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STRATEGY-PROOF RULES ON PARTIALLY SINGLE-PEAKED DOMAINS*

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Abstract

We consider domains that exhibit single-peakedness only over a subset of alternatives. We call such domains partially single-peaked and provide a characterization of the unanimous and strategy-proof social choice functions on these domains. As an application of this result, we obtain a characterization of the unanimous and strategy-proof social choice functions on multi-peaked domains (Stiglitz (1974), Shepsle (1979), Epple and Romano (1996a)), single-peaked domains with respect to a partial order (Chatterji and Massó (2015)), multiple single-peaked domains (Reffgen (2015)) and single-peaked domains on graphs (Schummer and Vohra (2002)). As a by-product of our results, it follows that strategy-proofness implies tops-onlyness on these domains. Further, we show that strategy-proofness and group strategy-proofness are equivalent on these domains.

KEYWORDS: Partially single-peaked domain, strategy-proofness, group strategy-proofness, partly dictatorial min-max rules.

JEL CLASSIFICATION CODES: D71, D82.

1. INTRODUCTION

1.1 BACKGROUND OF THE PROBLEM

This paper deals with the standard social choice problem where an alternative has to be chosen based on privately known preferences of the individuals in a society. A procedure that maps a

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collection of individual preferences to a social alternative is called a *social choice function* (SCF). In such a framework, it is natural to assume that individuals may misreport their preferences whenever it is strictly beneficial for them. An SCF is called (group) *strategy-proof* if no individual (group of individuals) finds it beneficial to misreport her preferences and is called *unanimous* if it always selects a unanimously agreed alternative whenever that exists.

Most of the subject matter of social choice theory concerns the study of the unanimous and strategy-proof SCFs for different admissible domains of preferences. In the seminal works by Gibbard (1973) and Satterthwaite (1975), it is shown that if a society has at least three alternatives and there is no particular restriction on the preferences of the individuals, then every unanimous and strategy-proof SCF is *dictatorial*, that is, a particular individual in the society determines the outcome regardless of the preferences of the others. The celebrated Gibbard-Satterthwaite theorem hinges crucially on the assumption that the admissible domain of each individual is unrestricted. However, it is well established that in many economic and political applications, there are natural restrictions on such domains. For instance, in the models of locating a firm in a unidimensional spatial market (Hotelling (1929)), setting the rate of carbon dioxide emissions (Black (1948)), setting the level of public expenditure (Romer and Rosenthal (1979)), and so on, preferences admit a natural restriction widely known as *single-peakedness*. Roughly speaking, the crucial property of a single-peaked preference is that there is a prior order over the alternatives such that the preference decreases as one moves away (with respect to the prior order) from her best alternative.

The study of single-peaked domains dates back to Black (1948), where it is shown that the pairwise majority rule is strategy-proof on such domains. Moulin (1980) and Weymark (2011) have characterized the unanimous and strategy-proof SCFs on such domains as *min-max rules*.[1,2] Recently, Achuthankutty and Roy (2017) characterize the domains where the set of unanimous and strategy-proof SCFs coincide with that of min-max rules.

1.2 Our Motivation

It is both experimentally and empirically established that in many political and economic scenarios (Niemi and Wright (1987), Feld and Grofman (1988), and Pappi and Eckstein (1998)), where the

[1] Barberà et al. (1993) and Ching (1997) provide equivalent presentations of this class of SCFs.

[2] A rich literature has developed around the single-peaked restriction by considering various generalizations and extensions (see Barberà et al. (1993), Demange (1982), Schummer and Vohra (2002), Nehring and Puppe (2007a), and Nehring and Puppe (2007b)).

preferences of individuals are normally assumed to be single-peaked, they are actually not. Nevertheless, such preferences have close resemblance with single-peakedness. In this paper, we model such preferences as *partially single-peaked*. Roughly speaking, partial single-peakedness requires the individual preferences to be single-peaked *only* over a subset of alternatives. It is worth noting that the structure of the unanimous and (group) strategy-proof rules on such domains are not explored in the literature. In view of this, our main motivation in this paper is to develop a general model for partially single-peaked domains and to provide a characterization of the unanimous and (group) strategy-proof rules on those. Below, we present some evidences of partially single-peaked domains in the literature. In Section 4, we will formally define these notions and show that they are special cases of partially single-peaked domains.

1.2.1 Multi-peaked Domains

In many practical scenarios in economics and politics, the preferences of the individuals often exhibit *multi-peakedness* as opposed to single-peakedness. As the name suggests, multi-peaked preferences admit multiple ideal points in a unidimensional policy space. We discuss a few settings where it is plausible to assume that individuals have multi-peaked preferences.

- *Preference for 'Do Something' in Politics*: Davis et al. (1970) and Egan (2014) consider public (decision) problems such as choosing alternate tax regimes, lowering health care costs, responding to foreign competition, reducing the national debt, etc. They show that a public problem is perceived to be poorly addressed by the status-quo policy, and consequently some individuals prefer both liberal and conservative policies to the moderate status quo. Clearly, such a preference will have two peaks, one on the left of the status quo and another one on the right.

- *Multi-stage Voting System*: Shepsle (1979), Denzau and Mackay (1981), Enelow and Hinich (1983), etc. deal with multi-stage voting system where individuals vote on a set of issues where each issue can be thought of as a unidimensional spectrum and voting is distributed over several stages considering one issue at a time. In such a model, preference of an individual over the present issue can be affected by her prediction of the outcome of the future issues. In other words, such a preference is not separable across issues. They show that the preferences of the individuals in such scenarios exhibit multi-peaked property.

- *Provision of Public Goods with Outside Options*: Barzel (1973), Stiglitz (1974), and Bearse et al. (2001) consider the problem of setting the level of tax rates to provide public funding in the education sector, and Ireland (1990) and Epple and Romano (1996a) consider the same problem in the health insurance market. They show that the preferences of individuals exhibit multi-peaked property due to the presence of outside options (i.e., the public good is also available in a competitive market as a private good). For instance, in the problem of determining educational subsidy, an individual with lower income may not prefer a moderate level of subsidy since she cannot afford to bear the remaining cost for higher education. Thus, her preference in such a scenario will have two peaks - one at a lower level of subsidy so that she can achieve primary education, and another one at a very high level of subsidy so that she can afford the remaining cost for higher education.
- *Provision of Excludable Public Goods*: Fernandez and Rogerson (1995) and Anderberg (1999) consider public good provision models such as health insurance, educational subsidies, pensions, etc. where the government provides the public good to a particular section of individuals, and show that individuals' preferences in such scenarios are multi-peaked.

1.2.2 Single-peaked Domains with respect to Partial Orders

In the literature, single-peaked domains are generally considered with respect to some (prior) linear order. Such a preference restriction requires an individual to order (*a priori*) the whole set of alternatives in a linear fashion. However, it is well-documented in psychology that in many situations individuals are unable to derive a complete ordering over the alternatives. For instance, in the political science literature, it may not be possible for the individuals to unambiguously order the parties who are moderate in their policies (center parties) over the policy spectrum. Similarly, in a public good provision problem where locations are distributed over different geographical regions, even though individuals can derive some prior ordering (based on traffic distance or so) over the locations that are in same region, but they may not be able to do the same for locations in different regions. Such a situation can only be modeled by considering single-peaked domains with respect to prior orderings that are incomplete (or partial). In this respect, our work is closely related to Chatterji and Massó (2015) who consider *semi-lattice single-peakedness* - preferences that are single-peaked with respect to a semi-lattice (which is a partial order).

1.2.3 Multiple Single-peaked Domains

Reffgen (2015) introduces the notion of *multiple single-peaked domains*. Such a domain is defined as a union of some domains each of which is single-peaked with respect to some prior orderings over the alternatives. A plausible justification for such a domain restriction is provided by Niemi (1969) who argues that the alternatives can be ordered differently using different criteria (which he calls an *impartial culture*) and it is not publicly known which individual uses what criterion. On one extreme, such a domain becomes an unrestricted domain if there is no consensus among the individuals on the prior order, and on the other extreme, it becomes a maximal single-peaked domain if all the individuals agree on a single prior order. It is worth noting that such domains can be seen as a special case of partially single-peaked domains.

1.2.4 Single-peaked Domains on Graphs

Schummer and Vohra (2002) considers domains that are based on some graph structure over the alternatives (e.g., locating a new station in a rail-road network). They assume that the individuals derive their preferences by using single-peakedness over some spanning tree of the underlying graph. In this paper, we show that when the underlying graph has some specific structure (involves a cycle or so), then the induced domains become partially single-peaked.

1.3 Our Contribution

In this paper, we develop a general model for partially single-peaked domains which capture the non-single-peaked domains that commonly arise in practical scenarios. Formally speaking, we assume that the whole interval of alternatives is divided into subintervals such that every preference in the domain is required to satisfy single-peakedness over each of those subintervals, and is allowed to violate the property outside those. We characterize the unanimous and strategy-proof SCFs on such domains as *partly dictatorial min-max rule (PDMMR)*. Loosely put, a PDMMR acts like a min-max rule over the subintervals where the domain respects single-peakedness and like a dictatorial rule everywhere else. We also establish the equivalence of strategy-proofness and group strategy-proofness on partially single-peaked domains. Barberà et al. (2010) provides a sufficient condition for the equivalence of strategy-proofness and group strategy-proofness on a domain. Partially single-peaked domains do not satisfy their condition. Therefore, we independently establish the equivalence of strategy-proofness and group strategy-proofness on these domains.

The class of partially single-peaked domains that we consider in this paper is quite large. It includes single-peaked domains on one extreme and unrestricted domains on the other. To corroborate this fact, we prove that partially single-peaked domains contain *almost* all domains on which (i) every unanimous and strategy-proof SCF is a PDMMR and (ii) every PDMMR is strategy-proof.

A crucial step in the proof of our characterization results is to establish the tops-onlyness property. Chatterji and Sen (2011) provide a sufficient condition for tops-onlyness, however partially single-peaked domains do not satisfy that condition.

To put our results in perspective, we conclude this section by comparing them with a few related articles. Chatterji et al. (2013) study a related restricted domain known as a *semi-single-peaked domain*. Such a domain violates single-peakedness around the *tails* of the prior order. They show that if a domain admits an anonymous (and hence non-dictatorial), tops-only, unanimous, and strategy-proof SCF, then it is a semi-single-peaked domain. However, we show that if single-peakedness is violated around the *middle* of the prior order, then there is *no* unanimous, strategy-proof, and anonymous SCF. Thus, our characterization result on partially single-peaked domains complements that in Chatterji et al. (2013). Recently, Arribillaga and Massó (2016) provide necessary and sufficient conditions for the comparability of two min-max rules in terms of their vulnerability to manipulation. However, our results identify the min-max rules that are manipulable if single-peakedness is violated over a subset of alternatives.

The rest of the paper is organized as follows. We describe the usual social choice framework in Section 2. In Section 3, we presents our main results. Section 4 provides a few applications of our results, and the last section concludes the paper. All the omitted proofs are collected in Appendix A.

2. Preliminaries

Let $N = \{1, \dots, n\}$ be a set of at least two agents, who collectively choose an element from a finite set $X = \{a, a+1, \dots, b-1, b\}$ of at least three alternatives, where a is an integer. For $x, y \in X$ such that $x \leq y$, we define the intervals $[x,y] = \{z \in X \mid x \leq z \leq y\}$, $[x,y) = [x,y] \setminus \{y\}$, $(x,y] = [x,y] \setminus \{x\}$, and $(x,y) = [x,y] \setminus \{x,y\}$. Throughout this paper, we denote by $\underline{x}$ and $\overline{x}$ two arbitrary but fixed alternatives such that $\underline{x} < \overline{x} - 1$. For notational convenience, whenever it is clear from the context, we do not use braces for singleton sets, i.e., we denote sets $\{i\}$ by i.

A *preference* P over X is a complete, transitive, and antisymmetric binary relation (also called a linear order) defined on X. We denote by $\mathbb{L}(X)$ the set of all preferences over X. An alternative $x \in X$ is called the k^{th} *ranked alternative* in a preference $P \in \mathbb{L}(X)$, denoted by $r_k(P)$, if $|\{a \in X \mid aPx\}| = k - 1$. For notational convenience, sometimes we denote by $P = xy\ldots$ a preference P with $r_1(P) = x$ and $r_2(P) = y$. A domain of admissible preferences, denoted by $\mathcal{D}$, is a subset of $\mathbb{L}(X)$. An element $P_N = (P_1, \ldots, P_n) \in \mathcal{D}^n$ is called a *preference profile*. The *top-set* of a preference profile P_N, denoted by $\tau(P_N)$, is defined as $\tau(P_N) = \{x \in X \mid r_1(P_i) = x \text{ for some } i \in N\}$.

2.1 Domains and Their Properties

In this subsection, we introduce a few properties of a domain and a class of domains.

Definition 2.1. A domain $\mathcal{D}$ of preferences is *regular* if for all $x \in X$, there exists a preference $P \in \mathcal{D}$ such that $r_1(P) = x$.

All the domains we consider in this paper are assumed to be regular.

Definition 2.2. A domain $\mathcal{D}$ satisfies the *top-connectedness* property if for all $x, y \in X$ with $|x - y| = 1$, there is $P \in \mathcal{D}$ such that $P = xy\ldots$.

2.1.1 Graph of a Domain

In this subsection, we introduce the notion of the graph of a domain. First, we introduce a few graph theoretic notions. A *directed graph* G is defined as a pair $\langle V, E\rangle$, where V is the set of *nodes* and $E \subseteq V \times V$ is the set of *directed edges*, and an *undirected graph* G is defined as a pair $\langle V, E\rangle$, where V is the set of nodes and $E \subseteq \{\{u, v\} \mid u, v \in V \text{ and } u \neq v\}$ is the set of *undirected edges*. For a graph (directed or undirected) $G = \langle V, E\rangle$, a *subgraph* G' of G is defined as a graph $G' = \langle V, E'\rangle$, where $E' \subseteq E$. For two graphs $G_1 = \langle V_1, E_1\rangle$ and $G_2 = \langle V_2, E_2\rangle$, the graph $G_1 \cup G_2$ is defined as $G_1 \cup G_2 = \langle V_1 \cup V_2, E_1 \cup E_2\rangle$.

All the graphs we consider in this paper are of the kind $G = \langle X, E\rangle$, i.e., whose node set is the set of alternatives.

Definition 2.3. A directed (undirected) graph $G = \langle X, E\rangle$ is called the *directed (undirected) line graph* on X if $(x, y) \in E$ ($\{x, y\} \in E$) if and only if $|x - y| = 1$.

Definition 2.4. A graph G is called a *directed (undirected) partial line graph* if G can be expressed as $G_1 \cup G_2$, where $G_1 = \langle X, E_1 \rangle$ is the directed (undirected) line graph on X and $G_2 = \langle [\underline{x}, \overline{x}], E_2 \rangle$ is a directed (undirected) graph such that $(\underline{x}, y), (\overline{x}, z) \in E_2$ ($\{\underline{x}, y\}, \{\underline{x}, z\} \in E_2$) for some $y \in (\underline{x}+1, \overline{x}]$ and $z \in [\underline{x}, \overline{x}-1)$.

In Figure 1, we present a directed partial line graph on $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ where $\underline{x} = x_3$ and $\overline{x} = x_6$.

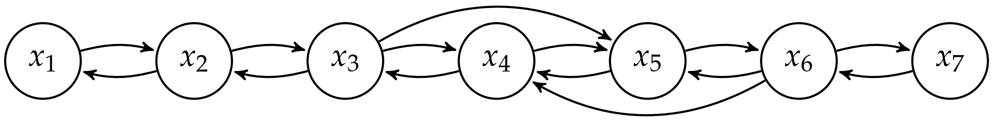


Figure 1: A directed partial line graph

Definition 2.5. The *top-graph* of a domain $\mathcal{D}$ is defined as the directed graph $\langle X, E \rangle$ such that $(x, y) \in E$ if and only if there exists a preference $P = xy \ldots \in \mathcal{D}$.

Note that a domain satisfies the top-connectedness property if and only if its top-graph is the directed line graph on X.

2.2 Single-peaked Domains

Definition 2.6. A preference $P \in \mathbb{L}(X)$ is called *single-peaked* if for all $x, y \in X$, $[x < y \leq r_1(P)$ or $r_1(P) \leq y < x]$ implies yPx. A domain is called *single-peaked* if each preference in it is single-peaked, and a domain is called *maximal single-peaked* if it contains all single-peaked preferences.

Definition 2.7. A domain is called *top-connected single-peaked* if it is both top-connected and single-peaked.

2.3 Partially Single-peaked Domains

In this section, we consider a class of domains that violates single-peaked property over the interval $[\underline{x}, \overline{x}]$ and exhibits the property everywhere else. We call such domains partially single-peaked domains which are formally defined below.

Definition 2.8. A domain $\tilde{\mathcal{S}}$ is said to *satisfy single-peakedness outside* $[\underline{x}, \overline{x}]$ if for all $P \in \tilde{\mathcal{S}}$, all $u \notin (\underline{x}, \overline{x})$, and all $v \in X$,

$$\big[v < u \leq r_1(P) \text{ or } r_1(P) \leq u < v\big] \text{ implies } uPv.$$

To gain more insight about Definition 2.8, first consider a preference with top-ranked alternative in $[\underline{x}, \overline{x}]$. Then, Definition 2.8 says that such a preference satisfies single-peakedness over the intervals $[a, \underline{x}]$ and $[\overline{x}, b]$. That is, the relative ordering of two alternatives u, v is derived by using single-peaked property whenever both of them are either in the interval $[a, \underline{x}]$ or in the interval $[\overline{x}, b]$. Note that Definition 2.8 does not impose any restriction on the relative ordering of an alternative in $[\underline{x}, \overline{x}]$ and any other alternative. Next, consider a preference P such that $r_1(P) \notin [\underline{x}, \overline{x}]$. Suppose, for instance, $r_1(P) \in [a, \underline{x})$. Then, Definition 2.8 says that P satisfies single-peakedness over the interval $[a, r_1(P)]$. It further says that if an alternative u lies in the interval $(r_1(P), \underline{x}]$ or in the interval $[\overline{x}, b]$, then, as required by single-peakedness, it is preferred to any alternative v in the interval $(u, b]$. Thus, Definition 2.8 does not impose on P any restriction on the relative ordering of an alternative in $(\underline{x}, \overline{x})$ and an alternative in $[\overline{x}, b]$. Therefore, in particular, Definition 2.8 does not impose any restriction on any preference on the relative ordering of two alternatives in the interval $(\underline{x}, \overline{x})$.

Definition 2.9. A domain $\tilde{\mathcal{S}}$ is said to *violate single-peakedness over* $[\underline{x}, \overline{x}]$ if there exist $Q = \underline{x}y\ldots, Q' = \overline{x}z\ldots \in \tilde{\mathcal{S}}$ such that either $\big[y \in (\underline{x}+1, \overline{x}) \text{ and } z \in (\underline{x}, \overline{x}-1)\big]$ or $\big[y = \overline{x} \text{ and } z = \underline{x}\big]$.

Note that since $r_2(Q) > r_1(Q) + 1$ and $r_2(Q') < r_1(Q') - 1$, both the preferences Q and Q' violate single-peakedness. This, together with the facts that $r_1(Q) = \underline{x}$, $r_1(Q') = \overline{x}$, and $r_2(Q), r_2(Q') \in (\underline{x}, \overline{x})$, implies that a domain with those two preferences violates single-peakedness over $[\underline{x}, \overline{x}]$. In Section 3.2, we show that the particular restrictions on the second-ranked alternatives of Q and Q' given in Definition 2.9 are necessary for the results we derive in this paper.

REMARK 2.1. Definition 2.9 considers violation of single-peakedness only over intervals. It may seem that the possibility of violating this over several intervals is excluded in this definition. However, as we argue in the following, that is not the case. Note that by Definition 2.9, if a domain violates single-peakedness over several intervals, then it also violates the same over the minimal interval that contains all those. Thus, for the notion of violation of single-peakedness that we consider in this paper, it is enough to consider it over an interval.

Definition 2.10. A domain $\tilde{S}$ is called *partially single-peaked* if

(i) it satisfies single-peakedness outside $[\underline{x}, \overline{x}]$ and violates it over $[\underline{x}, \overline{x}]$, and

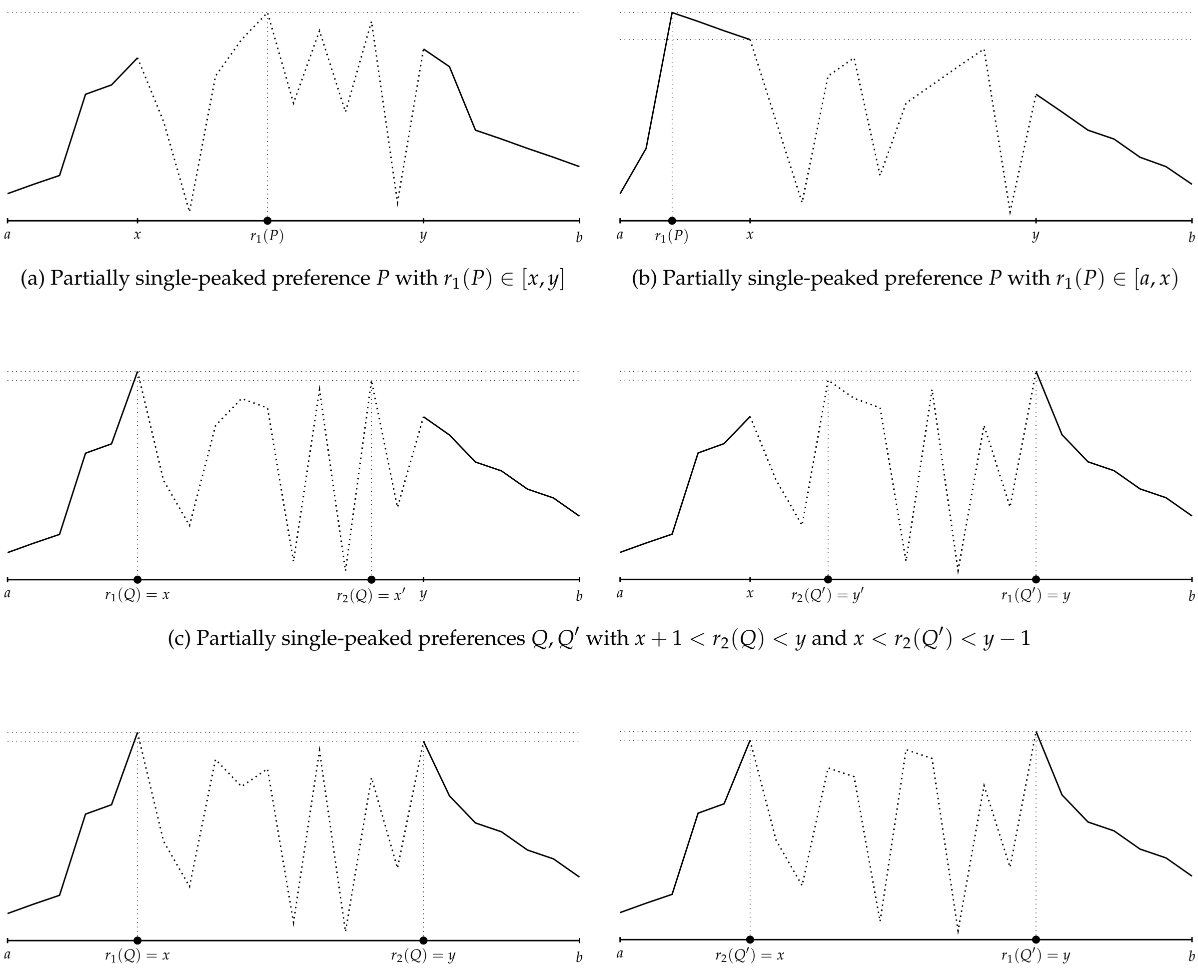
(ii) it contains a top-connected single-peaked domain.

REMARK 2.2. Condition (ii) in Definition 2.10 may not seem to be essential in modeling non-single-peaked preferences that arise in political and economic scenarios. However, we feel this is not the case. In most political and economic scenarios where a prior ordering over the alternatives exists (naturally), non-single-peaked preferences arise because some individuals may not use that ordering completely in deriving their preferences. However, there is no logical ground to rule out the possibility that some individuals may still use that ordering in deriving their preferences. Thus, one must allow for the single-peaked preferences in such domains.

We illustrate the notion of partially single-peaked domains in Figure 2. Figure 2(a) and Figure 2(b) present partially single-peaked preferences P with $r_1(P) \in [\underline{x}, \overline{x}]$ and $r_1(P) \in [a, \underline{x})$, respectively. Figure 2(c) presents partially single-peaked preferences $Q = \underline{x}y\ldots$ and $Q' = \overline{x}z\ldots$ when $y \in (\underline{x}+1, \overline{x})$ and $z \in (\underline{x}, \overline{x}-1)$, and Figure 2(d) presents those when $y = \overline{x}$ and $z = \underline{x}$. Note that, as explained before, all these preferences are single-peaked over the intervals $[a, \underline{x}]$ and $[\overline{x}, b]$. Furthermore, for the preference depicted in Figure 2(a), there is no restriction on the ranking of the alternatives in the interval $(\underline{x}, \overline{x})$, and for the one shown in Figure 2(b), there is no restriction on the ranking of the alternatives in the interval $(\underline{x}, \overline{x})$ except that $\underline{x}$ is preferred to all the alternatives in $(\underline{x}, b]$. Also, for the preferences in Figures 2(c) and 2(d), there is no restriction on the ranking of the alternatives in $(\underline{x}, \overline{x})$ other than that on the second-ranked alternatives.

Now, we interpret Definition 2.10 in terms of its top-graph. Let G be the top-graph of a partially single-peaked domain. Then, G can be written as $G_1 \cup G_2$, where $G_1 = \langle X, E_1 \rangle$ is the directed line graph on X and $G_2 = \langle [\underline{x}, \overline{x}], E_2 \rangle$ is a directed graph such that $(\underline{x}, r_2(Q)), (\overline{x}, r_2(Q')) \in E_2$ where $r_2(Q) \in (\underline{x}+1, \overline{x}]$ and $r_2(Q') \in [\underline{x}, \overline{x}-1)$. Therefore, G is a directed partial line graph. In Example 2.1, we present a partially single-peaked domain with seven alternatives, and in Figure 3, we present the top-graph of that domain.

Example 2.1. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, where $x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < x_7$, and let $\underline{x} = x_3$ and $\overline{x} = x_6$. Then, the domain in Table 1 is a partially single-peaked domain. To see this, first consider a preference with top-ranked alternative in the interval $[x_3, x_6]$, say P_7. Note that $x_3 P_7 x_2 P_7 x_1$ and $x_6 P_7 x_7$, which means P_7 is single-peaked over the intervals $[x_1, x_3]$ and $[x_6, x_7]$.



(a) Partially single-peaked preference P with $r_1(P) \in [x, y]$

(b) Partially single-peaked preference P with $r_1(P) \in [a, x)$

(c) Partially single-peaked preferences Q, Q' with $x + 1 < r_2(Q) < y$ and $x < r_2(Q') < y - 1$

(d) Partially single-peaked preferences Q, Q' with $r_2(Q) = y$ and $r_2(Q') = x$

Figure 2: Partially single-peaked preferences

Moreover, the position of x_5 is completely unrestricted (here at the bottom) in P_7. Next, consider a preference with top-ranked alternative in the interval $[x_1, x_3]$, say P_2. Once again, note that P_2 is single-peaked over the intervals $[x_1, x_3]$ and $[x_6, x_7]$. Further, x_3 is preferred to the alternatives x_4, x_5, x_6, x_7, and there is no restriction on the relative ordering of the alternatives x_4 and x_5 (here $x_5 P_2 x_4$). Thus, the domain in Table 1 satisfies single-peakedness outside the interval $[x_3, x_6]$. Now, consider the preferences Q and Q'. Since $r_1(Q) = x_3$, $r_2(Q) = x_5$, $r_1(Q') = x_6$, and $r_2(Q') = x_4$, this domain violates single-peakedness over $[x_3, x_6]$. Finally, note that the domain contains a top-connected single-peaked domain given by $P_1, P_3, P_4, P_5, P_6, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}$, and P_{14}.

The top-graph G of the domain in Example 2.1 is given in Figure 3. Note that G is a partial line graph since it can be written as $G_1 \cup G_2$, where G_1 is the directed line graph on $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ and G_2 is a directed graph on $\{x_3, x_4, x_5, x_6\}$ having edges (x_3, x_5) and (x_6, x_4).

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	Q	Q'
$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_6}$	$\mathbf{x_7}$	$\mathbf{x_3}$	$\mathbf{x_6}$
$\mathbf{x_2}$	$\mathbf{x_1}$	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_6}$	$\mathbf{x_3}$	$\mathbf{x_5}$	$\mathbf{x_4}$	$\mathbf{x_6}$	$\mathbf{x_5}$	$\mathbf{x_7}$	$\mathbf{x_6}$	$\mathbf{x_5}$	$\mathbf{x_4}$
x_3	x_3	x_3	x_1	x_4	x_2	x_3	x_5	x_3	x_3	x_4	x_4	x_5	x_5	x_2	x_3
x_4	x_6	x_4	x_4	x_5	x_5	x_2	x_2	x_2	x_6	x_3	x_3	x_4	x_4	x_6	x_7
x_5	x_5	x_5	x_5	x_6	x_6	x_1	x_6	x_1	x_7	x_2	x_2	x_3	x_3	x_1	x_2
x_6	x_7	x_6	x_6	x_7	x_1	x_7	x_1	x_6	x_2	x_7	x_7	x_2	x_2	x_7	x_1
x_7	x_4	x_7	x_7	x_1	x_7	x_5	x_7	x_7	x_1	x_1	x_1	x_1	x_1	x_4	x_5

Table 1: A partially single-peaked domain

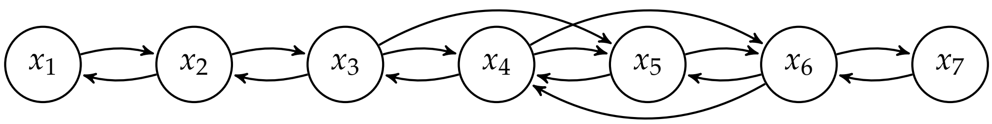


Figure 3: Top-graph of the domain in Example 2.1

2.4 Social Choice Functions and Their Properties

In this section, we introduce the notion of social choice functions and discuss their properties.

Definition 2.11. A *social choice function* (SCF) f on $\mathcal{D}^n$ is a mapping $f : \mathcal{D}^n \to X$.

Definition 2.12. An SCF $f : \mathcal{D}^n \to X$ is *unanimous* if for all $P_N \in \mathcal{D}^n$ such that $r_1(P_i) = x$ for all $i \in N$ and some $x \in X$, we have $f(P_N) = x$.

Definition 2.13. An SCF $f : \mathcal{D}^n \to X$ is *manipulable* if there exists $i \in N$, $P_N \in \mathcal{D}^n$, and $P_i^{'} \in \mathcal{D}$ such that $f(P_i^{'}, P_{N\setminus i}) P_i f(P_N)$. An SCF f is *strategy-proof* if it is not manipulable.

Definition 2.14. An SCF $f : \mathcal{D}^n \to X$ is called *dictatorial* if there exists $i \in N$ such that for all $P_N \in \mathcal{D}^n$, $f(P_N) = r_1(P_i)$.

Definition 2.15. A domain $\mathcal{D}$ is called *dictatorial* if every unanimous and strategy-proof SCF $f : \mathcal{D}^n \to X$ is dictatorial.

Definition 2.16. Two preference profiles P_N, P'_N are called *tops-equivalent* if $r_1(P_i) = r_1(P'_i)$ for all agents $i \in N$.

Definition 2.17. An SCF $f : \mathcal{D}^n \to X$ is called *tops-only* if for any two tops-equivalent $P_N, P'_N \in \mathcal{D}^n$, $f(P_N) = f(P'_N)$.

Definition 2.18. A domain $\mathcal{D}$ is called *tops-only* if every unanimous and strategy-proof SCF $f : \mathcal{D}^n \to X$ is tops-only.

Definition 2.19. An SCF $f : \mathcal{D}^n \to X$ is called *uncompromising* if for all $P_N \in \mathcal{D}^n$, all $i \in N$, and all $P'_i \in \mathcal{D}$:

(i) if $r_1(P_i) < f(P_N)$ and $r_1(P'_i) \leq f(P_N)$, then $f(P_N) = f(P'_i, P_{-i})$, and

(ii) if $f(P_N) < r_1(P_i)$ and $f(P_N) \leq r_1(P'_i)$, then $f(P_N) = f(P'_i, P_{-i})$.

REMARK 2.3. If an SCF satisfies uncompromisingness, then by definition, it is tops-only.

Definition 2.20. Let $\beta = (\beta_S)_{S \subseteq N}$ be a list of 2^n parameters satisfying: (i) $\beta_S \in X$ for all $S \subseteq N$, (ii) $\beta_\emptyset = b$, $\beta_N = a$, and (iii) for any $S \subseteq T$, $\beta_T \leq \beta_S$. Then, an SCF $f^\beta : \mathcal{D}^n \to X$ is called a *min-max rule with respect to* β if

$$f^\beta(P_N) = \min_{S \subseteq N}\{\max_{i \in S}\{r_1(P_i), \beta_S\}\}.$$

REMARK 2.4. Every min-max rule is uncompromising.[3]

Definition 2.21. A min-max rule $f^\beta : \mathcal{D}^n \to X$ with parameters $\beta = (\beta_S)_{S \subseteq N}$ is a *partly dictatorial min-max rule* (PDMMR) if there exists an agent $d \in N$, called the *partial dictator* of f^β, such that $\beta_d \in [a, \underline{x}]$ and $\beta_{N \setminus d} \in [\overline{x}, b]$.

In Lemma 3.1, we explain why the particular agent d is called the partial dictator of f^β.

REMARK 2.5. Reffgen (2015) defines *partly dictatorial generalized median voter scheme* (PDGMVS) on multiple single-peaked domains. It can be shown that PDMMR coincides with PDGMVS on those domains.[4]

3. RESULTS

3.1 UNANIMOUS AND STRATEGY-PROOF SCFS ON PARTIALLY SINGLE-PEAKED DOMAINS

In this subsection, we characterize the unanimous and strategy-proof SCFs on partially single-peaked domains as partly dictatorial generalized median voter schemes.

[3] For details, see Weymark (2011).

[4] For details see the proof of Theorem 3.1 in Reffgen (2015).

First, we present a lemma that justifies why the agent d in Definition 2.21 is called the partial dictator. It shows that a PDMMR chooses the top-ranked alternative of the partial dictator whenever that lies in the interval $[\underline{x}, \overline{x}]$. It further shows that it chooses an alternative in the interval $[a, \underline{x}]$ or $[\overline{x}, b]$ depending on whenever the top-ranked alternative of the partial dictator lies in that interval.

Lemma 3.1. *Let $f^\beta : \mathcal{D}^n \to X$ be a PDMMR. Suppose agent d is the partial dictator of f^β. Then,*

(i) $f^\beta(P_N) \in [a, \underline{x}]$ *if* $r_1(P_d) \in [a, \underline{x})$,

(ii) $f^\beta(P_N) \in [\overline{x}, b]$ *if* $r_1(P_d) \in (\overline{x}, b]$, *and*

(iii) $f^\beta(P_N) = r_1(P_d)$ *if* $r_1(P_d) \in [\underline{x}, \overline{x}]$.

Proof. First, we prove (i). The proof of (ii) can be established using symmetric arguments. Assume for contradiction that $r_1(P_d) \in [a, \underline{x})$ and $f^\beta(P_N) > \underline{x}$. Since f^β is a min-max rule, f^β is uncompromising. Therefore, $f^\beta(P'_d, P_{N \setminus d}) = f^\beta(P_N)$, where $r_1(P'_d) = a$. Again by uncompromisingness, we have $f^\beta(P'_N) \geq f^\beta(P_N)$, where $r_1(P'_i) = b$ for all $i \neq d$. Because $f^\beta(P_N) > \underline{x}$, this means $f^\beta(P'_N) > \underline{x}$. However, by the definition of f^β, $f^\beta(P'_N) = \beta_d$. Since $\beta_d \in [a, \underline{x}]$, this is a contradiction. This completes the proof of (i).

Now, we prove (iii). Without loss of generality, assume for contradiction that $r_1(P_d) \in [\underline{x}, \overline{x}]$ and $f^\beta(P_N) > r_1(P_d)$. Using a similar argument as for the proof of (i), we have $f^\beta(P'_N) \geq f^\beta(P_N)$, where $r_1(P'_d) = a$ and $r_1(P'_i) = b$ for all $i \neq d$. This, in particular, means $f^\beta(P'_N) > \underline{x}$. Since by the definition of f^β, $f^\beta(P'_N) = \beta_d$ and $\beta_d \in [a, \underline{x}]$, this is a contradiction. This completes the proof of (iii). ■

Now, we present a characterization of the the unanimous and strategy-proof SCFs on partially single-peaked domains.

Theorem 3.1. *Let $\tilde{\mathcal{S}}$ be a partially single-peaked domain. Then, an SCF $f : \tilde{\mathcal{S}}^n \to X$ is unanimous and strategy-proof if and only if it is a PDMMR.*

The proof of the Theorem 3.1 is relegated to Appendix A.

Our next corollary is a consequence of Lemma 3.1 and Theorem 3.1. It characterizes a class of dictatorial domains, and thereby it generalizes the celebrated Gibbard-Satterthwaite (Gibbard (1973), Satterthwaite (1975)) results. Note that our dictatorial result is independent of those in Aswal et al. (2003), Sato (2010), Pramanik (2015), and so on.

Corollary 3.1. *Let $\underline{x} = a$ and $\overline{x} = b$. Then, every partially single-peaked domain is dictatorial.*

3.2 A Result on Partial Necessity

In Subsection 3.1, we have focused on partially single-peaked domains and have shown that every unanimous and strategy-proof SCF on those is a PDMMR. In this subsection, we look at the converse of this problem, that is, we focus on PDMMR and investigate the class of domains where these rules are unanimous and strategy-proof. We show that the partially single-peaked domains are almost all domains with the said property. This indicates that our notion of partial single-peaked domains is quite general. A formal definition is as follows.

Definition 3.1. A domain $\mathcal{D}$ is called a *PDMMR domain* if

(i) every unanimous and strategy-proof SCF on $\mathcal{D}^n$ is a PDMMR, and

(ii) every PDMMR on $\mathcal{D}^n$ is strategy-proof.

Conditions (i), (ii), and (iii) in Definition 2.10 are obviously strong conditions. Are they necessary for PDMMR domains? The question appears to be extremely difficult to resolve completely. However, Lemma 3.2 shows that Conditions (i) and (ii) are necessary, and the subsequent discussion shows that Condition (iii) is also close to being necessary in an appropriate sense.

Lemma 3.2. *Let $\mathcal{D}$ be a PDMMR domain. Then, $\mathcal{D}$ satisfies single-peakedness outside $[\underline{x}, \overline{x}]$.*

Proof. First, we show that a preference with top-ranked alternative in $[\underline{x}, \overline{x}]$ satisfies single-peakedness outside $[\underline{x}, \overline{x}]$. Without loss of generality, assume for contradiction that there exists $\tilde{P} \in \mathcal{D}$ with $r_1(\tilde{P}) \in [\underline{x}, \overline{x}]$ such that $u\tilde{P}v$ for some $u < v \leq x$. Consider the PDMMR $f^{\beta} : \mathcal{D}^n \to X$, where

$$\beta_S = \begin{cases} v \text{ if } S = \{1\}, \\ a \text{ if } \{1\} \subsetneq S, \\ b \text{ if } 1 \notin S. \end{cases}$$

We show that f^{β} is not strategy-proof. Note that agent 1 is the partial dictator of f^{β}. Consider the preference profile $P_N \in \mathcal{D}^n$ such that $r_1(P_1) = a$, $P_2 = \tilde{P}$, and $r_1(P_j) = b$ for all $j \neq 1, 2$. Then, by the definition of f^{β}, $f^{\beta}(P_N) = v$. Let $P_2' \in \mathcal{D}$ be such that $r_1(P_2') = u$. Again, by the definition of f^{β}, $f^{\beta}(P_2', P_{N \setminus 2}) = u$. Since $u\tilde{P}v$, this means agent 2 manipulates at P_N via P_2'.

Now, we show that a preference with top-ranked alternative outside $[\underline{x}, \overline{x}]$ satisfies single-peakedness outside $[\underline{x}, \overline{x}]$. Without loss of generality, assume for contradiction that there exist $\tilde{P} \in \mathcal{D}$ with $r_1(\tilde{P}) \in [a, \underline{x})$ and $u, v \in X$ with $u \notin (\underline{x}, \overline{x})$ such that $\left[v < u \leq r_1(P) \text{ or } r_1(P) \leq u < v\right]$ and $v\tilde{P}u$. If $\left[v < u \leq r_1(\tilde{P})\right]$ and $v\tilde{P}u$, then using a similar argument as for the proof of the necessity of Condition (i), it follows that there is a PDMMR on $\mathcal{D}^n$ that is manipulable. Hence, assume $r_1(\tilde{P}) \leq u < v$ and $v\tilde{P}u$. We distinguish two cases.

CASE 1. Suppose $u \leq \underline{x}$.
Consider the PDMMR $f^\beta : \mathcal{D}^n \to X$, where

$$\beta_S = \begin{cases} u \text{ if } 1 \in S \text{ and } S \neq N, \\ b \text{ if } 1 \notin S. \end{cases}$$

We show that f^β is not strategy-proof. Let $P_N \in \mathcal{D}^n$ be such that $P_1 = \tilde{P}$ and $r_1(P_j) = b$ for all $j \neq 1$. Then, by the definition of f^β, $f^\beta(P_N) = u$. Let $P_1' \in \mathcal{D}$ be such that $r_1(P_1') = v$. Again, by the definition of f^β, $f^\beta(P_1', P_{N\setminus 1}) = v$. Since $v\tilde{P}u$, agent 1 manipulates at P_N via P_1'.

CASE 2. Suppose $\underline{x} < u$.
Since $u \notin (\underline{x}, \overline{x})$, this means $\overline{x} \leq u$. Consider the PDMMR $f^\beta : \mathcal{D}^n \to X$, where

$$\beta_S = \begin{cases} a \text{ if } 1 \in S, \\ u \text{ if } 1 \notin S \text{ and } S \neq \emptyset. \end{cases}$$

We show that f^β is not strategy-proof. Let $P_N \in \mathcal{D}^n$ be such that $P_2 = \tilde{P}$ and $r_1(P_j) = b$ for all $j \neq 2$. Then, by the definition of f^β, $f^\beta(P_N) = u$. Let $P_2' \in \mathcal{D}$ be such that $r_1(P_2') = v$. Again, by the definition of f^β, $f^\beta(P_2', P_{N\setminus 2}) = v$. Since $v\tilde{P}u$, agent 2 manipulates at P_N via P_2'. ∎

Coming to the violation of single-peakedness over $[\underline{x}, \overline{x}]$, that is, the requirement of the existence of two particular preferences Q, Q' as mentioned in Definition 2.9, it is to be noted that it can be violated in many ways. We consider those domains obtained through mild violations of the same and show that there do exist unanimous and strategy-proof SCFs on such domains that are not PDMMR.

Now, we discuss the necessity of the existence of two particular preferences Q, Q' as mentioned in Definition 2.9. Recall that Definition 2.9 requires two non-single-peaked preferences $Q = \underline{x}y\ldots$ and $Q' = \overline{x}z\ldots$ in $\mathcal{D}$ such that either $\left[y \in (\underline{x}+1, \overline{x}) \text{ and } z \in (\underline{x}, \overline{x}-1)\right]$ or $\left[y = \overline{x} \text{ and } z = \underline{x}\right]$.

Suppose a domain $\mathcal{D}$ satisfies single-peakedness outside $[\underline{x}, \overline{x}]$. Suppose further that it contains a non-single-peaked preference of the form Q, but no preference of the form Q'. In the following example, we construct a two-agent unanimous and strategy-proof SCF on such a domain that is not a PDMMR.

Example 3.1. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$. By $P = x_1x_2x_3x_4x_5$, we mean a preference P such that $x_1Px_2Px_3Px_4Px_5$. Consider the domain as follows:

$$\mathcal{D} = \{x_1x_2x_3x_4x_5, x_1x_3x_4x_5x_2, x_2x_1x_3x_4x_5, x_2x_3x_4x_5x_1, x_3x_2x_1x_4x_5, x_3x_4x_5x_2x_1, x_4x_3x_2x_1x_5, x_4x_5x_3x_2x_1, x_5x_4x_3x_2x_1\}.$$

Note that $\mathcal{D} \setminus \{x_1x_3x_4x_5x_2\}$ is a top-connected single-peaked domain and the preference $x_1x_3x_4x_5x_2$ is of the form Q where $\underline{x} = x_1$ and $\overline{x} \geq x_3$. However, there is no preference in $\mathcal{D}$ of the form Q', that is, no preference Q' with $r_1(Q') \geq x_3$ and $r_2(Q') \in [x_1, r_1(Q') - 1)$. In Table 2, we present a two-agent SCF that is unanimous and strategy-proof but not a PDMMR.

P_1 \ P_2	$x_1x_2x_3x_4x_5$	$x_1x_3x_4x_5x_2$	$x_2x_1x_3x_4x_5$	$x_2x_3x_4x_5x_1$	$x_3x_2x_1x_4x_5$	$x_3x_4x_5x_2x_1$	$x_4x_3x_2x_1x_5$	$x_4x_5x_3x_2x_1$	$x_5x_4x_3x_2x_1$
$x_1x_2x_3x_4x_5$	x_1	x_1	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_1x_3x_4x_5x_2$	x_1	x_1	x_2	x_2	x_3	x_3	x_3	x_3	x_3
$x_2x_1x_3x_4x_5$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_2x_3x_4x_5x_1$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_3x_2x_1x_4x_5$	x_2	x_3	x_2	x_2	x_3	x_3	x_3	x_3	x_3
$x_3x_4x_5x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_3	x_3	x_3
$x_4x_3x_2x_1x_5$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_4
$x_4x_5x_3x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_4
$x_5x_4x_3x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_5

Table 2: A unanimous and strategy-proof SCF which is not a PDMMR

It is left to the reader to verify that the SCF presented in Table 2 is unanimous and strategy-proof. Note that it violates tops-onlyness at the preference profiles $(x_3x_4x_5x_2x_1, x_1x_2x_3x_4x_5)$ and $(x_3x_4x_5x_2x_1, x_1x_3x_4x_5x_2)$, and hence it is not a PDMMR.

Now, suppose that $\mathcal{D}$ contains two non-single-peaked preferences Q and Q', however, they do *not* satisfy Definition 2.9 for their second-ranked alternatives. In the following example, we construct a two-agent unanimous and strategy-proof SCF on such a domain $\mathcal{D}$ that is not a PDMMR.

Example 3.2. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$. Let $\mathcal{D}$ be the domain given in Example 3.1. Consider the domain $\mathcal{D} \cup \{x_5x_1x_4x_3x_2\}$. As pointed out in Example 3.1,

$\mathcal{D} \setminus \{x_1x_3x_4x_5x_2\}$ is a top-connected single-peaked domain. Consider the non-single-peaked preferences $x_1x_3x_4x_5x_2$ and $x_5x_1x_4x_3x_2$. They can be considered as Q and Q' only if $\underline{x} = x_1$ and $\overline{x} = x_5$. However, since their second-ranked alternatives are x_3 and x_1, respectively, they do not satisfy Definition 2.9. In Table 3, we present a two-agent SCF that is unanimous and strategy-proof but not a PDMMR.

P_1 \ P_2	$x_1x_2x_3x_4x_5$	$x_1x_3x_4x_5x_2$	$x_2x_1x_3x_4x_5$	$x_2x_3x_4x_5x_1$	$x_3x_2x_1x_4x_5$	$x_3x_4x_5x_2x_1$	$x_4x_3x_2x_1x_5$	$x_4x_5x_3x_2x_1$	$x_5x_4x_3x_2x_1$	$x_5x_1x_4x_3x_2$
$x_1x_2x_3x_4x_5$	x_1	x_1	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_1
$x_1x_3x_4x_5x_2$	x_1	x_1	x_2	x_2	x_3	x_3	x_3	x_3	x_3	x_1
$x_2x_1x_3x_4x_5$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_2x_3x_4x_5x_1$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_3x_2x_1x_4x_5$	x_2	x_3	x_2	x_2	x_3	x_3	x_3	x_3	x_3	x_3
$x_3x_4x_5x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_3	x_3	x_3	x_3
$x_4x_3x_2x_1x_5$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_4	x_4
$x_4x_5x_3x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_4	x_4
$x_5x_4x_3x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_5	x_5
$x_5x_1x_4x_3x_2$	x_1	x_1	x_2	x_2	x_3	x_3	x_4	x_4	x_5	x_5

Table 3: A unanimous and strategy-proof SCF which is not a PDMMR

Note that the restriction of the SCF presented in Table 3 to $\mathcal{D}^2$ is same as the SCF presented in Table 2. It is left to the reader to verify that this SCF is unanimous and strategy-proof. However, as pointed out in Example 3.1, it violates tops-onlyness, and hence it is not a PDMMR.

3.3 Group Strategy-proofness

In this section, we consider group strategy-proofness and obtain a characterization of the unanimous and group strategy-proof SCFs on partially single-peaked domains. We begin with the definition of group strategy-proofness.

Definition 3.2. An SCF $f : \mathcal{D}^n \to X$ is called *group manipulable* if there is a preference profile P_N, a non-empty coalition $C \subseteq N$, and a preference profile $P'_C \in \mathcal{D}^{|C|}$ of the agents in C such that $f(P'_C, P_{N \setminus C}) P_i f(P_N)$ for all $i \in C$. An SCF $f : \mathcal{D}^n \to X$ is called *group strategy-proof* if it is not group manipulable.

In the following theorem, we present a characterization of the unanimous and group strategy-proof SCFs on partially single-peaked domains. It is worth mentioning that these domains do not satisfy the sufficient condition for the equivalence of strategy-proofness and group strategy-proofness provided in Barberà et al. (2010).

Theorem 3.2. *Let $\tilde{\mathcal{S}}$ be a partially single-peaked domain. Then, an SCF $f : \tilde{\mathcal{S}}^n \to X$ is unanimous and group strategy-proof if and only if it is a PDMMR.*

Proof. Let $\tilde{\mathcal{S}}$ be a partially single-peaked domain. Suppose $f : \tilde{\mathcal{S}}^n \to X$ is a PDMMR where agent d is the partial dictator. It is enough to show that f is group strategy-proof. Clearly, no group can manipulate f at a preference profile $P_N \in \tilde{\mathcal{S}}^n$ where $r_1(P_d) \in [\underline{x}, \overline{x}]$. Consider a preference profile $P_N \in \tilde{\mathcal{S}}^n$ such that $r_1(P_d) \in [a, \underline{x})$. We show that f is group strategy-proof at P_N. Since $r_1(P_d) \in [a, \underline{x})$, by the definition of PDMMR, $f(P_N) \in [a, \underline{x}]$. Let $C' = \{i \in N \mid r_1(P_i) \leq f(P_N)\}$ and let $C'' = \{i \in N \mid r_1(P_i) > f(P_N)\}$. Suppose a coalition C manipulates f at P_N. Then, there is $P'_C \in \tilde{\mathcal{S}}^{|C|}$ such that $f(P'_C, P_{N \setminus C}) P_i f(P_N)$ for all $i \in C$. If $f(P'_C, P_{N \setminus C}) < f(P_N)$, then by the definition of $\tilde{\mathcal{S}}$, we have $C \cap C'' = \emptyset$. However, by the definition of PDMMR, $f(P'_C, P_{N \setminus C}) \geq f(P_N)$ for all $C \subseteq C'$ and all $P'_C \in \tilde{\mathcal{S}}^{|C|}$, a contradiction. Again, if $f(P'_C, P_{N \setminus C}) > f(P_N)$, then by the definition of $\tilde{\mathcal{S}}$, we have $C \cap C' = \emptyset$. However, by the definition of PDMMR, $f(P'_C, P_{N \setminus C}) \leq f(P_N)$ for all $C \subseteq C''$ and all $P'_C \in \tilde{\mathcal{S}}^{|C|}$, a contradiction. The proof of the same for the case where $r_1(P_d) \in (\overline{x}, b]$ follows from a symmetric argument. This shows f is group strategy-proof, and hence completes the proof of the theorem. ■

4. Applications

In this section, we present a couple of examples of our main result.

4.1 Multi-peaked Domains

In Section 1, we have discussed the importance of multi-peaked domains in modeling preferences of individuals in certain economic and political scenarios. In this subsection, we formally define this notion and show that these are special cases of partially single-peaked domains.

Definition 4.1. A preference P is called *multi-peaked* if there are $d_0, p_1, d_1, p_2, d_2, \ldots, d_{k-1}, p_k, d_k$ with $a = d_0 \leq p_1 < d_1 < \ldots < p_k \leq d_k = b$ such that for all $i = 0, \ldots, k-1$ and all $x, y \in [d_i, d_{i+1}]$, $[x < y \leq p_{i+1}$ or $p_{i+1} \leq y < x]$ implies yPx. For such a preference P the alternatives $p_1, \ldots, p_k$ are called its *peaks*.

We present a multi-peaked domain in Figure 4.

Definition 4.2. Let c_1 and c_2 be such that $a \leq c_1 < c_2 - 1 \leq b$. Then, a domain $\mathcal{D}$ is called *multi-peaked with critical values* c_1, c_2 if each preference in $\mathcal{D}$ is either single-peaked or multi-peaked with all its peaks in the interval $[c_1, c_2]$.

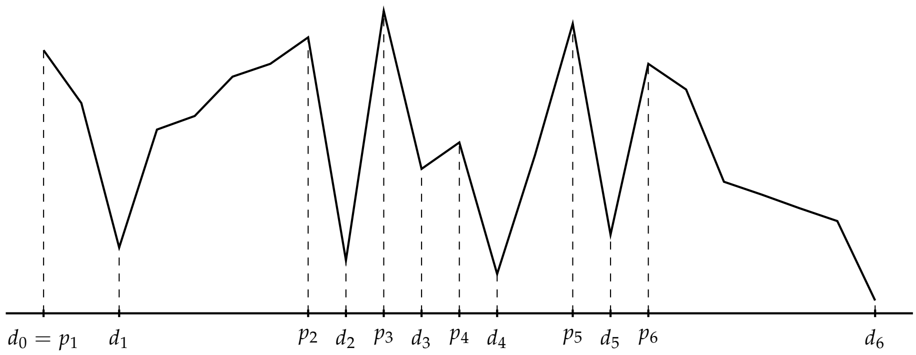


Figure 4: A multi-peaked preference

It is easy to verify that a multi-peaked domain with critical values $\underline{x}$ and $\overline{x}$ is a partially single-peaked domain. Thus, we have the following corollary.

Corollary 4.1. *Let $\mathcal{S}$ be a multi-peaked domain. Then, an SCF $f : \mathcal{S}^n \to X$ is unanimous and (group) strategy-proof if and only if it is a PDMMR.*

4.2 Single-peaked Domains with respect to Partial Orders

As discussed in Section 1, expecting individuals to have a complete prior order over the alternatives is a strong prerequisite. In view of this, we relax this condition by requiring the individuals to have a partial prior order over the alternatives and to derive preferences based on such a partial order. In this subsection, we argue that such a domain is partially single-peaked.

Definition 4.3. A binary relation is called a *partial order* if it is reflexive, antisymmetric, and transitive.

Note that a partial order need not be complete. We denote a partial order by $\triangleleft$.[5] Also, we write $a \trianglelefteq b$ to mean $a \triangleleft b$ or $a = b$.

Definition 4.4. A preference P is said to be *single-peaked with respect to a partial order* $\triangleleft$ over X if for all distinct $x, y \in X$,

$$[x \triangleleft y \trianglelefteq r_1(P) \text{ or } r_1(P) \trianglelefteq y \triangleleft x] \text{ implies } yPx.$$

A domain is called *single-peaked with respect to a partial order* $\triangleleft$ if it contains all single-peaked preferences with respect to $\triangleleft$.

[5] To be precise, the antisymmetric part of a partial order.

Since every partial order can be thought of a subset of a linear order (as a binary relation), it can be shown that a single-peaked domain with respect to a partial order is partially single-peaked. However, we do not provide a concrete proof of this since that is a bit technical.[6] Nevertheless, in what follows we provide a few examples of single-peaked domains with respect to partial orders and show that those domains are partially single-peaked.

Example 4.1. Suppose that the set of alternatives is partitioned into a number of subsets such that the designer knows how agents order (*a priori*) the alternatives in each of those subsets, but does not know how agents compare alternatives in two different subsets.

More formally, suppose that X is partitioned into the subsets $X_1, \ldots, X_k$. For all $i = 1, \ldots, k$, let $\prec_i \in \mathbb{L}(X_i)$ be a linear order over X_i. Consider the partial order $\lhd$ over X given by the union of $\prec_i$s, that is, $x \lhd y$ if and only if there is $i = 1, \ldots, k$ such that $x, y \in X_i$ and $x \prec_i y$. In what follows, we consider a simple such partial order and present the single-peaked domain with respect to the same.

Let the set of alternatives be $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Suppose that X is partitioned into the sets $\{x_1, x_2, x_3\}$ and $\{x_4, x_5, x_6\}$. Consider the partial order $\lhd$ given by $x_1 \lhd x_2 \lhd x_3$ and $x_4 \lhd x_5 \lhd x_6$. In Table 4, we present the single-peaked domain with respect to $\lhd$. Note that the domain has the property that its restriction on $\{x_1, x_2, x_3\}$ is single-peaked with respected to the prior order $x_1 \lhd x_2 \lhd x_3$ and on $\{x_4, x_5, x_6\}$ is single-peaked with respected to the prior order $x_4 \lhd x_5 \lhd x_6$. Since this domain is large, we provide only a few preferences that are significant for our purpose. Clearly, this domain is partially single-peaked with $\underline{x} = x_1$ and $\overline{x} = x_6$. Therefore, it follows from Theorem 3.1 that it is a dictatorial domain.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
$\mathbf{x_1}$	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_6}$
$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_3}$	$\mathbf{x_5}$	$\mathbf{x_4}$	$\mathbf{x_6}$	$\mathbf{x_5}$	$\mathbf{x_3}$
x_3	x_2	x_3	x_1	x_1	x_5	x_2	x_6	x_6	x_4	x_4	x_5
x_4	x_5	x_4	x_4	x_4	x_6	x_1	x_3	x_3	x_3	x_3	x_2
x_5	x_3	x_5	x_5	x_5	x_2	x_5	x_2	x_2	x_2	x_2	x_4
x_6	x_6	x_6	x_6	x_6	x_1	x_6	x_1	x_1	x_1	x_1	x_1

Table 4: A single-peaked domain with respect to the partial order $\lhd$

Example 4.2. In political science, it is often assumed that the parties can be ordered from left to right on the policy spectrum based on whether they are more liberal (left) or more conservative

[6] A proof of this fact is available on request.

(right) in their policies. Deriving such an ordering can be done unambiguously over the parties who are clearly identifiable as more left or more right. However, ordering parties who are moderate in their policies (i.e., having policies around the center of the spectrum) may not be possible. To model such a situation, one needs to assume that the prior ordering of the parties (on the political spectrum) is not complete around the center of the spectrum. In what follows, we consider a simple such partial order and present the single-peaked domain with respect to the same.

Suppose that the set of alternatives is given by $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Consider the partial order $\lhd$ obtained from the linear order $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5 \prec x_6$ by making x_3 and x_4 incomparable, that is, $\lhd$ is given by $x_1 \lhd x_2 \lhd x_3 \lhd x_5 \lhd x_6$ and $x_1 \lhd x_2 \lhd x_4 \lhd x_5 \lhd x_6$. The single-peaked domain with respect to $\lhd$ is given in Table 5. Note that this domain is partially single-peaked with $\underline{x} = x_2$ and $\overline{x} = x_5$. Therefore, it follows from Theorem 3.1 and Theorem 3.2 that any unanimous and (group) strategy-proof SCF on this domain is a PDMMR.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
$\mathbf{x_1}$	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_5}$	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_6}$
$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_3}$	$\mathbf{x_5}$	$\mathbf{x_4}$	$\mathbf{x_6}$	$\mathbf{x_3}$	$\mathbf{x_5}$	$\mathbf{x_5}$
x_3	x_4	x_3	x_1	x_3	x_1	x_5	x_5	x_3	x_6	x_4	x_4	x_4	x_3
x_4	x_3	x_4	x_4	x_1	x_4	x_6	x_6	x_6	x_3	x_3	x_2	x_3	x_4
x_5	x_5	x_5	x_5	x_5	x_5	x_2	x_2	x_2	x_2	x_2	x_6	x_2	x_2
x_6	x_6	x_6	x_6	x_6	x_6	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1

Table 5: A single-peaked domain with respect to the partial order $\lhd$

The following corollary summarizes the above discussion on single-peaked domains with respect to a partial order.

Corollary 4.2. *Let $\lhd$ be a partial order over X and let $\mathcal{S}$ be the single-peaked domain with respect to $\lhd$. Then, an SCF $f : \mathcal{S}^n \to X$ is unanimous and (group) strategy-proof if and only if it is a PDMMR.*

4.3 Multiple Single-peaked Domain

In this subsection, we consider a well-known class of domains called multiple single-peaked domains and show that they are special cases of partially single-peaked domains.

We begin with introducing the notion of a single-peaked domain with respect to an arbitrary order over X.

Definition 4.5. Let $\prec \in \mathbb{L}(X)$ be a prior order over X. Then, a preference $P \in \mathbb{L}(X)$ is *single-peaked with respect to* $\prec$ if for all $x, y \in X$, $[x \prec y \preceq r_1(P)$ or $r_1(P) \preceq y \prec x]$ implies yPx. A domain $\mathcal{S}_{\prec}$ is called a *single-peaked domain with respect to* $\prec$ if each preference in it is single-peaked with respect to $\prec$, and a domain $\bar{\mathcal{S}}_{\prec}$ is called *maximal single-peaked with respect to* $\prec$ if it contains all single-peaked preferences with respect to $\prec$.

Definition 4.6. Let $\mathcal{L} = \{\prec_1, \ldots, \prec_q\}$, where $\prec_k \in \mathbb{L}(X)$ for all $1 \leq k \leq q$, be a set of q prior orders over X. Then, a domain is called a *multiple single-peaked domain with respect to* $\mathcal{L}$, denoted by $\mathcal{S}_{\mathcal{L}}$, if $\mathcal{S}_{\mathcal{L}} = \bigcup\limits_{k \in \{1,\ldots,q\}} \bar{\mathcal{S}}_{\prec_k}$, where $\bar{\mathcal{S}}_{\prec_k}$ is the maximal single-peaked domain with respect to the prior order $\prec_k$. A multiple single-peaked domain with respect to $\mathcal{L}$ is called *trivial* if $\bar{\mathcal{S}}_{\prec} = \bar{\mathcal{S}}_{\prec'}$ for all $\prec, \prec' \in \mathcal{L}$.

For ease of presentation, for any multiple single-peaked domain with respect to $\mathcal{L}$, we assume without loss of generality that the integer ordering $<$ is in the set $\mathcal{L}$.

Definition 4.7. Let $\mathcal{S}_{\mathcal{L}}$ be a non-trivial multiple single-peaked domain with respect to a set of prior orders $\mathcal{L}$. Then, alternatives $u, v \in X$ with $u < v - 1$ are called *break-points* of $\mathcal{S}_{\mathcal{L}}$ if

(i) for all preferences $P \in \mathcal{S}_{\mathcal{L}}$ and all $c, d \in X \setminus (u, v)$, $\left[d < c \leq r_1(P) \text{ or } r_1(P) \leq c < d\right]$ implies cPd, and

(ii) there exist $P, P' \in \mathcal{S}_{\mathcal{L}}$ such that $r_1(P) = u$, $r_2(P) \in (u+1, v]$, $r_1(P') = v$, and $r_2(P') \in [u, v-1)$.

REMARK 4.1. The break points, say u, v, of a non-trivial multiple single-peaked domain $\mathcal{S}_{\mathcal{L}}$ induce the partition $\{X_L, X_M, X_R\}$ of X, where $X_L = [a, u)$, $X_M = [u, v]$, and $X_R = (v, b]$. Reffgen (2015) calls such a partition the *maximal common decomposition* of X and the sets X_L, X_M, and X_R as the *left component*, the *middle component*, and the *right component* of alternatives, respectively.

In the following, we illustrate the notion of break-points of a non-trivial multiple single-peaked domain by means of an example.

Example 4.3. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ be the set of alternatives. Consider the set of prior orders $\mathcal{L} = \{<, \prec_1, \prec_2, \prec_3\}$, where $<= x_1x_2x_3x_4x_5x_6x_7$, $\prec_1 = x_1x_2x_3x_5x_4x_6x_7$, $\prec_2 = x_1x_2x_5x_4x_3x_6$ x_7, and $\prec_3 = x_1x_2x_4x_3x_5x_6x_7$. Let $\mathcal{S}_{\mathcal{L}}$ be the multiple single-peaked domain with respect to $\mathcal{L}$. Clearly, $\mathcal{S}_{\mathcal{L}}$ is non-trivial since $\bar{\mathcal{S}}_{\prec_1} \neq \bar{\mathcal{S}}_{\prec_2}$. We claim $u = x_2$ and $v = x_6$ are the break points of

$\mathcal{S}_{\mathcal{L}}$. It is easy to verify that $\mathcal{S}_{\mathcal{L}}$ satisfies Condition (i) in Definition 4.7. For Condition (ii), note that we have preferences $P, P' \in \bar{\mathcal{S}}_{\prec_2} \subseteq \mathcal{S}_{\mathcal{L}}$ where $r_1(P) = x_2$, $r_2(P) = x_5$, $r_1(P') = x_6$, and $r_2(P') = x_3$. Further, note that the maximal common decomposition of X is given by $X_L = \{x_1\}$, $X_M = \{x_2, x_3, x_4, x_5, x_6\}$, and $X_R = \{x_7\}$.

It can be easily verified that every non-trivial multiple single-peaked domain is a partially single-peaked domain where $\underline{x}$ and $\overline{x}$ are the break-points. Thus, we have the following corollary.

Corollary 4.3 (Reffgen (2015))**.** *Let $\mathcal{S}_{\mathcal{L}}$ be a non-trivial multiple single-peaked domain with break-points $\underline{x}$ and $\overline{x}$. Then, an SCF $f : \mathcal{S}_{\mathcal{L}}^n \to X$ is unanimous and (group) strategy-proof if and only if it is a PDMMR.*

4.4 Single-peaked Domains on Graphs

In this subsection, we introduce the notion of single-peaked domains on graphs and show that such a domain is partially single-peaked if the underlying graph satisfies some condition. All the graphs we consider in this subsection are undirected.

Definition 4.8. A *path* in an undirected graph $G = \langle X, E \rangle$ from a node x to a node y, denoted by $\pi_G(x, y)$, is defined as a sequence of nodes $(x_1, \ldots, x_k)$ such that $\{x_i, x_{i+1}\} \in E$ for all $i = 1, \ldots, k-1$. An undirected graph $G = \langle X, E \rangle$ is called *connected* if for all $x, y \in X$, there is a path from x to y.

Definition 4.9. An undirected graph $G = \langle X, E \rangle$ is called a *tree* if for every two distinct nodes $x, y \in X$, there is a unique path from x to y. A *spanning tree* of an undirected connected graph G is defined as a connected subgraph of G that is a tree. For an undirected connected graph G, we denote by $\mathcal{T}_G$ the set of all spanning trees of G.

Definition 4.10. Let $T = \langle X, E \rangle$ be a tree. Then, a domain is called *single-peaked with respect to T*, denoted by $\mathcal{S}_T$, if for all $P \in \mathcal{S}_T$ and all distinct $x, y \in X$,

$$[x \in \pi_T(r_1(P), y)] \implies [xPy].$$

Definition 4.11. Let $G = \langle X, E \rangle$ be an undirected connected graph. Then, a domain is called *single-peaked with respect to G*, denoted by $\mathcal{S}_G$, if $\mathcal{S}_G = \cup_{T \in \mathcal{T}_G} \mathcal{S}_T$.

Note that if T is the undirected line graph on X, then $\mathcal{S}_T$ is the maximal single-peaked domain. In Lemma 4.1, we show that if a domain is single-peaked with respect to an undirected partial line graph as defined in Definition 2.4, then it is a partially single-peaked domain.

Lemma 4.1. *Let G be an undirected partial line graph. Then, $\mathcal{S}_G$ is a partially single-peaked domain.*

Proof. Let G be an undirected partial line graph. We show that $\mathcal{S}_G$ is a partially single-peaked domain. Let $G = G_1 \cup G_2$, where $G_1 = \langle X, E_1 \rangle$ is the undirected line graph on X and $G_2 = \langle [\underline{x}, \overline{x}], E_2 \rangle$ is an undirected graph such that $\{\underline{x}, y\}, \{\overline{x}, z\} \in E_2$ for some $y \in (\underline{x}+1, \overline{x}]$ and $z \in [\underline{x}, \overline{x}-1)$.

First, we show that $\mathcal{S}_G$ satisfies single-peakedness outside $[\underline{x}, \overline{x}]$. Take $P \in \mathcal{S}_G$ with $r_1(P) \in [\underline{x}, \overline{x}]$ and take $u, v \in X \setminus (\underline{x}, \overline{x})$. Suppose $[v < u \leq r_1(P)$ or $r_1(P) \leq u < v]$. Consider an arbitrary spanning tree T of G. Then, by the definition of G, $u \in \pi_T(r_1(P), v)$, and hence uPv. Therefore, P satisfies single-peakedness outside $[\underline{x}, \overline{x}]$. Using a similar argument, it can be shown that a preference P with $r_1(P) \notin [\underline{x}, \overline{x}]$ satisfies single-peakedness outside $[\underline{x}, \overline{x}]$.

Next, we show that $\mathcal{S}_G$ violates single-peakedness over $[\underline{x}, \overline{x}]$. Consider the tree $T = \langle X, E \rangle$ such that $E = (E_1 \setminus \{\underline{x}, \underline{x}+1\}) \cup \{\underline{x}, y\}$. Since $G_1 = \langle X, E_1 \rangle$ is the undirected line graph on X, T is a spanning tree of G. Because $\{\underline{x}, y\} \in E$, there is a preference $Q = \underline{x}y \ldots \in \mathcal{S}_T \subseteq \mathcal{S}_G$. Similarly, there is a preference $Q' = \overline{x}z \ldots \in \mathcal{S}_G$. If $y \neq \overline{x}$ and $z \neq \underline{x}$, then clearly Q and Q' satisfy Definition 2.9. On the other hand, if, for instance, $y = \overline{x}$, then that means there is an edge $\{\underline{x}, \overline{x}\}$ in G, and consequently, z can be chosen as $\underline{x}$. This shows $\mathcal{S}_G$ violates single-peakedness over $[\underline{x}, \overline{x}]$.

Now, we show that $\mathcal{S}_G$ contains a top-connected single-peaked domain. Since G_1 is the undirected line graph on X, $\mathcal{S}_{G_1}$ is the maximal single-peaked domain. Moreover, since G_1 is a spanning tree of G, $\mathcal{S}_{G_1} \subseteq \mathcal{S}_G$. This completes the proof of the lemma. ∎

Combining Theorem 3.1 and Theorem 3.2 with Lemma 4.1, we obtain the following characterization of the unanimous and strategy-proof SCFs on a single-peaked domain with respect to an undirected partial line graph.

Corollary 4.4. *Let $G = \langle X, E \rangle$ be an undirected partial line graph. Suppose $\mathcal{S}_G$ is the single-peaked domain with respect to G. Then, an SCF $f : \mathcal{S}_G^n \to X$ is unanimous and (group) strategy-proof if and only if it is a PDMMR.*

5. Conclusion

In this paper, we have considered non-single-peaked domains that arise in the literature of economics and political science. We have modelled them as partially single-peaked domains and have characterized all unanimous and (group) strategy-proof rules on those as PDMMR.

Appendix A. Proof of Theorem 3.1

We use the following theorem in Achuthankutty and Roy (2017) in the proof of Theorem 3.1. It characterizes the unanimous and strategy-proof SCFs on a top-connected single-peaked domain as min-max rules.

Theorem A.1 (Achuthankutty and Roy (2017))**.** *Let $\mathcal{S}$ be a top-connected single-peaked domain. Then, an SCF $f : \mathcal{S}^n \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

Proof of Theorem 3.1. (If part) Let $\tilde{\mathcal{S}}$ be a partially single-peaked domain. Suppose f^β be a PDMMR on $\tilde{\mathcal{S}}^n$. Then, f^β is unanimous by definition. We show that f^β is strategy-proof. Let d be the partial dictator of f^β. If $r_1(P_d) \in [\underline{x}, \overline{x}]$, then $f^\beta(P_N) = r_1(P_d)$, and hence f^β cannot be manipulated at a preference profile $P_N \in \tilde{\mathcal{S}}^n$. Take $P_N \in \tilde{\mathcal{S}}^n$ such that $r_1(P_d) \in [a, \underline{x})$. Then, by Lemma 3.1, $f^\beta(P_N) \in [a, \underline{x}]$. Take $i \in N$ such that $r_1(P_i) \leq f^\beta(P_N)$. By the definition of f^β, $f^\beta(P'_i, P_{N\setminus i}) \geq f^\beta(P_N)$ for all $P'_i \in \tilde{\mathcal{S}}$. Since $f^\beta(P_N) \leq \underline{x}$, by the definition of a partially single-peaked domain, $r_1(P_i) \leq f^\beta(P_N)$ means $f^\beta(P_N) P_i u$ for all $u > f^\beta(P_N)$. Therefore, agent i cannot manipulate f^β at P_N. By a symmetric argument, agent i cannot manipulate f^β at a preference profile where $r_1(P_i) \geq f^\beta(P_N)$. Using a similar argument, it follows that f^β cannot be manipulated at a preference profile P_N with $r_1(P_d) \in (\overline{x}, b]$. This completes the proof of the if part.

(Only-if part) Let $\tilde{\mathcal{S}}$ be a partially single-peaked domain. Suppose $f : \tilde{\mathcal{S}}^n \to X$ is a unanimous and strategy-proof SCF. We show that f is a PDMMR. Let $\mathcal{S}$ be a top-connected single-peaked domain contained in $\tilde{\mathcal{S}}$. Such a domain must exist by Definition 2.10. By Theorem A.1, f restricted to $\mathcal{S}^n$ must be a min-max rule. We establish a few properties of f in the following sequence of lemmas.

Our next lemma and its corollary show that f satisfies tops-onlyness for a particular type of preference profiles. It says the following. Let c be an arbitrary alternative. Consider a preference profile P_N such that for all $i \in N$, P_i is single-peaked and $r_1(P_i) \in \{\underline{x}, c\}$. Suppose the outcome of

f at P_N is c. Consider a tops-equivalent preference profile P'_N where the agents with top-ranked alternative c in P_N do not change their preferences in P'_N. Then, the outcome of f at P'_N must be c.

Lemma A.1. *Let $\emptyset \subsetneq S \subsetneq N$ and let $c \in X$. Suppose $(P_S, P_{N\setminus S}) \in \mathcal{S}^n$ and $(P'_S, P_{N\setminus S}) \in \tilde{\mathcal{S}}^n$ are two tops-equivalent preference profiles such that $r_1(P_i) = \underline{x}$ for all $i \in S$, and $r_1(P_j) = c$ for all $j \in N \setminus S$. Then, $f(P_S, P_{N\setminus S}) = c$ implies $f(P'_S, P_{N\setminus S}) = c$.*

Proof. Take S such that $\emptyset \subsetneq S \subsetneq N$. We prove the lemma using induction on $|c - \underline{x}|$. By unanimity, the lemma holds for $c = \underline{x}$. Suppose the lemma holds for all c such that $|c - \underline{x}| \leq k$. We prove the lemma for all c such that $|c - \underline{x}| = k + 1$. Take c such that $|c - \underline{x}| = k + 1$. Let $(P_S, P_{N\setminus S}) \in \mathcal{S}^n$ and $(P'_S, P_{N\setminus S}) \in \tilde{\mathcal{S}}^n$ be two tops-equivalent preference profiles such that $r_1(P_i) = \underline{x}$ for all $i \in S$, and $r_1(P_j) = c$ for all $j \in N \setminus S$. Suppose $f(P_S, P_{N\setminus S}) = c$. We show $f(P'_S, P_{N\setminus S}) = c$. We show this for $\underline{x} < c$, the proof for the case $\underline{x} > c$ is similar. Since $\underline{x} < c$ and $|c - \underline{x}| = k + 1$, we have $c = \underline{x} + k + 1$. Let $(P_S, \hat{P}_{N\setminus S}) \in \mathcal{S}^n$ be such that $\hat{P}_j = (\underline{x} + k)(\underline{x} + k + 1) \ldots$ for all $j \in N \setminus S$. Because f is a min-max rule on $\mathcal{S}^n$ and $f(P_S, P_{N\setminus S}) = \underline{x} + k + 1$, we have $f(P_S, \hat{P}_{N\setminus S}) = \underline{x} + k$. Since $(P_S, \hat{P}_{N\setminus S})$ and $(P'_S, \hat{P}_{N\setminus S})$ are tops-equivalent and $r_1(\hat{P}_j) = \underline{x} + k$ for all $j \in N \setminus S$, we have by the induction hypothesis, $f(P'_S, \hat{P}_{N\setminus S}) = \underline{x} + k$. For all $j \in N \setminus S$, let $\bar{P}_j = (\underline{x} + k + 1)(\underline{x} + k) \ldots \in \mathcal{S}$. Since $f(P'_S, \hat{P}_{N\setminus S}) = \underline{x} + k$, by moving the agents $j \in N \setminus S$ from $\hat{P}_j$ to $\bar{P}_j$ one-by-one and applying strategy-proofness at every step, we have $f(P'_S, \bar{P}_{N\setminus S}) \in \{\underline{x} + k, \underline{x} + k + 1\}$. We claim $f(P'_S, \bar{P}_{N\setminus S}) = \underline{x} + k + 1$. Assume for contradiction that $f(P'_S, \bar{P}_{N\setminus S}) = \underline{x} + k$. Recall that $P_i \in \mathcal{S}$ for all $i \in S$. Since $(\underline{x} + k) P_i (\underline{x} + k + 1)$ for all $i \in S$, by moving the agents $i \in S$ from P'_i to P_i one-by-one and applying strategy-proofness at every step, we have $f(P_S, \bar{P}_{N\setminus S}) \leq \underline{x} + k$. Since $r_1(P_j) = r_1(\bar{P}_j) = \underline{x} + k + 1$ for all $j \in N \setminus S$, by strategy-proofness, $f(P_S, P_{N\setminus S}) \neq \underline{x} + k + 1$. This contradicts our assumption that $f(P_S, P_{N\setminus S}) = \underline{x} + k + 1$. Therefore, $f(P'_S, \bar{P}_{N\setminus S}) = \underline{x} + k + 1$. Since $r_1(P_j) = r_1(\bar{P}_j) = \underline{x} + k + 1$ for all $j \in N \setminus S$, we have by strategy-proofness, $f(P'_S, P_{N\setminus S}) = \underline{x} + k + 1$. This completes the proof of the lemma. ■

Corollary A.1. *Let $\emptyset \subsetneq S \subsetneq N$ and let $c \in X$. Suppose $(P_S, P_{N\setminus S}) \in \mathcal{S}^n$ and $(P'_S, P_{N\setminus S}) \in \tilde{\mathcal{S}}^n$ are two tops-equivalent preference profiles such that $r_1(P_i) = \overline{x}$ for all $i \in S$, and $r_1(P_j) = c$ for all $j \in N \setminus S$. Then, $f(P_S, P_{N\setminus S}) = c$ implies $f(P'_S, P_{N\setminus S}) = c$.*

Our next lemma shows that the outcome of f at a boundary preference profile cannot be strictly in-between $\underline{x}$ and $\overline{x}$.[7]

[7] A *boundary preference profile* is one where the top-ranked alternative of each agent is either a or b.

Lemma A.2. *Let $P_N \in \tilde{\mathcal{S}}^n$ be such that $r_1(P_i) \in \{a, b\}$ for all $i \in N$. Then, $f(P_N) \notin (\underline{x}, \overline{x})$.*

Proof. Assume for contradiction that $f(P_N) = u \in (\underline{x}, \overline{x})$ for some $P_N \in \tilde{\mathcal{S}}^n$ such that $r_1(P_i) \in \{a, b\}$ for all $i \in N$. Let $S = \{i \in N \mid r_1(P_i) = a\}$. Then, it must be that $\emptyset \subsetneq S \subsetneq N$ as otherwise we are done by unanimity. Let $r_2(Q) = y$ and $r_2(Q') = z$, where $Q, Q' \in \tilde{\mathcal{S}}$ are as given in Definition 2.9. We distinguish three cases based on the relative positions of y, z, and u.

Case 1. Suppose $y \in (\underline{x}+1, \overline{x})$, $z \in (\underline{x}, \overline{x}-1)$, and $u \in (\underline{x}, z] \cup [y, \overline{x})$.
We consider the case where $u \in (\underline{x}, z]$, the proof for the case where $u \in [y, \overline{x})$ follows from a symmetric argument. Let $P'_N \in \mathcal{S}^n$ be such that $r_1(P'_i) = z$ for all $i \in S$, and $P'_j = (\overline{x}-1)(\overline{x})\ldots$ for all $j \in N \setminus S$. Further, let $\hat{P}_N \in \mathcal{S}^n$ be such that $r_1(\hat{P}_i) = \underline{x}$ for all $i \in S$ and $r_1(\hat{P}_j) = \underline{x}+1$ for all $j \in N \setminus S$. Because f is a min-max rule on $\mathcal{S}^n$ and $f(P_S, P_{N\setminus S}) = u$, we have $f(P'_S, P'_{N\setminus S}) = z$ and $f(\hat{P}_S, \hat{P}_{N\setminus S}) = \underline{x}+1$. As $f(\hat{P}_S, \hat{P}_{N\setminus S}) = \underline{x}+1$, by Lemma A.1, we have $f(Q_S, \hat{P}_{N\setminus S}) = \underline{x}+1$, where $Q_i = Q$ for all $i \in S$. Consider the preference profile $(Q'_S, P'_{N\setminus S})$, where $Q'_i = Q'$ for all $i \in S$. Note that $f(P'_S, P'_{N\setminus S}) = z$ and $Q' = \overline{x}z\ldots$. Therefore, by moving the agents $i \in S$ from P'_i to Q' one-by-one and using strategy-proofness at every step, we have $f(Q'_S, P'_{N\setminus S}) \in \{\overline{x}, z\}$. We claim $f(Q'_S, P'_{N\setminus S}) = \overline{x}$. Assume for contradiction that $f(Q'_S, P'_{N\setminus S}) = z$. Since $\overline{x}P'_j z$ for all $j \in N \setminus S$, by moving the agents $j \in N \setminus S$ from P'_j to Q' one-by-one and applying strategy-proofness at every step, we have $f(Q'_S, Q'_{N\setminus S}) \neq \overline{x}$. However, this contradicts unanimity. So, $f(Q'_S, P'_{N\setminus S}) = \overline{x}$. For all $i \in S$, let $\tilde{P}_i \in \mathcal{S}$ be such that $r_1(\tilde{P}_i) = \overline{x}$. By strategy-proofness, $f(\tilde{P}_S, P'_{N\setminus S}) = \overline{x}$. Since f is a min-max rule on $\mathcal{S}^n$, this means $f(\tilde{P}_S, \hat{P}_{N\setminus S}) = \overline{x}$. For all $i \in S$, let $\tilde{P}'_i \in \mathcal{S}$ be such that $r_1(\tilde{P}'_i) = y$. Because $(\tilde{P}_S, \hat{P}_{N\setminus S}), (\tilde{P}'_S, \hat{P}_{N\setminus S}) \in \mathcal{S}^n$ and f is a min-max rule on $\mathcal{S}^n$, $f(\tilde{P}_S, \hat{P}_{N\setminus S}) = \overline{x}$ implies $f(\tilde{P}'_S, \hat{P}_{N\setminus S}) = y$. Because $f(\tilde{P}'_S, \hat{P}_{N\setminus S}) = y$ and $Q = \underline{x}y\ldots$, by moving the agents $i \in S$ from $\tilde{P}'_i$ to Q one-by-one and applying strategy-proofness at every step, we have $f(Q_S, \hat{P}_{N\setminus S}) \in \{\underline{x}, y\}$. Since $\{\underline{x}+1\} \cap \{\underline{x}, y\} = \emptyset$ by our assumption, this is a contradiction to our earlier finding $f(Q_S, \hat{P}_{N\setminus S}) = \underline{x}+1$. This completes the proof of the lemma for Case 1.

Case 2. Suppose $y \in (\underline{x}+1, \overline{x})$, $z \in (\underline{x}, \overline{x}-1)$, $z < y-1$, and $u \in (z, y)$.
Let $P'_N, \hat{P}_N \in \mathcal{S}^n$ be such that $r_1(P'_i) = y$ and $r_1(\hat{P}_i) = \underline{x}$ for all $i \in S$, and $r_1(P'_j) = \overline{x}$ and $r_1(\hat{P}_j) = z$ for all $j \in N \setminus S$. Because f is a min-max rule on $\mathcal{S}^n$ and $f(P_S, P_{N\setminus S}) = u$, we have $f(P'_S, P'_{N\setminus S}) = y$ and $f(\hat{P}_S, \hat{P}_{N\setminus S}) = z$. As $f(\hat{P}_S, \hat{P}_{N\setminus S}) = z$, by Lemma A.1, we have $f(Q_S, \hat{P}_{N\setminus S}) = z$, where $Q_i = Q$ for all $i \in S$. Again, as $f(P'_S, P'_{N\setminus S}) = y$, by Corollary A.1, we have $f(P'_S, Q'_{N\setminus S}) = y$, where $Q'_j = Q'$ for all $j \in N \setminus S$. Because $f(Q_S, \hat{P}_{N\setminus S}) = z$ and $Q' = \overline{x}z\ldots$, by moving the agents $j \in N \setminus S$ from $\hat{P}_j$ to Q' one-by-one and using strategy-proofness at every

step, we have $f(Q_S, Q'_{N\setminus S}) \in \{\overline{x}, z\}$. Again, because $f(P'_S, Q'_{N\setminus S}) = y$, $Q = \underline{x}y\ldots$, by moving the agents $i \in S$ from P'_i to Q one-by-one and using strategy-proofness at every step, we have $f(Q_S, Q'_{N\setminus S}) \in \{\underline{x}, y\}$. Since $\{\underline{x}, y\} \cap \{\overline{x}, z\} = \emptyset$ by our assumption, this is a contradiction. This completes the proof of the lemma for Case 2.

CASE 3. Suppose $y = \overline{x}$, $z = \underline{x}$, and $u \in (z, y)$.
Let $P'_N \in \mathcal{S}^n$ be such that $r_1(P'_i) = \underline{x}$ for all $i \in S$ and $r_1(P'_j) = \overline{x}$ for all $j \in N \setminus S$. Because f is a min-max rule on $\mathcal{S}^n$ and $f(P_S, P_{N\setminus S}) = u$, we have $f(P'_S, P'_{N\setminus S}) = u$. Take $i \in N$ and consider the preference profile $(Q_i, P'_{S\setminus i}, P'_{N\setminus S})$, where $Q_i = Q$. Since $r_1(P'_i) = r_1(Q_i) = \underline{x}$ and $f(P'_S, P'_{N\setminus S}) \neq \underline{x}$, by strategy-proofness, $f(Q_i, P'_{S\setminus i}, P'_{N\setminus S}) \neq \underline{x}$. Continuing in this manner, it follows that $f(Q_S, P'_{N\setminus S}) \neq \underline{x}$, where $Q_i = Q$ for all $i \in S$. Moreover, since $r_2(Q_i) = \overline{x}$ for all $i \in S$ and $r_1(P'_j) = \overline{x}$ for all $j \in N \setminus S$, by unanimity and strategy-proofness, $f(Q_S, P'_{N\setminus S}) \in \{\underline{x}, \overline{x}\}$. Since $f(Q_S, P'_{N\setminus S}) \neq \underline{x}$, this means $f(Q_S, P'_{N\setminus S}) = \overline{x}$. Let $Q'_j = Q'$ for all $j \in N \setminus S$. As $f(Q_S, P'_{N\setminus S}) = \overline{x}$ and $r_1(Q') = \overline{x}$, by strategy-proofness, $f(Q_S, Q'_{N\setminus S}) = \overline{x}$. Now, if we first move the agents $j \in N \setminus S$ from P'_j to Q' and then move the agents $i \in S$ from P'_i to Q, then it follows from a similar argument that $f(Q_S, Q'_{N\setminus S}) = \underline{x}$. Since $\underline{x} \neq \overline{x}$, this is a contradiction to our earlier finding that $f(Q_S, Q'_{N\setminus S}) = \overline{x}$. This completes the proof of the lemma for Case 3.

Since Cases 1, 2 and 3 are exhaustive, this completes the proof of the lemma. ■

Let $(\beta_S)_{S\subseteq N}$ be the parameters of f restricted to $\mathcal{S}^n$. In Lemma A.3 and Lemma A.4, we establish a few properties of these parameters.

Lemma A.3. *For all $S \subseteq N$, $\beta_S \in [a, \underline{x}]$ if and only if $\beta_{N\setminus S} \in [\overline{x}, b]$.*

Proof. Take $S \subseteq N$. It is enough to show that $\beta_S \in [a, \underline{x}]$ implies $\beta_{N\setminus S} \in [\overline{x}, b]$. Assume for contradiction that $\beta_S, \beta_{N\setminus S} \in [a, \underline{x}]$. Let $Q' \in \tilde{\mathcal{S}}$ with $r_1(Q') = \overline{x}$ be as given in Definition 2.9. Suppose $r_2(Q') = z$. Take $u \in (z, \overline{x})$. Let $(P_S, P_{N\setminus S}) \in \mathcal{S}^n$ be such that $r_1(P_i) = a$ for all $i \in S$ and $r_1(P_j) = b$ for all $j \in N \setminus S$. Since f restricted to $\mathcal{S}^n$ is a min-max rule, $f(P_S, P_{N\setminus S}) = \beta_S \in [a, \underline{x}]$. Let $(P'_S, P'_{N\setminus S}) \in \mathcal{S}^n$ be such that $r_1(P'_i) = z$ for all $i \in S$ and $r_1(P'_j) = u$ for all $j \in N \setminus S$. Since $f(P_S, P_{N\setminus S}) \in [a, \underline{x}]$, by uncompromisingness of f restricted to $\mathcal{S}^n$, we have $f(P'_S, P'_{N\setminus S}) = z$. Because $Q' = \overline{x}z\ldots$, by moving the agents $i \in S$ one-by-one from P'_i to Q' and applying strategy-proofness at every step, we have $f(Q'_S, P'_{N\setminus S}) \in \{\overline{x}, z\}$, where $Q'_i = Q'$ for all $i \in S$.

Now, let $(\bar{P}_S, \bar{P}_{N\setminus S}) \in \mathcal{S}^n$ be such that $r_1(\bar{P}_i) = b$ for all $i \in S$ and $r_1(\bar{P}_j) = a$ for all $j \in N \setminus S$. Again, since f restricted to $\mathcal{S}^n$ is a min-max rule, $f(\bar{P}_S, \bar{P}_{N\setminus S}) = \beta_{N\setminus S} \in [a, \underline{x}]$. Recall that for

$j \in N \setminus S$, $P'_j \in \mathcal{S}$ with $r_1(P'_j) = u$. Consider $(P''_S, P'_{N\setminus S}) \in \mathcal{S}^n$ such that $r_1(P''_i) = \overline{x}$ for all $i \in S$. Since $f(\bar{P}_S, \bar{P}_{N\setminus S}) \in [a, \underline{x}]$, by uncompromisingness of f restricted to $\mathcal{S}^n$, we have $f(P''_S, P'_{N\setminus S}) = u$. Because $r_1(P''_i) = \overline{x} = r_1(Q')$ for all $i \in S$, by Corollary A.1, it follows that $f(Q'_S, P'_{N\setminus S}) = u$. However, as $u \notin \{\overline{x}, z\}$, this is a contradiction to our earlier finding that $f(Q'_S, P'_{N\setminus S}) \in \{\overline{x}, z\}$. This completes the proof of the lemma. ■

The following lemma says that there is exactly one agent i such that $\beta_i \in [a, \underline{x}]$.

Lemma A.4. *It must be that* $|\{i \in N \mid \beta_i \in [a, \underline{x}]\}| = 1$.

Proof. Suppose there are $i \neq j \in N$ such that $\beta_i, \beta_j \in [a, \underline{x}]$. By Lemma A.3, $\beta_i \in [a, \underline{x}]$ implies $\beta_{N\setminus i} \in [\overline{x}, b]$. Since $j \in N \setminus i$ and $\beta_T \leq \beta_S$ for all $S \subseteq T$, $\beta_{N\setminus i} \in [\overline{x}, b]$ implies $\beta_j \in [\overline{x}, b]$, a contradiction. Hence, there can be at most one agent $i \in N$ such that $\beta_i \in [a, \underline{x}]$.

Now, suppose $\beta_i \in [\overline{x}, b]$ for all $i \in N$. By Lemma A.3, this means $\beta_{N\setminus i} \in [a, \underline{x}]$ for all $i \in N$. Therefore, there must be $S \subseteq N$ such that $\beta_S \in [a, \underline{x}]$ and for all $S' \subsetneq S$, $\beta_{S'} \in [\overline{x}, b]$. By unanimity, $S \neq \emptyset$. If S is singleton, say $\{i\}$ for some $i \in N$, then $\beta_i \in [a, \underline{x}]$ and we are done. So assume that there are $j \neq k \in S$.

Consider the preference profile $P_N \in \mathcal{S}^n$ such that $r_1(P_j) = \underline{x} + 1$, $r_2(P_j) = \underline{x}$, $r_1(P_i) = y$ for all $i \notin S$, and $r_1(P_i) = \underline{x}$ for all $i \in S \setminus j$. Since $\beta_S \in [a, \underline{x}]$ and $\beta_{S'} \in [\overline{x}, b]$ for all $S' \subsetneq S$, it follows from the definition of a min-max rule that $f(P_N) = \underline{x} + 1$. Let $P'_k \in \mathcal{S}$ be such that $r_1(P'_k) = y$. Since $\beta_{S\setminus k} \in [\overline{x}, b]$ and f restricted to $\mathcal{S}^n$ is a min-max rule, it follows that $f(P'_k, P_{N\setminus k}) = y$. Consider the preference profile $(Q_k, P_{N\setminus k})$, where $Q_k = Q$. Because $f(P'_k, P_{N\setminus k}) = y$ and $Q_k = \underline{x}y\ldots$, by strategy-proofness, $f(Q_k, P_{N\setminus k}) \in \{\underline{x}, y\}$. Suppose $f(Q_k, P_{N\setminus k}) = \underline{x}$. Because $f(P_N) = \underline{x} + 1$ and $r_1(P_k) = \underline{x}$, this means agent k manipulates at P_N via Q_k. So, $f(Q_k, P_{N\setminus k}) = y$. Let $P'_j \in \mathcal{S}$ be such that $r_1(P'_j) = \underline{x}$. Since $\beta_S \in [a, \underline{x}]$ and $\underline{x}$ is the top-ranked alternative of the agents in S at preference profile $(P'_j, P_{N\setminus j})$, we have $f(P'_j, P_{N\setminus j}) = \underline{x}$. As $r_1(P_k) = r_1(Q_k) = \underline{x}$, this means $f(P'_j, Q_k, P_{N\setminus\{j,k\}}) = \underline{x}$. Because $f(Q_k, P_{N\setminus k}) = y$, $r_1(P_j) = \underline{x} + 1$, and $r_2(P_j) = \underline{x}$, agent j manipulates at $(Q_k, P_{N\setminus k})$ via P'_j. This completes the proof of the lemma. ■

REMARK A.1. By Lemma A.3 and Lemma A.4, it follows that f restricted to $\mathcal{S}^n$ is a PDMMR.

Our next lemma establishes that f is uncompromising.[8] First, we introduce few notations that we use in the proof of the lemma. For $P_N \in \tilde{\mathcal{S}}^n$, let $\tilde{N}(P_N) = \{i \in N \mid P_i \notin \mathcal{S}\}$ be the

[8] Since every SCF satisfying uncompromisingness is tops-only, Lemma A.5 shows that a partially single-peaked domain is a tops-only domain. It can be easily verified that partially single-peaked domains fail to satisfy the sufficient conditions for a domain to be tops-only identified in Chatterji and Sen (2011) and Chatterji and Zeng (2015).

set of agents who do not have single-peaked preferences at P_N. Moreover, for $0 \leq l \leq n$, let $\tilde{\mathcal{S}}^n_l = \{P_N \in \tilde{\mathcal{S}}^n \mid |\tilde{N}(P_N)| \leq l\}$ be the set of preference profiles where at most l agents have non-single-peaked preferences. Note that $\tilde{\mathcal{S}}^n_0 = \mathcal{S}^n$ and $\tilde{\mathcal{S}}^n_n = \tilde{\mathcal{S}}^n$.

Lemma A.5. *The SCF f is uncompromising.*

Proof. Since $\tilde{\mathcal{S}}^n_0 = \mathcal{S}^n$, f restricted to $\tilde{\mathcal{S}}^n_0$ is uncompromising. Suppose f restricted to $\tilde{\mathcal{S}}^n_k$ is uncompromising for some $k < n$. We show that f restricted to $\tilde{\mathcal{S}}^n_{k+1}$ is uncompromising. It is enough to show that f restricted to $\tilde{\mathcal{S}}^n_{k+1}$ is tops-only. To see this, note that if f restricted to $\tilde{\mathcal{S}}^n_{k+1}$ is tops-only, then f is uniquely determined on $\tilde{\mathcal{S}}^n_{k+1}$ by its outcomes on $\mathcal{S}^n$. Therefore, since f restricted to $\mathcal{S}^n$ is uncompromising, f is uncompromising on $\tilde{\mathcal{S}}^n_{k+1}$.

Take $P_N \in \tilde{\mathcal{S}}^n_{k+1}$ and $j \in \tilde{N}(P_N)$. Let $\hat{P}_j \in \mathcal{S}$ be such that $r_1(\hat{P}_j) = r_1(P_j)$. Then, P_N and $(\hat{P}_j, P_{N\setminus j})$ are tops-equivalent and $(\hat{P}_j, P_{N\setminus j}) \in \tilde{\mathcal{S}}^n_k$. It is sufficient to show that $f(P_N) = f(\hat{P}_j, P_{N\setminus j})$. Assume for contradiction that $f(P_N) \neq f(\hat{P}_j, P_{N\setminus j})$. Assume, without loss of generality, that the partial dictator of f restricted to $\mathcal{S}^n$ is agent 1. Then, by the induction hypothesis, agent 1 is the partial dictator of f restricted to $\tilde{\mathcal{S}}^n_k$, i.e., for all $P_N \in \tilde{\mathcal{S}}^n_k$, if $r_1(P_1) \in [a, \underline{x})$ then $f(P_N) \in [a, \underline{x}]$, if $r_1(P_1) \in (\overline{x}, b]$ then $f(P_N) \in [\overline{x}, b]$, and if $r_1(P_1) \in [\underline{x}, \overline{x}]$ then $f(P_N) = r_1(P_1)$. We distinguish two cases based on the position of the top-ranked alternative of agent 1.

CASE 1. Suppose $r_1(P_1) \in [a, \underline{x}) \cup (\overline{x}, b]$.

We consider the case where $r_1(P_1) \in [a, \underline{x})$, the proof for the case where $r_1(P_1) \in (\overline{x}, b]$ follows from symmetric arguments. Since $r_1(P_1) \in [a, \underline{x})$, we have $f(\hat{P}_j, P_{N\setminus j}) \in [a, \underline{x}]$. Because $\hat{P}_j$ is single-peaked, if $f(\hat{P}_j, P_{N\setminus j}) < f(P_N) \leq r_1(\hat{P}_j)$ or $r_1(\hat{P}_j) \leq f(P_N) < f(\hat{P}_j, P_{N\setminus j})$, then agent j manipulates at $(\hat{P}_j, P_{N\setminus j})$ via P_j. Moreover, since $f(\hat{P}_j, P_{N\setminus j}) \in [a, \underline{x}]$, if $f(P_N) < f(\hat{P}_j, P_{N\setminus j}) \leq r_1(\hat{P}_j)$ or $r_1(P_j) \leq f(\hat{P}_j, P_{N\setminus j}) < f(P_N)$, then by the definition of a partially single-peaked domain, agent j manipulates at $(P_j, P_{N\setminus j})$ via $\hat{P}_j$. Now, suppose $f(\hat{P}_j, P_{N\setminus j}) < r_1(\hat{P}_j) < f(P_N)$. Let $\bar{P}_j \in \mathcal{S}$ be such that $r_1(\bar{P}_j) = f(P_N)$. Since f restricted to $\tilde{\mathcal{S}}^n_k$ is uncompromising and $f(\hat{P}_j, P_{N\setminus j}) < r_1(\hat{P}_j) < r_1(\bar{P}_j)$, we have $f(\bar{P}_j, P_{N\setminus j}) = f(\hat{P}_j, P_{N\setminus j})$. Because $r_1(\bar{P}_j) = f(P_N)$, it follows that agent j manipulates at $(\bar{P}_j, P_{N\setminus j})$ via P_j. Using a similar argument, it can be shown that $f(P_N) < r_1(\hat{P}_j) < f(\hat{P}_j, P_{N\setminus j})$ leads to a manipulation by agent j. Therefore, $f(P_N) = f(\hat{P}_j, P_{N\setminus j})$ when $r_1(P_1) \in [a, \underline{x})$. This completes the proof of the lemma for Case 1.

CASE 2. Suppose $r_1(P_1) \in [\underline{x}, \overline{x}]$.

Since agent 1 is the partial dictator, $f(\hat{P}_j, P_{N\setminus j}) = r_1(P_1)$. Consider $\bar{P}_j \in \mathcal{S}$ such that $r_1(\bar{P}_j) = f(P_N)$. Since $(\bar{P}_j, P_{N\setminus j}) \in \tilde{\mathcal{S}}^n_k$, by the induction hypothesis, we have $f(\bar{P}_j, P_{N\setminus j}) = r_1(P_1)$. Because

$r_1(\bar{P}_j) = f(P_N)$ and $f(\bar{P}_j, P_{N\setminus j}) = r_1(P_1) \neq f(P_N)$, agent j manipulates at $(\bar{P}_j, P_{N\setminus j})$ via P_j. Therefore, $f(P_N) = f(\hat{P}_j, P_{N\setminus j})$ when $r_1(P_1) \in [\underline{x}, \overline{x}]$. This completes the proof of the lemma for Case 2.

Since Cases 1 and 2 are exhaustive, this completes the proof of the lemma by induction. ■

Now, we complete the proof of the only-if part of Theorem 3.1. Since f is uncompromising on $\tilde{\mathcal{S}}^n$ and f restricted to $\mathcal{S}^n$ is a min-max rule with parameters $(\beta_S)_{S\subseteq N}$ satisfying the properties as stated in Lemma A.3 and Lemma A.4, it follows that f is a PDMMR. ■

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