \\ &= \sum_{x\in\mathfrak{X}_\mathcal{R}} b_x p_x. \end{aligned} \tag{105}$$

Secondly, by the Robbons-Fubini theorem (see in [22]) a certainty of co∼event $\mathcal{R}$ coincides with (106):

$$\begin{aligned} \mathcal{R} &= \sum_{x\in\mathfrak{X}_\mathcal{R}} \langle x|x\rangle\,, \\ \mathbf{\Phi}(\mathcal{R}) &= \sum_{x\in\mathfrak{X}_\mathcal{R}} \mathbf{\Phi}(\langle x|x\rangle), \\ &= \sum_{x\in\mathfrak{X}_\mathcal{R}} \mathbf{B}(\langle x|)\mathbf{P}(|x\rangle), \\ &= \sum_{x\in\mathfrak{X}_\mathcal{R}} b_x p_x = \mathbf{E}_{\mathbf{B}}(\,\mathbf{P}(|\mathcal{R}\rangle)\,), \end{aligned} \tag{106}$$

as required.

Definition 26 *(probability distance)*. The *probability distance* of the ket-event $|w\rangle \in |\mathcal{A}\rangle$ till the experienced ket-element $|\mathcal{R}\rangle$ is the mean-believable probability of their symmetrical difference:

$$\rho\big(\,|\mathcal{R}\rangle\,,|w\rangle\,\big) = \mathbf{E}_{\mathbf{B}}\Big(\mathbf{P}\big(\,|\mathcal{R}\rangle\,\Delta\,|w\rangle\,\big)\Big) = \sum_{x\in\mathfrak{X}_{\mathcal{R}}}\mathbf{P}\big(\,|x\rangle\,\Delta\,|w\rangle\,\big)\mathbf{B}\big(\,\langle x|\,\big).$$

Theorem 6 *(extremal properties of the mean-probable ket-event)*. *The mean-probable ket-event* $\mathcal{E}(|K_{\mathfrak{X}_{\mathcal{R}}}\rangle)$ *of the experienced ket-element* $|\mathcal{R}\rangle$ *minimizes its probability distance*

$$\rho\big(\,|\mathcal{R}\rangle\,,|\mathcal{E}\mathcal{R}\rangle\,\big) = \min_{\substack{|w\rangle\in|\mathcal{A}\rangle\\ \mathbf{P}(|w\rangle)=\mathbf{E}_{\mathbf{B}}(\mathbf{P}(|\mathcal{R}\rangle))}}\rho\big(\,|\mathcal{R}\rangle\,,|w\rangle\,\big)$$

till the experienced ket-element $|\mathcal{R}\rangle$ *among ket-events* $|w\rangle\in|\mathcal{A}\rangle$*, probability of which is equal to the mean-believable probability of the experienced ket-element.*

Proof differs from the proof of Lemma 2 only in denotations.

6.2 Dual co∼event means of ordered co∼event

Let

- $\big(\,\langle\Omega|\Omega\rangle\,,\langle\mathcal{A}|\mathcal{A}\rangle\,,\mathbf{\Phi}\big)$ is the certainty bra-ket-space with certainty measure $\mathbf{\Phi}$,
- $\mathcal{R}\subseteq\langle\Omega|\Omega\rangle$ is the co∼event generating the element-set labelling $\langle\mathfrak{X}_{\mathcal{R}}|\mathbf{\mathfrak{S}}^{\mathfrak{X}_{\mathcal{R}}}\rangle$,
- $\langle\mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}|\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}\rangle \subseteq \langle\mathcal{A}|\mathcal{A}\rangle$ is the sigma-subalgebra of bra-events $\langle x|\in\langle\mathfrak{X}_{\mathcal{R}}|\subseteq\langle\mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}|\subseteq\langle\mathcal{A}|$ and terraced ket-events $|\mathrm{ter}(X/\!\!/\mathfrak{X}_{\mathcal{R}})\rangle\in|\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}\rangle\subseteq|\mathcal{A}\rangle$,
- $\mathcal{R}_{\mathbf{B}}\subseteq\langle\Omega|\Omega\rangle$ is the $\mathbf{B}$-quotient-projection of co∼event $\mathcal{R}$ generating the element-set labelling $\langle\mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}|\mathbf{\mathfrak{S}}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}}\rangle$,
- $|\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}}\rangle\subseteq|\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}\rangle\subseteq|\mathcal{A}\rangle$ is the sigma-subalgebra terraced ket-events $|\mathrm{ter}(X^{\mathbf{B}}/\!\!/\mathfrak{X}^{\mathbf{B}}_{\mathcal{R}})\rangle\in|\mathbf{\mathfrak{S}}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}}\rangle$,
- $\mathcal{R}_{\mathbf{P}}\subseteq\langle\Omega|\Omega\rangle$ is the $\mathbf{P}$-quotient-projection of co∼event $\mathcal{R}$ generating the element-set labelling $\langle\mathfrak{X}^{\mathbf{P}}_{\mathcal{R}}|\mathbf{\mathfrak{S}}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{P}}\rangle$,
- $\langle\mathcal{A}^{\mathbf{P}}_{\mathfrak{X}_{\mathcal{R}}}|\subseteq\langle\mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}|\subseteq\langle\mathcal{A}|$ is the sigma-subalgebra of bra-events $\langle x^{\mathbf{P}}|\in\langle\mathfrak{X}^{\mathbf{P}}_{\mathcal{R}}|$,
- $|\mathfrak{R}_{\mathbf{B}}\rangle_{\leqslant 2}\subseteq\langle\mathcal{A}|\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}}\rangle\subseteq\langle\mathcal{A}|\mathcal{A}\rangle$ is the set of ket-monoplet and ket-doublet ordered co∼events from the sigma-subalgebra $\langle\mathcal{A}|\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}}\rangle$,
- $\langle\mathfrak{R}_{\mathbf{P}}|_{\leqslant 2}\subseteq\langle\mathcal{A}^{\mathbf{P}}_{\mathfrak{X}_{\mathcal{R}}}|\mathcal{A}\rangle\subseteq\langle\mathcal{A}|\mathcal{A}\rangle$ is the set of bra-monoplet and bra-doublet ordered co∼events from the sigma-subalgebra $\langle\mathcal{A}^{\mathbf{P}}_{\mathfrak{X}_{\mathcal{R}}}|\mathcal{A}\rangle$.

Dual co∼events means are defined within the framework of the certainty space $\langle\Omega,\mathcal{A},\mathbf{B}|\Omega,\mathcal{A},\mathbf{P}\rangle$ firstly for the ordered co∼event $\mathcal{R}\subseteq\langle\Omega|\Omega\rangle$ to define them for an arbitrary co∼event as event means of two its quotient-projections $\mathcal{R}_{\mathbf{B}}\subseteq\langle\Omega|\Omega\rangle$ and $\mathcal{R}_{\mathbf{P}}\subseteq\langle\Omega|\Omega\rangle$ (see Definition 17 on page 162).

Note 1 *(on «labellability» and «unlabellability» of events and co∼events)*. In the theory of experience and chance studying co∼events as $\langle\mathcal{A}|\mathcal{A}\rangle$-measurable binary relations $\mathcal{R}\in\langle\mathcal{A}|\mathcal{A}\rangle$ one of the central roles the element-set labelling $\langle\mathfrak{X}_{\mathcal{R}}|\mathbf{\mathfrak{S}}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ of the space $\langle\Omega|\Omega\rangle$ plays which is generated by every co∼event $\mathcal{R}$ such that $\mathcal{R}\in\langle\mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}|\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}\rangle$. In other words, every co∼event $\mathcal{R}$ is $\langle\mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}|\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}\rangle$-measurable, i.e. measurable with respect to the sigma-algebra $\langle\mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}|\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}\rangle$, or *bra-ket-measurable*. For the description of quotient-projections of co∼events it needs also the $\langle\mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}|\mathcal{A}\rangle$-measurable co∼events, or *bra-measurable*, and the $\langle\mathcal{A}|\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}\rangle$-measurable co∼events, or *ket-measurable*. However, to emphasize the importance of the element-set labelling for the theory of experience and chance along with standard terms «measurable events» and «measurable co∼events» I will use their synonyms «labelled events» and «labelled co∼events», Understanding by them events and co∼events, *labelled* within the framework of corresponding sigma-algebras. For example, $\langle\mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}|$*-labelled bra-event,* $|\mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}\rangle$*-labelled ket-event,* $\langle\mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}|\mathcal{A}\rangle$*-labelled, or bra-labelled*

co∼event is a co∼event, bra-events of which are $\langle\mathcal{A}_{\mathfrak{X}_\mathcal{R}}|$-labelled, and ket-events are $|\mathcal{A}^{\mathfrak{X}_\mathcal{R}}\rangle$-unlabelled, $\langle\mathcal{A}|\mathcal{A}^{\mathfrak{X}_\mathcal{R}}\rangle$***-labelled, or ket-labelled co∼event*** is a co∼event, bra-events of which are $\langle\mathcal{A}_{\mathfrak{X}_\mathcal{R}}|$***-unlabelled***, and ket-events are $|\mathcal{A}^{\mathfrak{X}_\mathcal{R}}\rangle$***-labelled*** etc.

Note 2 *(on «labellability» and «unlabellability» of event and co∼event means).* You can define the dual co∼event means as a experienced-certainty co∼event $\langle\mathcal{ER}|\Omega\rangle \subseteq \langle\Omega|\Omega\rangle$ and a full-believability-random co∼event $\langle\Omega|\mathcal{ER}\rangle \subseteq \langle\Omega|\Omega\rangle$ relying on definitions of event means $\langle\mathcal{ER}| \in \langle\mathcal{A}|$ and $|\mathcal{ER}\rangle \in |\mathcal{A}\rangle$ and, without philosophizing slyly, stop at this. However, due to the fact that event means are so defined, they can remain without «convenient» labelling, i.e. without the labelling within the framework of the sigma-algebras $\langle\mathcal{A}_{\mathfrak{X}_\mathcal{R}}| \subseteq \langle\mathcal{A}|$ and $|\mathcal{A}^{\mathfrak{X}_\mathcal{R}}\rangle \subseteq |\mathcal{A}\rangle$ then the co∼event means defined on their basis, of course, inherit the property of «unlabellability»[15]. Here I define the dual co∼event means slightly differently, returning the «labellability» of co∼event bra-means in the sigma-algebra $\langle\mathcal{A}_{\mathfrak{X}_\mathcal{R}}|\mathcal{A}\rangle$ and the «labellability» of co∼event ket-means in the sigma-algebra $\langle\mathcal{A}|\mathcal{A}^{\mathfrak{X}_\mathcal{R}}\rangle$.

6.2.1 Bra-mean and ket-mean of an ordered co∼event

I'm going to give the definitions of the bra-mean and ket-mean of an ordered co∼event, picking on their roles such a *bra-labellable* bra-mean and *ket-labellable* ket-mean that would have all the characteristic properties of the corresponding event means, with the exception of the undesirable *unlabellability* property. By charactristic properties, I understand the properties of event means, including their extreme properties, formulated in the form of lemmas 1 and 2 on page 147 and 148 and theorems 1, 2, 5, and 6 on page 151, 153, 169, and 171.

Definition 27 *(β-bra-event and π-ket-event).* The β***-bra-event*** is the $\langle\mathcal{A}|$-measurable bra-event $\langle\lambda^\beta| \subseteq \langle\Omega|$ which is experienced with believability $\beta = \mathbf{B}(\langle\lambda^\beta|)$; the π***-ket-event*** is the $|\mathcal{A}\rangle$-measurable ket-event $|\lambda^\pi\rangle \subseteq |\Omega\rangle$ which happens with probability $\pi = \mathbf{P}(|\lambda^\pi\rangle)$.

Definition 28 *(bra-family and ket-family of monoplet and doublet co∼events).* The $\langle\mathcal{A}|\mathcal{A}\rangle$-measurable co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ defines the following bra-family and ket-family of monoplet and doublet co∼events:

$$\langle\mathfrak{R}|^{\mathcal{R}}_{\leqslant 2} = \{\mathcal{L}(X,x) \in \mathfrak{R}_{\leqslant 2} \colon X \subset \mathfrak{X}_\mathcal{R}, x \in \mathfrak{X}_\mathcal{R} - X\} \subseteq \langle\mathcal{A}_{\mathfrak{X}_\mathcal{R}}|\mathcal{A}\rangle\,, \tag{107}$$

$$|\mathfrak{R}\rangle^{\mathcal{R}}_{\leqslant 2} = \{\mathcal{T}(\mathcal{Z},X) \in \mathfrak{R}_{\leqslant 2} \colon \mathcal{Z} \subset \mathbf{Z}^{\mathfrak{X}_\mathcal{R}}, X \in \mathbf{Z}^{\mathfrak{X}_\mathcal{R}} - \mathcal{Z}\} \subseteq \langle\mathcal{A}|\mathcal{A}^{\mathfrak{X}_\mathcal{R}}\rangle\,, \tag{108}$$

where

$$\mathcal{L}(X,x) = \langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}_\mathcal{R}}|\Omega\rangle + \langle x|\lambda\rangle \in \langle\mathcal{A}_{\mathfrak{X}_\mathcal{R}}|\mathcal{A}\rangle\,, \tag{109}$$

$$\mathcal{T}(\mathcal{Z},X) = \left\langle \Omega \,\middle|\, \sum_{Y\in\mathcal{Z}} \mathsf{ter}(Y /\!\!/ \mathfrak{X}_\mathcal{R}) \right\rangle + \langle\lambda|\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle \in \langle\mathcal{A}|\mathcal{A}^{\mathfrak{X}_\mathcal{R}}\rangle\,. \tag{110}$$

Property 15 *(bra-family and ket-family of monoplet and doublet co∼events).* Bra-labellable co∼events $\mathcal{L}(X,x)$ and ket-labellable co∼events $\mathcal{T}(\mathcal{Z},X)$ from (108) and (109) are either monolet or doublet co∼events depending on whether the subsets $X \subset \mathfrak{X}_\mathcal{R}$ and $\mathcal{Z} \subset \mathbf{Z}^{\mathfrak{X}_\mathcal{R}}$ are empty or not, and which are representable in the form:

$$\mathcal{L}(X,x) = \begin{cases} \langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}_\mathcal{R}}|\Omega\rangle + \langle x|\lambda\rangle\,, & X \neq \emptyset, \\ \langle x|\lambda\rangle\,, & X = \emptyset, \end{cases} \tag{111}$$

$$\mathcal{T}(\mathcal{Z},X) = \begin{cases} \left\langle \Omega \,\middle|\, \sum_{Y\in\mathcal{Z}} \mathsf{ter}(Y /\!\!/ \mathfrak{X}_\mathcal{R}) \right\rangle + \langle\lambda|\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle\,, & \mathcal{Z} \neq \emptyset, \\ \langle\lambda|\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle\,, & \mathcal{Z} = \emptyset, \end{cases} \tag{112}$$

becuase $\langle \mathsf{Ter}_{\emptyset /\!\!/ \mathfrak{X}_\mathcal{R}}| = \varnothing_{\langle\Omega|}$ and $\sum_{Y\in\emptyset_{\mathbf{Z}^{\mathfrak{X}_\mathcal{R}}}} |\mathsf{ter}(Y /\!\!/ \mathfrak{X}_\mathcal{R})\rangle = \varnothing_{|\Omega\rangle}$.

[15] In Appendix on page 257 the representatives of the «unlabelled» co∼event means are shown in dotted lines.

Reminder 1 *(denotations in element-set labellings of N-tuple ordered co∼events).* For the $\langle\mathcal{A}|\mathcal{A}\rangle$-measurable N-tuple ordered co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ with the labelling $\langle\mathfrak{X}_\mathcal{R}|\mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}\rangle$ we introduce one new (114) and remind three old denotations:

$$\mathfrak{Z}_n = \{X_n, X_{n+1}, \ldots, X_N\} \subseteq \mathfrak{Z}^{\mathfrak{X}_\mathcal{R}}, \tag{113}$$

$$|x_n\rangle = \sum_{X \in \mathfrak{Z}_n} |\mathsf{ter}(X /\!\!/ \mathfrak{X}_\mathcal{R})\rangle = \sum_{i=n}^{N} |\mathsf{ter}(X_i /\!\!/ \mathfrak{X}_\mathcal{R})\rangle \in |\mathcal{A}_{\mathfrak{X}_\mathcal{R}}\rangle\,, \tag{114}$$

$$X_n = \{x_1, \ldots, x_n\} \subseteq \mathfrak{X}_\mathcal{R}, \tag{115}$$

$$\langle\mathsf{Ter}_{X_n /\!\!/ \mathfrak{X}_\mathcal{R}}| = \sum_{x \in X_n} \langle x| = \sum_{i=1}^{n} \langle x_i| \in \langle\mathcal{A}^{\mathfrak{X}_\mathcal{R}}|\,, \tag{116}$$

where $n = 1, \ldots, N$. These denotations will be used in the following definition.

Definition 29 *(bra-mean and ket-mean of an ordered co∼event).* For the ordered co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ (see Definition 9 on page 153) with certainty $\mathbf{\Phi}(\mathcal{R}) = \mathbf{B}(\langle\mathcal{E}\mathcal{R}|) = \mathbf{P}(|\mathcal{E}\mathcal{R}\rangle)$

1) the *bra-mean* is defined as bra-labellable monoplet or doublet co∼event $\langle\mathcal{E}\mathcal{R}|^\pi\rangle = \mathcal{L}(X_n, x_{n+1}) \in \langle\mathfrak{R}|^{\mathcal{R}}_{\leqslant 2}$ like that
$$\langle\mathcal{E}\mathcal{R}|^\pi\rangle = \langle\mathsf{Ter}_{X_n /\!\!/ \mathfrak{X}_\mathcal{R}}|\Omega\rangle + \langle x_{n+1}|\lambda^\pi\rangle \tag{117}$$
for some $n = 0, 1, \ldots, N-1$ and $\pi \in (0,1)$ with which $\mathbf{\Phi}(\mathcal{R})$ satisfies the inequalities:
$$b(X_n /\!\!/ \mathfrak{X}_\mathcal{R}) < \mathbf{\Phi}(\mathcal{R}) \leqslant b(X_n /\!\!/ \mathfrak{X}_\mathcal{R}) + \pi b_{x_{n+1}}, \tag{118}$$
where the value
$$\pi = \frac{\mathbf{\Phi}(\mathcal{R}) - b(X_n /\!\!/ \mathfrak{X}_\mathcal{R})}{b_{x_{n+1}}}, \tag{119}$$
is called the *residual probability* of the bra-mean;

2) the *ket-mean* is defined as ket-labellable monoplet or doublet co∼event $\langle^\beta|\mathcal{E}\mathcal{R}\rangle = \mathcal{T}(\mathfrak{Z}_{n+1}, X_n) \in |\mathfrak{R}\rangle^{\mathcal{R}}_{\leqslant 2}$ like that
$$\langle^\beta|\mathcal{E}\mathcal{R}\rangle = \langle\Omega|x_{n+1}\rangle + \langle\lambda^\beta|\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_\mathcal{R})\rangle \tag{120}$$
for some $n = 1, \ldots, N$ and $\beta \in (0,1)$ with which $\mathbf{\Phi}(\mathcal{R})$ satisfies inequalities:
$$p_{x_{n+1}} < \mathbf{\Phi}(\mathcal{R}) \leqslant p_{x_{n+1}} + \beta p(X_n /\!\!/ \mathfrak{X}_\mathcal{R}), \tag{121}$$
where the value
$$\beta = \frac{\mathbf{\Phi}(\mathcal{R}) - p_{n+1}}{p(X_n /\!\!/ \mathfrak{X}_\mathcal{R})}, \tag{122}$$
is called the *residual believability* of the ket-mean.

Remark 1 *(interpretation of bra-mean of an ordered co∼event).* The certainty distribution of the ordered co∼event $\mathcal{R}$, occurring with certainty $\mathbf{\Phi}(\mathcal{R})$, defines its *bra-mean* for some $n = 0, 1, \ldots, N-1$ with which $b(X_n /\!\!/ \mathfrak{X}_\mathcal{R}) < \mathbf{\Phi}(\mathcal{R}) \leqslant b(X_{n+1} /\!\!/ \mathfrak{X}_\mathcal{R})$, as the monoplet or the doublet co∼event

$$\langle\mathcal{E}\mathcal{R}|^\pi\rangle = \langle\mathsf{Ter}_{X_n /\!\!/ \mathfrak{X}_\mathcal{R}}|\Omega\rangle + \langle x_{n+1}|\lambda^\pi\rangle\,, \tag{123}$$

which occurs with the same certainty $\mathbf{\Phi}(\langle\mathcal{E}\mathcal{R}|^\pi\rangle) = \mathbf{\Phi}(\mathcal{R})$ then, when the experienced-certainty co∼event $\langle\mathsf{Ter}_{X_n /\!\!/ \mathfrak{X}_\mathcal{R}}|\Omega\rangle$ occurs with certainty $\mathbf{\Phi}(\langle\mathsf{Ter}_{X_n /\!\!/ \mathfrak{X}_\mathcal{R}}|\Omega\rangle) = b(X_n /\!\!/ \mathfrak{X}_\mathcal{R})$ $\big($is experienced with believability $b(X_n /\!\!/ \mathfrak{X}_\mathcal{R}) = \mathbf{B}(\langle\mathsf{Ter}_{X_n /\!\!/ \mathfrak{X}_\mathcal{R}}|)$ and happens with probability $\mathbf{P}(|\Omega\rangle) = 1\big)$, and the co∼event $\langle x_{n+1}|\lambda^\pi\rangle$ occurs with certainty $\mathbf{\Phi}(\langle x_{n+1}|\lambda^\pi\rangle) = b_{x_{n+1}} \times \pi$ $\big($is experienced with believability $b_{x_{n+1}} = \mathbf{B}(\langle x_{n+1}|)$ and happens with the residual probability $\pi = \mathbf{P}(|\lambda^\pi\rangle)\big)$.

Remark 2 *(interpretation of the ket-mean of an ordered co∼event).* The certainty distribution of the ordered co∼event $\mathcal{R}$, occurring with certainty $\mathbf{\Phi}(\mathcal{R})$, defines its *ket-mean* for some $n = 1, \ldots, N$ with which $p_{x_{n+1}} < \mathbf{\Phi}(\mathcal{R}) \leqslant p_{x_n}$, as the monoplet or the doublet co∼event

$$\langle^{\beta}|\mathcal{ER}\rangle = \langle\Omega|x_{n+1}\rangle + \langle\lambda^{\beta}|\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle\,, \tag{124}$$

which occurs with the same certainty $\mathbf{\Phi}(\langle^{\beta}|\mathcal{ER}\rangle) = \mathbf{\Phi}(\mathcal{R})$ then, when the full-believable-random co∼event $\langle\Omega|x_{n+1}\rangle$ occurs with certainty $\mathbf{\Phi}(\langle\Omega|x_{n+1}\rangle) = p_{x_{n+1}}$ (is experienced with believability $\mathbf{B}(\langle\Omega|) = 1$ and happens with probability $p_{x_{n+1}} = \mathbf{P}(|x_{n+1}\rangle)$), and the co∼event $\langle\lambda^{\beta}|\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle$ occurs with certainty $\mathbf{\Phi}(\langle\lambda^{\beta}|\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle) = \beta \times p(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})$ (is experienced with the residual believability $\beta = \mathbf{B}(\langle\lambda^{\beta}|)$ and happens with probability $p(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}) = \mathbf{P}(|\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle)$).

`Theorem 7` *(residual probability and residual believability of co∼event means of an ordered co∼event).* 1) The *residual probability π of the bra-mean $\langle\mathcal{ER}|^{\pi}\rangle$ has a sense of conditional believability of mean-believable bra-events $\langle\mathcal{ER}|$ under the condition that the bra-event $\langle x_1| \subseteq \langle\Omega|$ or $\langle x_{n+1}| \subseteq \langle\Omega|$ is experienced correspondingly*. 2) The *residual believability β of the ket-mean $\langle^{\beta}|\mathcal{ER}\rangle$ has a sense of conditional probability of mean-probable ket-events $|\mathcal{ER}\rangle$ under the condition that the terraced ket-event $|ter(X_1 /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle \subseteq \langle\Omega|$ or $|ter(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle$ happens correspondingly*.

Proof. By the Robbins-Fubini theorem[16] (see Footnote[3] on page 147) and by the definitions of mean-believable bra-events (88) and mean-probable ket-events (99) we have

$$\begin{aligned}\mathbf{\Phi}(\mathcal{R}) &= \mathbf{E}_{\mathbf{P}}\Big(\mathbf{B}(\langle\mathcal{R}|)\Big) = \mathbf{B}(\langle\mathcal{ER}|),\\ \mathbf{\Phi}(\mathcal{R}) &= \mathbf{E}_{\mathbf{B}}\Big(\mathbf{P}(|\mathcal{R}\rangle)\Big) = \mathbf{P}(|\mathcal{ER}\rangle).\end{aligned} \tag{125}$$

From the definitions of mean-believable bra-events (88) and mean-probable ket-events (99), and also from the additivity of measures $\mathbf{B}$ and $\mathbf{P}$ we get

$$\mathbf{\Phi}(\mathcal{R}) \leqslant b_{x_1} \quad\Longrightarrow\quad 0 < \mathbf{B}\big(\langle\mathcal{ER}|\big) \leqslant b_{x_1}, \tag{126}$$

$$b(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}) < \mathbf{\Phi}(\mathcal{R}) \leqslant b(X_{n+1} /\!\!/ \mathfrak{X}_{\mathcal{R}}) \quad\Longrightarrow\quad b(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}) < \mathbf{B}\big(\langle\mathcal{ER}|\big) \leqslant b(X_{n+1} /\!\!/ \mathfrak{X}_{\mathcal{R}}), \tag{127}$$

$$\mathbf{\Phi}(\mathcal{R}) \leqslant p(X_N /\!\!/ \mathfrak{X}_{\mathcal{R}}) \quad\Longrightarrow\quad 0 < \mathbf{P}\big(|\mathcal{ER}\rangle\big) \leqslant p(X_N /\!\!/ \mathfrak{X}_{\mathcal{R}}), \tag{128}$$

$$p_{x_{n+1}} < \mathbf{\Phi}(\mathcal{R}) \leqslant p_{x_n} \quad\Longrightarrow\quad p_{x_{n+1}} < \mathbf{P}\big(|\mathcal{ER}\rangle\big) \leqslant p_{x_n}. \tag{129}$$

From here it follows that the residual probability of the bra-mean is interpreted as the conditional believability:

$$\begin{aligned}(127) \quad\Longrightarrow\quad \pi &= \frac{\mathbf{B}\big(\langle\mathcal{ER}|\big)}{b_{x_1}} = \frac{\mathbf{B}\big(\langle\mathcal{ER}| \cap \langle x_1|\big)}{b_{x_1}} = \mathbf{B}\big(\langle\mathcal{ER}| \,/\, \langle x_1|\big),\\ (128) \quad\Longrightarrow\quad \pi &= \frac{\mathbf{B}\big(\langle\mathcal{ER}|\big) - b(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})}{b_{x_{n+1}}} = \frac{\mathbf{B}\big(\langle\mathcal{ER}| \cap \langle x_{n+1}|\big)}{b_{x_{n+1}}} = \mathbf{B}\big(\langle\mathcal{ER}| \,/\, \langle x_{n+1}|\big),\end{aligned} \tag{130}$$

and the residual believability of the ket-mean is interpreted as the conditional probability:

$$\begin{aligned}(129) \quad\Longrightarrow\quad \beta &= \frac{\mathbf{P}\big(|\mathcal{ER}\rangle\big)}{p(X_N /\!\!/ \mathfrak{X}_{\mathcal{R}})} = \frac{\mathbf{P}\big(|\mathcal{ER}\rangle \cap |\mathsf{ter}(X_N /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle\big)}{p(X_N /\!\!/ \mathfrak{X}_{\mathcal{R}})} = \mathbf{P}\big(|\mathcal{ER}\rangle \,/\, |\mathsf{ter}(X_N /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle\big),\\ (130) \quad\Longrightarrow\quad \beta &= \frac{\mathbf{P}\big(|\mathcal{ER}\rangle\big) - p_{x_{n+1}}}{p(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})} = \frac{\mathbf{P}\big(|\mathcal{ER}\rangle \cap |\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle\big)}{p(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})} = \mathbf{P}\big(|\mathcal{ER}\rangle \,/\, |\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle\big).\end{aligned} \tag{131}$$

`Property 16` *(certainty of bra-mean and ket-mean of an ordered co∼event).* The certainty of the bra-mean $\langle\mathcal{ER}|^{\pi}\rangle \subseteq \langle\Omega|\Omega\rangle$ and of the ket-mean $\langle^{\beta}|\mathcal{ER}\rangle \subseteq \langle\Omega|\Omega\rangle$ ordered co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ coincides with certainty of the co∼event:

$$\mathbf{\Phi}(\langle\mathcal{ER}|^{\pi}\rangle) = \mathbf{\Phi}(\langle^{\beta}|\mathcal{ER}\rangle) = \mathbf{\Phi}(\mathcal{R}). \tag{132}$$

[16] For details, see [22].

Proof. 1) The certainty of the bra-mean. If $b(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}) < \mathbf{\Phi}(\mathcal{R}) \leqslant b(X_{n+1} /\!\!/ \mathfrak{X}_{\mathcal{R}})$ for some $n = 0, 1, \ldots, N-1$, then

$$\begin{aligned} \mathbf{\Phi}(\langle \mathcal{E}\mathcal{R}|^{\pi}\rangle) &= \mathbf{\Phi}\big(\,\langle \mathsf{Ter}_{X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}}|\Omega\rangle + \langle x_{n+1}|\lambda^{\pi}\rangle\,\big) = \\ &= \mathbf{B}\big(\,\langle \mathsf{Ter}_{X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}}|\,\big) + \mathbf{B}\big(\,\langle x_{n+1}|\,\big)\mathbf{P}\big(\,|\lambda^{\pi}\rangle\,\big) \\ &= b(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}) + \big(b(X_{n+1} /\!\!/ \mathfrak{X}_{\mathcal{R}}) - b(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\big)\pi \\ &= b(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}) + \big(b(X_{n+1} /\!\!/ \mathfrak{X}_{\mathcal{R}}) - b(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\big)\frac{\mathbf{\Phi}(\mathcal{R}) - b(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})}{b(X_{n+1} /\!\!/ \mathfrak{X}_{\mathcal{R}}) - b(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})} = \mathbf{\Phi}(\mathcal{R}). \end{aligned} \tag{133}$$

2) The certainty of the ket-mean. If $p_{x_{n+1}} < \mathbf{\Phi}(\mathcal{R}) \leqslant p_{x_n}$ for some $n = 1, \ldots, N$, then

$$\begin{aligned} \mathbf{\Phi}(\langle^{\beta}|\mathcal{E}\mathcal{R}\rangle) &= \mathbf{\Phi}(\langle\Omega|x_{n+1}\rangle + \langle\lambda^{\beta}|\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle) \\ &= p_{x_{n+1}} + \mathbf{B}(\langle\lambda^{\beta}|)\mathbf{P}(\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})) \\ &= p_{x_{n+1}} + \beta p(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}) \\ &= p_{x_{n+1}} + \beta(p_{x_n} - p_{x_{n+1}}) \\ &= p_{x_{n+1}} + \frac{\mathbf{\Phi}(\mathcal{R}) - p_{x_{n+1}}}{p_{x_n} - p_{x_{n+1}}}\big(p_{x_n} - p_{x_{n+1}}\big) = \mathbf{\Phi}(\mathcal{R}). \end{aligned} \tag{134}$$

`Property 17` *(bra-mean).* The ***bra-mean*** $\langle \mathcal{E}\mathcal{R}|^{\pi}\rangle$ of the ordered co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$
1) occurs with certainty $\mathbf{\Phi}(\langle \mathcal{E}\mathcal{R}|^{\pi}\rangle) = \mathbf{\Phi}(\mathcal{R})$;
2) is concluded between bra-events from the sigma-algebra $\langle \mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}|\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}\rangle$:

$$\langle \mathcal{E}_n\mathcal{R}|\Omega\rangle \ \subset\ \langle \mathcal{E}\mathcal{R}|^{\pi}\rangle \ \subseteq\ \langle \mathcal{E}_{n+1}\mathcal{R}|\Omega\rangle\,, \tag{135}$$

for some $n = 0, 1, \ldots, N-1$ and $\pi \in (0,1)$ with which $\mathbf{\Phi}(\mathcal{R})$ satisfies the inequalities:

$$b(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}) \ <\ \mathbf{\Phi}(\mathcal{R}) \ \leqslant\ b(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}) + \pi b_{x_{n+1}}; \tag{136}$$

3) is bra-labellable, i.e., $\langle \mathcal{E}\mathcal{R}|^{\pi}\rangle \in \langle \mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}|\mathcal{A}\rangle$;
4) is representable in the from:

$$\langle \mathcal{E}\mathcal{R}|^{\pi}\rangle = \langle \mathcal{E}_n\mathcal{R}|\Omega\rangle + \langle x_{n+1}|\lambda^{\pi}\rangle \subseteq \langle \mathcal{E}_{n+1}\mathcal{R}|\Omega\rangle\,, \tag{137}$$

for some $n = 0, 1, \ldots, N-1$, with which $\mathbf{\Phi}(\mathcal{R})$ satisfies the inequalities (137), where the co∼event $\langle x_{n+1}|\lambda^{\pi}\rangle \in \langle \mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}|\mathcal{A}\rangle$ has a certainty $\mathbf{\Phi}(\langle x_{n+1}|\lambda^{\pi}\rangle) = \mathbf{\Phi}(\mathcal{R}) - b(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})$.

`Property 18` *(ket-mean).* The ***ket-mean*** $\langle\Omega|\mathcal{E}\mathcal{R}\rangle$ of the ordered co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$
1) occurs with certainty $\mathbf{\Phi}(\langle^{\beta}|\mathcal{E}\mathcal{R}\rangle) = \mathbf{\Phi}(\mathcal{R})$;
2) concluded between the full-believable-random co∼events from the sigma-algebra $\langle \mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}|\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}\rangle$:

$$\langle\Omega|\mathcal{E}_{n+1}\mathcal{R}\rangle \ \subset\ \langle^{\beta}|\mathcal{E}\mathcal{R}\rangle \ \subseteq\ \langle\Omega|\mathcal{E}_n\mathcal{R}\rangle \tag{138}$$

for some $n = 1, \ldots, N$ and $\beta \in (0,1)$, with which $\mathbf{\Phi}(\mathcal{R})$ satisfies the inequalities:

$$p_{x_{n+1}} \ <\ \mathbf{\Phi}(\mathcal{R}) \ \leqslant\ p_{x_{n+1}} + \beta p(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}). \tag{139}$$

3) is ket-labellable, i.e., $\langle^{\beta}|\mathcal{E}\mathcal{R}\rangle \in \langle \mathcal{A}|\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}\rangle$;
4) is presentable in the form

$$\langle^{\beta}|\mathcal{E}\mathcal{R}\rangle = \langle\Omega|\mathcal{E}_{n+1}\mathcal{R}\rangle + \langle\lambda^{\beta}|\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle\,, \tag{140}$$

where the co∼event $\langle\lambda^{\beta}|\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle$ has a certainty $\mathbf{\Phi}(\langle\lambda^{\beta}|\mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle) = \mathbf{\Phi}(\mathcal{R}) - p_{x_{n+1}}$.

`Theorem 8` *(extremal properties of bra-mean and ket-mean of an ordered co∼event). For the ordered co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ with certainty $\varphi = \mathbf{\Phi}(\mathcal{R})$*

- *the bra-mean* $\langle \mathcal{E}\mathcal{R}|^{\pi}\rangle \subseteq \langle\Omega|\Omega\rangle$ *minimizes the certainty distance*[17] *till the co$\sim$event* $\mathcal{R}$*:*

$$\mathbf{\Phi}(\langle \mathcal{E}\mathcal{R}|^{\pi}\rangle \Delta \mathcal{R}) = \min_{\substack{\mathcal{L}\in\langle\mathfrak{R}|^{\mathcal{R}}_{\leqslant 2}\\ \mathbf{\Phi}(\mathcal{L})=\varphi}} \mathbf{\Phi}(\mathcal{L}\Delta\mathcal{R}), \tag{141}$$

among bra-labellable monoplet and doublet co$\sim$events $\mathcal{L} \in \langle\mathfrak{R}|^{\mathcal{R}}_{\leqslant 2} \subseteq \langle \mathcal{A}_{\mathfrak{X}_{\mathcal{R}}}|\mathcal{A}\rangle$ *with wich* $\mathbf{\Phi}(\mathcal{L}) = \varphi$*;*

- *the ket-mean* $\langle^{\beta}|\mathcal{E}\mathcal{R}\rangle \subseteq \langle\Omega|\Omega\rangle$ *minimizes the certainty distance till the co$\sim$event* $\mathcal{R}$*:*

$$\mathbf{\Phi}(\langle^{\beta}|\mathcal{E}\mathcal{R}\rangle \Delta \mathcal{R}) = \min_{\substack{\mathcal{T}\in|\mathfrak{R}\rangle^{\mathcal{R}}_{\leqslant 2}\\ \mathbf{\Phi}(\mathcal{T})=\varphi}} \mathbf{\Phi}(\mathcal{T}\Delta\mathcal{R}), \tag{142}$$

among ket-labellable monoplet and doublet co$\sim$events $\mathcal{T} \in |\mathfrak{R}\rangle^{\mathcal{R}}_{\leqslant 2} \subseteq \langle \mathcal{A}|\mathcal{A}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ *with which* $\mathbf{\Phi}(\mathcal{T}) = \varphi$*.*

Proof follows from analogous assertions for mean-believable bra-events and mean-probable ket-events (see theorems 5 and 6 on page 169 and 171). In fact, the statement of this theorem is actually a «translation» of the formulation of theorems 5 and 6 from the event to the co$\sim$event language. With this «translation» the mean-believable bra-event $\langle\mathcal{E}\mathcal{R}| \subseteq \langle\Omega|$ and the mean-probable ket-event $|\mathcal{E}\mathcal{R}\rangle \subseteq |\Omega\rangle$ is replaced correspondingly by co$\sim$events $\langle\mathcal{E}\mathcal{R}|\Omega\rangle \subseteq \langle\Omega|\Omega\rangle$ and $\langle\Omega|\mathcal{E}\mathcal{R}\rangle \subseteq \langle\Omega|\Omega\rangle$ for which, obviously, the assertions of theorems 5 and 6 remain true. To prove the theorem, we can only note the relationships:

$$\mathbf{\Phi}(\langle \mathcal{E}\mathcal{R}|^{\pi}\rangle \Delta \mathcal{R}) = \mathbf{\Phi}(\langle \mathcal{E}\mathcal{R}|\Omega\rangle \Delta \mathcal{R}), \tag{143}$$
$$\mathbf{\Phi}(\langle^{\beta}|\mathcal{E}\mathcal{R}\rangle \Delta \mathcal{R}) = \mathbf{\Phi}(\langle \Omega|\mathcal{E}\mathcal{R}\rangle \Delta \mathcal{R}), \tag{144}$$

which follows directly from Definition 29 of the bra-mean and the ket-mean of an ordered co$\sim$event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$.

Definition 30 *(bra-variance and ket-variance of co$\sim$event). The bra-variance and the ket-variance of the ordered co$\sim$event* $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ are defined as values equal to certainty distances of the co$\sim$event $\mathcal{R}$ to its corresponding co$\sim$event means:

$$\text{bra-var}\,\mathcal{R} = \mathbf{\Phi}(\langle \mathcal{E}\mathcal{R}|^{\pi}\rangle \Delta \mathcal{R}), \tag{145}$$
$$\text{ket-var}\,\mathcal{R} = \mathbf{\Phi}(\langle^{\beta}|\mathcal{E}\mathcal{R}\rangle \Delta \mathcal{R}). \tag{146}$$

6.3 Dual co$\sim$event means of an arbitrary co$\sim$event

Definition 31 *(bra-mean and ket-mean of an arbitrary co$\sim$event).* For the arbitrary co$\sim$event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$

1) the *bra-mean* $\langle \mathcal{E}\mathcal{R}|^{\pi}\rangle$ is defined as the bra-mean of its $\mathbf{P}$-quotient-projections[18] $\mathcal{R}_{\mathbf{P}} \subseteq \langle\Omega|\Omega\rangle$:

$$\langle \mathcal{E}\mathcal{R}|^{\pi}\rangle = \langle \mathcal{E}\mathcal{R}|^{\pi}_{\mathbf{P}}\rangle \in \langle\mathfrak{R}_{\mathbf{P}}|_{\leqslant 2}\,, \tag{147}$$

where π is the residual probability of co$\sim$event $\mathcal{R}$;

2) the *ket-mean* $\langle^{\beta}|\mathcal{E}\mathcal{R}\rangle$ is defined as the ket-mean of its $\mathbf{B}$-quotient-projections[19] $\mathcal{R}_{\mathbf{B}} \subseteq \langle\Omega|\Omega\rangle$:

$$\langle^{\beta}|\mathcal{E}\mathcal{R}\rangle = \langle^{\beta}|\mathcal{E}\mathcal{R}\rangle_{\mathbf{B}} \in |\mathfrak{R}_{\mathbf{B}}\rangle_{\leqslant 2}\,, \tag{148}$$

where β is the residual believability of co$\sim$event $\mathcal{R}$.

Definition 32 *(bra-variance and ket-variance of an arbitrary co$\sim$event). The bra-variance and the ket-variance of the arbitrary co$\sim$event* $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ are defined as values equal to certainty distances of quotient-projections of co$\sim$event $\mathcal{R}$ to its corresponding co$\sim$event means:

$$\text{bra-var}\,\mathcal{R} = \mathbf{\Phi}(\langle \mathcal{E}\mathcal{R}|^{\pi}_{\mathbf{P}}\rangle \Delta \mathcal{R}_{\mathbf{P}}), \tag{149}$$
$$\text{ket-var}\,\mathcal{R} = \mathbf{\Phi}(\langle^{\beta}|\mathcal{E}\mathcal{R}\rangle_{\mathbf{B}} \Delta \mathcal{R}_{\mathbf{B}}). \tag{150}$$

[17] A certainty of symmetrical difference of two co$\sim$events from $\langle\Omega|\Omega\rangle$.
[18] see Definition 17 on page 162.

7 Two examples of an experienced-random experiment

7.1 Approval voting in elections

Approval (soft rating) voting is «one of the most interesting applications of the modern theory of social choice» — wrote Brahms and Nagel [7, 1991]. «Approval voting is the voting procedure proposed by independent experts in the 1970s, in which voters can vote or approve as many candidates as they want in multi-member elections (i.e. with more than two candidates). Each candidate who receives the approval of one voter receives one vote, and the candidate with the most votes wins». (see Brams and Fishburn [6, 1992], and also [4, 1978],[5, 1983]).

It is clear that in addition to the main application of the approval voting to identify winners in election, it can also be used to understand the uncertainty in behavior of voters and candidates in elections. The purpose of this article is to clarify the answers to the questions, some of which have arisen earlier, but have not been properly articulated[19]:

- with what believabilities each voter is experienced an approval of various subsets of candidates? (the *believability distribution* of a set of voters),
- with what probabilities each candidate *happens* to be approved by voters? (*probability distribution* of a set of candidates),
- what is the «typical» or «mean» set of candidates which *happens* to be approved by voters and with what *probability*? (the *mean-probable set* of candidates or the *ket-mean co∼event* «election»),
- what is the «typical» or «mean» set of voters which *is experienced* an approval for candidates and with what *believability*? (the *mean-believable set* of voters and the *bra-mean co∼event* «election»),
- with what *certainty a co∼event «election» occurs*?,
- how is certainty of the co∼event «election» distributed over the space «voters–candidates»? (the *certainty distribution of the co∼event* «election»);
- with what *certainty* do the extreme deviations of voters and candidates of its corresponding «typical» or «mean» *co∼events* and so on *occur*?

In addition to these questions, the theory of experience and chance [22] allows us to answer questions that have never been put before, for example,

- with what *certainty* a co∼event «non-election» being a complementary to the co∼event «election», does occur?
- how is certainty of the co∼event «nonelection» distributed over the space «voters–candidates»? (the *certainty distribution of the co∼event* «non-election»),

as well as many others questions associated with a co∼event «non-selection», the answers to which expand the understanding of the behavior of the tandem «voters–candidates» and leave a promising space for co∼event imagination.

7.1.1 The first statistics of approval voting

Brams and Fishburn analyzed the 55827x4 statistics (Table 3 in [6]) given in the table 4, the 1988 IEEE elections, which were conducted in accordance with the approval voting. Our main idea is an experienced-random analysis of this statistic, assuming that each candidate is approved by the subset X of the voters $x \in \mathfrak{X}$ which collectively form the set $\mathfrak{X} = \{x_1, \dots, x_N\}$ from $N = |\mathfrak{X}|$ voters ordered with respect to the number of candidates approved by them. The subsets $X \subseteq \mathfrak{X}$ of voters that approve each candidate form a set $\mathfrak{S}^{\mathfrak{X}} = \{X_1, \dots, X_{\mathfrak{S}^N}\} \subseteq \mathcal{P}(\mathfrak{X})$ from $\mathfrak{S}^N = |\mathfrak{S}^{\mathfrak{X}}|$, the set of candidates labels, ordered with respect to the number of approval voters. The total number of voters and the number of candidates in statistics were $N = 55,827$ and $\mathfrak{S}^N = 4$, so the labelling sets look like: $\mathfrak{X} = \{x_1, \dots, x_{55827}\}$, $\mathfrak{S}^{\mathfrak{X}} = \{X_1, \dots, X_4\} = \{C, D, B, A\}$.

Numbers of voters for subsets of candidates:					
		$N(\emptyset) = 1,100$			
	$N(A) = 10,738$	$N(B) = 6,561$	$N(C) = 7,626$	$N(D) = 8,521$	
$N(AB) = 3,578$	$N(AC) = 659$	$N(AD) = 6,679$	$N(BC) = 1,425$	$N(BD) = 1,824$	$N(CD) = 608$
	$N(ABC) = 148$	$N(ABD) = 5,605$	$N(ACD) = 143$	$N(BCD) = 89$	
		$N(\mathfrak{S}^{\mathfrak{X}}) = 523$			

Totals:			
$N_A = 28,073$	$N_B = 19,753$	$N_C = 11,221$	$N_D = 23,992$

Table 4: Numbers of voters $N(\alpha)$ who voted for the 16 different subsets $\alpha \in \mathcal{P}(\mathfrak{S}^{\mathfrak{X}})$ of candidates from a power set of candidates $\mathfrak{S}^{\mathfrak{X}}$ in the 1988 IEEE election and AV totals. (Taken from Brams and Fishburn [6, 1992, Table 3], with $N(\emptyset)$ is a number of voters with "None votes", $N(\mathfrak{S}^{\mathfrak{X}})$ is a number of voters with "All votes", for example $N(ABC) = N(\{A, B, C\})$ is a number of voters with the subset $\{A, B, C\}$ of votes where $\{A, B, C\} \subseteq \mathfrak{S}^{\mathfrak{X}}$, N_A is a number of voters for the candidate $A \in \mathfrak{S}^{\mathfrak{X}}$, and at last $N = \sum_{\alpha \in \mathcal{P}(\mathfrak{S}^{\mathfrak{X}})} N(\alpha) = 55,827$ is a total number of voters.)

Candidates	X_4	X_3	X_2	X_1	Total	No. of Voters
0-voters	0	0	0	0	0	1,100
1-Voters	10,738	8,521	6,561	7,626	33,446	33,446
2-Voters	10,916	9,111	6,827	2,692	29,546	14,773
3-Voters	5,896	5,837	5,842	380	27,955	5,985
4-Voters	523	523	523	523	2,092	523
Total	28,073	23,992	19,753	11,221	93,039	55,827

Table 5: Numbers of votes and 55827 voters who voted for the 4 candidates: AV Vote Totals in 1988 IEEE (Calculated from Tab. 4).

Our *experienced-random approach* supposes that we deal with statistics resulting from the *experienced-random experiment* in the form of the co$\sim$event «election» («approval voting of voters for candidates»), the binary relation $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$, generating the element-set labelling $\langle\mathfrak{X}_{\mathcal{R}}|\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}\rangle = \langle\mathfrak{X}|\mathfrak{S}^{\mathfrak{X}}\rangle$. In other words, the labelling set of labels $\mathfrak{X}_{\mathcal{R}}$ is interpreted as the set of voters: $\mathfrak{X}_{\mathcal{R}} = \mathfrak{X}$, and the labelling set of set labels $\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}$ is interpreted as the set of candidates: $\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} = \mathfrak{S}^{\mathfrak{X}}$. The element-set labelling $\langle\mathfrak{X}|\mathfrak{S}^{\mathfrak{X}}\rangle$ by means of relations of $\mathbf{P}$-equivalence «$\overset{\mathbf{P}}{\sim}$» and $\mathbf{B}$-equivalence «$\overset{\mathbf{B}}{\sim}$» defines the $\mathbf{PB}$-quotient-labelling $\langle\mathfrak{X}^{\mathbf{P}}|\mathfrak{S}^{\mathfrak{X}}_{\mathbf{B}}\rangle$, which is called the $\mathbf{PB}$-quotient-space «voters–candidates». The labelling sets of this quotient-space have a form: $\mathfrak{X}^{\mathbf{P}} = \{x^{\mathbf{P}}_1, \dots, x^{\mathbf{P}}_5\}$, $\mathfrak{S}^{\mathfrak{X}}_{\mathbf{B}} = \{X^{\mathbf{B}}_1, \dots, X^{\mathbf{B}}_4\}$, where voters $x^{\mathbf{P}} \in \mathfrak{X}^{\mathbf{P}}$ are ordered with respect to decreasing the number of their approving votes for all candidates, $x^{\mathbf{P}}_5 = \Omega^{\mathbf{P}}$ are the voters who did not approve candidates, and candidates $X^{\mathbf{B}} \in \mathfrak{S}^{\mathfrak{X}}_{\mathbf{B}}$ are ordered with respect to increasing the number of approval votes, received from voters. In the statistics $X^{\mathbf{B}}_4$ is the set label of the 28073-candidate (i.e. of candidate with 28073 approval votes), $X^{\mathbf{B}}_3$ is the set of the 23992-candidate, $X^{\mathbf{B}}_2$ is the set label of the 19753-candidate, $X^{\mathbf{B}}_1$ is the set label of the 11221-candidate.

Bra-mean of the co$\sim$event «election». The mean value of voters, voting for each candidate, 20759.750. The value and the experienced-random statistics of voting lead to the bra-mean co$\sim$event (see Figures 13, 14 and 17)

$$\langle\mathcal{ER}|^{0.965}\rangle = \langle\mathsf{Ter}_{X_2/\!\!/\mathfrak{X}_{\mathcal{R}}}|\Omega\rangle + \langle x_3|\mathsf{ter}(W_2/\!\!/\{W_1, W_2\})\rangle\,, \tag{151}$$

which occurs with certainty $\mathbf{\Phi}(\langle\mathcal{ER}|^{0.965}\rangle) = \mathbf{\Phi}(\mathcal{R}) = 0.372 = 20759.750/55827$ then, when the co$\sim$event $\langle\mathsf{Ter}_{X_2/\!\!/\mathfrak{X}_{\mathcal{R}}}|\Omega\rangle$ occurs with certainty $\mathbf{\Phi}(\langle\mathsf{Ter}_{X_2/\!\!/\mathfrak{X}_{\mathcal{R}}}|\Omega\rangle) = 0.117$ $\big($is experienced with believability $\mathbf{B}(\langle\mathsf{Ter}_{X_2/\!\!/\mathfrak{X}_{\mathcal{R}}}|) = 0.117$ and happens with probability $\mathbf{P}(|\Omega\rangle) = 1\big)$, and the co$\sim$event $\langle x_3|\mathsf{ter}(W_2/\!\!/\{W_1, W_2\})\rangle$ occurs with certainty $\mathbf{\Phi}(\langle x_3|\mathsf{ter}(W_2/\!\!/\{W_1, W_2\})\rangle) = 0.255 = 0.265 \times 0.965$ $\big($is experienced with believability $\mathbf{B}(\langle x_3|) = 0.265$ and happens with the residual probability $\mathbf{P}(|\mathsf{ter}(W_2/\!\!/\{W_1, W_2\})\rangle) = 0.965\big)$.

Ket-mean of the co$\sim$event «election». The mean value of candidates, for which each voter voted, 1.487. The value and the experienced-random statistics lead to the ket-mean co$\sim$event (see Figures 13, 14 and 17)

$$\langle^{0.487}|\mathcal{ER}\rangle = \langle\Omega|x_1\rangle + \langle w_2|\mathsf{ter}(X_2/\!\!/\mathfrak{X}_{\mathcal{R}})\rangle\,, \tag{152}$$

which occurs with certainty $\mathbf{\Phi}(\langle^{0.487}|\mathcal{ER}\rangle) = \mathbf{\Phi}(\mathcal{R}) = 0.372 = 1.487/4$ then, when co$\sim$event $\langle\Omega|x_1\rangle$ occurs with certainty $\mathbf{\Phi}(\langle\Omega|x_1\rangle) = 0.250$ $\big($is experienced with believability $\mathbf{B}(\langle\Omega|) = 1$ and happens

[19] In the text below, the terms of the *theory of experience and chance* [22] which opens the way to achieving these goals, are highlighted in italics.

with probability $\mathbf{P}(|x_4\rangle) = 0.250\big)$, and the co∼event $\langle w_2|\mathsf{ter}(X_2 /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle$ occurs with certainty $\mathbf{\Phi}(\langle w_2|\mathsf{ter}(X_5 /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle) = 0.122 = 0.487 \times 0.250$ $\big($is experienced with residual believability $\mathbf{B}(\langle w_2|) = 0.487$ and happens with probability $\mathbf{P}(|\mathsf{ter}(X_2 /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle) = 0.250\big)$.

7.1.2 The second statistics of approval voting

The following 3933x5-statistics which was analyzed by Brams and Fichburn (see Table 2 in [6]) is given in Table 6. It also describes results of the experienced-random experiment of election with approval voting. Unfortunately, this Brahms and Fishburn statistics contains only total sample values which are insufficient for estimating the certainty distribution of co∼event $\mathcal{R}$ on the space of «candidate–voters», as well as its event and co∼-event means. We took the liberty to «restore»[20] a full 3933x5-statistics which, having the same total sample values as Brams and Fishburn, would allow us to evaluate all the event and co∼event characteristics of the co∼event $\mathcal{R}$ «election» (see Table 7).

Our main idea is the same. It is an experienced-random analysis of this statistics, assuming that each candidate is approved by the subset X of the voters $x \in \mathfrak{X}$ which collectively form the set $\mathfrak{X} = \{x_1, \dots, x_N\}$ from $N = |\mathfrak{X}|$ voters ordered with respect to the number of candidates approved by them. The subsets $X \subseteq \mathfrak{X}$ of voters that approve each candidate form the set $\mathfrak{S}^{\mathfrak{X}} = \{X_1, \dots, X_{\mathfrak{S}^N}\} \subseteq \mathcal{P}(\mathfrak{X})$ and $\mathfrak{S}^N = |\mathfrak{S}^{\mathfrak{X}}|$ of set labels of candidates ordered with respect to the number of approval votes, submitted for them by voters. The total number of voters and the number of candidates in statistics were $N = 3933$ and $\mathfrak{S}^N = 5$ so that the labelling sets have the form: $\mathfrak{X} = \{x_1, \dots, x_{3933}\}$, $\mathfrak{S}^{\mathfrak{X}} = \{X_1, \dots, X_5\} = \{A, B, D, C, E\}$.

Candidates	A	B	C	D	E	Total	No. of Voters
0-voters	0	0	0	0	0	0	0
1-Voters	848	618	652	660	303	3,081	3,081
2-Voters	276	275	264	273	132	1,220	610
3-Voters	122	127	134	118	87	588	196
4-Voters	21	32	34	31	30	148	37
5-voters	9	9	9	9	9	45	9
Total	1,276	1,093	1,061	1,091	561	5,082	3,933

Table 6: Numbers of votes and 3933 voters who voted for the 5 candidates: AV Vote Totals in 1987 MAA Election (Taken from Brams and Fishburn [6, 1992, Table 2]).

Numbers of voters for subsets of candidates:

		$N(\emptyset) = 0$		
$N(A) = 848$	$N(B) = 618$	$N(C) = 652$	$N(D) = 660$	$N(E) = 303$
$N(AB) = 84$	$N(AC) = 77$	$N(AD) = 82$	$N(AE) = 33$	$N(BC) = 77$
$N(BD) = 81$	$N(BE) = 33$	$N(DC) = 77$	$N(CE) = 33$	$N(DE) = 33$
$N(ABC) = 30$	$N(BCD) = 30$	$N(ACD) = 27$	$N(ABD) = 22$	$N(BEC) = 16$
$N(ACE) = 10$	$N(ABE) = 16$	$N(DEC) = 15$	$N(DEB) = 13$	$N(ADE) = 11$
$N(BCDE) = 16$	$N(ACDE) = 5$	$N(ABDE) = 3$	$N(ABCE) = 6$	$N(ABCD) = 7$
		$N(ABCDE) = 9$		

Totals:

$N_A = 1,256$	$N_B = 1,093$	$N_C = 1,061$	$N_D = 1,091$	$N_E = 561$

Table 7: Numbers of voters $N(\alpha)$ who voted for the 16 different subsets $\alpha \in \mathcal{P}(\mathfrak{S}^{\mathfrak{X}})$ of candidates from a power set of candidates $\mathfrak{S}^{\mathfrak{X}}$ in 1987 MAA Election and AV totals. (Taken from Brams and Fishburn [6, 1992, Table 2], with $N(\emptyset)$ is a number of voters with "None votes", $N(\mathfrak{S}^{\mathfrak{X}})$ is a number of voters with "All votes", for example $N(ABC) = N(\{A, B, C\})$ is a number of voters with the subset $\{A, B, C\}$ of votes where $\{A, B, C\} \subseteq \mathfrak{S}^{\mathfrak{X}}$, N_A is a number of voters for the candidate $A \in \mathfrak{S}^{\mathfrak{X}}$, and at last $N = \sum_{\alpha \in \mathcal{P}(\mathfrak{S}^{\mathfrak{X}})} N(\alpha) = 3,933$ is a total number of voters.)

Our *experienced-random approach* assumed that we deal with a statistics resulting from the *experienced-random experiemnt* in the form of the co∼event «election» («approval voting of voters for candidates»), the binary relation $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$, generating the element-set labelling $\langle \mathfrak{X}_{\mathcal{R}}|\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}\rangle = \langle \mathfrak{X}|\mathfrak{S}^{\mathfrak{X}}\rangle$. In other words, the labelling set of labels $\mathfrak{X}_{\mathcal{R}}$ is interpreted as the set of voters: $\mathfrak{X}_{\mathcal{R}} = \mathfrak{X}$, and the labelling set of set labels $\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}$ is interpreted as the set of candidates: $\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} = \mathfrak{S}^{\mathfrak{X}}$. The element-set labelling $\langle \mathfrak{X}|\mathfrak{S}^{\mathfrak{X}}\rangle$ and

[20] Of course, in this example, the results of the «restoration» of the votes «2-voters» and «3-voters» on 2-subsets and 3-subsets of candidates are not may be unique.

relations of $\mathbf{P}$-equivalence «$\overset{\mathbf{P}}{\sim}$» and $\mathbf{B}$-equivalence «$\overset{\mathbf{B}}{\sim}$» define the $\mathbf{PB}$-quotient-labelling $\langle \mathfrak{X}^{\mathbf{P}} | \mathfrak{S}^{\mathfrak{X}}_{\mathbf{B}} \rangle$, which is called the $\mathbf{PB}$-quotient-space «voters–candidates». The labelling sets of this quotient-space have a form: $\mathfrak{X}^{\mathbf{P}} = \{x^{\mathbf{P}}_1, \dots, x^{\mathbf{P}}_5\}$, $\mathfrak{S}^{\mathfrak{X}}_{\mathbf{B}} = \{X^{\mathbf{B}}_1, \dots, X^{\mathbf{B}}_5\}$, where voters $x^{\mathbf{P}} \in \mathfrak{X}^{\mathbf{P}}$ ordered in descending the number of their approving votes for all candidates, and candidates $X^{\mathbf{B}} \in \mathfrak{S}^{\mathfrak{X}}_{\mathbf{B}}$ ordered in increasing number of approving votes received by them from voters. In the statistics it turned out that $X^{\mathbf{B}}_5$ is the set label of 1276-candidate (i.e. of candidate with 1276 approval votes), $X^{\mathbf{B}}_4$ is the set label of 1093-candidate, $X^{\mathbf{B}}_3$ is the set label of 1091-candidate, $X^{\mathbf{B}}_2$ is the set label of 1061-candidate, $X^{\mathbf{B}}_1$ is the set label of 561-candidate.

Bra-mean of the co$\sim$event «election». The mean value of voters, which voted for each candidate, 1016.400. The value and the experienced-random statistics lead to the bra-mean of the co$\sim$event (see Figures 15, 16 and 18)

$$\langle \mathcal{ER} |^{0.053} \rangle = \langle \mathsf{Ter}_{X_2 /\!\!/ \mathfrak{X}_{\mathcal{R}}} | \Omega \rangle + \langle x_3 | \mathsf{ter}(W_2 /\!\!/ \{W_1, W_2\}) \rangle \,, \tag{153}$$

which occurs with certainty $\mathbf{\Phi}(\langle \mathcal{ER}|^{0.053}\rangle) = \mathbf{\Phi}(\mathcal{R}) = 0.258 = 1016.400/3933$ then, when the co$\sim$event $\langle \mathsf{Ter}_{X_4 /\!\!/ \mathfrak{X}_{\mathcal{R}}} | \Omega \rangle$ occurs with certainty $\mathbf{\Phi}(\langle \mathsf{Ter}_{X_4 /\!\!/ \mathfrak{X}_{\mathcal{R}}} | \Omega \rangle) = 0.217$ $\big($is experienced with believability $\mathbf{B}(\langle \mathsf{Ter}_{X_4 /\!\!/ \mathfrak{X}_{\mathcal{R}}} |) = 0.217$ and happens with probability $\mathbf{P}(|\Omega\rangle) = 1\big)$, and the co$\sim$event $\langle x_5 | \mathsf{ter}(W_2 /\!\!/ \{W_1, W_2\})\rangle$ occurs with certainty $\mathbf{\Phi}(\langle x_5 | \mathsf{ter}(W_2 /\!\!/ \{W_1, W_2\})\rangle) = 0.041 = 0.783 \times 0.053$ $\big($is experienced with believability $\mathbf{B}(\langle x_5|) = 0.783$ and happens with the residual probability $\mathbf{P}(|\mathsf{ter}(W_2 /\!\!/ \{W_1, W_2\})\rangle) = 0.053\big)$.

Ket-mean of the co$\sim$event «election». The mean value of candidates, for which each voter voted, 1.292. The value and the experienced-random statistics lead to the ket-mean of the co$\sim$event (see Figures 15, 16 and 18)

$$\langle^{0.292} | \mathcal{ER} \rangle = \langle \Omega | x_1 \rangle + \langle w_2 | \mathsf{ter}(X_2 /\!\!/ \mathfrak{X}_{\mathcal{R}}) \rangle \,, \tag{154}$$

which occurs with certainty $\mathbf{\Phi}(\langle^{0.487} | \mathcal{ER}\rangle) = \mathbf{\Phi}(\mathcal{R}) = 0.258 = 1.292/5$ then, when the co$\sim$event $\langle \Omega | x_1 \rangle$ occurs with certainty $\mathbf{\Phi}(\langle \Omega | x_1 \rangle) = 0.200$ $\big($is experienced with believability $\mathbf{B}(\langle \Omega |) = 1$ and happens with probability $\mathbf{P}(|x_4\rangle) = 0.200\big)$, and the co$\sim$event $\langle w_2 | \mathsf{ter}(X_2 /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle$ occurs with certainty $\mathbf{\Phi}(\langle w_2 | \mathsf{ter}(X_5 /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle) = 0.058 = 0.292 \times 0.200$ $\big($is experienced with the residual believability $\mathbf{B}(\langle w_2 |) = 0.292$ and happens with probability $\mathbf{P}(|\mathsf{ter}(X_2 /\!\!/ \mathfrak{X}_{\mathcal{R}})\rangle) = 0.200\big)$.

7.2 «Approval voting» in forestry

The statistics (see Figures 1) survey trees by foresters to label trees for felling, which was carried out similarly to the procedure of approval voting, was considered in [13]. The authors of this work came to the problem of approving voting in the context of the annex to forestry [2], where foresters were encouraged to classify trees as «good» or «bad» for killing them (to fell). Foresters had to give each tree a label, following instructions known in advance, which they more or less followed because of their experience. In terms of the theory of social choice, foresters are voters, and trees are candidates; «good» can mean «approved». The purpose of this experiment in forestry was to study the psychology of foresters' behavior in the procedure for selecting trees. In everyday forestry, one forester walks through the forest and labels trees that must be cut down based on his professional experience. One of the reasons for such experiments is the modeling of the forester's work in modern forest models that describe the evolution of forests over time, when the human element is an important element of the model. However, the authors of [13] also see a more general problem of understanding foresters' psychology, their interaction with trees and their environment.

Our main idea is in the experienced-random analysis of this statistics under assumption that each tree is labelled for cutting by the subset X of foresters $x \in \mathfrak{X}$ which all together form the set $\mathfrak{X} = \{x_1, \dots, x_N\}$ of $N = |\mathfrak{X}|$ foresters ordered with respect to the number of trees labelled by them. The subset $X \subseteq \mathfrak{X}$ of foresters, which label each tree, form the set $\mathfrak{S}^{\mathfrak{X}} = \{X_1, \dots, X_{\mathfrak{S}^N}\} \subseteq \mathcal{P}(\mathfrak{X})$ from $\mathfrak{S}^N = |\mathfrak{S}^{\mathfrak{X}}|$ of set labels on trees ordered respect to the number of foresters labelled them. The total number of foresters and the number of trees in statistics were $N = 15$ and $\mathfrak{S}^N = 387$, so that the labelling sets have the form: $\mathfrak{X} = \{x_1, \dots, x_{15}\}$, $\mathfrak{S}^{\mathfrak{X}} = \{X_1, \dots, X_{387}\}$.

Our *experienced-random approach* assumes that we deal with statistics resulting from the *experienced-random experiment* in the form of the co$\sim$event «choosing by foresters the trees for felling», the binary relation $\mathcal{R} \subseteq \langle \Omega | \Omega \rangle$, generating the element-set labelling $\langle \mathfrak{X}_{\mathcal{R}} | \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} \rangle = \langle \mathfrak{X} | \mathfrak{S}^{\mathfrak{X}} \rangle$. In other words, the labelling set of labels $\mathfrak{X}_{\mathcal{R}}$ is interpreted as the set of foresters: $\mathfrak{X}_{\mathcal{R}} = \mathfrak{X}$, and the labelling set of set labels $\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}$ is interpreted as the set of trees: $\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} = \mathfrak{S}^{\mathfrak{X}}$. The element-set labelling $\langle \mathfrak{X} | \mathfrak{S}^{\mathfrak{X}} \rangle$ and relations of $\mathbf{P}$-equivalence

«$\overset{\mathbf{P}}{\sim}$» and $\mathbf{B}$-equivalence «$\overset{\mathbf{B}}{\sim}$» define the $\mathbf{PB}$-quotient-labelling $\langle \mathfrak{X}^{\mathbf{P}} | \mathcal{X}_{\mathbf{B}}^{\mathfrak{X}} \rangle$, which is called the $\mathbf{PB}$-quotient-space «foresters–trees». The labelling sets of this quotient-space have the form: $\mathfrak{X}^{\mathbf{P}} = \{x_1^{\mathbf{P}}, \ldots, x_{15}^{\mathbf{P}}\}$, $\mathcal{X}_{\mathbf{B}}^{\mathfrak{X}} = \{X_1^{\mathbf{B}}, \ldots, X_{14}^{\mathbf{B}}\}$, where foresters $x^{\mathbf{P}} \in \mathfrak{X}^{\mathbf{P}}$ are ordered in descending order of the number of labels they put on all trees, and trees $X^{\mathbf{B}} \in \mathcal{X}_{\mathbf{B}}^{\mathfrak{X}}$ are ordered in ascending the number of labels placed on them by foresters. In the statistics it turned out that $X_{13}^{\mathbf{B}}$ is the set label of 13-trees (i.e. of trees labelled by 13 labels), $X_{12}^{\mathbf{B}}$ is the set label of 12-trees, ..., $X_{10}^{\mathbf{B}}$ is the set label of 10-trees, ..., $X_1^{\mathbf{B}}$ is the set label of 1-trees, and $\emptyset^{\mathbf{B}}$ is the empty set label of 0-trees, i.e. of trees, which were left without labels.

The new approach is significantly set-theoretical and assumes that terraced bra-events $\langle \mathsf{Ter}_{X /\!\!/ \mathfrak{X}_{\mathcal{R}}} | \subseteq \langle \Omega |$ are realizations of a *random set*, and ket-events $| x \rangle \subseteq | \Omega \rangle$ are realization of an *experienced set* (see [22]), for anlysis of which an experienced-random generalization of some elements of the statistical theory of compact random sets is suitable [14, 11, 8]. In addition, we describe new forms of analysis of experienced-random data, based on the theory of experience and chance [22].

m	13	12	11	10	9	8	7	6	5	4	3	2	1	0
$N(m)$	2	3	2	4	10	11	20	38	40	57	62	50	58	30

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$M(n)$	184	134	130	115	112	105	104	101	93	86	84	57	54	46	37

Figure 1: At the top: the statistics of choosing by 15 foresters (rows) from 387 trees (columns) the trees for felling, among which 30 trees were unchosen, i.e. were left without labels (green shows even numbers of labels on the trees in the columns of the table). Below: the numbers $N(m)$ of trees with m labels, $m = 13, 12, \ldots, 1, 0$. Even lower: the «activity» $M(n)$ of the forester with number $n = 1, 2, \ldots, 15$ (the number of labelled trees in the rows of the table above).

Bra-mean of the co∼event «choosing by foresters the trees for felling». The mean number of foresters, label each tree, 3.726. The value and the experienced-random statistics of choosing lead to the bra-mean of the co∼event (see Figures 3 — 11, and also 12)

$$\langle \mathcal{E}\mathcal{R} |^{0.726} \rangle = \langle \mathsf{Ter}_{X_3 /\!\!/ \mathfrak{X}_{\mathcal{R}}} | \Omega \rangle + \langle x_4 | \mathsf{ter}(W_2 /\!\!/ \{W_1, W_2\}) \rangle \,, \tag{155}$$

which occurs with certainty $\mathbf{\Phi}(\langle \mathcal{E}\mathcal{R} |^{0.726} \rangle) = \mathbf{\Phi}(\mathcal{R}) = 0.248 = 3.726/15$ then, when the co∼event $\langle \mathsf{Ter}_{X_3 /\!\!/ \mathfrak{X}_{\mathcal{R}}} | \Omega \rangle$ occurs with certainty $\mathbf{\Phi}(\langle \mathsf{Ter}_{X_3 /\!\!/ \mathfrak{X}_{\mathcal{R}}} | \Omega \rangle) = 0.200$ $\big($is experienced with believability $\mathbf{B}(\langle \mathsf{Ter}_{X_3 /\!\!/ \mathfrak{X}_{\mathcal{R}}} |) = 0.200$ and happens with probability $\mathbf{P}(| \Omega \rangle) = 1\big)$, and the co∼event $\langle x_4 | \mathsf{ter}(W_2 /\!\!/ \{W_1, W_2\}) \rangle$ occurs with certainty $\mathbf{\Phi}(\langle x_4 | \mathsf{ter}(W_2 /\!\!/ \{W_1, W_2\}) \rangle) = 0.048 = 0.067 \times 0.726$ $\big($is experienced with believability $\mathbf{B}(\langle x_4 |) = 0.067$ and happens with the residual probability $\mathbf{P}(| \mathsf{ter}(W_2 /\!\!/ \{W_1, W_2\}) \rangle) = 0.726$).

Ket-mean of the co∼event «choosing by foresters the trees for felling». The mean number of trees, labelled by each forester, 96.133. The value and the experienced-random statistics of choosing lead to the ket-mean of the co∼event (see Figures 3 — 11, and also 12)

$$\langle^{0.153} | \mathcal{E}\mathcal{R} \rangle = \langle \Omega | x_4 \rangle + \langle w_2 | \mathsf{ter}(X_5 /\!\!/ \mathfrak{X}_{\mathcal{R}}) \rangle \,, \tag{156}$$

which occurs with certainty $\mathbf{\Phi}(\langle^{0.153} | \mathcal{E}\mathcal{R} \rangle) = \mathbf{\Phi}(\mathcal{R}) = 0.248 = 96.133/387$ then, when the co∼event $\langle \Omega | x_4 \rangle$ occurs with certainty $\mathbf{\Phi}(\langle \Omega | x_4 \rangle) = 0.233$ $\big($is experienced with believability $\mathbf{B}(\langle \Omega |) = 1$ and happens with probability $\mathbf{P}(| x_4 \rangle) = 0.233\big)$, and the co∼event $\langle w_2 | \mathsf{ter}(X_n /\!\!/ \mathfrak{X}_{\mathcal{R}}) \rangle$ occurs with certainty $\mathbf{\Phi}(\langle w_2 | \mathsf{ter}(X_5 /\!\!/ \mathfrak{X}_{\mathcal{R}}) \rangle) = 0.015 = 0.153 \times 0.103$ $\big($is experienced with the residual believability $\mathbf{B}(\langle w_2 |) = 0.153$ and happens with probability $\mathbf{P}(| \mathsf{ter}(X_5 /\!\!/ \mathfrak{X}_{\mathcal{R}}) \rangle) = 0.103\big)$.

8 Appendix

8.1 Results of processing the statistics 55827x4 «voters–candidates» and 15x387 «foresters–trees» and the transposed statistics 4x55827 «candidates–voters» and 387x15 «trees–foresters»

	$\vert X^{\mathbf{B}}_{15}\rangle$	$\vert X^{\mathbf{B}}_{14}\rangle$	$\vert X^{\mathbf{B}}_{13}\rangle$	$\vert X^{\mathbf{B}}_{12}\rangle$	$\vert X^{\mathbf{B}}_{11}\rangle$	$\vert X^{\mathbf{B}}_{10}\rangle$	$\vert X^{\mathbf{B}}_{9}\rangle$	$\vert X^{\mathbf{B}}_{8}\rangle$	$\vert X^{\mathbf{B}}_{7}\rangle$	$\vert X^{\mathbf{B}}_{6}\rangle$	$\vert X^{\mathbf{B}}_{5}\rangle$	$\vert X^{\mathbf{B}}_{4}\rangle$	$\vert X^{\mathbf{B}}_{3}\rangle$	$\vert X^{\mathbf{B}}_{2}\rangle$	$\vert X^{\mathbf{B}}_{1}\rangle$
$\langle x^{\mathbf{P}}_{1}\vert$	0,00	0,00	0,50	0,00	0,00	0,00	0,50	0,00	0,00	0,50	0,00	0,00	0,50	0,00	0,00
$\langle x^{\mathbf{P}}_{2}\vert$	0,00	0,33	0,00	0,00	0,00	0,33	0,33	0,00	0,33	0,33	0,33	0,00	0,67	0,00	0,33
$\langle x^{\mathbf{P}}_{3}\vert$	0,00	0,00	0,00	0,50	0,50	0,00	0,50	1,00	0,00	0,00	0,00	0,50	0,00	0,50	0,50
$\langle x^{\mathbf{P}}_{4}\vert$	0,25	0,25	0,00	0,00	0,50	0,50	0,50	0,25	0,50	0,50	0,75	0,50	0,00	0,00	0,50
$\langle x^{\mathbf{P}}_{5}\vert$	0,50	0,40	0,30	0,10	0,40	0,40	0,30	0,30	0,30	0,40	0,10	0,50	0,70	0,50	0,80
$\langle x^{\mathbf{P}}_{6}\vert$	0,27	0,09	0,18	0,64	0,36	0,18	0,55	0,27	0,36	0,64	0,55	0,91	0,73	0,73	0,55
$\langle x^{\mathbf{P}}_{7}\vert$	0,20	0,45	0,40	0,45	0,45	0,10	0,70	0,50	0,70	0,65	0,35	0,80	0,60	0,80	0,85
$\langle x^{\mathbf{P}}_{8}\vert$	0,39	0,58	0,39	0,53	0,63	0,45	0,66	0,58	0,68	0,55	0,53	0,74	0,71	0,79	0,79
$\langle x^{\mathbf{P}}_{9}\vert$	0,23	0,55	0,55	0,68	0,55	0,57	0,70	0,60	0,75	0,75	0,72	0,82	0,82	0,82	0,88
$\langle x^{\mathbf{P}}_{10}\vert$	0,51	0,65	0,63	0,60	0,65	0,81	0,70	0,82	0,70	0,72	0,77	0,79	0,82	0,91	0,91
$\langle x^{\mathbf{P}}_{11}\vert$	0,48	0,65	0,74	0,84	0,73	0,87	0,66	0,74	0,74	0,82	0,94	0,92	0,95	0,94	0,98
$\langle x^{\mathbf{P}}_{12}\vert$	0,58	0,74	0,74	0,82	0,90	0,90	0,82	0,88	0,90	0,84	0,94	0,96	0,98	1,00	1,00
$\langle x^{\mathbf{P}}_{13}\vert$	0,83	0,84	0,98	0,86	0,90	0,97	0,86	0,93	0,91	1,00	0,98	0,95	1,00	1,00	0,98
$\langle \varnothing^{\mathbf{P}}_{\Omega}\vert$	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00

Figure 2: Quotient-indicator of the M-complementary co$\sim$event on the 14x15 quotient-space for 387x15 statistics (maximum value = 1).

	$\vert X^{\mathbf{B}}_{15}\rangle$	$\vert X^{\mathbf{B}}_{14}\rangle$	$\vert X^{\mathbf{B}}_{13}\rangle$	$\vert X^{\mathbf{B}}_{12}\rangle$	$\vert X^{\mathbf{B}}_{11}\rangle$	$\vert X^{\mathbf{B}}_{10}\rangle$	$\vert X^{\mathbf{B}}_{9}\rangle$	$\vert X^{\mathbf{B}}_{8}\rangle$	$\vert X^{\mathbf{B}}_{7}\rangle$	$\vert X^{\mathbf{B}}_{6}\rangle$	$\vert X^{\mathbf{B}}_{5}\rangle$	$\vert X^{\mathbf{B}}_{4}\rangle$	$\vert X^{\mathbf{B}}_{3}\rangle$	$\vert X^{\mathbf{B}}_{2}\rangle$	$\vert X^{\mathbf{B}}_{1}\rangle$
$\langle x^{\mathbf{P}}_{1}\vert$	1,000	1,000	0,500	1,000	1,000	1,000	0,500	1,000	1,000	0,500	1,000	1,000	0,500	1,000	1,000
$\langle x^{\mathbf{P}}_{2}\vert$	1,000	0,667	1,000	1,000	1,000	0,667	0,667	1,000	0,667	0,667	0,667	1,000	0,333	1,000	0,667
$\langle x^{\mathbf{P}}_{3}\vert$	1,000	1,000	1,000	0,500	0,500	1,000	0,500	0,000	1,000	1,000	1,000	0,500	1,000	0,500	0,500
$\langle x^{\mathbf{P}}_{4}\vert$	0,750	0,750	1,000	1,000	0,500	0,500	0,500	0,750	0,500	0,500	0,250	0,500	1,000	1,000	0,500
$\langle x^{\mathbf{P}}_{5}\vert$	0,500	0,600	0,700	0,900	0,600	0,600	0,700	0,700	0,700	0,600	0,900	0,500	0,300	0,500	0,200
$\langle x^{\mathbf{P}}_{6}\vert$	0,727	0,909	0,818	0,364	0,636	0,818	0,455	0,727	0,636	0,364	0,455	0,091	0,273	0,273	0,455
$\langle x^{\mathbf{P}}_{7}\vert$	0,800	0,550	0,600	0,550	0,550	0,900	0,300	0,500	0,300	0,350	0,650	0,200	0,400	0,200	0,150
$\langle x^{\mathbf{P}}_{8}\vert$	0,605	0,421	0,605	0,474	0,368	0,553	0,342	0,421	0,316	0,447	0,474	0,263	0,289	0,211	0,211
$\langle x^{\mathbf{P}}_{9}\vert$	0,775	0,450	0,450	0,325	0,450	0,425	0,300	0,400	0,250	0,250	0,275	0,175	0,175	0,175	0,125
$\langle x^{\mathbf{P}}_{10}\vert$	0,491	0,351	0,368	0,404	0,351	0,193	0,298	0,175	0,298	0,281	0,228	0,211	0,175	0,088	0,088
$\langle x^{\mathbf{P}}_{11}\vert$	0,516	0,355	0,258	0,161	0,274	0,129	0,339	0,258	0,258	0,177	0,065	0,081	0,048	0,065	0,016
$\langle x^{\mathbf{P}}_{12}\vert$	0,420	0,260	0,260	0,180	0,100	0,100	0,180	0,120	0,100	0,160	0,060	0,040	0,020	0,000	0,000
$\langle x^{\mathbf{P}}_{13}\vert$	0,172	0,155	0,017	0,138	0,103	0,034	0,138	0,069	0,086	0,000	0,017	0,052	0,000	0,000	0,017
$\langle \varnothing^{\mathbf{P}}_{\Omega}\vert$	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000

Figure 3: Quotient-indicator of the co$\sim$event on 14x15 quotient-space for the 387x15 statistics (maximum value = 1).

	$\vert X^{\mathbf{B}}_{15}\rangle$	$\vert X^{\mathbf{B}}_{14}\rangle$	$\vert X^{\mathbf{B}}_{13}\rangle$	$\vert X^{\mathbf{B}}_{12}\rangle$	$\vert X^{\mathbf{B}}_{11}\rangle$	$\vert X^{\mathbf{B}}_{10}\rangle$	$\vert X^{\mathbf{B}}_{9}\rangle$	$\vert X^{\mathbf{B}}_{8}\rangle$	$\vert X^{\mathbf{B}}_{7}\rangle$	$\vert X^{\mathbf{B}}_{6}\rangle$	$\vert X^{\mathbf{B}}_{5}\rangle$	$\vert X^{\mathbf{B}}_{4}\rangle$	$\vert X^{\mathbf{B}}_{3}\rangle$	$\vert X^{\mathbf{B}}_{2}\rangle$	$\vert X^{\mathbf{B}}_{1}\rangle$
$\langle x^{\mathbf{P}}_{1}\vert$	0,032	0,032	0,016	0,032	0,032	0,032	0,016	0,032	0,032	0,016	0,032	0,032	0,016	0,032	0,032
$\langle x^{\mathbf{P}}_{2}\vert$	0,048	0,032	0,048	0,048	0,048	0,032	0,032	0,048	0,032	0,032	0,032	0,048	0,016	0,048	0,032
$\langle x^{\mathbf{P}}_{3}\vert$	0,032	0,032	0,032	0,016	0,016	0,032	0,016	0,000	0,032	0,032	0,032	0,016	0,032	0,016	0,016
$\langle x^{\mathbf{P}}_{4}\vert$	0,048	0,048	0,065	0,065	0,032	0,032	0,032	0,048	0,032	0,032	0,016	0,032	0,065	0,065	0,032
$\langle x^{\mathbf{P}}_{5}\vert$	0,081	0,097	0,113	0,145	0,097	0,097	0,113	0,113	0,113	0,097	0,145	0,081	0,048	0,081	0,032
$\langle x^{\mathbf{P}}_{6}\vert$	0,129	0,161	0,145	0,065	0,113	0,145	0,081	0,129	0,113	0,065	0,081	0,016	0,048	0,048	0,081
$\langle x^{\mathbf{P}}_{7}\vert$	0,258	0,177	0,194	0,177	0,177	0,290	0,097	0,161	0,097	0,113	0,210	0,065	0,129	0,065	0,048
$\langle x^{\mathbf{P}}_{8}\vert$	0,371	0,258	0,371	0,290	0,226	0,339	0,210	0,258	0,194	0,274	0,290	0,161	0,177	0,129	0,129
$\langle x^{\mathbf{P}}_{9}\vert$	0,500	0,290	0,290	0,210	0,290	0,274	0,194	0,258	0,161	0,161	0,177	0,113	0,113	0,113	0,081
$\langle x^{\mathbf{P}}_{10}\vert$	0,452	0,323	0,339	0,371	0,323	0,177	0,274	0,161	0,274	0,258	0,210	0,194	0,161	0,081	0,081
$\langle x^{\mathbf{P}}_{11}\vert$	0,516	0,355	0,258	0,161	0,274	0,129	0,339	0,258	0,258	0,177	0,065	0,081	0,048	0,065	0,016
$\langle x^{\mathbf{P}}_{12}\vert$	0,339	0,210	0,210	0,145	0,081	0,081	0,145	0,097	0,081	0,129	0,048	0,032	0,016	0,000	0,000
$\langle x^{\mathbf{P}}_{13}\vert$	0,161	0,145	0,016	0,129	0,097	0,032	0,129	0,065	0,081	0,000	0,016	0,048	0,000	0,000	0,016
$\langle \varnothing^{\mathbf{P}}_{\Omega}\vert$	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000

Figure 4: Quotient-c.d. of the co$\sim$event on 14x15 quotient-space for the 387x15 statistics (maximum value $\approx$ 0,010680448).

	$\lvert X^{\mathbf{B}}_{15}\rangle$	$\lvert X^{\mathbf{B}}_{14}\rangle$	$\lvert X^{\mathbf{B}}_{13}\rangle$	$\lvert X^{\mathbf{B}}_{12}\rangle$	$\lvert X^{\mathbf{B}}_{11}\rangle$	$\lvert X^{\mathbf{B}}_{10}\rangle$	$\lvert X^{\mathbf{B}}_{9}\rangle$	$\lvert X^{\mathbf{B}}_{8}\rangle$	$\lvert X^{\mathbf{B}}_{7}\rangle$	$\lvert X^{\mathbf{B}}_{6}\rangle$	$\lvert X^{\mathbf{B}}_{5}\rangle$	$\lvert X^{\mathbf{B}}_{4}\rangle$	$\lvert X^{\mathbf{B}}_{3}\rangle$	$\lvert X^{\mathbf{B}}_{2}\rangle$	$\lvert X^{\mathbf{B}}_{1}\rangle$
$\langle x^{\mathbf{P}}_{1}\rvert$	0,032	0,032	0,032	0,032	0,032	0,032	0,032	0,032	0,032	0,032	0,032	0,032	0,032	0,032	0,032
$\langle x^{\mathbf{P}}_{2}\rvert$	0,048	0,048	0,048	0,048	0,048	0,048	0,048	0,048	0,048	0,048	0,048	0,048	0,048	0,048	0,048
$\langle x^{\mathbf{P}}_{3}\rvert$	0,032	0,032	0,032	0,032	0,032	0,032	0,032	0,032	0,032	0,032	0,032	0,032	0,032	0,032	0,032
$\langle x^{\mathbf{P}}_{4}\rvert$	0,065	0,065	0,065	0,065	0,065	0,065	0,065	0,065	0,065	0,065	0,065	0,065	0,065	0,065	0,065
$\langle x^{\mathbf{P}}_{5}\rvert$	0,161	0,161	0,161	0,161	0,161	0,161	0,161	0,161	0,161	0,161	0,161	0,161	0,161	0,161	0,161
$\langle x^{\mathbf{P}}_{6}\rvert$	0,177	0,177	0,177	0,177	0,177	0,177	0,177	0,177	0,177	0,177	0,177	0,177	0,177	0,177	0,177
$\langle x^{\mathbf{P}}_{7}\rvert$	0,323	0,323	0,323	0,323	0,323	0,323	0,323	0,323	0,323	0,323	0,323	0,323	0,323	0,323	0,323
$\langle x^{\mathbf{P}}_{8}\rvert$	0,613	0,613	0,613	0,613	0,613	0,613	0,613	0,613	0,613	0,613	0,613	0,613	0,613	0,613	0,613
$\langle x^{\mathbf{P}}_{9}\rvert$	0,645	0,645	0,645	0,645	0,645	0,645	0,645	0,645	0,645	0,645	0,645	0,645	0,645	0,645	0,645
$\langle x^{\mathbf{P}}_{10}\rvert$	0,919	0,919	0,919	0,919	0,919	0,919	0,919	0,919	0,919	0,919	0,919	0,919	0,919	0,919	0,919
$\langle x^{\mathbf{P}}_{11}\rvert$	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000
$\langle x^{\mathbf{P}}_{12}\rvert$	0,806	0,806	0,806	0,806	0,806	0,806	0,806	0,806	0,806	0,806	0,806	0,806	0,806	0,806	0,806
$\langle x^{\mathbf{P}}_{13}\rvert$	0,935	0,935	0,935	0,935	0,935	0,935	0,935	0,935	0,935	0,935	0,935	0,935	0,935	0,935	0,935
$\langle \varnothing^{\mathbf{P}}_{\Omega}\rvert$	0,484	0,484	0,484	0,484	0,484	0,484	0,484	0,484	0,484	0,484	0,484	0,484	0,484	0,484	0,484

Figure 5: Quotient-c.d. on 14x15 quotient-space for the 387x15 statistics (maximum value ≈ 0,010680448).

	$\lvert X^{\mathbf{B}}_{15}\rangle$	$\lvert X^{\mathbf{B}}_{14}\rangle$	$\lvert X^{\mathbf{B}}_{13}\rangle$	$\lvert X^{\mathbf{B}}_{12}\rangle$	$\lvert X^{\mathbf{B}}_{11}\rangle$	$\lvert X^{\mathbf{B}}_{10}\rangle$	$\lvert X^{\mathbf{B}}_{9}\rangle$	$\lvert X^{\mathbf{B}}_{8}\rangle$	$\lvert X^{\mathbf{B}}_{7}\rangle$	$\lvert X^{\mathbf{B}}_{6}\rangle$	$\lvert X^{\mathbf{B}}_{5}\rangle$	$\lvert X^{\mathbf{B}}_{4}\rangle$	$\lvert X^{\mathbf{B}}_{3}\rangle$	$\lvert X^{\mathbf{B}}_{2}\rangle$	$\lvert X^{\mathbf{B}}_{1}\rangle$
$\langle x^{\mathbf{P}}_{1}\rvert$	0,000	0,000	0,016	0,000	0,000	0,000	0,016	0,000	0,000	0,016	0,000	0,000	0,016	0,000	0,000
$\langle x^{\mathbf{P}}_{2}\rvert$	0,000	0,016	0,000	0,000	0,000	0,016	0,016	0,000	0,016	0,016	0,016	0,000	0,032	0,000	0,016
$\langle x^{\mathbf{P}}_{3}\rvert$	0,000	0,000	0,000	0,016	0,016	0,000	0,016	0,032	0,000	0,000	0,000	0,016	0,000	0,016	0,016
$\langle x^{\mathbf{P}}_{4}\rvert$	0,016	0,016	0,000	0,000	0,032	0,032	0,032	0,016	0,032	0,032	0,048	0,032	0,000	0,000	0,032
$\langle x^{\mathbf{P}}_{5}\rvert$	0,081	0,065	0,048	0,016	0,065	0,065	0,048	0,048	0,048	0,065	0,016	0,081	0,113	0,081	0,129
$\langle x^{\mathbf{P}}_{6}\rvert$	0,048	0,016	0,032	0,113	0,065	0,032	0,097	0,048	0,065	0,113	0,097	0,161	0,129	0,129	0,097
$\langle x^{\mathbf{P}}_{7}\rvert$	0,065	0,145	0,129	0,145	0,145	0,032	0,226	0,161	0,226	0,210	0,113	0,258	0,194	0,258	0,274
$\langle x^{\mathbf{P}}_{8}\rvert$	0,242	0,355	0,242	0,323	0,387	0,274	0,403	0,355	0,419	0,339	0,323	0,452	0,435	0,484	0,484
$\langle x^{\mathbf{P}}_{9}\rvert$	0,145	0,355	0,355	0,435	0,355	0,371	0,452	0,387	0,484	0,484	0,468	0,532	0,532	0,532	0,565
$\langle x^{\mathbf{P}}_{10}\rvert$	0,468	0,597	0,581	0,548	0,597	0,742	0,645	0,758	0,645	0,661	0,710	0,726	0,758	0,839	0,839
$\langle x^{\mathbf{P}}_{11}\rvert$	0,484	0,645	0,742	0,839	0,726	0,871	0,661	0,742	0,742	0,823	0,935	0,919	0,952	0,935	0,984
$\langle x^{\mathbf{P}}_{12}\rvert$	0,468	0,597	0,597	0,661	0,726	0,726	0,661	0,710	0,726	0,677	0,758	0,774	0,790	0,806	0,806
$\langle x^{\mathbf{P}}_{13}\rvert$	0,774	0,790	0,919	0,806	0,839	0,903	0,806	0,871	0,855	0,935	0,919	0,887	0,935	0,935	0,919
$\langle \varnothing^{\mathbf{P}}_{\Omega}\rvert$	0,484	0,484	0,484	0,484	0,484	0,484	0,484	0,484	0,484	0,484	0,484	0,484	0,484	0,484	0,484

Figure 6: Quotient-c.d. of the M-complement co∼event on 14x15 quotient-space for the 387x15 statistics (maximum value ≈ 0,010680448).

	$\lvert X^{\mathbf{B}}_{13}\rangle$	$\lvert X^{\mathbf{B}}_{12}\rangle$	$\lvert X^{\mathbf{B}}_{11}\rangle$	$\lvert X^{\mathbf{B}}_{10}\rangle$	$\lvert X^{\mathbf{B}}_{9}\rangle$	$\lvert X^{\mathbf{B}}_{8}\rangle$	$\lvert X^{\mathbf{B}}_{7}\rangle$	$\lvert X^{\mathbf{B}}_{6}\rangle$	$\lvert X^{\mathbf{B}}_{5}\rangle$	$\lvert X^{\mathbf{B}}_{4}\rangle$	$\lvert X^{\mathbf{B}}_{3}\rangle$	$\lvert X^{\mathbf{B}}_{2}\rangle$	$\lvert X^{\mathbf{B}}_{1}\rangle$	$\lvert \emptyset^{\mathbf{B}}\rangle$
$\langle x^{\mathbf{P}}_{1}\rvert$	1,00	1,00	1,00	0,75	0,50	0,73	0,80	0,61	0,78	0,49	0,52	0,42	0,17	0,00
$\langle x^{\mathbf{P}}_{2}\rvert$	1,00	0,67	1,00	0,75	0,60	0,91	0,55	0,42	0,45	0,35	0,35	0,26	0,16	0,00
$\langle x^{\mathbf{P}}_{3}\rvert$	0,50	1,00	1,00	1,00	0,70	0,82	0,60	0,61	0,45	0,37	0,26	0,26	0,02	0,00
$\langle x^{\mathbf{P}}_{4}\rvert$	1,00	1,00	0,50	1,00	0,90	0,36	0,55	0,47	0,33	0,40	0,16	0,18	0,14	0,00
$\langle x^{\mathbf{P}}_{5}\rvert$	1,00	1,00	0,50	0,50	0,60	0,64	0,55	0,37	0,45	0,35	0,27	0,10	0,10	0,00
$\langle x^{\mathbf{P}}_{6}\rvert$	1,00	0,67	1,00	0,50	0,60	0,82	0,90	0,55	0,42	0,19	0,13	0,10	0,03	0,00
$\langle x^{\mathbf{P}}_{7}\rvert$	0,50	0,67	0,50	0,50	0,70	0,45	0,30	0,34	0,30	0,30	0,34	0,18	0,14	0,00
$\langle x^{\mathbf{P}}_{8}\rvert$	1,00	1,00	0,00	0,75	0,70	0,73	0,50	0,42	0,40	0,18	0,26	0,12	0,07	0,00
$\langle x^{\mathbf{P}}_{9}\rvert$	1,00	0,67	1,00	0,50	0,70	0,64	0,30	0,32	0,25	0,30	0,26	0,10	0,09	0,00
$\langle x^{\mathbf{P}}_{10}\rvert$	0,50	0,67	1,00	0,50	0,60	0,36	0,35	0,45	0,25	0,28	0,18	0,16	0,00	0,00
$\langle x^{\mathbf{P}}_{11}\rvert$	1,00	0,67	1,00	0,25	0,90	0,45	0,65	0,47	0,28	0,23	0,06	0,06	0,02	0,00
$\langle x^{\mathbf{P}}_{12}\rvert$	1,00	1,00	0,50	0,50	0,50	0,09	0,20	0,26	0,17	0,21	0,08	0,04	0,05	0,00
$\langle x^{\mathbf{P}}_{13}\rvert$	0,50	0,33	1,00	1,00	0,30	0,27	0,40	0,29	0,17	0,18	0,05	0,02	0,00	0,00
$\langle x^{\mathbf{P}}_{14}\rvert$	1,00	1,00	0,50	1,00	0,50	0,27	0,20	0,21	0,17	0,09	0,06	0,00	0,00	0,00
$\langle x^{\mathbf{P}}_{15}\rvert$	1,00	0,67	0,50	0,50	0,20	0,45	0,15	0,21	0,13	0,09	0,02	0,00	0,02	0,00

Figure 7: Quotient-indicator of the co∼event on the 15x14 quotient-space for 15x387 statistics (maximum value = 1).

	$\vert X^{\mathbf{B}}_{13}\rangle$	$\vert X^{\mathbf{B}}_{12}\rangle$	$\vert X^{\mathbf{B}}_{11}\rangle$	$\vert X^{\mathbf{B}}_{10}\rangle$	$\vert X^{\mathbf{B}}_{9}\rangle$	$\vert X^{\mathbf{B}}_{8}\rangle$	$\vert X^{\mathbf{B}}_{7}\rangle$	$\vert X^{\mathbf{B}}_{6}\rangle$	$\vert X^{\mathbf{B}}_{5}\rangle$	$\vert X^{\mathbf{B}}_{4}\rangle$	$\vert X^{\mathbf{B}}_{3}\rangle$	$\vert X^{\mathbf{B}}_{2}\rangle$	$\vert X^{\mathbf{B}}_{1}\rangle$	$\vert \emptyset^{\mathbf{B}}\rangle$
$\langle x^{\mathbf{P}}_{1}\vert$	1,000	1,000	1,000	0,750	0,500	0,727	0,800	0,605	0,775	0,491	0,516	0,420	0,172	0,000
$\langle x^{\mathbf{P}}_{2}\vert$	1,000	0,667	1,000	0,750	0,600	0,909	0,550	0,421	0,450	0,351	0,355	0,260	0,155	0,000
$\langle x^{\mathbf{P}}_{3}\vert$	0,500	1,000	1,000	1,000	0,700	0,818	0,600	0,605	0,450	0,368	0,258	0,260	0,017	0,000
$\langle x^{\mathbf{P}}_{4}\vert$	1,000	1,000	0,500	1,000	0,900	0,364	0,550	0,474	0,325	0,404	0,161	0,180	0,138	0,000
$\langle x^{\mathbf{P}}_{5}\vert$	1,000	1,000	0,500	0,500	0,600	0,636	0,550	0,368	0,450	0,351	0,274	0,100	0,103	0,000
$\langle x^{\mathbf{P}}_{6}\vert$	1,000	0,667	1,000	0,500	0,600	0,818	0,900	0,553	0,425	0,193	0,129	0,100	0,034	0,000
$\langle x^{\mathbf{P}}_{7}\vert$	0,500	0,667	0,500	0,500	0,700	0,455	0,300	0,342	0,300	0,298	0,339	0,180	0,138	0,000
$\langle x^{\mathbf{P}}_{8}\vert$	1,000	1,000	0,000	0,750	0,700	0,727	0,500	0,421	0,400	0,175	0,258	0,120	0,069	0,000
$\langle x^{\mathbf{P}}_{9}\vert$	1,000	0,667	1,000	0,500	0,700	0,636	0,300	0,316	0,250	0,298	0,258	0,100	0,086	0,000
$\langle x^{\mathbf{P}}_{10}\vert$	0,500	0,667	1,000	0,500	0,600	0,364	0,350	0,447	0,250	0,281	0,177	0,160	0,000	0,000
$\langle x^{\mathbf{P}}_{11}\vert$	1,000	0,667	1,000	0,250	0,900	0,455	0,650	0,474	0,275	0,228	0,065	0,060	0,017	0,000
$\langle x^{\mathbf{P}}_{12}\vert$	1,000	1,000	0,500	0,500	0,500	0,091	0,200	0,263	0,175	0,211	0,081	0,040	0,052	0,000
$\langle x^{\mathbf{P}}_{13}\vert$	0,500	0,333	1,000	1,000	0,300	0,273	0,400	0,289	0,175	0,175	0,048	0,020	0,000	0,000
$\langle x^{\mathbf{P}}_{14}\vert$	1,000	1,000	0,500	1,000	0,500	0,273	0,200	0,211	0,175	0,088	0,065	0,000	0,000	0,000
$\langle x^{\mathbf{P}}_{15}\vert$	1,000	0,667	0,500	0,500	0,200	0,455	0,150	0,211	0,125	0,088	0,016	0,000	0,017	0,000

Figure 8: Quotient-indicator of the co∼event on 15x14 quotient-space for the 15x387 statistics (maximum value = 1).

	$\vert X^{\mathbf{B}}_{13}\rangle$	$\vert X^{\mathbf{B}}_{12}\rangle$	$\vert X^{\mathbf{B}}_{11}\rangle$	$\vert X^{\mathbf{B}}_{10}\rangle$	$\vert X^{\mathbf{B}}_{9}\rangle$	$\vert X^{\mathbf{B}}_{8}\rangle$	$\vert X^{\mathbf{B}}_{7}\rangle$	$\vert X^{\mathbf{B}}_{6}\rangle$	$\vert X^{\mathbf{B}}_{5}\rangle$	$\vert X^{\mathbf{B}}_{4}\rangle$	$\vert X^{\mathbf{B}}_{3}\rangle$	$\vert X^{\mathbf{B}}_{2}\rangle$	$\vert X^{\mathbf{B}}_{1}\rangle$	$\vert \emptyset^{\mathbf{B}}\rangle$
$\langle x^{\mathbf{P}}_{1}\vert$	0,032	0,048	0,032	0,048	0,081	0,129	0,258	0,371	0,500	0,452	0,516	0,339	0,161	0,000
$\langle x^{\mathbf{P}}_{2}\vert$	0,032	0,032	0,032	0,048	0,097	0,161	0,177	0,258	0,290	0,323	0,355	0,210	0,145	0,000
$\langle x^{\mathbf{P}}_{3}\vert$	0,016	0,048	0,032	0,065	0,113	0,145	0,194	0,371	0,290	0,339	0,258	0,210	0,016	0,000
$\langle x^{\mathbf{P}}_{4}\vert$	0,032	0,048	0,016	0,065	0,145	0,065	0,177	0,290	0,210	0,371	0,161	0,145	0,129	0,000
$\langle x^{\mathbf{P}}_{5}\vert$	0,032	0,048	0,016	0,032	0,097	0,113	0,177	0,226	0,290	0,323	0,274	0,081	0,097	0,000
$\langle x^{\mathbf{P}}_{6}\vert$	0,032	0,032	0,032	0,032	0,097	0,145	0,290	0,339	0,274	0,177	0,129	0,081	0,032	0,000
$\langle x^{\mathbf{P}}_{7}\vert$	0,016	0,032	0,016	0,032	0,113	0,081	0,097	0,210	0,194	0,274	0,339	0,145	0,129	0,000
$\langle x^{\mathbf{P}}_{8}\vert$	0,032	0,048	0,000	0,048	0,113	0,129	0,161	0,258	0,258	0,161	0,258	0,097	0,065	0,000
$\langle x^{\mathbf{P}}_{9}\vert$	0,032	0,032	0,032	0,032	0,113	0,113	0,097	0,194	0,161	0,274	0,258	0,081	0,081	0,000
$\langle x^{\mathbf{P}}_{10}\vert$	0,016	0,032	0,032	0,032	0,097	0,065	0,113	0,274	0,161	0,258	0,177	0,129	0,000	0,000
$\langle x^{\mathbf{P}}_{11}\vert$	0,032	0,032	0,032	0,016	0,145	0,081	0,210	0,290	0,177	0,210	0,065	0,048	0,016	0,000
$\langle x^{\mathbf{P}}_{12}\vert$	0,032	0,048	0,016	0,032	0,081	0,016	0,065	0,161	0,113	0,194	0,081	0,032	0,048	0,000
$\langle x^{\mathbf{P}}_{13}\vert$	0,016	0,016	0,032	0,065	0,048	0,048	0,129	0,177	0,113	0,161	0,048	0,016	0,000	0,000
$\langle x^{\mathbf{P}}_{14}\vert$	0,032	0,048	0,016	0,065	0,081	0,048	0,065	0,129	0,113	0,081	0,065	0,000	0,000	0,000
$\langle x^{\mathbf{P}}_{15}\vert$	0,032	0,032	0,016	0,032	0,032	0,081	0,048	0,129	0,081	0,081	0,016	0,000	0,016	0,000

Figure 9: Quotient-c.d. of the co∼event on 15x14 quotient-space for the 15x387 statistics (maximum value ≈ 0,010680448).

	$\vert X^{\mathbf{B}}_{13}\rangle$	$\vert X^{\mathbf{B}}_{12}\rangle$	$\vert X^{\mathbf{B}}_{11}\rangle$	$\vert X^{\mathbf{B}}_{10}\rangle$	$\vert X^{\mathbf{B}}_{9}\rangle$	$\vert X^{\mathbf{B}}_{8}\rangle$	$\vert X^{\mathbf{B}}_{7}\rangle$	$\vert X^{\mathbf{B}}_{6}\rangle$	$\vert X^{\mathbf{B}}_{5}\rangle$	$\vert X^{\mathbf{B}}_{4}\rangle$	$\vert X^{\mathbf{B}}_{3}\rangle$	$\vert X^{\mathbf{B}}_{2}\rangle$	$\vert X^{\mathbf{B}}_{1}\rangle$	$\vert \emptyset^{\mathbf{B}}\rangle$
$\langle x^{\mathbf{P}}_{1}\vert$	0,032	0,048	0,032	0,065	0,161	0,177	0,323	0,613	0,645	0,919	1,000	0,806	0,935	0,484
$\langle x^{\mathbf{P}}_{2}\vert$	0,032	0,048	0,032	0,065	0,161	0,177	0,323	0,613	0,645	0,919	1,000	0,806	0,935	0,484
$\langle x^{\mathbf{P}}_{3}\vert$	0,032	0,048	0,032	0,065	0,161	0,177	0,323	0,613	0,645	0,919	1,000	0,806	0,935	0,484
$\langle x^{\mathbf{P}}_{4}\vert$	0,032	0,048	0,032	0,065	0,161	0,177	0,323	0,613	0,645	0,919	1,000	0,806	0,935	0,484
$\langle x^{\mathbf{P}}_{5}\vert$	0,032	0,048	0,032	0,065	0,161	0,177	0,323	0,613	0,645	0,919	1,000	0,806	0,935	0,484
$\langle x^{\mathbf{P}}_{6}\vert$	0,032	0,048	0,032	0,065	0,161	0,177	0,323	0,613	0,645	0,919	1,000	0,806	0,935	0,484
$\langle x^{\mathbf{P}}_{7}\vert$	0,032	0,048	0,032	0,065	0,161	0,177	0,323	0,613	0,645	0,919	1,000	0,806	0,935	0,484
$\langle x^{\mathbf{P}}_{8}\vert$	0,032	0,048	0,032	0,065	0,161	0,177	0,323	0,613	0,645	0,919	1,000	0,806	0,935	0,484
$\langle x^{\mathbf{P}}_{9}\vert$	0,032	0,048	0,032	0,065	0,161	0,177	0,323	0,613	0,645	0,919	1,000	0,806	0,935	0,484
$\langle x^{\mathbf{P}}_{10}\vert$	0,032	0,048	0,032	0,065	0,161	0,177	0,323	0,613	0,645	0,919	1,000	0,806	0,935	0,484
$\langle x^{\mathbf{P}}_{11}\vert$	0,032	0,048	0,032	0,065	0,161	0,177	0,323	0,613	0,645	0,919	1,000	0,806	0,935	0,484
$\langle x^{\mathbf{P}}_{12}\vert$	0,032	0,048	0,032	0,065	0,161	0,177	0,323	0,613	0,645	0,919	1,000	0,806	0,935	0,484
$\langle x^{\mathbf{P}}_{13}\vert$	0,032	0,048	0,032	0,065	0,161	0,177	0,323	0,613	0,645	0,919	1,000	0,806	0,935	0,484
$\langle x^{\mathbf{P}}_{14}\vert$	0,032	0,048	0,032	0,065	0,161	0,177	0,323	0,613	0,645	0,919	1,000	0,806	0,935	0,484
$\langle x^{\mathbf{P}}_{15}\vert$	0,032	0,048	0,032	0,065	0,161	0,177	0,323	0,613	0,645	0,919	1,000	0,806	0,935	0,484

Figure 10: Quotient-c.d. on 15x14 quotient-space for the 15x387 statistics (maximum value ≈ 0,010680448).

	$\vert x^{\mathbf{B}}_{13}\rangle$	$\vert x^{\mathbf{B}}_{12}\rangle$	$\vert x^{\mathbf{B}}_{11}\rangle$	$\vert x^{\mathbf{B}}_{10}\rangle$	$\vert x^{\mathbf{B}}_{9}\rangle$	$\vert x^{\mathbf{B}}_{8}\rangle$	$\vert x^{\mathbf{B}}_{7}\rangle$	$\vert x^{\mathbf{B}}_{6}\rangle$	$\vert x^{\mathbf{B}}_{5}\rangle$	$\vert x^{\mathbf{B}}_{4}\rangle$	$\vert x^{\mathbf{B}}_{3}\rangle$	$\vert x^{\mathbf{B}}_{2}\rangle$	$\vert x^{\mathbf{B}}_{1}\rangle$	$\vert\emptyset^{\mathbf{B}}\rangle$
$\langle x^{\mathbf{P}}_{1}\vert$	0,000	0,000	0,000	0,016	0,081	0,048	0,065	0,242	0,145	0,468	0,484	0,468	0,774	0,484
$\langle x^{\mathbf{P}}_{2}\vert$	0,000	0,016	0,000	0,016	0,065	0,016	0,145	0,355	0,355	0,597	0,645	0,597	0,790	0,484
$\langle x^{\mathbf{P}}_{3}\vert$	0,016	0,000	0,000	0,000	0,048	0,032	0,129	0,242	0,355	0,581	0,742	0,597	0,919	0,484
$\langle x^{\mathbf{P}}_{4}\vert$	0,000	0,000	0,016	0,000	0,016	0,113	0,145	0,323	0,435	0,548	0,839	0,661	0,806	0,484
$\langle x^{\mathbf{P}}_{5}\vert$	0,000	0,000	0,016	0,032	0,065	0,065	0,145	0,387	0,355	0,597	0,726	0,726	0,839	0,484
$\langle x^{\mathbf{P}}_{6}\vert$	0,000	0,016	0,000	0,032	0,065	0,032	0,032	0,274	0,371	0,742	0,871	0,726	0,903	0,484
$\langle x^{\mathbf{P}}_{7}\vert$	0,016	0,016	0,016	0,032	0,048	0,097	0,226	0,403	0,452	0,645	0,661	0,661	0,806	0,484
$\langle x^{\mathbf{P}}_{8}\vert$	0,000	0,000	0,032	0,016	0,048	0,048	0,161	0,355	0,387	0,758	0,742	0,710	0,871	0,484
$\langle x^{\mathbf{P}}_{9}\vert$	0,000	0,016	0,000	0,032	0,048	0,065	0,226	0,419	0,484	0,645	0,742	0,726	0,855	0,484
$\langle x^{\mathbf{P}}_{10}\vert$	0,016	0,016	0,000	0,032	0,065	0,113	0,210	0,339	0,484	0,661	0,823	0,677	0,935	0,484
$\langle x^{\mathbf{P}}_{11}\vert$	0,000	0,016	0,000	0,048	0,016	0,097	0,113	0,323	0,468	0,710	0,935	0,758	0,919	0,484
$\langle x^{\mathbf{P}}_{12}\vert$	0,000	0,000	0,016	0,032	0,081	0,161	0,258	0,452	0,532	0,726	0,919	0,774	0,887	0,484
$\langle x^{\mathbf{P}}_{13}\vert$	0,016	0,032	0,000	0,000	0,113	0,129	0,194	0,435	0,532	0,758	0,952	0,790	0,935	0,484
$\langle x^{\mathbf{P}}_{14}\vert$	0,000	0,000	0,016	0,000	0,081	0,129	0,258	0,484	0,532	0,839	0,935	0,806	0,935	0,484
$\langle x^{\mathbf{P}}_{15}\vert$	0,000	0,016	0,016	0,032	0,129	0,097	0,274	0,484	0,565	0,839	0,984	0,806	0,919	0,484

Figure 11: Quotient-c.d. of the M-complement co∼event on 15x14 quotient-space for the 15x387 statistics (maximum value ≈ 0,010680448).

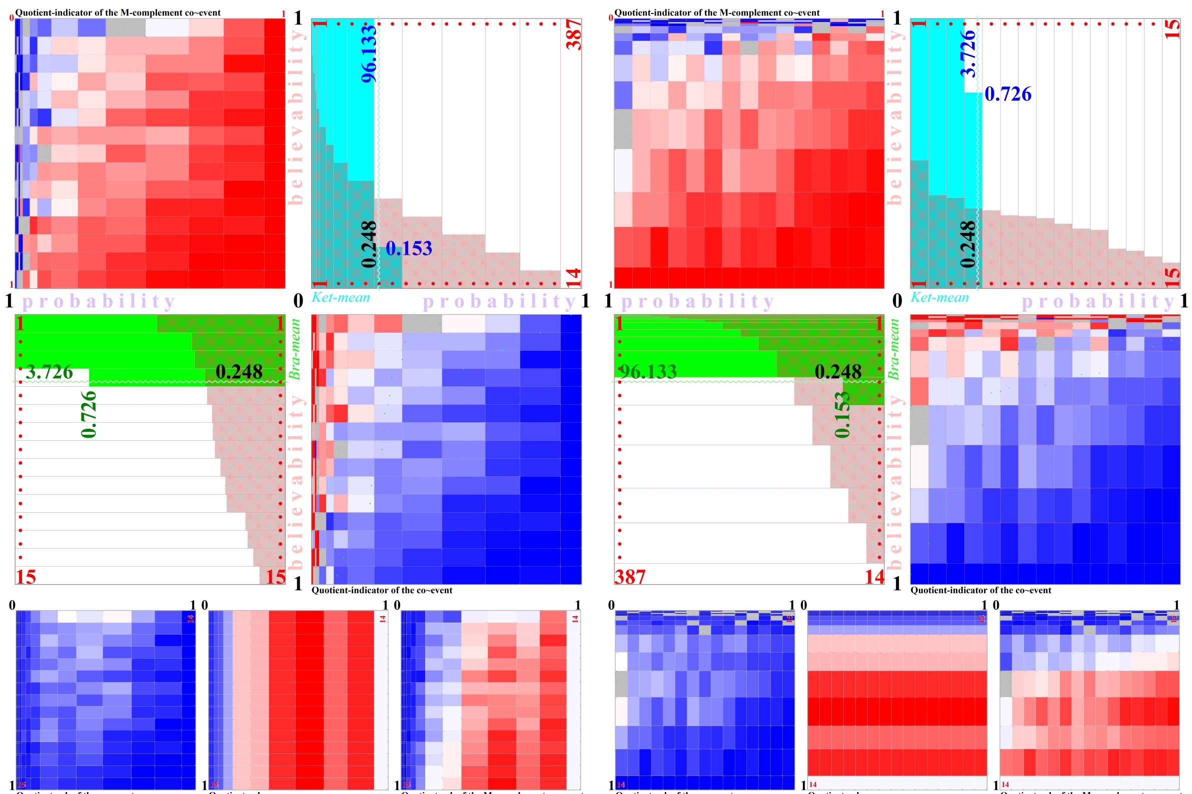

Figure 12: Bra-mean and ket-mean: for 15x387-statistics «foresters–trees», left; for 387x15-statistics «trees–foresters», right.

	$\vert X_4^{\mathbf{B}}\rangle$	$\vert X_3^{\mathbf{B}}\rangle$	$\vert X_2^{\mathbf{B}}\rangle$	$\vert X_1^{\mathbf{B}}\rangle$
$\langle x_1^{\mathbf{P}}\vert$	0,00	0,00	0,00	0,00
$\langle x_2^{\mathbf{P}}\vert$	0,01	0,02	0,02	0,94
$\langle x_3^{\mathbf{P}}\vert$	0,26	0,38	0,54	0,82
$\langle x_4^{\mathbf{P}}\vert$	0,68	0,75	0,80	0,77
$\langle \varnothing_\Omega^{\mathbf{P}}\vert$	1,00	1,00	1,00	1,00

Quotient-indicator
of the M-complementary co∼event

Ket-mean
co∼event

Bra-mean
co∼event

	$\vert X_4^{\mathbf{B}}\rangle$	$\vert X_3^{\mathbf{B}}\rangle$	$\vert X_2^{\mathbf{B}}\rangle$	$\vert X_1^{\mathbf{B}}\rangle$
$\langle x_1^{\mathbf{P}}\vert$	1,00	1,00	1,00	1,00
$\langle x_2^{\mathbf{P}}\vert$	0,99	0,98	0,98	0,06
$\langle x_3^{\mathbf{P}}\vert$	0,74	0,62	0,46	0,18
$\langle x_4^{\mathbf{P}}\vert$	0,32	0,25	0,20	0,23
$\langle \varnothing_\Omega^{\mathbf{P}}\vert$	0,00	0,00	0,00	0,00

Quotient-indicator
of the co∼event

	$\vert X_4^{\mathbf{B}}\rangle$	$\vert X_3^{\mathbf{B}}\rangle$	$\vert X_2^{\mathbf{B}}\rangle$	$\vert X_1^{\mathbf{B}}\rangle$
$\langle x_1^{\mathbf{P}}\vert$	0,02	0,02	0,02	0,02
$\langle x_2^{\mathbf{P}}\vert$	0,18	0,17	0,17	0,01
$\langle x_3^{\mathbf{P}}\vert$	0,33	0,27	0,20	0,08
$\langle x_4^{\mathbf{P}}\vert$	0,32	0,25	0,20	0,23
$\langle \varnothing_\Omega^{\mathbf{P}}\vert$	0,00	0,00	0,00	0,00

Quotient-c.d. of the co∼event

	$\vert X_4^{\mathbf{B}}\rangle$	$\vert X_3^{\mathbf{B}}\rangle$	$\vert X_2^{\mathbf{B}}\rangle$	$\vert X_1^{\mathbf{B}}\rangle$
$\langle x_1^{\mathbf{P}}\vert$	0,02	0,02	0,02	0,02
$\langle x_2^{\mathbf{P}}\vert$	0,18	0,18	0,18	0,18
$\langle x_3^{\mathbf{P}}\vert$	0,44	0,44	0,44	0,44
$\langle x_4^{\mathbf{P}}\vert$	1,00	1,00	1,00	1,00
$\langle \varnothing_\Omega^{\mathbf{P}}\vert$	0,03	0,03	0,03	0,03

Quotient-c.d.

	$\vert X_4^{\mathbf{B}}\rangle$	$\vert X_3^{\mathbf{B}}\rangle$	$\vert X_2^{\mathbf{B}}\rangle$	$\vert X_1^{\mathbf{B}}\rangle$
$\langle x_1^{\mathbf{P}}\vert$	0,00	0,00	0,00	0,00
$\langle x_2^{\mathbf{P}}\vert$	0,00	0,00	0,00	0,17
$\langle x_3^{\mathbf{P}}\vert$	0,12	0,17	0,24	0,36
$\langle x_4^{\mathbf{P}}\vert$	0,68	0,75	0,80	0,77
$\langle \varnothing_\Omega^{\mathbf{P}}\vert$	0,03	0,03	0,03	0,03

Quotient-c.d.
of the M-complementary co∼event

Figure 13: The scheme of results on the Figure 17 for input 55827x4 statistics on the 5x4 quotient-space: two co∼event means, two quotient-indicators and three distributions of the co∼event («voters–candidates 55827x4 statistics»). The color palette shows values from $[0, 1]$ (normalized by the maximum value to better visualize small values; maximum value visualize as a red) by the following way: red ∼ 1, white ∼ 1/2, blue ∼ 0. For the quotient-indicators maximum value = 1, for three quotient-c.d. maximum value = 0,14978 ...

	$\vert X_4^{\mathbf{B}}\rangle$	$\vert X_3^{\mathbf{B}}\rangle$	$\vert X_2^{\mathbf{B}}\rangle$	$\vert X_1^{\mathbf{B}}\rangle$	$\vert \emptyset^{\mathbf{B}}\rangle$
$\langle x_1^{\mathbf{P}}\vert$	0,00	0,01	0,26	0,68	1,00
$\langle x_2^{\mathbf{P}}\vert$	0,00	0,02	0,38	0,75	1,00
$\langle x_3^{\mathbf{P}}\vert$	0,00	0,02	0,54	0,80	1,00
$\langle \varnothing_\Omega^{\mathbf{P}}\vert$	0,00	0,94	0,82	0,77	1,00

Quotient-indicator
of the M-complementary co∼event

Ket-mean
co∼event

Bra-mean
co∼event

	$\vert X_4^{\mathbf{B}}\rangle$	$\vert X_3^{\mathbf{B}}\rangle$	$\vert X_2^{\mathbf{B}}\rangle$	$\vert X_1^{\mathbf{B}}\rangle$	$\vert \emptyset^{\mathbf{B}}\rangle$
$\langle x_1^{\mathbf{P}}\vert$	1,00	0,99	0,74	0,32	0,00
$\langle x_2^{\mathbf{P}}\vert$	1,00	0,98	0,62	0,25	0,00
$\langle x_3^{\mathbf{P}}\vert$	1,00	0,98	0,46	0,20	0,00
$\langle \varnothing_\Omega^{\mathbf{P}}\vert$	1,00	0,06	0,18	0,23	0,00

Quotient-indicator
of the co∼event

	$\vert X_4^{\mathbf{B}}\rangle$	$\vert X_3^{\mathbf{B}}\rangle$	$\vert X_2^{\mathbf{B}}\rangle$	$\vert X_1^{\mathbf{B}}\rangle$	$\vert \emptyset^{\mathbf{B}}\rangle$
$\langle x_1^{\mathbf{P}}\vert$	0,02	0,18	0,33	0,32	0,00
$\langle x_2^{\mathbf{P}}\vert$	0,02	0,17	0,27	0,25	0,00
$\langle x_3^{\mathbf{P}}\vert$	0,02	0,17	0,20	0,20	0,00
$\langle \varnothing_\Omega^{\mathbf{P}}\vert$	0,02	0,01	0,08	0,23	0,00

Quotient-c.d.
of the co∼event

	$\vert X_4^{\mathbf{B}}\rangle$	$\vert X_3^{\mathbf{B}}\rangle$	$\vert X_2^{\mathbf{B}}\rangle$	$\vert X_1^{\mathbf{B}}\rangle$	$\vert \emptyset^{\mathbf{B}}\rangle$
$\langle x_1^{\mathbf{P}}\vert$	0,02	0,18	0,44	1,00	0,03
$\langle x_2^{\mathbf{P}}\vert$	0,02	0,18	0,44	1,00	0,03
$\langle x_3^{\mathbf{P}}\vert$	0,02	0,18	0,44	1,00	0,03
$\langle \varnothing_\Omega^{\mathbf{P}}\vert$	0,02	0,18	0,44	1,00	0,03

Quotient-c.d.

	$\vert X_4^{\mathbf{B}}\rangle$	$\vert X_3^{\mathbf{B}}\rangle$	$\vert X_2^{\mathbf{B}}\rangle$	$\vert X_1^{\mathbf{B}}\rangle$	$\vert \emptyset^{\mathbf{B}}\rangle$
$\langle x_1^{\mathbf{P}}\vert$	0,00	0,00	0,12	0,68	0,03
$\langle x_2^{\mathbf{P}}\vert$	0,00	0,00	0,17	0,75	0,03
$\langle x_3^{\mathbf{P}}\vert$	0,00	0,00	0,24	0,80	0,03
$\langle \varnothing_\Omega^{\mathbf{P}}\vert$	0,00	0,17	0,36	0,77	0,03

Quotient-c.d.
of the M-complementary co∼event

Figure 14: The scheme of results on the Figure 17 for input 4x55827 statistics on the 4x5 quotient-space: two co∼event means, two quotient-indicators and three distributions of the co∼event 4x55827 statistics»). The color palette shows values from $[0, 1]$ (normalized by the maximum value to better visualize small values; maximum value visualize as a red) by the following way: red ∼ 1, white ∼ 1/2, blue ∼ 0. For the two quotient-indicators maximum value = 1, for three quotient-c.d. maximum value = 0,14978 ...

	$\|X^{\mathbf{B}}_5\rangle$	$\|X^{\mathbf{B}}_4\rangle$	$\|X^{\mathbf{B}}_3\rangle$	$\|X^{\mathbf{B}}_2\rangle$	$\|X^{\mathbf{B}}_1\rangle$
$\langle x^{\mathbf{P}}_1\|$	0,00	0,00	0,00	0,00	0,00
$\langle x^{\mathbf{P}}_2\|$	0,43	0,08	0,16	0,14	0,19
$\langle x^{\mathbf{P}}_3\|$	0,38	0,32	0,40	0,35	0,56
$\langle x^{\mathbf{P}}_4\|$	0,55	0,55	0,57	0,55	0,78
$\langle x^{\mathbf{P}}_5\|$	0,72	0,79	0,79	0,80	0,90

Quotient-indicator
of the M-complementary co∼event

Ket-mean
co∼event

Bra-mean
co∼event

	$\|X^{\mathbf{B}}_5\rangle$	$\|X^{\mathbf{B}}_4\rangle$	$\|X^{\mathbf{B}}_3\rangle$	$\|X^{\mathbf{B}}_2\rangle$	$\|X^{\mathbf{B}}_1\rangle$
$\langle x^{\mathbf{P}}_1\|$	1,00	1,00	1,00	1,00	1,00
$\langle x^{\mathbf{P}}_2\|$	0,57	0,92	0,84	0,86	0,81
$\langle x^{\mathbf{P}}_3\|$	0,62	0,68	0,60	0,65	0,44
$\langle x^{\mathbf{P}}_4\|$	0,45	0,45	0,43	0,45	0,22
$\langle x^{\mathbf{P}}_5\|$	0,28	0,21	0,21	0,20	0,10

Quotient-indicator
of the co∼event

	$\|X^{\mathbf{B}}_5\rangle$	$\|X^{\mathbf{B}}_4\rangle$	$\|X^{\mathbf{B}}_3\rangle$	$\|X^{\mathbf{B}}_2\rangle$	$\|X^{\mathbf{B}}_1\rangle$
$\langle x^{\mathbf{P}}_1\|$	0,00	0,00	0,00	0,00	0,00
$\langle x^{\mathbf{P}}_2\|$	0,01	0,01	0,01	0,01	0,01
$\langle x^{\mathbf{P}}_3\|$	0,04	0,04	0,04	0,04	0,03
$\langle x^{\mathbf{P}}_4\|$	0,09	0,09	0,09	0,09	0,04
$\langle x^{\mathbf{P}}_5\|$	0,28	0,21	0,21	0,20	0,10

Quotient-c.d. of the co∼event

	$\|X^{\mathbf{B}}_5\rangle$	$\|X^{\mathbf{B}}_4\rangle$	$\|X^{\mathbf{B}}_3\rangle$	$\|X^{\mathbf{B}}_2\rangle$	$\|X^{\mathbf{B}}_1\rangle$
$\langle x^{\mathbf{P}}_1\|$	0,00	0,00	0,00	0,00	0,00
$\langle x^{\mathbf{P}}_2\|$	0,01	0,01	0,01	0,01	0,01
$\langle x^{\mathbf{P}}_3\|$	0,06	0,06	0,06	0,06	0,06
$\langle x^{\mathbf{P}}_4\|$	0,20	0,20	0,20	0,20	0,20
$\langle x^{\mathbf{P}}_5\|$	1,00	1,00	1,00	1,00	1,00

Quotient-c.d.

	$\|X^{\mathbf{B}}_5\rangle$	$\|X^{\mathbf{B}}_4\rangle$	$\|X^{\mathbf{B}}_3\rangle$	$\|X^{\mathbf{B}}_2\rangle$	$\|X^{\mathbf{B}}_1\rangle$
$\langle x^{\mathbf{P}}_1\|$	0,00	0,00	0,00	0,00	0,00
$\langle x^{\mathbf{P}}_2\|$	0,01	0,00	0,00	0,00	0,00
$\langle x^{\mathbf{P}}_3\|$	0,02	0,02	0,03	0,02	0,04
$\langle x^{\mathbf{P}}_4\|$	0,11	0,11	0,11	0,11	0,16
$\langle x^{\mathbf{P}}_5\|$	0,72	0,79	0,79	0,80	0,90

Quotient-c.d.
of the M-complementary co∼event

Figure 15: The scheme of results on the Figure 18 for input 3933x5 statistics on the 5x5 quotient-space: two co∼event means, two quotient-indicators and three distributions of the co∼event («voters–candidates 3933x5 statistics»). The color palette shows values from $[0,1]$ (normalized by the maximum value to better visualize small values; maximum value visualize as a red) by the following way: red $\sim$ 1, white $\sim$ 1/2, blue $\sim$ 0. For the quotient-indicators maximum value = 1, for three quotient-c.d. maximum value = 0,15667 ...

	$\|X^{\mathbf{B}}_5\rangle$	$\|X^{\mathbf{B}}_4\rangle$	$\|X^{\mathbf{B}}_3\rangle$	$\|X^{\mathbf{B}}_2\rangle$	$\|X^{\mathbf{B}}_1\rangle$
$\langle x^{\mathbf{P}}_1\|$	0,00	0,43	0,38	0,55	0,72
$\langle x^{\mathbf{P}}_2\|$	0,00	0,08	0,32	0,55	0,79
$\langle x^{\mathbf{P}}_3\|$	0,00	0,16	0,40	0,57	0,79
$\langle x^{\mathbf{P}}_4\|$	0,00	0,14	0,35	0,55	0,80
$\langle x^{\mathbf{P}}_5\|$	0,00	0,19	0,56	0,78	0,90

Quotient-indicator
of the M-complementary co∼event

Ket-mean
co∼event

Bra-mean
co∼event

	$\|X^{\mathbf{B}}_5\rangle$	$\|X^{\mathbf{B}}_4\rangle$	$\|X^{\mathbf{B}}_3\rangle$	$\|X^{\mathbf{B}}_2\rangle$	$\|X^{\mathbf{B}}_1\rangle$
$\langle x^{\mathbf{P}}_1\|$	1,00	0,57	0,62	0,45	0,28
$\langle x^{\mathbf{P}}_2\|$	1,00	0,92	0,68	0,45	0,21
$\langle x^{\mathbf{P}}_3\|$	1,00	0,84	0,60	0,43	0,21
$\langle x^{\mathbf{P}}_4\|$	1,00	0,86	0,65	0,45	0,20
$\langle x^{\mathbf{P}}_5\|$	1,00	0,81	0,44	0,22	0,10

Quotient-indicator
of the co∼event

	$\|X^{\mathbf{B}}_5\rangle$	$\|X^{\mathbf{B}}_4\rangle$	$\|X^{\mathbf{B}}_3\rangle$	$\|X^{\mathbf{B}}_2\rangle$	$\|X^{\mathbf{B}}_1\rangle$
$\langle x^{\mathbf{P}}_1\|$	0,00	0,01	0,04	0,09	0,28
$\langle x^{\mathbf{P}}_2\|$	0,00	0,01	0,04	0,09	0,21
$\langle x^{\mathbf{P}}_3\|$	0,00	0,01	0,04	0,09	0,21
$\langle x^{\mathbf{P}}_4\|$	0,00	0,01	0,04	0,09	0,20
$\langle x^{\mathbf{P}}_5\|$	0,00	0,01	0,03	0,04	0,10

Quotient-c.d.
of the co∼event

	$\|X^{\mathbf{B}}_5\rangle$	$\|X^{\mathbf{B}}_4\rangle$	$\|X^{\mathbf{B}}_3\rangle$	$\|X^{\mathbf{B}}_2\rangle$	$\|X^{\mathbf{B}}_1\rangle$
$\langle x^{\mathbf{P}}_1\|$	0,00	0,01	0,06	0,20	1,00
$\langle x^{\mathbf{P}}_2\|$	0,00	0,01	0,06	0,20	1,00
$\langle x^{\mathbf{P}}_3\|$	0,00	0,01	0,06	0,20	1,00
$\langle x^{\mathbf{P}}_4\|$	0,00	0,01	0,06	0,20	1,00
$\langle x^{\mathbf{P}}_5\|$	0,00	0,01	0,06	0,20	1,00

Quotient-c.d.

	$\|X^{\mathbf{B}}_5\rangle$	$\|X^{\mathbf{B}}_4\rangle$	$\|X^{\mathbf{B}}_3\rangle$	$\|X^{\mathbf{B}}_2\rangle$	$\|X^{\mathbf{B}}_1\rangle$
$\langle x^{\mathbf{P}}_1\|$	0,00	0,01	0,02	0,11	0,72
$\langle x^{\mathbf{P}}_2\|$	0,00	0,00	0,02	0,11	0,79
$\langle x^{\mathbf{P}}_3\|$	0,00	0,00	0,03	0,11	0,79
$\langle x^{\mathbf{P}}_4\|$	0,00	0,00	0,02	0,11	0,80
$\langle x^{\mathbf{P}}_5\|$	0,00	0,00	0,04	0,16	0,90

Quotient-c.d.
of the M-complementary co∼event

Figure 16: The scheme of results on the Figure 18 for input 5x3933 statistics on the 5x5 quotient-space: two co∼event means, two quotient-indicators and three distributions of the co∼event («voters–candidates 5x3933 statistics»). The color palette shows values from $[0,1]$ (normalized by the maximum value to better visualize small values; maximum value visualize as a red) by the following way: red $\sim$ 1, white $\sim$ 1/2, blue $\sim$ 0. For the two quotient-indicators maximum value = 1, for three quotient-c.d. maximum value = 0,15667 ...

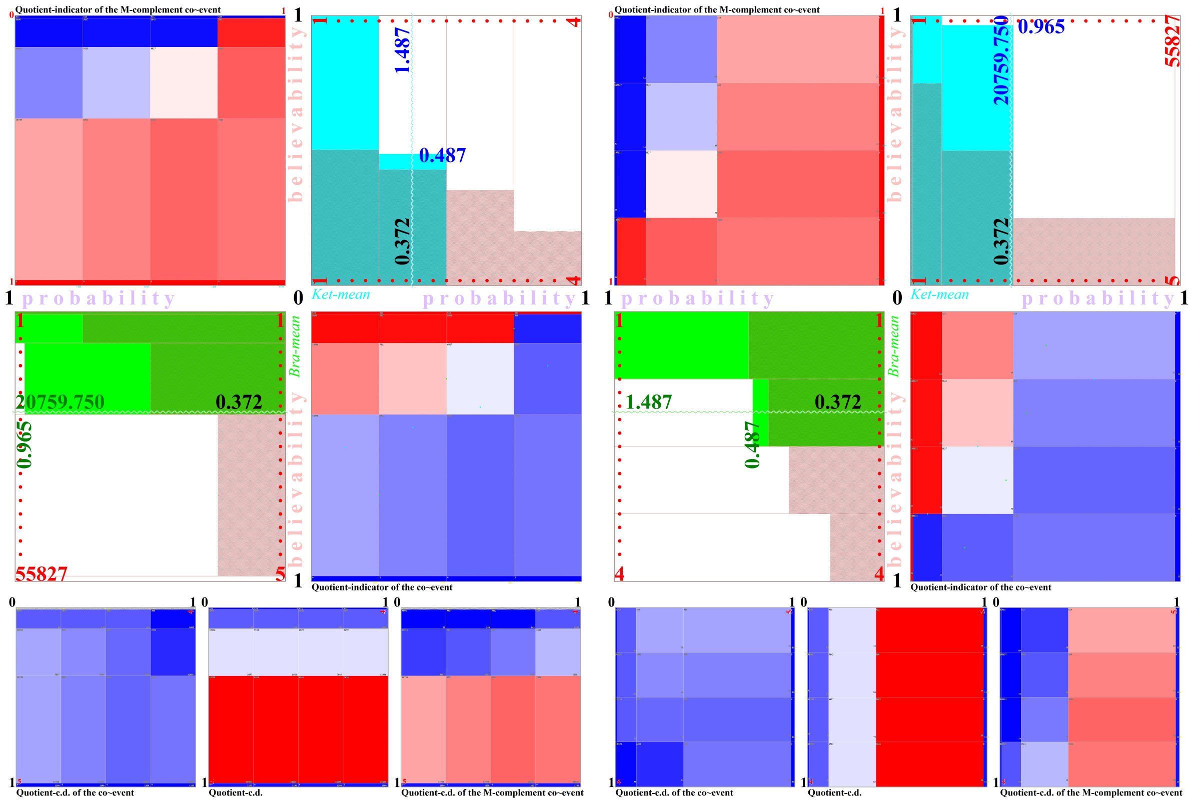

Figure 17: Bra-mean and ket-mean: for 55827x4-statistics «voters–candidates», left; for 4x55827-statistics «candidates–voters», right.

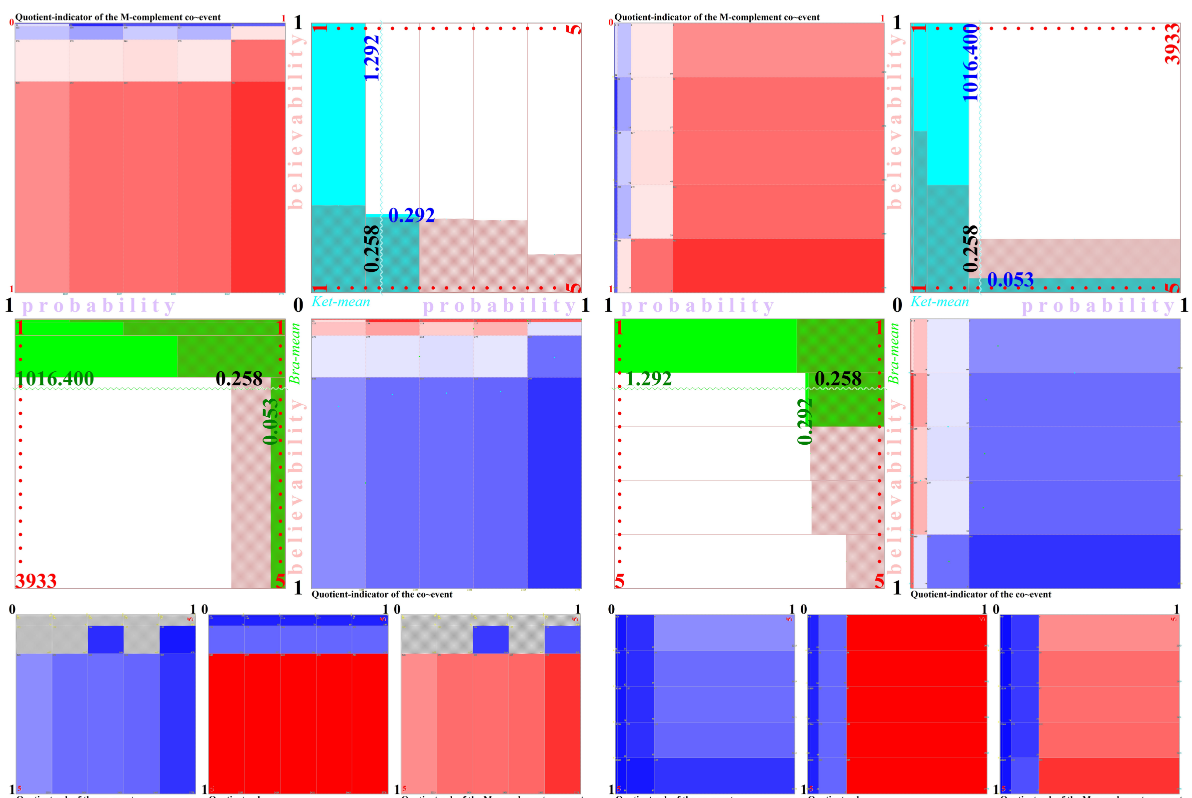

Figure 18: Bra-mean and ket-mean: for 3933x5-statistics «voters–candidates», left; for 5x3933-statistics «candidates–voters», right.

8.2 Venn diagrams of bra-means and ket-means for some co∼events

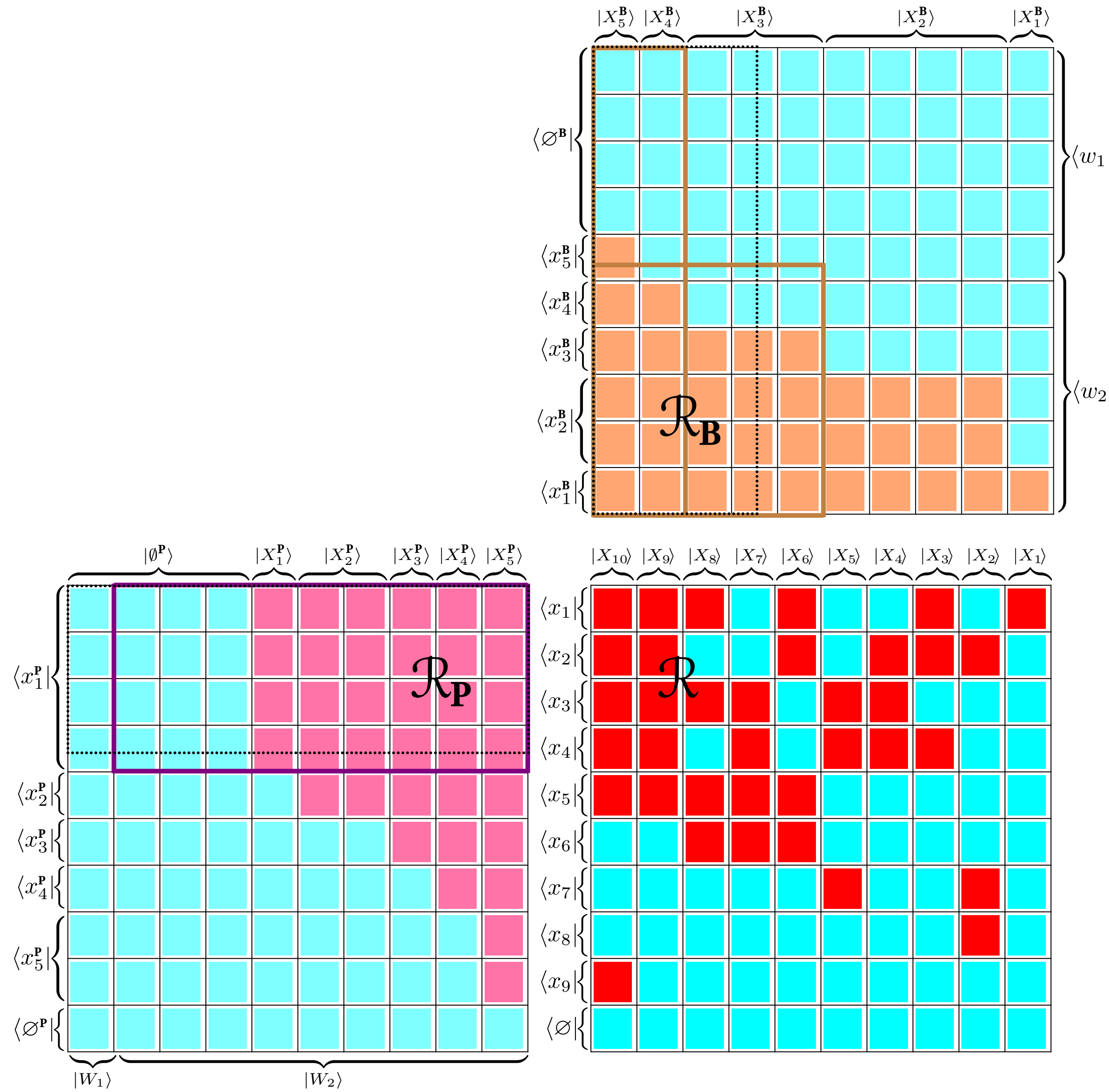

Table 8: Venn diagrams of the co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ (red) and of two its quotient-projections: $\mathcal{R}_{\mathbf{P}} \subseteq \langle\Omega|\Omega\rangle$ (magenta) and $\mathcal{R}_{\mathbf{B}} \subseteq \langle\Omega|\Omega\rangle$ (orange) on the bra-ket-space $\langle\Omega|\Omega\rangle$ with the labellings $\langle\mathfrak{X}_{\mathcal{R}}|\mathfrak{z}^{\mathfrak{X}_{\mathcal{R}}}\rangle$, $\langle\mathfrak{X}_{\mathcal{R}}^{\mathbf{P}}|\mathfrak{z}_{\mathbf{P}}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ and $\langle\mathfrak{X}_{\mathcal{R}}^{\mathbf{B}}|\mathfrak{z}_{\mathbf{B}}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ correspondingly; $N = 9$, $\mathfrak{z}^N = 10$, $N_{\mathbf{P}} = 5$, $N_{\mathbf{B}} = 5$; $\mathbf{\Phi}(\mathcal{R}) = \mathbf{\Phi}(\mathcal{R}_{\mathbf{P}}) = \mathbf{\Phi}(\mathcal{R}_{\mathbf{B}}) = 0.36$; the bra-mean (violet): $\langle\mathcal{E}\mathcal{R}|^{\pi}\rangle = \langle \mathrm{Ter}_{X_1^{\mathbf{P}}}|\mathrm{ter}(W_2)\rangle$, $p(W_1) = 0.1, p(W_2) = 0.9$; the residual probability $\pi = 0.9$; the ket-mean (brown): $\langle^{\beta}|\mathcal{E}\mathcal{R}\rangle = \langle w_1|x_4^{\mathbf{B}}\rangle + \langle w_2|x_5^{\mathbf{B}}\rangle$, $b_{w_1} = 0.47, b_{w_2} = 0.53$; the residual believability $\beta = 0.53$. The representatives of unlabellable co∼event means $\langle\mathcal{E}\mathcal{R}_{\mathbf{P}}|\Omega\rangle$ and $\langle\Omega|\mathcal{E}\mathcal{R}_{\mathbf{B}}\rangle$ are shown by dotted line.

$$
\begin{aligned}
\mathfrak{X}_{\mathcal{R}} &= \{x_1, \ldots, x_9\}, \quad N = 9,\\
|x_1\rangle &= |X_{10}\rangle+|X_9\rangle+|X_8\rangle+|X_6\rangle+|X_3\rangle+|X_1\rangle\,,\\
|x_2\rangle &= |X_{10}\rangle+|X_9\rangle+|X_6\rangle+|X_4\rangle+|X_3\rangle+|X_2\rangle\,,\\
|x_3\rangle &= |X_{10}\rangle+|X_9\rangle+|X_8\rangle+|X_7\rangle+|X_5\rangle+|X_4\rangle\,,\\
|x_4\rangle &= |X_{10}\rangle+|X_9\rangle+|X_7\rangle+|X_5\rangle+|X_4\rangle+|X_3\rangle\,,\\
|x_5\rangle &= |X_{10}\rangle+|X_9\rangle+|X_8\rangle+|X_7\rangle+|X_6\rangle\,,\\
|x_6\rangle &= |X_8\rangle+|X_7\rangle+|X_6\rangle\,,\\
|x_7\rangle &= |X_5\rangle+|X_2\rangle\,,\\
|x_8\rangle &= |X_2\rangle\,,\\
|x_9\rangle &= |X_{10}\rangle\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{z}^{\mathfrak{X}_{\mathcal{R}}} &= \{X_{10}, \ldots, X_1\}, \quad \mathfrak{z}^N = 10,\\
X_{10} &= \{x_1, x_2, x_3, x_4, x_5, x_9\},\\
X_9 &= \{x_1, x_2, x_3, x_4, x_5\},\\
X_8 &= \{x_1, x_3, x_5, x_6\},\\
X_7 &= \{x_3, x_4, x_5, x_6\},\\
X_6 &= \{x_1, x_2, x_5, x_6\},\\
X_5 &= \{x_3, x_4, x_7\},\\
X_4 &= \{x_2, x_3, x_4\},\\
X_3 &= \{x_1, x_2, x_4\},\\
X_2 &= \{x_2, x_7, x_8\},\\
X_1 &= \{x_1\},
\end{aligned}
$$

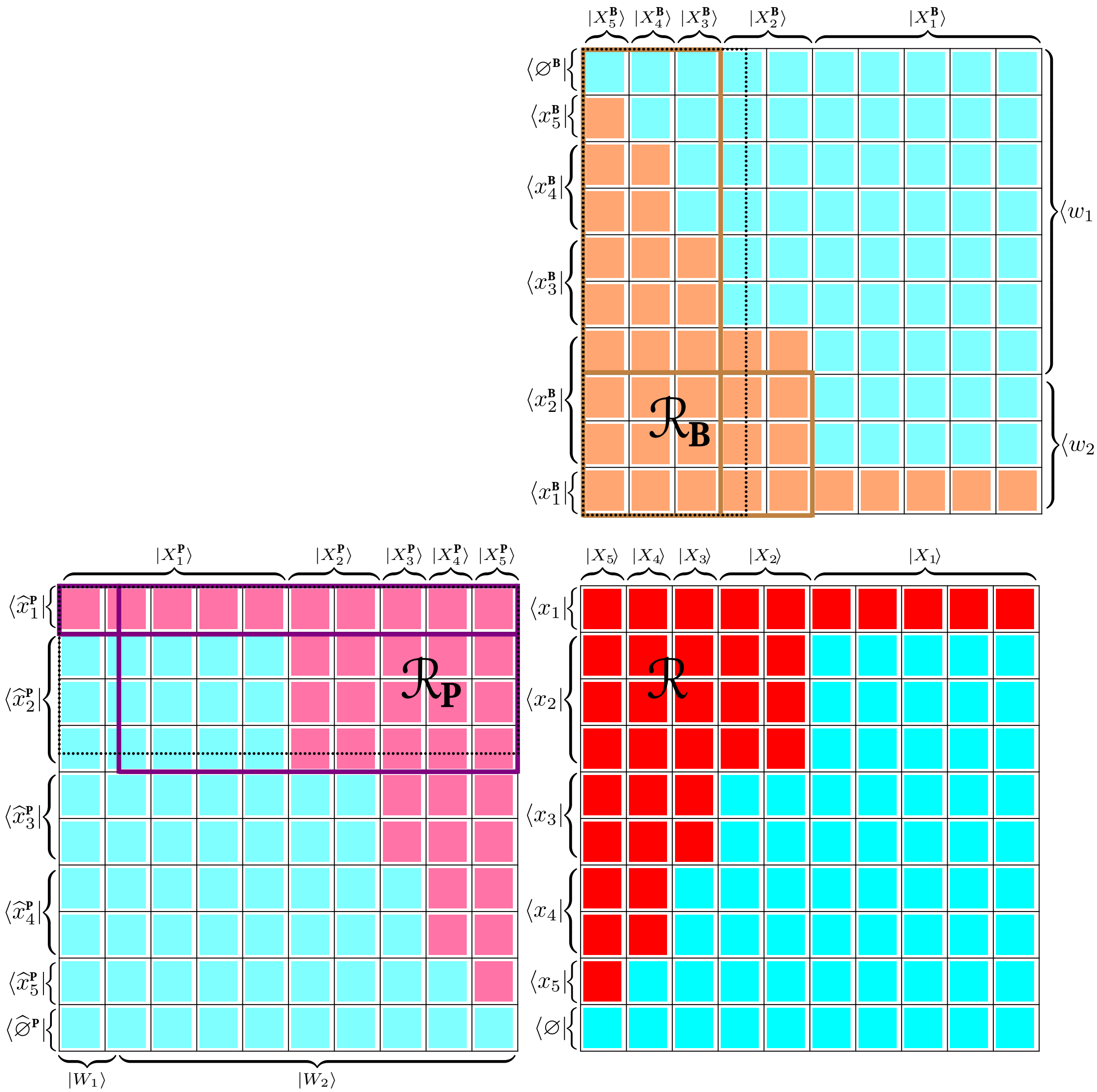

Table 9: Venn diagrams of the co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ (red) and of two its quotient-projections $\mathcal{R}_{\mathbf{P}} \subseteq \langle\Omega|\Omega\rangle$ (magenta) and $\mathcal{R}_{\mathbf{B}} \subseteq \langle\Omega|\Omega\rangle$ (orange) on the bra-ket-space $\langle\Omega|\Omega\rangle$ with the labellings $\langle\mathfrak{X}_{\mathcal{R}}|\mathfrak{Z}^{\mathfrak{X}_{\mathcal{R}}}\rangle$, $\langle\mathfrak{X}_{\mathcal{R}}^{\mathbf{P}}|\mathfrak{Z}_{\mathbf{P}}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ and $\langle\mathfrak{X}_{\mathcal{R}}^{\mathbf{B}}|\mathfrak{Z}_{\mathbf{B}}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ correspondingly; $N = 5, \mathfrak{Z}^N = 5, N_{\mathbf{P}} = 5, N_{\mathbf{B}} = 5$; $\mathbf{\Phi}(\mathcal{R}) = \mathbf{\Phi}(\mathcal{R}_{\mathbf{P}}) = \mathbf{\Phi}(\mathcal{R}_{\mathbf{B}}) = 0.36$; the bra-mean (violet) $\langle\mathcal{E}\mathcal{R}|^{\pi}\rangle = \langle\mathsf{Ter}_{X_1^{\mathbf{P}}}|\mathsf{ter}(W_1)\rangle + \langle\mathsf{Ter}_{X_2^{\mathbf{P}}}|\mathsf{ter}(W_2)\rangle$, $p(W_1) = 0.13$, $p(W_2) = 0.87$; the residual probability $\pi = 0.87$; the ket-mean (brown) $\langle^{\beta}|\mathcal{E}\mathcal{R}\rangle = \langle w_1|x_3^{\mathbf{B}}\rangle + \langle w_2|x_2^{\mathbf{B}}\rangle$, $b_{w_1} = 0.70$, $b_{w_2} = 0.30$; the residual believability $\beta = 0.30$. The representatives of unlabellable co∼event means $\langle\mathcal{E}\mathcal{R}_{\mathbf{P}}|\Omega\rangle$ and $\langle\Omega|\mathcal{E}\mathcal{R}_{\mathbf{B}}\rangle$ are shown by dotted line.

$$
\begin{aligned}
\mathfrak{X}_{\mathcal{R}} &= \{x_1, \ldots, x_5, \varnothing\}, \quad N = 5, \\
|x_1\rangle &= |X_5\rangle + |X_4\rangle + |X_3\rangle + |X_2\rangle + |X_1\rangle, \\
|x_2\rangle &= |X_5\rangle + |X_4\rangle + |X_3\rangle + |X_2\rangle, \\
|x_3\rangle &= |X_5\rangle + |X_4\rangle + |X_3\rangle, \\
|x_4\rangle &= |X_5\rangle + |X_4\rangle, \\
|x_5\rangle &= |X_5\rangle,
\end{aligned}
\qquad
\begin{aligned}
\mathfrak{Z}^{\mathfrak{X}_{\mathcal{R}}} &= \{X_5, \ldots, X_1, \emptyset\}, \quad \mathfrak{Z}^N = 5, \\
X_5 &= \{x_1, x_2, x_3, x_4, x_5\}, \\
X_4 &= \{x_1, x_2, x_3, x_4\}, \\
X_3 &= \{x_1, x_2, x_3\}, \\
X_2 &= \{x_1, x_2\}, \\
X_1 &= \{x_1\}.
\end{aligned}
$$

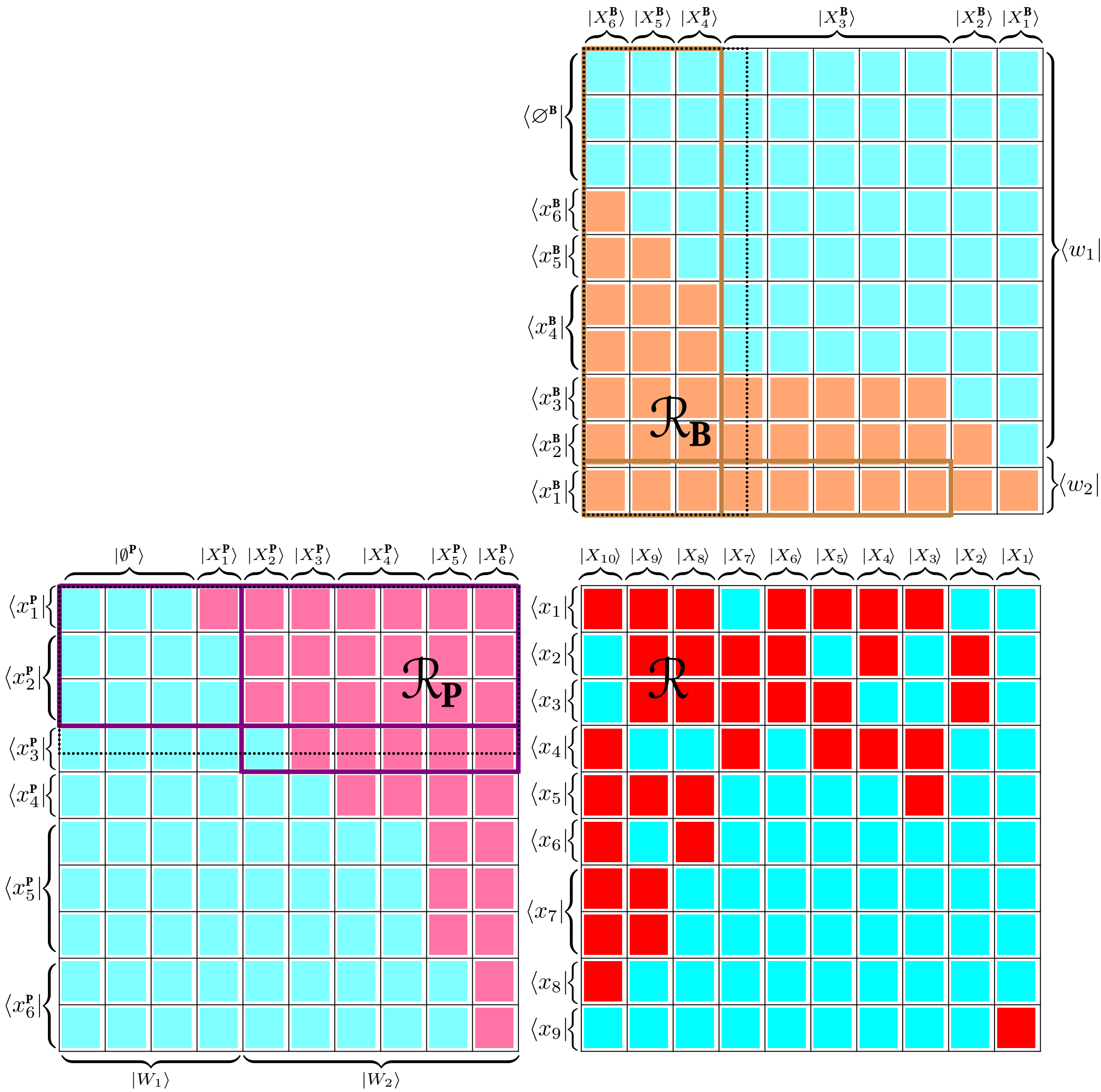

Table 10: Venn diagrams of the co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ (red) and of two its quotient-projections $\mathcal{R}_{\mathbf{P}} \subseteq \langle\Omega|\Omega\rangle$ (magenta) and $\mathcal{R}_{\mathbf{B}} \subseteq \langle\Omega|\Omega\rangle$ (orange) on the bra-ket-space $\langle\Omega|\Omega\rangle$ with the labellings $\langle\mathfrak{X}_{\mathcal{R}}|\mathfrak{x}^{\mathfrak{X}_{\mathcal{R}}}\rangle$, $\langle\mathfrak{X}^{\mathbf{P}}_{\mathcal{R}}|\mathfrak{x}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{P}}\rangle$ and $\langle\mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}|\mathfrak{x}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}}\rangle$ correspondingly; $N = 9, \mathfrak{x}^N = 10, N_{\mathbf{P}} = 6, N_{\mathbf{B}} = 6$; $\mathbf{\Phi}(\mathcal{R}) = \mathbf{\Phi}(\mathcal{R}_{\mathbf{P}}) = \mathbf{\Phi}(\mathcal{R}_{\mathbf{B}}) = 0.36$; the bra-mean (violet) $\langle\mathcal{E}\mathcal{R}|^{\beta}\rangle = \langle\mathsf{Ter}_{X_2^{\mathbf{P}}}|\mathsf{ter}(W_1)\rangle + \langle\mathsf{Ter}_{X_3^{\mathbf{P}}}|\mathsf{ter}(W_2)\rangle$, $p(W_1) = 0.4, p(W_2) = 0.6$; the residual probability $\pi = 0.6$; the ket-mean (brown) $\langle^{\pi}|\mathcal{E}\mathcal{R}\rangle = \langle w_1|x_4^{\mathbf{B}}\rangle + \langle w_2|x_3^{\mathbf{B}}\rangle$, $b_{w_1} = 0.88, b_{w_2} = 0.12$; the residual believability $\beta = 0.12$. The representatives of unlabellable co∼event means $\langle\mathcal{E}\mathcal{R}_{\mathbf{P}}|\Omega\rangle$ and $\langle\Omega|\mathcal{E}\mathcal{R}_{\mathbf{B}}\rangle$ are shown by dotted line.

$$
\begin{aligned}
\mathfrak{X}_{\mathcal{R}} &= \{x_1, \dots, x_9\}, \quad N = 9,\\
|x_1\rangle &= |X_{10}\rangle+|X_9\rangle+|X_8\rangle+|X_6\rangle+|X_5\rangle+|X_4\rangle+|X_3\rangle\,,\\
|x_2\rangle &= |X_9\rangle+|X_8\rangle+|X_7\rangle+|X_6\rangle+|X_4\rangle+|X_2\rangle\,,\\
|x_3\rangle &= |X_9\rangle+|X_8\rangle+|X_7\rangle+|X_6\rangle+|X_5\rangle+|X_2\rangle\,,\\
|x_4\rangle &= |X_{10}\rangle+|X_7\rangle+|X_5\rangle+|X_4\rangle+|X_3\rangle\,,\\
|x_5\rangle &= |X_{10}\rangle+|X_9\rangle+|X_8\rangle+|X_3\rangle\,,\\
|x_6\rangle &= |X_{10}\rangle+|X_8\rangle\,,\\
|x_7\rangle &= |X_{10}\rangle+|X_9\rangle\,,\\
|x_8\rangle &= |X_{10}\rangle\,,\\
|x_9\rangle &= |X_1\rangle\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{x}^{\mathfrak{X}_{\mathcal{R}}} &= \{X_{10}, \dots, X_1\}, \quad \mathfrak{x}^N = 10,\\
X_{10} &= \{x_1, x_4, x_5, x_6, x_7, x_8\},\\
X_9 &= \{x_1, x_2, x_3, x_5, x_7, x_8\},\\
X_8 &= \{x_1, x_2, x_3, x_5, x_6\},\\
X_7 &= \{x_2, x_3, x_4\},\\
X_6 &= \{x_1, x_2, x_3\},\\
X_5 &= \{x_1, x_3, x_4\},\\
X_4 &= \{x_1, x_2, x_4\},\\
X_3 &= \{x_1, x_4, x_5\},\\
X_2 &= \{x_2, x_3\},\\
X_1 &= \{x_{10}\},
\end{aligned}
$$

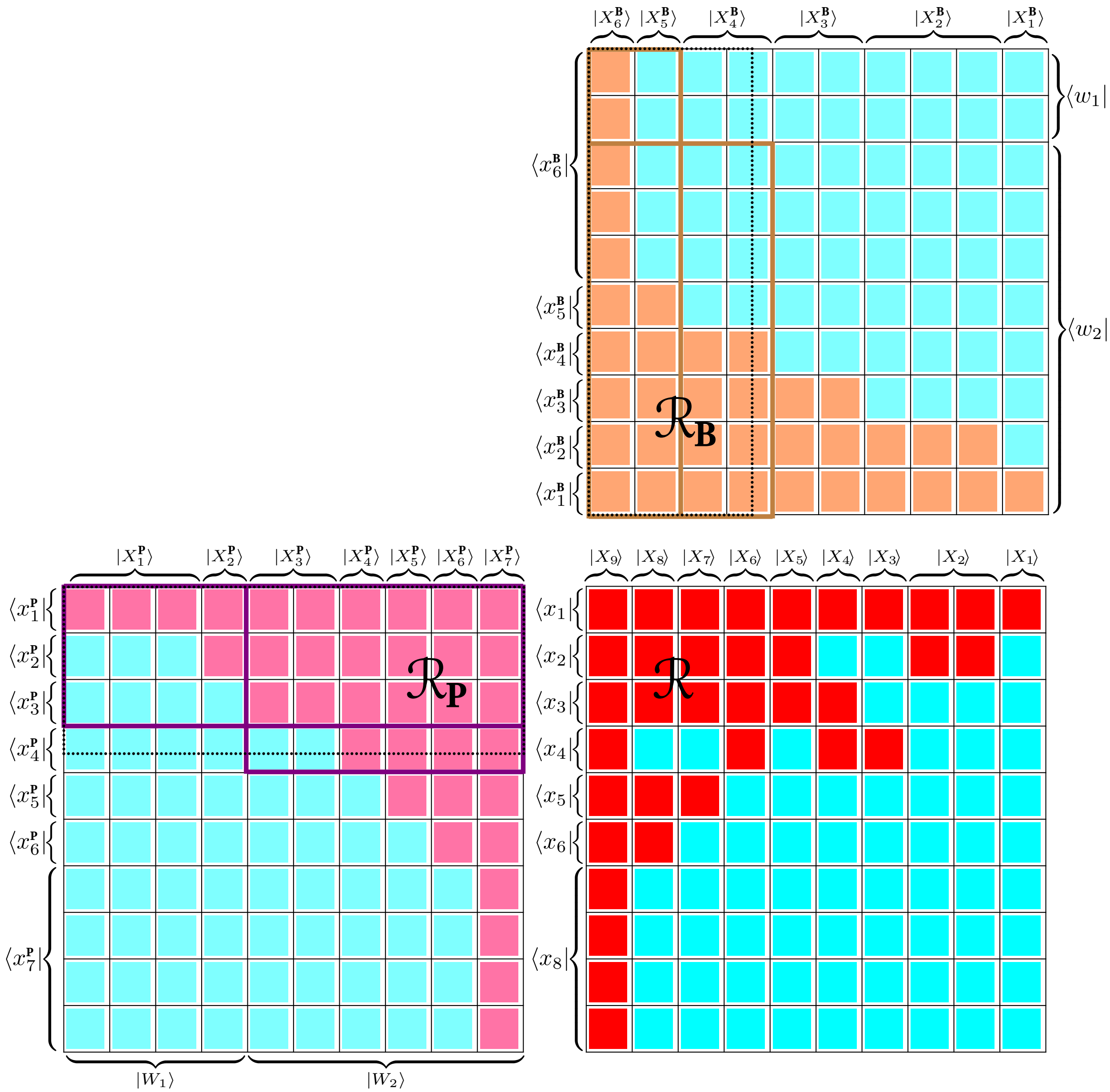

Table 11: Venn diagrams of the co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ (red) and of two its quotient-projections $\mathcal{R}_{\mathbf{P}} \subseteq \langle\Omega|\Omega\rangle$ (magenta) and $\mathcal{R}_{\mathbf{B}} \subseteq \langle\Omega|\Omega\rangle$ (orange) on the bra-ket-space $\langle\Omega|\Omega\rangle$ with the labellings $\langle\mathfrak{X}_{\mathcal{R}}|\mathfrak{z}^{\mathfrak{X}_{\mathcal{R}}}\rangle$, $\langle\mathfrak{X}^{\mathbf{P}}_{\mathcal{R}}|\mathfrak{z}_{\mathbf{P}}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ and $\langle\mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}|\mathfrak{z}_{\mathbf{B}}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ correspondingly; $N = 7, \mathfrak{z}^N = 9, N_{\mathbf{P}} = 7, N_{\mathbf{B}} = 6$; $\mathbf{\Phi}(\mathcal{R}) = \mathbf{\Phi}(\mathcal{R}_{\mathbf{P}}) = \mathbf{\Phi}(\mathcal{R}_{\mathbf{B}}) = 0.36$; the bra-mean (violet) $\langle\mathcal{ER}|^{\pi}\rangle = \langle\mathsf{Ter}_{X_3^{\mathbf{P}}}|\mathsf{ter}(W_1)\rangle + \langle\mathsf{Ter}_{X_4^{\mathbf{P}}}|\mathsf{ter}(W_2)\rangle$, $p(W_1) = 0.4, p(W_2) = 0.6$; the residual probability $\pi = 0.6$; the ket-mean (brown) $\langle^{\beta}|\mathcal{ER}\rangle = \langle w_1|x_5^{\mathbf{B}}\rangle + \langle w_2|x_4^{\mathbf{B}}\rangle$, $b_{w_1} = 0.2, b_{w_2} = 0.8$; the residual believability $\beta = 0.8$. The representatives of unlabellable co∼event means $\langle\mathcal{ER}_{\mathbf{P}}|\Omega\rangle$ and $\langle\Omega|\mathcal{ER}_{\mathbf{B}}\rangle$ are shown by dotted line.

$$
\begin{aligned}
\mathfrak{X}_{\mathcal{R}} &= \{x_1, \dots, x_7\}, \quad N = 7,\\
|x_1\rangle &= |X_{10}\rangle+|X_9\rangle+|X_8\rangle+|X_7\rangle+|X_6\rangle+|X_5\rangle+|X_4\rangle+|X_3\rangle+|X_2\rangle+|X_1\rangle,\\
|x_2\rangle &= |X_9\rangle+|X_8\rangle+|X_7\rangle+|X_6\rangle+|X_5\rangle+|X_2\rangle\,,\\
|x_3\rangle &= |X_9\rangle+|X_8\rangle+|X_7\rangle+|X_6\rangle+|X_5\rangle+|X_4\rangle\,,\\
|x_4\rangle &= |X_9\rangle+|X_6\rangle+|X_4\rangle+|X_3\rangle\,,\\
|x_5\rangle &= |X_9\rangle+|X_8\rangle+|X_7\rangle\,,\\
|x_6\rangle &= |X_9\rangle+|X_8\rangle\,,\\
|x_7\rangle &= |X_9\rangle\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{z}^{\mathfrak{X}_{\mathcal{R}}} &= \{X_9, \dots, X_1\}, \quad \mathfrak{z}^N = 9,\\
X_9 &= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\},\\
X_8 &= \{x_1, x_2, x_3, x_5, x_6\},\\
X_7 &= \{x_1, x_2, x_3, x_5\},\\
X_6 &= \{x_1, x_2, x_3, x_4\},\\
X_5 &= \{x_1, x_2, x_3\},\\
X_4 &= \{x_1, x_3, x_4\},\\
X_3 &= \{x_1, x_4\},\\
X_2 &= \{x_1, x_2\},\\
X_1 &= \{x_1\}.
\end{aligned}
$$

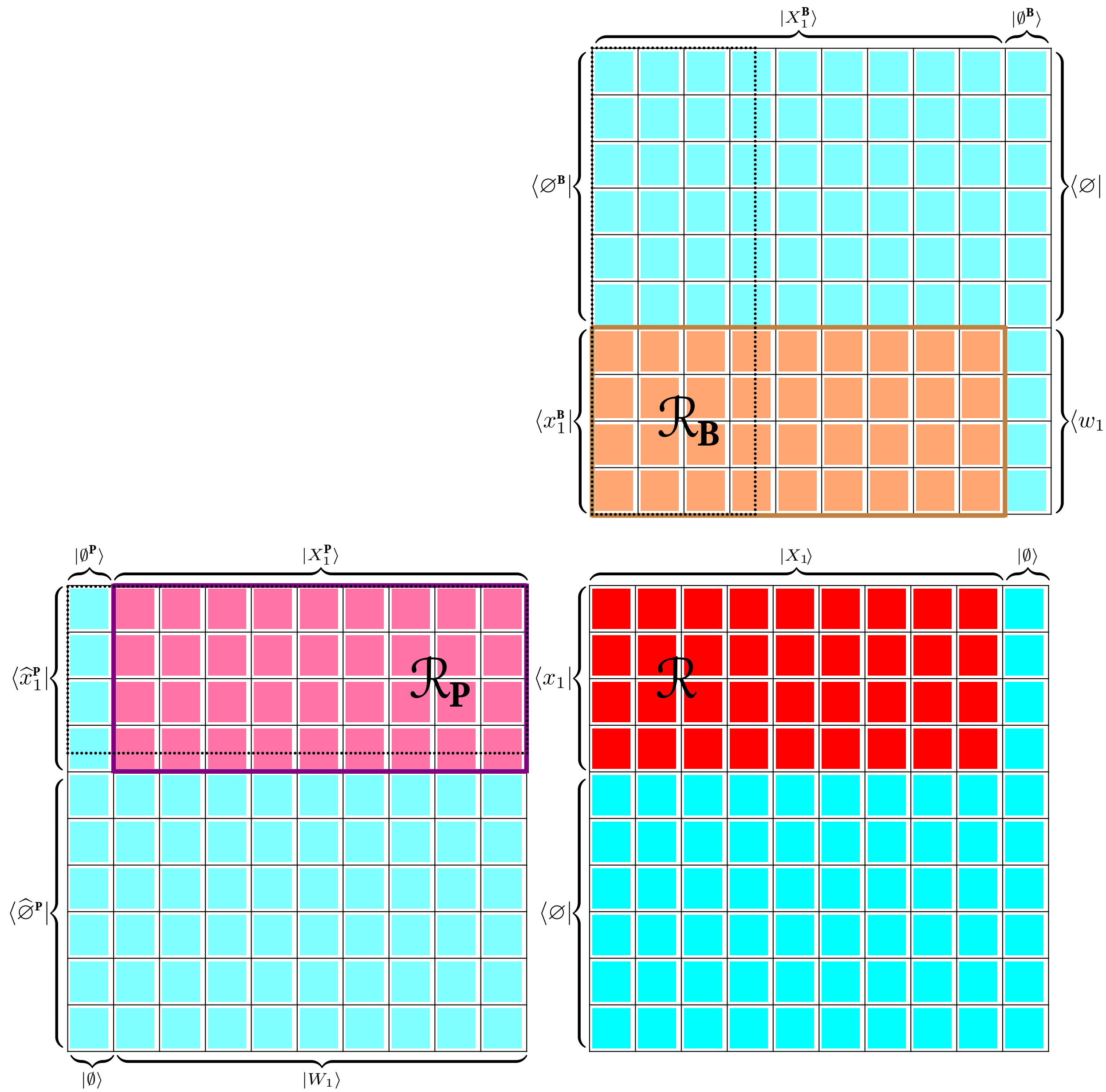

Table 12: Venn diagrams of the co∼event $\mathcal{R} \subseteq \langle\Omega|\Omega\rangle$ (red) and of two its quotient-projections $\mathcal{R}_{\mathbf{P}} \subseteq \langle\Omega|\Omega\rangle$ (magenta) and $\mathcal{R}_{\mathbf{B}} \subseteq \langle\Omega|\Omega\rangle$ (orange) on the bra-ket-space $\langle\Omega|\Omega\rangle$ with the labellings $\langle\mathfrak{X}_{\mathcal{R}}|\mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}}\rangle$, $\langle\mathfrak{X}_{\mathcal{R}}^{\mathbf{P}}|\mathfrak{S}_{\mathbf{P}}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ and $\langle\mathfrak{X}_{\mathcal{R}}^{\mathbf{B}}|\mathfrak{S}_{\mathbf{B}}^{\mathfrak{X}_{\mathcal{R}}}\rangle$ correspondingly; $N = 1, \mathfrak{S}^N = 1, N_{\mathbf{P}} = 1, N_{\mathbf{B}} = 1$; $\mathbf{\Phi}(\mathcal{R}) = \mathbf{\Phi}(\mathcal{R}_{\mathbf{P}}) = \mathbf{\Phi}(\mathcal{R}_{\mathbf{B}}) = 0.36$; the bra-mean (violet) $\langle\mathcal{E}\mathcal{R}|^{\pi}\rangle = \langle\mathrm{Ter}_{X_1^{\mathbf{P}}}|\mathrm{ter}(W_1)\rangle$, $p(\emptyset) = 0.1, p(W_1) = 0.9$; the residual probability $\pi = 0.9$; the ket-mean (brown) $\langle^{\beta}|\mathcal{E}\mathcal{R}\rangle = \langle w_1|x_1^{\mathbf{B}}\rangle$, $b_{\varnothing} = 0.6$, $b_{w_1} = 0.4$; the residual believability $\beta = 0.4$. The representatives of unlabellable co∼event means $\langle\mathcal{E}\mathcal{R}_{\mathbf{P}}|\Omega\rangle$ and $\langle\Omega|\mathcal{E}\mathcal{R}_{\mathbf{B}}\rangle$ are shown by dotted line.

$$\mathfrak{X}_{\mathcal{R}} = \{x_1, \varnothing\}, \quad N = 1, \qquad \mathfrak{S}^{\mathfrak{X}_{\mathcal{R}}} = \{X_1, \emptyset\}, \quad \mathfrak{S}^N = 1,$$
$$|x_1\rangle = |X_1\rangle\,, \qquad X_1 = \{x_1\}.$$

Above in the tables 10 — 12 some types of dual bra-means $\langle \mathcal{ER}|^{\pi}\rangle$ and ket-means $\langle^{\beta}|\mathcal{ER}\rangle$ of co∼event $\mathcal{R}$, its quotient-projections $\mathcal{R}_{\mathbf{B}}$ and $\mathcal{R}_{\mathbf{P}}$, and also representatives of unlabellable co∼event means $\langle \mathcal{ER}|\Omega\rangle$ and $\langle\Omega|\mathcal{ER}\rangle$ (by dotted line) are shown. Due to lack of space in these tables the abbreviation $|X\rangle$ for the terraced ket-events $|\mathrm{ter}(X/\!\!/\mathfrak{X}_{\mathcal{R}})\rangle$ is used, and also the following standard denotations are used:

$\mathbf{B}$-quotient-labelling	$\mathbf{P}$-quotient-labelling
$\left\langle \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}} \middle| \mathcal{Z}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}} \right\rangle,$	$\left\langle \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}} \middle| \mathcal{Z}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{P}} \right\rangle,$
$\mathfrak{X}^{\mathbf{B}}_{\mathcal{R}} = \begin{cases} \{x^{\mathbf{B}}_1, \dots, x^{\mathbf{B}}_{N_{\mathbf{B}}}\}, \\ \{x^{\mathbf{B}}_1, \dots, x^{\mathbf{B}}_{N_{\mathbf{B}}}, \varnothing^{\mathbf{B}}\}, \end{cases}$	$\mathfrak{X}^{\mathbf{P}}_{\mathcal{R}} = \begin{cases} \{x^{\mathbf{P}}_1, \dots, x^{\mathbf{P}}_{N_{\mathbf{P}}}\}, \\ \{x^{\mathbf{P}}_1, \dots, x^{\mathbf{P}}_{N_{\mathbf{P}}}, \varnothing^{\mathbf{P}}\}, \end{cases}$
$\mathcal{Z}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}} = \begin{cases} \{X^{\mathbf{B}}_1, \dots, X^{\mathbf{B}}_{N_{\mathbf{B}}}\}, \\ \{X^{\mathbf{B}}_1, \dots, X^{\mathbf{B}}_{N_{\mathbf{B}}}, \emptyset^{\mathbf{B}}\}, \end{cases}$	$\mathcal{Z}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{P}} = \begin{cases} \{X^{\mathbf{P}}_1, \dots, X^{\mathbf{P}}_{N_{\mathbf{P}}}\}, \\ \{X^{\mathbf{P}}_1, \dots, X^{\mathbf{P}}_{N_{\mathbf{P}}}, \emptyset^{\mathbf{P}}\}, \end{cases}$
$\|\varnothing^{\mathbf{B}}\rangle \subseteq \|\Omega\rangle, \quad \emptyset^{\mathbf{B}} \subseteq \mathfrak{X}^{\mathbf{B}}_{\mathcal{R}},$	$\langle\varnothing^{\mathbf{P}}\| \subseteq \langle\Omega\|, \quad \emptyset^{\mathbf{P}} \subseteq \mathfrak{X}^{\mathbf{P}}_{\mathcal{R}},$
$\|x^{\mathbf{B}}_i\rangle = \sum_{j=i}^{N_{\mathbf{B}}} \|X^{\mathbf{B}}_j\rangle,\ i = 1, ..., N_{\mathbf{B}}.$	$\|x^{\mathbf{P}}_i\rangle = \sum_{j=i}^{N_{\mathbf{P}}} \|X^{\mathbf{P}}_j\rangle,\ i = 1, ..., N_{\mathbf{P}}.$
$X^{\mathbf{B}}_j = \sum_{i=1}^{N_{\mathbf{B}}} \{x^{\mathbf{B}}_i\},\ j = 1, ..., N_{\mathbf{B}},$	$X^{\mathbf{P}}_j = \sum_{i=1}^{N_{\mathbf{P}}} \{x^{\mathbf{P}}_i\},\ j = 1, ..., N_{\mathbf{P}},$
$N_{\mathbf{B}} = \|\mathfrak{X}^{\mathbf{B}}_{\mathcal{R}}\|_{\neq\varnothing} = \left\|\mathcal{Z}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{B}}\right\|_{\neq\emptyset} = \mathcal{Z}^{N_{\mathbf{B}}}_{\mathbf{B}},$	$N_{\mathbf{P}} = \|\mathfrak{X}^{\mathbf{P}}_{\mathcal{R}}\|_{\neq\varnothing} = \left\|\mathcal{Z}^{\mathfrak{X}_{\mathcal{R}}}_{\mathbf{P}}\right\|_{\neq\emptyset} = \mathcal{Z}^{N_{\mathbf{P}}}_{\mathbf{P}}.$

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