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Abstract

We will show that Sion's minimax theorem is equivalent to the existence of Nash equilibrium in a symmetric multi-person zero-sum game. If a zero-sum game is asymmetric, maximin strategies and minimax strategies of players do not correspond to Nash equilibrium strategies. However, if it is symmetric, the maximin strategy and the minimax strategy constitute a Nash equilibrium.

Keywords:
multi-person zero-sum game, Nash equilibrium, Sion's minimax theorem.

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1. Introduction

We consider the relation between Sion's minimax theorem and the existence of Nash equilibrium in a symmetric multi-person zero-sum game. We will show that they are equivalent. An example of such a game is a relative profit maximization game in a Cournot oligopoly. Suppose that there are $n \geq 3$ firms in an oligopolistic industry. Let $\bar{\pi}_i$ be the absolute profit of the i-th firm. Then, its relative profit is

$$\pi_i = \bar{\pi}_i - \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} \bar{\pi}_j.$$

We see

$$\sum_{i=1}^{n} \pi_i = \sum_{i=1}^{n} \bar{\pi}_i - \frac{1}{n-1}(n-1) \sum_{j=1}^{n} \bar{\pi}_j = 0.$$

Thus, the relative profit maximization game in a Cournot oligopoly is a zero-sum game[2]. If the oligopoly is asymmetric because the demand function is not symmetric or firms have different cost functions, maximin strategies and minimax strategies of firms do not correspond to Nash equilibrium strategies. However, if the demand function is symmetric and the firms have the same cost function, the maximin strategy and the minimax strategy constitute a Nash equilibrium.

2. The model

Consider a symmetric n-person zero-sum game with $n \geq 3$ as follows. There are n players, $1, 2, \dots, n$. The set of players is denoted by N. A vector of strategic variables is $(s_1, s_2, \dots, s_n) \in S_1 \times S_2 \times \cdots \times S_n$. S_i is a convex and compact set in a linear topological space for each $i \in N$. The payoff functions of the players are $u_i(s_1, s_2, \dots, s_n)$ for $i \in N$. We assume

> u_i for each $i \in N$ is upper semi-continuous and quasi-concave on S_i for each $s_j \in S_j$, $j \in N$, $j \neq i$. It is lower semi-continuous and quasi-convex on S_j for $j \in N$, $j \neq i$ for each $s_i \in S_i$.

[2] About relative profit maximization under imperfect competition please see Matsumura, Matsushima and Cato (2013), Satoh and Tanaka (2013), Satoh and Tanaka (2014a), Satoh and Tanaka (2014b), Tanaka (2013a), Tanaka (2013b) and Vega-Redondo (1997)

Since the game is symmetric, all players have the same payoff function, and we have

$$\sum_{i=1}^{n} u_i(s_1, s_2, \dots, s_n) = 0, \tag{1}$$

for given $(s_1, s_2, \dots, s_n)$ because we consider a zero-sum game. Also all S_i's are identical. Denote them by S.

3. The main results

Sion's minimax theorem (Sion (1958), Komiya (1988), Kindler (2005)) is stated as follows.

Lemma 1 (Sion's minimax theorem)**.** *Let X and Y be non-void convex and compact subsets of two linear topological spaces, and let $f : X \times Y \to \mathbb{R}$ be a function that is upper semi-continuous and quasi-concave in the first variable and lower semi-continuous and quasi-convex in the second variable. Then*

$$\max_{x\in X} \min_{y\in Y} f(x,y) = \min_{y\in Y} \max_{x\in X} f(x,y).$$

We follow the description of this theorem in Kindler (2005).

Suppose that $s_k \in S_k$ for all $k \in N$ other than i and j, $j \neq i$ are given. Denote a vector of such s_k's by $\mathbf{s}_{-i,j}$. Then, $u_i(s_1, s_2, \dots, s_n)$ is written as $u_i(s_i, s_j, \mathbf{s}_{-i,j})$, and it is a function of s_i and s_j. We can apply Lemma 1 to such a situation, and get the following lemma.

Lemma 2. *Let $j \neq i$, and S_i and S_j be non-void convex and compact subsets of two linear topological spaces, and let $u_i : S_i \times S_j \to \mathbb{R}$ given $\mathbf{s}_{-i,j}$ be a function that is upper semi-continuous and quasi-concave on S_i and lower semi-continuous and quasi-convex on S_j. Then*

$$\max_{s_i\in S_i} \min_{s_j\in S_j} u_i(s_i, s_j, \mathbf{s}_{-i,j}) = \min_{s_j\in S_j} \max_{s_i\in S_i} u_i(s_i, s_j, \mathbf{s}_{-i,j}).$$

We assume that $\arg\max_{s_i\in S_i} \min_{s_j\in S_j} u_i(s_i, s_j, \mathbf{s}_{-i,j})$ and $\arg\min_{s_j\in S_j} \max_{s_i\in S_i} u_i(s_i, s_j, \mathbf{s}_{-i,j})$ are single-valued for any pair of i and j. By the maximum theorem they are continuous in $\mathbf{s}_{-i,j}$.

Consider the following function;

$$\begin{pmatrix} s_1 \\ s_2 \\ \dots \\ s_n \end{pmatrix} \to \begin{pmatrix} \arg\max_{s_1 \in S_1} \min_{s_2 \in S_2} u_1(s_1, s_2, \mathbf{s}_{-1,2}) \\ \arg\max_{s_2 \in S_2} \min_{s_3 \in S_3} u_2(s_2, s_3, \mathbf{s}_{-2,3}) \\ \dots \\ \arg\max_{s_n \in S_n} \min_{s_1 \in S_1} u_n(s_n, s_1, \mathbf{s}_{-1,n}) \end{pmatrix},$$

given $(s_1, s_2, \dots, s_n)$. This function is continuous, and each S_i is convex and compact. Therefore, there exists a fixed point $(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n)$ (by Glicksberg's fixed point theorem (Glicksberg (1952))).

Similarly, we can consider the following function;

$$\begin{pmatrix} s_1 \\ s_2 \\ \dots \\ s_n \end{pmatrix} \to \begin{pmatrix} \arg\min_{s_1 \in S_1} \max_{s_2 \in S_1} u_2(s_1, s_2, \mathbf{s}_{-1,2}) \\ \arg\min_{s_2 \in S_2} \max_{s_3 \in S_3} u_3(s_2, s_3, \mathbf{s}_{-2,3}) \\ \dots \\ \arg\min_{s_n \in S_n} \max_{s_1 \in S_1} u_1(s_n, s_1, \mathbf{s}_{-1,n}) \end{pmatrix},$$

given $(s_1, s_2, \dots, s_n)$. This function also has a fixed point, $(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_n)$.

Since we consider a symmetric game in which all players have the same payoff function, we can assume that when $\mathbf{s}_{-i,j} = \mathbf{s}_{-k,l}$,

$$\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}) = \max_{s_k \in S} \min_{s_l \in S} u_k(s_k, s_l, \mathbf{s}_{-k,l}) = \min_{s_l \in S} \max_{s_k \in S} u_k(s_k, s_l, \mathbf{s}_{-k,l})$$
$$= \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}),$$

and

$$\arg\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}) = \arg\max_{s_k \in S} \min_{s_l \in S} u_k(s_k, s_l, \mathbf{s}_{-k,l}) = \arg\min_{s_l \in S} \max_{s_k \in S} u_k(s_k, s_l, \mathbf{s}_{-k,l})$$
$$= \arg\min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}) \text{ for } i, j, k, l \in N.$$

They mean

$$\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}) = \max_{s_j \in S} \min_{s_i \in S} u_j(s_i, s_j, \mathbf{s}_{-i,j})$$
$$= \min_{s_i \in S} \max_{s_j \in S} u_j(s_i, s_j, \mathbf{s}_{-i,j}) = \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}),$$

and

$$\arg\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}) = \arg\max_{s_j \in S} \min_{s_i \in S} u_j(s_i, s_j, \mathbf{s}_{-i,j})$$
$$= \arg\min_{s_i \in S} \max_{s_j \in S} u_j(s_i, s_j, \mathbf{s}_{-i,j}) = \arg\min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}) \text{ for any } i, j.$$

Then, we find $(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_n) = (\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n)$. Let $\mathbf{s} = (s, s, \dots, s)$. If $(s_1, s_2, \dots, s_n) = \mathbf{s}$, all $\arg\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, s, \dots, s)$'s and all $\arg\min_{s_i \in S} \max_{s_j \in S} u_j(s_i, s_j, s, \dots, s)$'s are the same. Thus, the fixed point obtained from above two functions is symmetric. Denote it by $\tilde{\mathbf{s}} = (\tilde{s}, \tilde{s}, \dots, \tilde{s})$. These arguments ensure the existence of symmetric maximin and minimax strategies.

Summarizing the results, Sion's minimax theorem for a symmetric multi-person zero-sum game is stated as follows.

Theorem 1. *Let S_i's for $i \in N$ be non-void convex and compact subsets of linear topological spaces, let $u_i : S_i \times S_j \to \mathbb{R}$ given $\tilde{\mathbf{s}}_{-i,j}$ be a function that is upper semi-continuous and quasi-concave on S_i and lower semi-continuous and quasi-convex on S_j for all $j \neq i$ and $i \in N$, and $S_i = S$ for all $i \in N$. Then, there exists $\tilde{\mathbf{s}} = (\tilde{s}, \tilde{s}, \dots, \tilde{s})$ such that*

$$\begin{aligned} &\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \max_{s_j \in S} \min_{s_i \in S} u_j(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) \\ &= \min_{s_i \in S} \max_{s_j \in S} u_j(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}), \end{aligned}$$

and

$$\begin{aligned} &\arg\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \arg\max_{s_j \in S} \min_{s_i \in S} u_j(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) \\ &= \arg\min_{s_i \in S} \max_{s_j \in S} u_j(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \arg\min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) \text{ for any } i, j, \end{aligned}$$

where $\tilde{\mathbf{s}}_{-i,j} = (\tilde{s}, \tilde{s}, \dots, \tilde{s})$ for $k \in N,\ k \neq i, j$.

Now we consider a Nash equilibrium of a symmetric multi-person zero-sum game. Let $s_i^*,\ i \in N$, be the values of s_i's which, respectively, maximize $u_i,\ i \in N$, given $s_j^*, j \neq i$, in a neighborhood around $(s_1^*, s_2^*, \dots, s_n^*)$ in $S_1 \times S_2 \times \cdots \times S_n$. Then,

$$u_i(s_1^*, \dots, s_i^*, \dots, s_n^*) \geq u_i(s_1^*, \dots, s_i, \dots, s_n^*) \text{ for all } s_i \neq s_i^*,\ i \in N. \tag{2}$$

Since the game is symmetric, we consider a symmetric equilibrium such that all s_i^*'s are equal at equilibria. Thus, $u_i(s_1^*, \dots, s_i^*, \dots, s_n^*)$'s for all i are equal, and by the property of zero-sum game they are zero. By symmetry of the game we have

$$u_j(s_1^*, \dots, s_i, \dots, s_n^*) = u_k(s_1^*, \dots, s_i, \dots, s_n^*) \text{ for } j \neq i, k \neq i, j \neq k.$$

From this and (1)

$$-\sum_{j=1, j\neq i}^{n} u_j(s_1^*, \dots, s_i, \dots, s_n^*) = -(n-1)u_j(s_1^*, \dots, s_i, \dots, s_n^*) = u_i(s_1^*, \dots, s_i, \dots, s_n^*).$$

Therefore, from (2)

$$u_j(s_1^*, \dots, s_i, \dots, s_n^*) \geq u_j(s_1^*, \dots, s_i^*, \dots, s_n^*) \text{ for } j \neq i.$$

By symmetry

$$u_i(s_1^*, \dots, s_j, \dots, s_n^*) \geq u_i(s_1^*, \dots, s_i^*, \dots, s_n^*) \text{ for } j \neq i.$$

Combining this and (2)

$$u_i(s_1^*, \dots, s_i, \dots, s_n^*) \leq u_i(s_1^*, \dots, s_i^*, \dots, s_n^*) \leq u_i(s_1^*, \dots, s_j, \dots, s_n^*)$$
$$\text{for all } s_i \neq s_i^* \text{ and all } s_j \neq s_j^*,\ j \neq i,\ i \in N.$$

This is equivalent to

$$u_i(s_1^*, \dots, s_i^*, \dots, s_n^*) = \max_{s_i} u_i(s_1^*, \dots, s_i, \dots, s_n^*) = \min_{s_j} u_i(s_1^*, \dots, s_j, \dots, s_n^*),$$
$$j \neq i \text{ given } s_k^*,\ k \neq i, j,$$

Denote the symmetric Nash equilibrium of the zero-sum game by $\mathbf{s}^* = (s^*, s^*, \dots, s^*)$. Let $\tilde{\mathbf{s}}^*_{-i,j} = (\tilde{s}^*, \tilde{s}^*, \dots, \tilde{s}^*)$ for $k \in N,\ k \neq i, j$, and $\tilde{\mathbf{s}}_{-i,j} = (\tilde{s}, \tilde{s}, \dots, \tilde{s})$ for $k \in N,\ k \neq i, j$. We can show the following result.

Theorem 2. *The following three statements are equivalent.*

(1) *There exists a symmetric Nash equilibrium in a symmetric multi-person zero-sum game.*

(2) *There exists* $\tilde{\mathbf{s}} = (\tilde{s}, \tilde{s}, \dots, \tilde{s})$ *such that the following relation holds.*

$$\mathbf{v}_i^s \equiv \max_{s_i} \min_{s_j} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \min_{s_j} \max_{s_i} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) \equiv \mathbf{v}_j^s \text{ for any pair of } i \text{ and } j.$$

(3) *There exists a real number* $\mathbf{v}_s$, s_i^m *and* s_j^m *such that*

$$u_i(s_i^m, s_j, \tilde{\mathbf{s}}_{-i,j}) \geq \mathbf{v}_s \text{ for any } s_j, \text{ and } u_i(s_i, s_j^m, \tilde{\mathbf{s}}_{-i,j}) \leq \mathbf{v}_s \text{ for any } s_i, \tag{3}$$
$$\text{for any pair of } i \text{ and } j.$$

Proof. $(1 \to 2)$

Set $\tilde{\mathbf{s}} = \mathbf{s}^*$. Then,

$$\mathbf{v}_j^s = \min_{s_j} \max_{s_i} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) \leq \max_{s_i} u_i(s_i, s_j^*, \mathbf{s}^*_{-i,j}) = u_i(s_i^*, s_j^*, \mathbf{s}^*_{-i,j})$$
$$= \min_{s_j} u_i(s_i^*, s_j, \mathbf{s}^*_{-i,j}) \leq \max_{s_i} \min_{s_j} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \mathbf{v}_i^s.$$

On the other hand, $\min_{s_j} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) \le u_i(s_i, s_j, \mathbf{s}^*_{-i,j})$, then $\max_{s_i}\min_{s_j} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) \le \max_{s_i} u_i(s_i, s_j, \mathbf{s}^*_{-i,j})$, and so $\max_{s_i}\min_{s_j} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) \le \min_{s_j}\max_{s_i} u_i(s_i, s_j, \mathbf{s}^*_{-i,j})$. Thus, $\mathbf{v}^s_i \le \mathbf{v}^s_j$, and we have $\mathbf{v}^s_i = \mathbf{v}^s_j$.

$(2 \to 3)$

Set $\tilde{\mathbf{s}} = \mathbf{s}^*$. Let $s^m_i = \arg\max_{s_i}\min_{s_j} u_i(s_i, s_j, \mathbf{s}^*_{-i,j})$ (the maximin strategy), $s^m_j = \arg\min_{s_j}\max_{s_i} u_i(s_i, s_j, \mathbf{s}^*_{-i,j})$ (the minimax strategy), and let $\mathbf{v}_s = \mathbf{v}^s_i = \mathbf{v}^s_j$. Then, we have

$$u_i(s^m_i, s_j, \mathbf{s}^*_{-i,j}) \ge \min_{s_j} u_i(s^m_i, s_j, \mathbf{s}^*_{-i,j}) = \max_{s_i}\min_{s_j} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \mathbf{v}_s$$
$$= \min_{s_j}\max_{s_i} u_i(s_i, s_j, \mathbf{s}^*_{-i,j}) = \max_{s_i} u_i(s_i, s^m_j, \mathbf{s}^*_{-i,j}) \ge u_i(s_i, s^m_j, \mathbf{s}^*_{-i,j}).$$

By Theorem 1 $s^m_i = \tilde{s} = s^*$ and $s^m_j = \tilde{s} = s^*$.

$(3 \to 1)$

Set $\tilde{\mathbf{s}} = \mathbf{s}^*$. Since $s^m_i = s^m_j = s^*$ from (3) we get

$$u_i(s^*, s_j, \mathbf{s}^*_{-i,j}) \ge \mathbf{v}_s \ge u_i(s_i, s^*, \mathbf{s}^*_{-i,j}) \text{ for all } s_i \in S_i,\ s_j \in S_j.$$

Putting $s_i = s^*_i$ and $s_j = s^*_j$, we see $\mathbf{v}_s = u_i(s^*, s^*, \mathbf{s}^*_{-i,j})$ and $\mathbf{s}^* = (s^*, s^*, \dots, s^*)$ is an equilibrium. □

Therefore, Sion's minimax theorem is equivalent to the existence of Nash equilibrium of a symmetric multi-person zero-sum game.

4. Example of asymmetric multi-person zero-sum game

Consider a three-person game. Suppose that the payoff functions of players are

$$\pi_1 = (a-s_1-(s_2+s_3)s_1-c_1s_1-\frac{1}{2}[(a-s_2-(s_1+s_3)s_2-c_2s_2+(a-s_3-(s_2+s_1)s_3-c_3s_3],$$

$$\pi_2 = (a-s_2-(s_1+s_3)s_2-c_2s_2-\frac{1}{2}[(a-s_1-(s_2+s_3)s_1-c_1s_1+(a-s_3-(s_2+s_1)s_3-c_3s_3],$$

and

$$\pi_3 = (a-s_3-(s_2+s_1)s_3-c_3s_3-\frac{1}{2}[(a-s_1-(s_2+s_3)s_1-c_1s_1+(a-s_2-(s_1+s_3)s_2-c_2s_2].$$

This is a model of relative profit maximization in a three firms Cournot oligopoly with constant marginal cost and zero fixed cost producing a homogeneous good.

$s_i, i = 1, 2, 3$ are the outputs of the firms. The conditions for maximization of $\pi_i,\ i = 1, 2, 3$ are

$$\frac{\partial \pi_1}{\partial s_1} = a - 2s_1 - (s_2 + s_3) - c_1 + \frac{1}{2}(s_2 + s_3) = 0,$$

$$\frac{\partial \pi_2}{\partial s_2} = a - 2s_2 - (s_1 + s_3) - c_2 + \frac{1}{2}(s_1 + s_3) = 0,$$

and

$$\frac{\partial \pi_3}{\partial s_3} = a - 2s_3 - (s_2 + s_1) - c_3 + \frac{1}{2}(s_2 + s_1) = 0.$$

The Nash equilibrium strategies are

$$s_1 = \frac{3a - 5c_1 + c_2 + c_3}{9},\ s_2 = \frac{3a - 5c_2 + c_1 + c_3}{9},\ s_3 = \frac{3a - 5c_3 + c_2 + c_1}{9}. \quad (4)$$

We consider maximin and minimax strategy about Player 1 and 2. The condition for minimization of π_1 with respect to s_2 is $\frac{\partial \pi_1}{\partial s_2} = 0$. Denote s_2 which satisfies this condition by $s_2(s_1, s_3)$, and substitute it into π_1. Then, the condition for maximization of π_1 with respect to s_1 given $s_2(s_1, s_3)$ and s_3 is

$$\frac{\partial \pi_1}{\partial s_1} + \frac{\partial \pi_1}{\partial s_2}\frac{ds_2}{ds_1} = 0.$$

We call the strategy of Player 1 obtained from these conditions the maximin strategy of Player 1 to Player 2. It is denoted by $\arg\max_{s_1}\min_{s_2}\pi_1$. The condition for maximization of π_1 with respect to s_1 is $\frac{\partial \pi_1}{\partial s_1} = 0$. Denote s_1 which satisfies this condition by $s_1(s_2, s_3)$, and substitute it into π_1. Then, the condition for minimization of π_1 with respect to s_2 given $s_1(s_2, s_3)$ is

$$\frac{\partial \pi_1}{\partial s_2} + \frac{\partial \pi_1}{\partial s_1}\frac{ds_1}{d_2} = 0.$$

We call the strategy of Player 2 obtained from these conditions the minimax strategy of Player 2 to Player 1. It is denoted by $\arg\min_{s_2}\max_{s_1}\pi_1$. In our example we obtain

$$\arg\max_{s_1}\min_{s_2}\pi_1 = \frac{3a - 4c_1 + c_2}{9},\ \arg\min_{s_2}\max_{s_1}\pi_1 = \frac{6a - 9s_3 - 2c_1 - 4c_2}{9}.$$

Similarly, we get the following results.

$$\arg\max_{s_2}\min_{s_1}\pi_2 = \frac{3a - 4c_2 + c_1}{9},\ \arg\min_{s_1}\max_{s_2}\pi_2 = \frac{6a - 9s_3 - 2c_2 - 4c_1}{9},$$

$$\arg\max_{s_1}\min_{s_3}\pi_1 = \frac{3a-4c_1+c_3}{9},\ \arg\min_{s_3}\max_{s_1}\pi_1 = \frac{6a-9s_2-2c_1-4c_3}{9},$$
$$\arg\max_{s_3}\min_{s_1}\pi_3 = \frac{3a-4c_3+c_1}{9},\ \arg\min_{s_1}\max_{s_3}\pi_3 = \frac{6a-9s_2-2c_3-4c_1}{9},$$
$$\arg\max_{s_2}\min_{s_3}\pi_2 = \frac{3a-4c_2+c_3}{9},\ \arg\min_{s_3}\max_{s_2}\pi_2 = \frac{6a-9s_1-2c_2-4c_3}{9},$$
$$\arg\max_{s_3}\min_{s_2}\pi_3 = \frac{3a-4c_3+c_2}{9},\ \arg\min_{s_2}\max_{s_3}\pi_3 = \frac{6a-9s_1-2c_3-4c_2}{9}.$$

If the game is asymmetric, for example, $c_2 \neq c_3$, $\arg\max_{s_1}\min_{s_2}\pi_1 \neq \arg\max_{s_1}\min_{s_3}\pi_1$, $\arg\max_{s_2}\min_{s_3}\pi_2 \neq \arg\max_{s_3}\min_{s_2}\pi_3$, $\arg\min_{s_3}\max_{s_2}\pi_2 \neq \arg\min_{s_2}\max_{s_3}\pi_3$and so on. However, if the game is symmetric, we have $c_2 = c_3 = c_1$ and

$$\arg\max_{s_1}\min_{s_2}\pi_1 = \arg\max_{s_2}\min_{s_1}\pi_2 = \arg\max_{s_1}\min_{s_3}\pi_1 = \arg\max_{s_3}\min_{s_1}\pi_3$$
$$= \arg\max_{s_2}\min_{s_3}\pi_2 = \arg\max_{s_3}\min_{s_2}\pi_3 = \frac{a-c_1}{3}.$$

All of the Nash equilibrium strategies of the players in (4) are also equal to $\frac{a-c_1}{3}$. Assume $s_2 = s_3 = s_1$ as well as $c_2 = c_3 = c_1$. Then,

$$\arg\min_{s_2}\max_{s_1}\pi_1 = \arg\min_{s_1}\max_{s_2}\pi_2 = \arg\min_{s_3}\max_{s_1}\pi_1 = \arg\min_{s_1}\max_{s_3}\pi_3$$
$$= \arg\min_{s_3}\max_{s_2}\pi_2 = \arg\min_{s_2}\max_{s_3}\pi_3 = \frac{2a-3s_1-2c_1}{3}.$$

Further, if

$$s_1 = \arg\min_{s_1}\max_{s_2}\pi_2 = \arg\min_{s_1}\max_{s_3}\pi_3,$$

we obtain

$$\arg\min_{s_2}\max_{s_1}\pi_1 = \arg\min_{s_1}\max_{s_2}\pi_2 = \arg\min_{s_3}\max_{s_1}\pi_1 = \arg\min_{s_1}\max_{s_3}\pi_3$$
$$= \arg\min_{s_3}\max_{s_2}\pi_2 = \arg\min_{s_2}\max_{s_3}\pi_3 = \frac{a-c_1}{3}.$$

Therefore, the maximin strategy, the minimax strategy and the Nash equilibrium strategy for all players are equal.

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