Public debt, pollution and environmental taxes: Nash and Stackelberg equilibria

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by

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Abstract
Public debt accumulation and pollution result to disutility while time path must be sustainable. Policy weapons available to the government with regard to public debt is the generation of primary surpluses to sustain public debt while concerning pollution environmental taxation is expected to reduce emissions. In this paper, we address these factors in a simple dynamic game in order to find ways at which the notions of public debt, pollution, and taxation are interrelated. The starting point of the model is the identity of current account as the equation of motion of public debt, while public debt is considering as a stock and the stress of the regulator is to raise the nation's primary surplus. Nash and Stackelberg differential game solutions are used to explore the strategic interactions. In the Nash equilibrium establishment of cyclical strategies, during the game between the polluters in one hand and the government on the other, requires that the discount rate of the polluters must be greater than government's discount rate. That is the polluters must be more impatient than the government. In the case of hierarchical setting, the analytical expressions of the strategic variables and the steady state value of public debt stock are provided. Furthermore, we found the analytical expressions of the value functions, making, therefore, the policy implications an easy task. Finally, we found the conditions under which the conflict is more intensive, in the two cases of equilibrium, according to the shadow price of the environmental damages.

Keywords: Public debt; Pollution; Taxation; Dynamic games; Nash equilibrium; Stackelberg equilibrium.

JEL Codes: C72; H23; Q52; H62.

1. Introduction

In this paper we deal with the dynamics of accumulation of a nation's public debt which harms prosperity of the economic agents of a country or a nation. As it is already known public debt accumulation produces disutility, therefore is detrimental to nations' households, because it reduces their consumption in order to meet their future tax burden (Greiner and Fincke, 2009)

Considering methodology, game theory may be used as an appropriate tool in order to design an efficient action against accumulation of the public debt, because the regulator has to take into consideration the response of victims e.g. the honest taxpayers and so on. As it happens in most cases, every socially undesirable stock is an irreversible fact, and therefore one main concern of the social planer should be the discovery of effective ways to reduce the sources which are responsible for the unwished stock accumulation. We use both Nash and Stackelberg differential game approaches to study the intertemporal strategic interactions between the group of corrupt officials and the tax evaders on one side and the government on the other.

The major problem of the public debt accumulation requires to finding ways to effectively reduce the unwished public debt stock, maintaining, at the same time, the standards of the economic process within a country. In the macroeconomic literature the same problem is addressed as whether the time path of the public debt is sustainable. Nowadays modern models about the sustainability of public debt does not involve the central bank of an economy, because the central banks are independent and therefore the nation's government should not rest on central banks when deciding about real public debt reduction. As it is well known the main goal of the government is to achieve sustainable debt policies. In order to saw how public debt and (primary) surpluses are (crucially) connected, we resort in the recent literature. In macroeconomics literature and in the case the interest rate exceeds the growth rate of the gross domestic product (GDP), a given debt policy is said sustainable if the primary surplus relative to GDP is a positive and linearly increasing function of the debt to GDP ratio. If a government pursues a sustainable policy the debt ratio remains constant in the long-run or it converges to zero. As pointed out by Greiner and Fincke (2009) a stationary debt to GDP ratio tends to be sustainable if the government raises its own primary surpluses as the public debt rises.

Moreover, it is known that there exists a statistically significant positive correlation between government debt levels and environmental quality (see, among others, Fohda and Seegmuller, 2011). In the present model, we introduce the possibility that firms will engage in activities such as emission of pollutants. On the other hand the government, in order to maintain a sustainable time path of the public debt, has to attain primary surpluses by taxing the polluting activities but at the same time has to take care of environmental quality by financing abatement processes. Therefore there exists a conflict between the two parties, i.e. the government and the polluters, and this conflict is modeled as an intertemporal dynamic game.

Returning to the model solution problems, one of the main concerns should be the irregularity of multiple equilibrium points. Finding multiple equilibrium points in economic models is not an attractive solution for the policy makers. But the recognition of multiple optimal stable equilibrium points may be crucial in order to locate the thresholds separating the basins of attraction surrounding these different equilibria. Starting at a threshold, a rational economic agent is indifferent between moving toward one or the other equilibrium, but a small movement away from the threshold can "destroy" this indifference, leading in a unique optimal course of action.

The introductory one sector, with a convex – concave production function, optimal growth model of Skiba (Skiba, 1978) was the start of a fast growing literature towards the cyclical solution strategies generated in intertemporal dynamic economic models. Wirl (1995), in resources stock literature, reconsidering a model of Clark *et al* (1979), concludes that equilibrium that falls below the maximum sustainable yield but that exceeds the intertemporal harvest rule due to the positive spillovers allows for optimal, long run, cyclical harvest strategies.

As it is already made clear, the purpose of the present paper is to uncover principles underpinning efficient design of countermeasures against the sources of the undesired public debt accumulation. In particular, we model the optimal balance of competing parties, and we intend to find the implications of misspecification at the level for success or failure. An important aim of the first part of our research is the identification of mechanisms generating oscillations of both responsible (for the public debt) agents' illegal activities and periodic countermeasures taken by the government. The discussion of a threshold occurrence does not only limited in the well known (S, s) policies in inventory management, but there are however, other nonlinearities implying oscillatory behavior. We intend to study this issue by using the methodology of stable limit cycles. Limit cycles, has the intuitive explanation which says that if a trajectory of a continuous dynamical system stays in a bounded region forever, it has to approach a point or a cycle. Cycles gives rise to cyclical policies in economic models, e.g. if a policy trajectory say a higher primary surplus policy, which is restricted in a bounded planar space then this policy sooner or later will retrace its previous steps. Moreover, in higher than the two dimensional systems, sufficient conditions for the existence of limit cycles of nonlinear dynamical systems are provident. Arithmetically the sufficient conditions requires that a pair of purely imaginary eigenvalues exists, for a particular value of the bifurcation parameter, and the real part of this pair of eigenvalues changes smoothly its sign as the parameter is altered from below its actual value to above.

The stability of limit cycles is of great importance for the long run behavior of a dynamical system. Economic mechanisms that may be a source of limit cycles, as mentioned by Dockner and Feichtinger (1995) are: (i) complementarity over time, (ii) dominated cross effects with respect to capital stocks, and (iii) positive growth of equilibrium.

The contribution of the paper, in the public economics field, is that it considers the accumulation of the public debt as a conflict between two rivals. One is the government policy imposing measures in order to augment primary surplus, which in turn ensures sustainability of public debt, while the other is the group of the polluting firms and by the large the polluters. As it is mentioned above, the problem is modeled first as a Nash differential game, for which we explore at equilibrium the possibility of limit cycles and second as a Stackelberg differential game for which we calculate the equilibrium strategies. Such stock accumulation and regulation control models can be found, among others, in Forster (1980) concerning optimal energy use model; in Xepapadeas (1992) regarding environmental policy design and non-point source pollution and so on.

The remainder of the paper is organized as follows. Section 2 comments on cyclical policies of the two sides, while Section 3 introduces the Nash differential game and gives a necessary condition for cyclical strategies. Section 4 investigates the Stackelberg differential game between the government and polluters and calculates the equilibrium strategies and the players' value functions. The last section concludes the paper.

2. Intuition of the cyclical policies between taxation and emissions

An intuitive explanation of cyclical policies in the proposed public debt model, between polluters and the government, may be the following. The group of polluters derives utility from the higher intensity of their polluting activities, such as emissions, while the other side (i.e. the government) derives utility from the taxation revenues taken against the polluting activities (e.g. taxation of emissions).

Let us start with rather low and declining stock of public debt. A farsighted regulator, which only gains benefits from the reduction of public debt, will curb its tax measures against pollution since a further reduction of the public debt's stock would only be possible at high costs. As a consequence, the stock of public debt begins to grow again. Now the polluters have to react by increasing the intensity of their emissions but only moderately since taxation undertook by the government would not be still very efficient with higher costs and moderate benefits for the government.

Moreover, this would stabilize the stock of public debt and the dynamical system would approach a stable steady state. If the polluters have a high discount rate, which is a realistic assumption, they behave myopically reacting strongly, i.e. they intensify their emissions. This provokes incremental taxation measures on the government's side which in turn lead to an increasing reduction of public debt. To avoid further taxation payments the group of polluters has to reduce the intensity of their emissions, so the cycle would close.

3. The Nash Differential Game

Let us denote by $B(t)$ the instantaneous public debt of a country at time t. Without any measures taken by the government and also without any other actions on the side of the polluters, the stock of public debt grows according to the function $G(B)$, which is considered as growth function, obviously dependent on the interest rate, satisfying the conditions $G(0)=0$, $G(B)>0$ for all $B\in(0,K)$, $G(B)<0$ for all $B\in(K,\infty)$, $G''(B)\leq 0$. Starting up the polluting mechanisms is costly for the polluters, due to for instance the compliance costs. These costs reduce their capital available to their polluting activities. The growth reduction of public debt, however, does not only depend on the intensity of the polluting activities, i.e. emissions $E(t)$, but is also influenced by the taxation revenues $T(t)$ undertaken by the government. We set as instrument variables for the polluters' side the intensity of emissions $E(t)$ and the government's taxation $T(t)$ undertaken, which are assumed non-negatives $E(t)\geq 0,\ \ T(t)\geq 0$.

According to the recent proposed models, one way for the public debt accumulation reduction is the function of primary surplus. In our case the surplus function is denoted by $S(E,T)$ and it is a function of the two above control variables, instead of a single time function. Combining the growth of public debt $G(B)$ with the surplus function $S(E,T)$ the state dynamics can be written as

$$\dot{B}=G(B)-S(E,T), \qquad B(0)=B_0>0 \tag{1}$$

Along a trajectory the non negativity constraint is imposed, that is

$$B(t)\geq 0 \quad \forall t\geq 0 \tag{2}$$

With the assumption of the compliance costs and the damages incurred in the group of polluters, a higher intensity of emissions and also the government taxation measures, cause a stronger reduction of their capital resources and therefore we assume the partial derivatives of the surplus' function $S(E,T)$ to be positive, i.e. $S_E>0$, $S_T>0$.

Moreover the law of diminishing returns is applied only for the government actions undertaken, that is $S_{TT} < 0$ and for simplicity we assume $S_{EE} = 0$

The utility functions the two players need to maximize defined as follows:

Player 1, the government, derives instantaneous utility on one hand from the primary surplus $S(E,T)$, while on the other hand their taxation measures effort $T(t)$ gives rise to increasing and convex costs $a(T)$. Moreover the disutility derived by a high level of public debt is described by the increasing function $\delta(B)$. Summing up, the present value of player's 1 utility is described by the following functional

$$J_1 = \int_0^\infty e^{-\rho_1 t}\left[S(E,T) - \delta(B) - a(T)\right]dt \tag{3}$$

Player 2, the group of polluters, could enjoy potential utility $W(B)$ from the given level of public debt $B(t)$, and utility, as well, from the intensity E of their emissions realization, which is described by the function $U(E)$. The argument that a given level of public debt offers potential utility on the polluters seems somewhat confusing, although it could happen in real life. According to some environmental policies, the central regulator has the ability to turn the optimal allocation of his effort between the pollutants abatement and the polluters' taxation (Halkos and Papageorgiou, 2017; Halkos, 1992). If the choice of the regulator is the pollutants abatement the polluters avoid the taxation punishment and therefore enjoy potential utility. For the utilities $W(B)$ and $U(E)$ we assume that are monotonically increasing functions with decreasing marginal returns, therefore for the first derivatives we have $W'(B) > 0,\ U'(E) > 0$ and for the second derivatives $W''(B) < 0,\ U''(E) < 0$.

So, player's 2 utility function is defined, in its additively separable form, as:

$$J_2 = \int_0^{\infty} e^{-\rho_2 t}\left[W(B)+U(E)\right]dt \tag{4}$$

3.1. Equilibrium analysis

We begin analysis with the concept of an open-loop Nash equilibrium, which is based on the fact that every player's strategy is the best reply to the opponent's exogenously given strategy. Obviously, equilibrium holds if both strategies are simultaneously best replies.

The current value Hamiltonians for both players, are defined as follows

$$H_1 = S(E,T)-\delta(B)-a(T)+\lambda\big(G(B)-S(E,T)\big)$$

$$H_2 = W(B)+U(E)+\mu\big(G(B)-S(E,T)\big)$$

The first order conditions, for the maximization problem, are the following system of differential equations for both players:

First, the maximized Hamiltonians are

$$\frac{\partial H_1}{\partial T} = (1-\lambda)S_T(E,T)-a'(T)=0 \tag{5}$$

$$\frac{\partial H_2}{\partial E} = U'(E)-\mu S_E(E,T)=0 \tag{6}$$

and second the costate variables are defined by the equations

$$\dot{\lambda} = \rho_1\lambda-\frac{\partial H_1}{\partial B} = \lambda\left[\rho_1-G'(B)\right]+\delta'(B) \tag{7}$$

$$\dot{\mu} = \rho_2\mu-\frac{\partial H_2}{\partial B} = \mu\left[\rho_2-G'(B)\right]-W'(B) \tag{8}$$

3.2. Stability of equilibrium

An interior steady state $\left(B^*,\lambda^*,\mu^*\right)$ with the optimal controls $\left(E^*,T^*\right)$ is a solution of the following system (taking steady states):

$$\begin{aligned}
&G(B)=S(E,T)\\
&\lambda\left(\rho_1-G'(B)\right)=-\delta'(B)\\
&\mu\left(\rho_2-G'(B)\right)=W'(B)\\
&(1-\lambda)S_T(E,T)=a'(T)\\
&\mu S_E(E,T)=U'(E)
\end{aligned}$$

and the Jacobian matrix, evaluated at the steady state, is

$$J=\begin{pmatrix}
\frac{\partial}{\partial B}\left[G(B)-S(E,T)\right] & \frac{\partial}{\partial \lambda}\left[G(B)-S(E,T)\right] & \frac{\partial}{\partial \mu}\left[G(B)-S(E,T)\right]\\
\frac{\partial}{\partial B}\left[\lambda\left(\rho_1-G'(B)\right)+\delta'(B)\right] & \frac{\partial}{\partial \lambda}\left[\lambda\left(\rho_1-G'(B)\right)+\delta'(B)\right] & \frac{\partial}{\partial \mu}\left[\lambda\left(\rho_1-G'(B)\right)+\delta'(B)\right]\\
\frac{\partial}{\partial B}\left[\mu\left(\rho_2-G'(B)\right)-W'(B)\right] & \frac{\partial}{\partial \lambda}\left[\mu\left(\rho_2-G'(B)\right)-W'(B)\right] & \frac{\partial}{\partial \mu}\left[\mu\left(\rho_2-G'(B)\right)-W'(B)\right]
\end{pmatrix}$$

which after the simple calculations, takes the following final form:

$$J=\begin{pmatrix}
G'(B) & -\partial S(E,T)/\partial\lambda & -\partial S(E,T)/\partial\mu\\
-\lambda G''(B)+\delta''(B) & \rho_1-G'(B) & 0\\
-\mu G''(B)-W''(B) & 0 & \rho_2-G'(B)
\end{pmatrix} \qquad (9)$$

The main stability analysis is focused in periodic solutions, and therefore we make use of the Hopf bifurcations. Thus, computing determinants and trace of the Jacobian matrix (9) we have

$$\text{tr } J=\rho_1+\rho_2-G'(B)$$

$$\begin{aligned}
\det J = {} & G'(B)\left[\rho_1-G'(B)\right]\left[\rho_2-G'(B)\right]-\left[\lambda G''(B)-\delta''(B)\right]\left[\rho_2-G'(B)\right]\frac{\partial S(E,T)}{\partial\lambda}-\\
& -\left[\mu G''(B)+W''(B)\right]\left[\rho_1-G'(B)\right]\frac{\partial S(E,T)}{\partial\mu}
\end{aligned}$$

The Jacobian (9) possesses two purely imaginary eigenvalues $\pm i\sqrt{\omega}$ if the condition $\frac{\det J}{\text{tr}J}=\omega>0$ holds.

In the following we compute the value of ω as:

$$\omega = \rho_1\rho_2 - [G'(B)]^2 - [\lambda G''(B) - \delta''(B)]\frac{\partial S(E,T)}{\partial \lambda} - [\mu G''(B) + W''(B)]\frac{\partial S(E,T)}{\partial \mu}$$

A Hopf bifurcation can therefore only occur if the conditions $\omega > 0$ and the following

$$\rho_1[\lambda G''(B) - \delta''(B)]\frac{\partial S(E,T)}{\partial \lambda} + \rho_2[\mu G''(B) + W''(B)]\frac{\partial S(E,T)}{\partial \mu} = \\ = \rho_1\rho_2[\rho_1 + \rho_2 - 2G'(B)] \quad (10)$$

hold. In what follows we give specific forms in the functions of the model in order to extract some useful conclusions for periodic solutions.

3.3. Specifications of the model

We specify the functions involved as

Growth function of public debt : $G(B) = rB(1-B)$ (11.1)

The primary surplus function as a Cobb – Douglas type

$$S(E,T) = T^{\pi}E \quad (11.2)$$

The government's cost function due to the taxation as a linear function

$$a(T) = aT \quad (11.3)$$

The government's damage function $\delta(B)$ and the polluters' potential utility derived from the debt $W(B)$ in linear forms, respectively

$$\delta(B) = \delta B \quad (11.4)$$

$$W(B) = wB \quad (11.5)$$

Finally, the utility function derived from the emissions realization (on behalf of the group of polluters), we assume to be in the form

$$U(E) = \gamma - \frac{E^{\eta-1}}{1-\eta} \quad (11.6)$$

with $r,\ a,\ \delta,\ w,\ \gamma,\ b > 0,\ \ \pi,\ \eta \in (0,1)$

For the above specifications, the necessary condition for cyclical strategies is given from the next proposition.

Proposition 1

Given the specifications (11.1)-(11.6) *for the functions of the model, a necessary condition for cyclical strategies is that the government is more farsighted than the polluters; therefore the condition* $\rho_2 > \rho_1$ *has to hold.*

Proof

In the appendix A

4. The Stackelberg Setting

In this section we analyze the case in which the two players of the game move hierarchically and the rate of abatement pollution is chosen by the government before the group of polluters decides on the rate of their methods, thus the government is the leader.

4.1. The polluters as follower

We first consider the optimization problem for the follower, i.e. the group of polluters, which takes the action of the leader as given. Therefore the polluters face the following objective which is maximized, that is

$$\max_E \int_0^\infty e^{-\rho_2 t} \left(W(B) + U(E)\right) dt$$

Note that the maximization takes place with respect to the intensity of the emissions utilization, which means higher intensive use. The state variable evolves according to (1), for which the growth function is a linear function of the interest payment i.e. $G(B) = rB$, r being the interest rate. Therefore, the equation of motion of the public debt becomes $\dot{B} = rB - S(E,T)$. Moreover, we assume separability of the model through the separable utility function of the group of polluters. Therefore we assume that the potential utility enjoyed by the polluters would be in the form, $W(B) = wB$ and the utility derived from the intensive use of their emissions would be in the linear form $U(E) = \beta E$, as well.

The polluters' Hamiltonian current value, after the above simplifications is given by the following expression

$$H_2 = wB + \beta E + \mu\left[rB - S(E,T)\right]$$

The first order conditions for an interior solution w.r.t. the control E is therefore,

$$\frac{\partial H_2}{\partial E} = \beta - \mu S_E(E,T) = 0$$

and the specification for the primary surplus function the following: $S(E,T) = T^{\pi}E^{\varepsilon}$, $\varepsilon > 1$.

Next we get the optimal control function for the polluters (i.e. the emissions function), as follows

$$S_E = \varepsilon T^{\pi} E^{\varepsilon-1} = \frac{\beta}{\mu} \quad \Rightarrow \quad E^*(T) = \left[\frac{\beta}{\varepsilon\mu}\right]^{\frac{1}{\varepsilon-1}} T^{\frac{\pi}{1-\varepsilon}} \tag{12}$$

Now, the adjoint variable μ has to follow the differential equation

$$\dot{\mu} = \rho_2\mu - \frac{\partial H_2}{\partial B} = \mu\left[\rho_2 - r\right] - w \tag{13}$$

Substituting the follower's optimal control function (12) into the primary surplus function, we take the analytical form of the primary surplus function, as:

$$S\left(E^*(T),T\right) = T^{\pi}\left(E^*(T)\right)^{\varepsilon} = T^{\frac{\pi}{1-\varepsilon}}\left[\frac{\varepsilon\mu}{\beta}\right]^{\frac{\varepsilon}{1-\varepsilon}} \tag{14}$$

The readable expressions of (12) and (14) leads to the conclusion which says that, since $\varepsilon > 1$ and therefore $\frac{\pi}{1-\varepsilon} < 0$, an increase in the taxation measures on behalf the government T, results in a more cautious control on behalf the followers, $E^*(T)$. Similarly, (14) leads to a lower reduction of the public debt, i.e. $S(T, E^*(T))$ has a lower value.

4.2. The Government as regulator

Following Dockner *et al.* (2000) (mainly Chapter 5) we formulate the government's problem, for which the leader has to take into account the dynamics of the optimal decisions of the follower, expressed by the adjoint equation (13). Equation (13) now describes the evolution of the second state variable, therefore the leader's problem now is treated as an optimal control problem with two state variables. Moreover, combining with the early calculated analytical form (14), the leader's objective functional becomes (we assume that the damage function $\psi(E)$, due to emissions realization, in the form $\psi(E) = \psi E$)

$$\max_T \int_0^\infty e^{-\rho_1 t} \left(T^{\frac{\pi}{1-\varepsilon}} \left[\frac{\varepsilon\mu}{\beta} \right]^{\frac{\varepsilon}{1-\varepsilon}} - aT - \delta B - \psi \left[\frac{\varepsilon\mu}{\beta} \right]^{\frac{1}{1-\varepsilon}} T^{\frac{\pi}{1-\varepsilon}} \right) dt \qquad (15)$$

which is subject to both state dynamics, i.e. firstly the dynamics of the public debt and secondly the dynamics of the shadow price of emissions (due to the follower's maximization problem), i.e. maximization (15) is made subject to the following differential equations

$$\dot{B} = rB - T^{\frac{\pi}{1-\varepsilon}} \left[\frac{\varepsilon\mu}{\beta} \right]^{\frac{\varepsilon}{1-\varepsilon}} \qquad (16)$$

$$\dot{\mu} = \mu[\rho_2 - r] - w \tag{17}$$

The Hamiltonian current value of the above system (15) — (17) becomes

$$H_1 = T^{\frac{\pi}{1-\varepsilon}} \left\{ \left[\frac{\varepsilon\mu}{\beta}\right]^{\frac{\varepsilon}{1-\varepsilon}} (1-\lambda) - \psi \left[\frac{\varepsilon\mu}{\beta}\right]^{\frac{1}{1-\varepsilon}} \right\} - aT - \delta B + \lambda r B + \xi\left[\mu(\rho_2 - r) - w\right]$$

with λ, ξ to denote the adjoint variables of the state variables B, μ respectively. We note again that the shadow price μ for the polluters now becomes the new state variable for the government's problem.

Taking first order conditions we are able to express analytically the leader's optimal control T^* as a function of the adjoint variables λ, ξ, that is,

$$\frac{\partial H_1}{\partial T} = \frac{\pi}{1-\varepsilon} T^{\frac{\pi+\varepsilon-1}{1-\varepsilon}} \left[\frac{\varepsilon\mu}{\beta}\right]^{\frac{\varepsilon}{1-\varepsilon}} \left(1 - \lambda - \frac{\psi\varepsilon\mu}{\beta}\right) - a = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \; T^*(\lambda, \xi) = \left[\frac{a\beta(\varepsilon-1)}{\pi\left[\psi\varepsilon\mu + \beta(\lambda-1)\right]}\right]^{\frac{1-\varepsilon}{\pi+\varepsilon-1}} \left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{\varepsilon}{\pi+\varepsilon-1}} \tag{18}$$

Moreover, the above adjoint variables evolve according to the following differential equations

$$\dot{\lambda} = \rho_1\lambda - \frac{\partial H_1}{\partial B} = \lambda(\rho_1 - r) + \delta \tag{19}$$

$$\dot{\xi} = \rho_1\xi - \frac{\partial H_1}{\partial \mu} = \xi(\rho_1 - \rho_2 + r) - \frac{aT}{\pi\mu} \tag{20}$$

Note that, due to state separability of the model, the government's adjoint variable ξ, with respect to the follower's adjoint variable μ, has no influence on the leader's optimization problem.

The findings in the Stackelberg game are summarized in the following proposition.

Proposition 2.

In the Stackelberg game with the government as leader and the polluters as follower, a feasible solution exists, iff ψ *is sufficient large, i.e. iff*

$$\psi > \frac{\beta(1-\lambda)}{\varepsilon\mu} = \frac{\beta(\rho_1 - r + \delta)(\rho_2 - r)}{\varepsilon w(\rho_1 - r)} \quad \text{or} \quad \varepsilon > \frac{\beta(1-\lambda)}{\psi\mu} \tag{21}$$

The optimal strategies are then given by

$$T_S^* = \left[\frac{a\beta(\varepsilon-1)}{\pi\left[\psi\varepsilon\mu + \beta(\lambda-1)\right]}\right]^{\frac{1-\varepsilon}{\pi+\varepsilon-1}} \left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{\varepsilon}{\pi+\varepsilon-1}} \tag{22}$$

$$E_S^* = \left[\frac{a\beta(\varepsilon-1)}{\pi\left[\psi\varepsilon\mu + \beta(\lambda-1)\right]}\right]^{\frac{\pi}{\pi+\varepsilon-1}} \left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{1-\pi}{\pi+\varepsilon-1}} \tag{23}$$

the optimal primary surplus function is given by

$$S_S^*(E,T) = \left[\frac{a\beta(\varepsilon-1)}{\pi\left[\psi\varepsilon\mu + \beta(\lambda-1)\right]}\right]^{\frac{\pi}{\pi+\varepsilon-1}} \left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{\varepsilon}{\pi+\varepsilon-1}} \tag{24}$$

The steady state value of the public debt is given by

$$B_S^\infty(E,T) = \frac{1}{r}\left[\frac{a\beta(\varepsilon-1)}{\pi\left[\psi\varepsilon\mu + \beta(\lambda-1)\right]}\right]^{\frac{\pi}{\pi+\varepsilon-1}} \left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{\varepsilon}{\pi+\varepsilon-1}} \tag{25}$$

Proof

The values $(22)-(25)$ follow from further substitutions of (18) from the maximization condition $\frac{\partial H_1}{\partial E} = 0$ and from the steady state condition $\dot{B} = 0 \iff rB - S_S^*(E,T) = 0$.

The proposition below calculates the analytical expressions of the value functions for both players.

Proposition 3.

In the Stackelberg game the analytic forms of the objective functionals are given by

$$J_S^1 = \frac{a(1-\pi-\varepsilon)}{\pi\rho_1}\left[\frac{a\beta(\varepsilon-1)}{\pi\left[\psi\varepsilon\mu+\beta(\lambda-1)\right]}\right]^{\frac{1-\varepsilon}{\pi+\varepsilon-1}}\left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{\varepsilon}{\pi+\varepsilon-1}} - \frac{\delta B_0}{\rho_1 - r} \quad (26)$$

$$J_S^2 = \frac{\beta(\varepsilon-1)}{\varepsilon\rho_2}\left[\frac{a\beta(\varepsilon-1)}{\pi\left[\psi\varepsilon\mu+\beta(\lambda-1)\right]}\right]^{\frac{\pi}{\pi+\varepsilon-1}}\left(\frac{\beta}{\varepsilon\mu}\right)^{\frac{1-\pi}{\pi+\varepsilon-1}} + \frac{wB_0}{\rho_2 - r} \quad (27)$$

Proof

On request.

The next result is a policy making decision, according to which the shadow price of the emissions plays the regulatory role.

Proposition 4.

For the values of ψ such that

$$\frac{\beta(1-\lambda)}{\varepsilon\mu} < \psi < \frac{\beta(1-\lambda)}{\mu}$$

the regulator as leader act more cautiously and the follower more aggressively compared to the Nash case[1]. This leads to a higher value of the primary surplus and a higher profit of the Stackelberg follower, compared to the Nash case. For values of ψ larger than $\frac{\beta(1-\lambda)}{\mu}$, i.e. $\left(\psi > \frac{\beta(1-\lambda)}{\mu}\right)$, the government acts more aggressively and the polluters more cautiously compared to the Nash case, leading to a lower primary surplus function and a lower objective value for the follower compared to the Nash case.

Proof

On request.

[1] For the Nash differential game exposition see Halkos and Papageorgiou (2011).

Since ψ is the crucial variable which measures damages due to the intensive use of polluting mechanisms, it is obvious (from proposition 4) that for large values of ψ the regulator follows more truculent policy, but for small values of ψ the leader's policy is holding back.

4. Conclusions

The purpose of this paper was to investigate the dynamics of the public debt accumulation together with the actions undertaken in order to reduce debt's accumulation. For this purpose we setup a simple model of accumulation. We model the public debt as an accumulated stock first in a simultaneous (Nash) game. The Nash game takes place between the government which uses as control its taxation policy and the group of polluting firms using the intensity of their emissions as their control variable.

The economic analysis that follows in the game's solution, focused on cyclical policies, reveals the possibility of limit cycles between emissions rate and taxation rate. As a result, we found the sufficient condition for the cyclical policies existence. According to that result, it suffices, assuming different discount rates, the discount rate of the group of polluting firms to be greater than the discount rate of the government.

In the second setting, we extend the simultaneous move in a hierarchical (Stackelberg) differential game, for which the government undertakes the role of leader, while the polluting agents undertake the follower's role. In the above Stackelberg game, we first calculate the analytical expressions of each player's strategies and the analytical expression of the function that reduces the public debt, i.e. the primary surplus function.

We also found the analytical expressions of the value functions. The last proposition of the paper concerns the behavior of the primary surplus function. To be more precise, we found the interval between one crucial parameter of the model lies. If this parameter lies between certain values the primary surplus function takes higher values, leading therefore to higher level of utility for the follower, compared with the Nash case.

On the other hand, if the parameter takes a higher value than the threshold, the government acts more aggressively and the polluters more cautiously, leading to a lower abatement function and therefore to a lower objective value for the follower, in comparison to the Nash case.

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Appendix A

Proof of proposition 1.

With the specifications, given in subsection entitled "**3.3. Specifications of the model**", one can compute

$G'(B)=r(1-2B)$, $G''(B)=-2r$, $S_T(E,T)=\pi T^{\pi-1}$, $S_E(E,T)=T^{\pi}$, $a'(T)=a$, $U'(E)=E^{\eta-2}$, $\delta'(B)=\delta$, $W'(B)=w$

$$\frac{\partial H_1}{\partial T}=0 \Leftrightarrow (1-\lambda)S_T(E,T)=a'(T) \Leftrightarrow (1-\lambda)\pi T^{\pi-1}E=a \qquad (A.1)$$

$$\frac{\partial H_2}{\partial E}=0 \Leftrightarrow U'(E)=\mu S_E(E,T) \Leftrightarrow \mu T^{\pi}=E^{\eta-2} \qquad (A.2)$$

Combining $(A.1)$ and $(A.2)$ the optimal strategies take the following forms

$$T^*=\mu^{-1/[1+(1-\pi)(1-\eta)]}\left[\frac{a}{\pi(1-\lambda)}\right]^{(\eta-2)/[1+(1-\eta)(1-\pi)]} \qquad (A.3),$$

$$E^*=\mu^{(\pi-1)/[1+(1-\pi)(1-\eta)]}\left[\frac{a}{\pi(1-\lambda)}\right]^{\pi/[1+(1-\pi)(1-\eta)]} \qquad (A.4)$$

and the optimal surplus function becomes

$$S(E^*,T^*)=\mu^{-1/[1+(1-\pi)(1-\eta)]}\left[\frac{a}{\pi(1-\lambda)}\right]^{\pi(\eta-1)/[1+(1-\pi)(1-\eta)]} \qquad (A.5)$$

with the following partial derivatives

$$\frac{\partial S}{\partial\lambda}=\frac{\mu^{-1/[1+(1-\pi)(1-\eta)]}\left[\frac{a}{\pi(1-\lambda)}\right]^{\pi(\eta-1)/[1+(1-\pi)(1-\eta)]}}{(1-\lambda)}\frac{\pi(\eta-1)}{1+(1-\eta)(1-\pi)}= \qquad (A.6)$$
$$=\frac{S(E^*,T^*)}{(1-\lambda)}\frac{\pi(\mu-1)}{1+(1-\eta)(1-\pi)}$$

$$\frac{\partial S}{\partial\mu}=\frac{\mu^{-1/[1+(1-\pi)(1-\eta)]}\left[\frac{a}{\pi(1-\lambda)}\right]^{\pi(\eta-1)/[1+(1-\pi)(1-\eta)]}}{\lambda_2}\frac{-1}{1+(1-\eta)(1-\pi)}= \qquad (A.7)$$
$$=\frac{S(E^*,T^*)}{\mu}\frac{-1}{1+(1-\eta)(1-\pi)}$$

Both derivatives $(A.6)$, $(A.7)$ are negatives due to the assumptions on the parameters $\pi,\ \eta\in(0,1)$ and on the signs of the functions derivates, that is

$S_T > 0,\ S_E > 0,\ W'(B) > 0,\ \delta'(B) > 0$, which ensures the positive sign of the adjoints $\lambda,\ \mu$.

The bifurcation condition $\omega = \dfrac{\det(J)}{\mathrm{tr}(J)}$ now becomes

$\rho_1\rho_2\left[\rho_1+\rho_2-2G'(B)\right]=\lambda\rho_1G''(B)\dfrac{\partial S}{\partial\lambda}+\mu\rho_2G''(B)\dfrac{\partial S}{\partial\mu}$, which after substituting the values from $(\mathrm{A}.6)$, $(\mathrm{A}.7)$ and making the rest of algebraic manipulations, finally yields (at the steady states)

$$\frac{S(E_\infty,T_\infty)G''(B)}{1+(1-\eta)(1-\pi)}\left[\rho_1\pi(1-\eta)\frac{\delta}{\delta+G'(B)-\rho_1}-\rho_2\right]-\rho_1\rho_2\left[\rho_1+\rho_2-2G'(B)\right]=0 \quad (\mathrm{A}.8)$$

Where we have set $\dfrac{\lambda}{1-\lambda}=\dfrac{\delta}{\rho_1-G'(B)-\delta}$ stemming from the adjoint equation $\dot{\lambda}=\lambda\left(\rho_1-G'(B)\right)-\delta'(B)$, which at the steady states reduces into $\lambda=\delta'(B)/\left(\rho_1-G'(B)\right)$.

Condition $w>0$ after substitution the values from $(\mathrm{A}.6)$, $(\mathrm{A}.7)$ becomes

$$w=\rho_1\rho_2-\left[G'(B)\right]^2+\frac{S(E,T)G''(B)}{1+(1-\eta)(1-\pi)}\left[\pi(1-\eta)\frac{-\delta}{G'(B)+\delta-\rho_1}+1\right]>0 \qquad (\mathrm{A}.9)$$

The division of $(\mathrm{A}.8)$ by ρ_1 yields

$$\frac{S(E_\infty,T_\infty)G''(B)}{1+(1-\eta)(1-\pi)}\left[\pi(1-\eta)\frac{\delta}{\delta+G'(B)-\rho_1}-\frac{\rho_2}{\rho_1}\right]-\rho_2\left[\rho_1+\rho_2-2G'(B)\right]=0 \quad (\mathrm{A}.10)$$

The sum $(\mathrm{A}.9)+(\mathrm{A}.10)$ must be positive, thus after simplifications and taking into account that (at the steady state) $S(E_\infty,T_\infty)=G(B)$, we have:

$G(B)G''(B)\dfrac{\rho_1-\rho_2}{\rho_1\left[1+(1-\eta)(1-\pi)\right]}>\left[\rho_2-G'(B)\right]^2$ and the result $\rho_2>\rho_1$ follows from the strict concavity of the growth function $G''<0$, since it is assumed $0<\eta<1$ and $0<\pi<1$, as well.