Sion’s minimax theorem and Nash equilibrium of symmetric multi-person zero-sum game

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Abstract

We will show that Sion’s minimax theorem is equivalent to the existence of Nash equilibrium in a symmetric multi-person zero-sum game. If a zero-sum game is asymmetric, maximin strategies and minimax strategies of players do not correspond to Nash equilibrium strategies. However, if it is symmetric, the maximin strategy and the minimax strategy constitute a Nash equilibrium.

Keywords:
multi-person zero-sum game, Nash equilibrium, Sion’s minimax theorem.

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1. Introduction

We consider the relation between Sion’s minimax theorem and the existence of Nash equilibrium in a symmetric multi-person zero-sum game. We will show that they are equivalent. An example of such a game is a relative profit maximization game in a Cournot oligopoly. Suppose that there are \( n \geq 3 \) firms in an oligopolistic industry. Let \( \bar{\pi}_i \) be the absolute profit of the \( i \)-th firm. Then, its relative profit is

\[
\pi_i = \bar{\pi}_i - \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} \bar{\pi}_j.
\]

We see

\[
\sum_{i=1}^{n} \pi_i = \sum_{i=1}^{n} \bar{\pi}_i - \frac{1}{n-1} (n-1) \sum_{j=1}^{n} \bar{\pi}_j = 0.
\]

Thus, the relative profit maximization game in a Cournot oligopoly is a zero-sum game\(^2\). If the oligopoly is asymmetric because the demand function is not symmetric or firms have different cost functions, maximin strategies and minimax strategies of firms do not correspond to Nash equilibrium strategies. However, if the demand function is symmetric and the firms have the same cost function, the maximin strategy and the minimax strategy constitute a Nash equilibrium.

2. The model

Consider a symmetric \( n \)-person zero-sum game with \( n \geq 3 \) as follows. There are \( n \) players, \( 1, 2, \ldots, n \). The set of players is denoted by \( N \). A vector of strategic variables is \((s_1, s_2, \ldots, s_n) \in S_1 \times S_2 \times \cdots \times S_n \). \( S_i \) is a convex and compact set in a linear topological space for each \( i \in N \). The payoff functions of the players are \( u_i(s_1, s_2, \ldots, s_n) \) for \( i \in N \). We assume

\( u_i \) for each \( i \in N \) is upper semi-continuous and quasi-concave on \( S_i \)

for each \( s_j \in S_j, j \in N, j \neq i \). It is lower semi-continuous and quasi-convex on \( S_j \) for \( j \in N, j \neq i \) for each \( s_i \in S_i \).

Since the game is symmetric, all players have the same payoff function, and we have
\[ \sum_{i=1}^{n} u_i(s_1, s_2, \ldots, s_n) = 0, \] for given \((s_1, s_2, \ldots, s_n)\) because we consider a zero-sum game. Also all \(S_i\)'s are identical. Denote them by \(S\).

3. The main results

Sion’s minimax theorem (Sion (1958), Komiya (1988), Kindler (2005)) is stated as follows.

**Lemma 1** (Sion’s minimax theorem). Let \(X\) and \(Y\) be non-void convex and compact subsets of two linear topological spaces, and let \(f : X \times Y \to \mathbb{R}\) be a function that is upper semi-continuous and quasi-concave in the first variable and lower semi-continuous and quasi-convex in the second variable. Then
\[
\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).
\]

We follow the description of this theorem in Kindler (2005).

Suppose that \(s_k \in S_k\) for all \(k \in \mathbb{N}\) other than \(i\) and \(j, j \neq i\) are given. Denote a vector of such \(s_k\)'s by \(s_{-i,j}\). Then, \(u_i(s_1, s_2, \ldots, s_n)\) is written as \(u_i(s_i, s_j, s_{-i,j})\), and it is a function of \(s_i\) and \(s_j\). We can apply Lemma 1 to such a situation, and get the following lemma.

**Lemma 2.** Let \(j \neq i\), and \(S_i\) and \(S_j\) be non-void convex and compact subsets of two linear topological spaces, and let \(u_i : S_i \times S_j \to \mathbb{R}\) be a function that is upper semi-continuous and quasi-concave on \(S_i\) and lower semi-continuous and quasi-convex on \(S_j\). Then
\[
\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, s_{-i,j}) = \min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j, s_{-i,j}).
\]

We assume that \(\arg \max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, s_{-i,j})\) and \(\arg \min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j, s_{-i,j})\) are single-valued for any pair of \(i\) and \(j\). By the maximum theorem they are continuous in \(s_{-i,j}\).
Consider the following function;
\[
\begin{align*}
\left(\begin{array}{c}
s_1 \\
s_2 \\
\vdots \\
s_n \\
\end{array}\right) & \rightarrow \left\{ \begin{array}{c}
\arg \max_{s_1 \in S_1} \min_{s_2 \in S_2} u_1(s_1, s_2, s_{-1,2}) \\
\arg \max_{s_2 \in S_2} \min_{s_3 \in S_3} u_2(s_2, s_3, s_{-2,3}) \\
\vdots \\
\arg \max_{s_n \in S_n} \min_{s_{i-1} \in S_{i-1}} u_n(s_n, s_{1,i}, s_{-1,n}) \\
\end{array} \right\},
\end{align*}
\]
given \((s_1, s_2, \ldots, s_n)\). This function is continuous, and each \(S_i\) is convex and compact. Therefore, there exists a fixed point \((\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n)\) (by Glicksberg’s fixed point theorem (Glicksberg (1952))).

Similarly, we can consider the following function;
\[
\begin{align*}
\left(\begin{array}{c}
s_1 \\
s_2 \\
\vdots \\
s_n \\
\end{array}\right) & \rightarrow \left\{ \begin{array}{c}
\arg \min_{s_1 \in S_1} \max_{s_2 \in S_2} u_2(s_1, s_2, s_{-1,2}) \\
\arg \min_{s_2 \in S_2} \max_{s_3 \in S_3} u_3(s_2, s_3, s_{-2,3}) \\
\vdots \\
\arg \min_{s_n \in S_n} \max_{s_{i-1} \in S_{i-1}} u_n(s_n, s_{1,i}, s_{-1,n}) \\
\end{array} \right\},
\end{align*}
\]
given \((s_1, s_2, \ldots, s_n)\). This function also has a fixed point, \((\tilde{s}_1', \tilde{s}_2', \ldots, \tilde{s}_n')\).

Since we consider a symmetric game in which all players have the same payoff function, we can assume that when \(s_{-i,j} = s_{-k,l}\),
\[
\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, s_{-i,j}) = \max_{s_k \in S} \min_{s_l \in S} u_k(s_k, s_l, s_{-k,l}) = \min_{s_i \in S} \max_{s_j \in S} u_i(s_i, s_j, s_{-i,j}),
\]
and
\[
\arg \max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, s_{-i,j}) = \arg \max_{s_k \in S} \min_{s_l \in S} u_k(s_k, s_l, s_{-k,l}) = \arg \min_{s_i \in S} \max_{s_j \in S} u_i(s_i, s_j, s_{-i,j})
\]
for \(i, j, k, l \in N\).

They mean
\[
\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, s_{-i,j}) = \max_{s_j \in S} \min_{s_i \in S} u_j(s_i, s_j, s_{-i,j})
\]
and
\[
\arg \max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, s_{-i,j}) = \arg \max_{s_j \in S} \min_{s_i \in S} u_j(s_i, s_j, s_{-i,j})
\]
for any \(i, j\).
Then, we find \((\tilde{s}_1', \tilde{s}_2', \ldots, \tilde{s}_n') = (\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n)\). Let \(s = (s, s, \ldots, s)\). If \((s_1, s_2, \ldots, s_n) = s\), all arg max \(s_j \in S \min_{s_j \in S} u_i(s, s_j, s, \ldots, s)\)'s and all arg min \(s_j \in S \max_{s_j \in S} u_j(s, s_j, s, \ldots, s)\)'s are the same. Thus, the fixed point obtained from above two functions is symmetric. Denote it by \(\tilde{s} = (\tilde{s}, \tilde{s}, \ldots, \tilde{s})\). These arguments ensure the existence of symmetric maximin and minimax strategies.

Summarizing the results, Sion’s minimax theorem for a symmetric multi-person zero-sum game is stated as follows.

**Theorem 1.** Let \(S_i^j's \) for \(i \in N \) be non-void convex and compact subsets of linear topological spaces, let \(u_i : S_i \times S_j \rightarrow \mathbb{R} \) given \(\tilde{s}_{-i,j} \) be a function that is upper semi-continuous and quasi-concave on \(S_i \) and lower semi-continuous and quasi-convex on \(S_j \) for all \(j \neq i \) and \(i \in N \), and \(S_i = S \) for all \(i \in N \). Then, there exists \(\tilde{s} = (\tilde{s}, \tilde{s}, \ldots, \tilde{s})\) such that

\[
\max_{s \in S} \min_{s_j \in S} u_i(s, s_j, \tilde{s}_{-i,j}) = \max_{s_j \in S} \min_{s \in S} u_j(s, s_j, \tilde{s}_{-i,j})
\]

and

\[
\arg \max_{s \in S} \min_{s_j \in S} u_i(s, s_j, \tilde{s}_{-i,j}) = \arg \max_{s \in S} \min_{s_j \in S} u_j(s, s_j, \tilde{s}_{-i,j})
\]

where \(\tilde{s}_{-i,j} = (\tilde{s}, \tilde{s}, \ldots, \tilde{s})\) for \(k \in N \), \(k \neq i, j\).

Now we consider a Nash equilibrium of a symmetric multi-person zero-sum game. Let \(s_i^*, i \in N\), be the values of \(s_i\)'s which, respectively, maximize \(u_i\), \(i \in N\), given \(s_j, j \neq i\), in a neighborhood around \((s_1^*, s_2^*, \ldots, s_n^*)\) in \(S_1 \times S_2 \times \cdots \times S_n\). Then,

\[
u_i(s_1^*, s_2^*, \ldots, s_n^*) \geq u_i(s_1^*, s_i, s_2^*, \ldots, s_n^*) \text{ for all } s_i \neq s_i^*, i \in N.
\]

Since the game is symmetric, we consider a symmetric equilibrium such that all \(s_i^*\)'s are equal at equilibria. Thus, \(u_i(s_1^*, s_2^*, \ldots, s_n^*)\)'s for all \(i\) are equal, and by the property of zero-sum game they are zero. By symmetry of the game we have

\[
u_j(s_1^*, s_i, \ldots, s_n^*) = u_k(s_1^*, s_i, \ldots, s_n^*) \text{ for } j \neq i, k \neq i, j \neq k.
\]

From this and (1)

\[
- \sum_{j=1, j \neq i}^n u_j(s_1^*, \ldots, s_i, \ldots, s_n^*) = -(n-1)u_j(s_1^*, \ldots, s_i, \ldots, s_n^*) = u_i(s_1^*, \ldots, s_i, \ldots, s_n^*).
\]
Therefore, from (2)

\[ u_j(s_1^r, \ldots, s_i, \ldots, s_n^r) \geq u_j(s_1^r, \ldots, s_i^r, \ldots, s_n^r) \text{ for } j \neq i. \]

By symmetry

\[ u_i(s_1^r, \ldots, s_j, \ldots, s_n^r) \geq u_i(s_1^r, \ldots, s_j^r, \ldots, s_n^r) \text{ for } j \neq i. \]

Combining this and (2)

\[ u_i(s_1^r, \ldots, s_i, \ldots, s_n^r) \leq u_i(s_1^r, \ldots, s_i^r, \ldots, s_n^r) \leq u_i(s_1^r, \ldots, s_j, \ldots, s_n^r) \]

for all \( s_i \neq s_i^* \) and all \( s_j \neq s_j^* \), \( j \neq i, \ i \in N \).

This is equivalent to

\[ u_i(s_1^r, \ldots, s_i^r, \ldots, s_n^r) = \max_{s_i} u_i(s_1^r, \ldots, s_i, \ldots, s_n^r) = \min_{s_j} u_i(s_1^r, \ldots, s_j, \ldots, s_n^r), \]

\( j \neq i \) given \( s_i^* \), \( k \neq i, j \).

Denote the symmetric Nash equilibrium of the zero-sum game by \( s^* = (s^*, s^*, \ldots, s^*) \). Let \( \bar{s}_{-i,j} = (\bar{s}^*, \bar{s}^*, \ldots, \bar{s}^*) \) for \( k \in N, k \neq i, j \), and \( \bar{s}_{-i,j} = (\bar{s}, \bar{s}, \ldots, \bar{s}) \) for \( k \in N, k \neq i, j \). We can show the following result.

**Theorem 2.** The following three statements are equivalent.

1. There exists a symmetric Nash equilibrium in a symmetric multi-person zero-sum game.
2. There exists \( \bar{s} = (\bar{s}, \bar{s}, \ldots, \bar{s}) \) such that the following relation holds.

\[ v_i^j \equiv \max_{s_i} \min_{s_j} u_i(s_i, s_j, \bar{s}_{-i,j}) = \min_{s_j} \max_{s_i} u_i(s_i, s_j, \bar{s}_{-i,j}) \equiv v_j^i \text{ for any pair of } i \text{ and } j. \]

3. There exists a real number \( v_s, s_i^m \) and \( s_j^m \) such that

\[ u_i(s_i^m, s_j, \bar{s}_{-i,j}) \geq v_s \text{ for any } s_j, \text{ and } u_i(s_i, s_j^m, \bar{s}_{-i,j}) \leq v_s \text{ for any } s_i, \]

for any pair of \( i \) and \( j \).

**Proof.** \((1 \rightarrow 2)\)

Set \( \bar{s} = s^* \). Then,

\[ v_i^j = \min_{s_j} \max_{s_i} u_i(s_i, s_j, s_{-i,j}^*) \leq \max_{s_i} u_i(s_i, s_j, s_{-i,j}^*) = u_i(s_i^*, s_j, s_{-i,j}^*) \]

\[ = \min_{s_j} u_i(s_i^*, s_j, s_{-i,j}^*) \leq \max_{s_i} u_i(s_i, s_j, s_{-i,j}^*) = v_i^j. \]
On the other hand, \( \min_{s_j} u_i(s_i, s_j, s_{-i,j}^*) \leq u_i(s_i, s_j, s_{-i,j}^*) \), then \( \max_{s_i} \min_{s_j} u_i(s_i, s_j, s_{-i,j}^*) \leq \max_{s_i} u_i(s_i, s_j, s_{-i,j}^*) \) and so \( \max_{s_i} \min_{s_j} u_i(s_i, s_j, s_{-i,j}^*) \leq \min_{s_j} \max_{s_i} u_i(s_i, s_j, s_{-i,j}^*) \).

Thus, \( v_i^* \leq v_j^* \), and we have \( v_i^* = v_j^* \).

(2 \( \rightarrow \) 3)

Set \( \bar{s} = s^* \). Let \( s_i^m = \arg \max_{s_i} \min_{s_j} u_i(s_i, s_j, s_{-i,j}^*) \) (the maximin strategy), \( s_j^m = \arg \min_{s_j} \max_{s_i} u_i(s_i, s_j, s_{-i,j}^*) \) (the minimax strategy), and let \( v_s = v_i^* = v_j^* \).

Then, we have

\[
\begin{align*}
 u_i(s_i^m, s_j, s_{-i,j}^*) & \geq \min_{s_j} u_i(s_i^m, s_j, s_{-i,j}^*) = \max_{s_i} \min_{s_j} u_i(s_i, s_j, s_{-i,j}^*) = v_s \\
 & = \min_{s_j} \max_{s_i} u_i(s_i, s_j, s_{-i,j}^*) = \max_{s_i} \min_{s_j} u_i(s_i, s_j, s_{-i,j}^*) \geq u_i(s_i, s_j^m, s_{-i,j}^*).
\end{align*}
\]

By Theorem 1 \( s_i^m = \bar{s} = s^* \) and \( s_j^m = \bar{s} = s^* \).

(3 \( \rightarrow \) 1)

Set \( \bar{s} = s^* \). Since \( s_i^m = s_j^m = s^* \) from (3) we get

\[
 u_i(s^*, s_j, s_{-i,j}^*) \geq v_s \geq u_i(s_i, s^*, s_{-i,j}^*) \quad \text{for all} \ s_i \in S_i, \ s_j \in S_j.
\]

Putting \( s_i = s_i^* \) and \( s_j = s_j^* \), we see \( v_s = u_i(s^*, s^*, s_{-i,j}^*) \) and \( s^* = (s^*, s^*, \ldots, s^*) \) is an equilibrium. \( \square \)

Therefore, Sion’s minimax theorem is equivalent to the existence of Nash equilibrium of a symmetric multi-person zero-sum game.

4. Example of asymmetric multi-person zero-sum game

Consider a three-person game. Suppose that the payoff functions of players are

\[
\begin{align*}
\pi_1 &= (a-s_1-(s_2+s_3)s_1-c_1s_1-\frac{1}{2}[(a-s_2-(s_1+s_3)s_2-c_2s_2+(a-s_3-(s_2+s_1)s_3-c_3s_3)], \\
\pi_2 &= (a-s_2-(s_1+s_3)s_2-c_2s_2-\frac{1}{2}[(a-s_1-(s_2+s_3)s_1-c_1s_1+(a-s_3-(s_2+s_1)s_3-c_3s_3)], \\
\pi_3 &= (a-s_3-(s_2+s_1)s_3-c_3s_3-\frac{1}{2}[(a-s_1-(s_2+s_3)s_1-c_1s_1+(a-s_2-(s_1+s_3)s_2-c_2s_2].
\end{align*}
\]

This is a model of relative profit maximization in a three firms Cournot oligopoly with constant marginal cost and zero fixed cost producing a homogeneous good.
\( s_i, i = 1, 2, 3 \) are the outputs of the firms. The conditions for maximization of \( \pi_i, i = 1, 2, 3 \) are

\[
\frac{\partial \pi_1}{\partial s_1} = a - 2s_1 - (s_2 + s_3) - c_1 + \frac{1}{2}(s_2 + s_3) = 0,
\]

\[
\frac{\partial \pi_2}{\partial s_2} = a - 2s_2 - (s_1 + s_3) - c_2 + \frac{1}{2}(s_1 + s_3) = 0,
\]

and

\[
\frac{\partial \pi_3}{\partial s_3} = a - 2s_3 - (s_2 + s_1) - c_3 + \frac{1}{2}(s_2 + s_1) = 0.
\]

The Nash equilibrium strategies are

\[
s_1 = \frac{3a - 5c_1 + c_2 + c_3}{9}, \quad s_2 = \frac{3a - 5c_2 + c_1 + c_3}{9}, \quad s_3 = \frac{3a - 5c_3 + c_2 + c_1}{9}. \quad (4)
\]

We consider maximin and minimax strategy about Player 1 and 2. The condition for minimization of \( \pi_1 \) with respect to \( s_2 \) is \( \frac{\partial \pi_1}{\partial s_2} = 0 \). Denote \( s_2 \) which satisfies this condition by \( s_2(s_1, s_3) \), and substitute it into \( \pi_1 \). Then, the condition for maximization of \( \pi_1 \) with respect to \( s_1 \) given \( s_2(s_1, s_3) \) and \( s_3 \) is

\[
\frac{\partial \pi_1}{\partial s_1} + \frac{\partial \pi_1}{\partial s_2} \frac{ds_2}{ds_1} = 0.
\]

We call the strategy of Player 1 obtained from these conditions the maximin strategy of Player 1 to Player 2. It is denoted by \( \arg \max_{s_1} \min_{s_2} \pi_1 \). The condition for maximization of \( \pi_1 \) with respect to \( s_1 \) is \( \frac{\partial \pi_1}{\partial s_1} = 0 \). Denote \( s_1 \) which satisfies this condition by \( s_1(s_2, s_3) \), and substitute it into \( \pi_1 \). Then, the condition for minimization of \( \pi_1 \) with respect to \( s_2 \) given \( s_1(s_2, s_3) \) is

\[
\frac{\partial \pi_1}{\partial s_2} + \frac{\partial \pi_1}{\partial s_1} \frac{ds_1}{ds_2} = 0.
\]

We call the strategy of Player 2 obtained from these conditions the minimax strategy of Player 2 to Player 1. It is denoted by \( \arg \min_{s_2} \max_{s_1} \pi_1 \). In our example we obtain

\[
\arg \max_{s_1} \min_{s_2} \pi_1 = \frac{3a - 4c_1 + c_2}{9}, \quad \arg \min_{s_2} \max_{s_1} \pi_1 = \frac{6a - 9s_3 - 2c_1 - 4c_2}{9}.
\]

Similarly, we get the following results.

\[
\arg \max_{s_2} \min_{s_1} \pi_2 = \frac{3a - 4c_2 + c_1}{9}, \quad \arg \min_{s_1} \max_{s_2} \pi_2 = \frac{6a - 9s_3 - 2c_2 - 4c_1}{9},
\]

\[
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\]
Further, if
\[
\arg\max_{s_1} \arg\min_{s_3} \pi_1 = \frac{3a - 4c_1 + c_3}{9}, \quad \arg\min_{s_1} \arg\max_{s_3} \pi_1 = \frac{6a - 9s_2 - 2c_1 - 4c_3}{9}, \\
\arg\max_{s_3} \arg\min_{s_1} \pi_3 = \frac{3a - 4c_3 + c_1}{9}, \quad \arg\min_{s_3} \arg\max_{s_1} \pi_3 = \frac{6a - 9s_2 - 2c_3 - 4c_1}{9}, \\
\arg\max_{s_2} \arg\min_{s_3} \pi_2 = \frac{3a - 4c_2 + c_3}{9}, \quad \arg\min_{s_2} \arg\max_{s_3} \pi_2 = \frac{6a - 9s_1 - 2c_2 - 4c_3}{9}, \\
\arg\max_{s_3} \arg\min_{s_2} \pi_3 = \frac{3a - 4c_3 + c_2}{9}, \quad \arg\min_{s_3} \arg\max_{s_2} \pi_3 = \frac{6a - 9s_1 - 2c_3 - 4c_2}{9}.
\]
If the game is asymmetric, for example, \(c_2 \neq c_3\), \(\arg\max_{s_1} \min_{s_2} \pi_1 \neq \arg\max_{s_1} \min_{s_3} \pi_1\), \(\arg\max_{s_1} \min_{s_2} \pi_2 \neq \arg\max_{s_1} \min_{s_2} \pi_3\), \(\arg\min_{s_3} \max_{s_2} \pi_2 \neq \arg\min_{s_3} \max_{s_2} \pi_3\) and so on. However, if the game is symmetric, we have \(c_2 = c_3 = c_1\) and
\[
\arg\max_{s_1} \arg\min_{s_2} \pi_1 = \arg\max_{s_2} \arg\min_{s_1} \pi_2 = \arg\max_{s_1} \arg\min_{s_3} \pi_3 = \arg\max_{s_3} \arg\min_{s_1} \pi_2
\]
\[
= \arg\max_{s_2} \arg\min_{s_1} \pi_2 = \arg\max_{s_1} \arg\min_{s_3} \pi_3 = \frac{a - c_1}{3}.
\]
All of the Nash equilibrium strategies of the players in (4) are also equal to \(\frac{a-c_1}{3}\).
Assume \(s_2 = s_3 = s_1\) as well as \(c_2 = c_3 = c_1\). Then,
\[
\arg\min_{s_2} \arg\max_{s_1} \pi_1 = \arg\min_{s_1} \arg\max_{s_2} \pi_2 = \arg\min_{s_1} \arg\max_{s_3} \pi_1 = \arg\min_{s_3} \arg\max_{s_1} \pi_3
\]
\[
= \arg\min_{s_3} \arg\max_{s_2} \pi_2 = \arg\min_{s_2} \arg\max_{s_3} \pi_3 = \frac{2a - 3s_1 - 2c_1}{3}.
\]
Further, if
\[
s_1 = \arg\min_{s_1} \arg\max_{s_2} \pi_2 = \arg\min_{s_1} \arg\max_{s_3} \pi_3,
\]
we obtain
\[
\arg\min_{s_2} \arg\max_{s_1} \pi_1 = \arg\min_{s_1} \arg\max_{s_2} \pi_2 = \arg\min_{s_1} \arg\max_{s_3} \pi_1 = \arg\min_{s_3} \arg\max_{s_1} \pi_3
\]
\[
= \arg\min_{s_3} \arg\max_{s_2} \pi_2 = \arg\min_{s_2} \arg\max_{s_3} \pi_3 = \frac{a - c_1}{3}.
\]
Therefore, the maximin strategy, the minimax strategy and the Nash equilibrium strategy for all players are equal.

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References


