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Do Inada Conditions imply Cobb-Douglas Asymptotic Behavior or only a Elasticity of Substitution equal to one

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Abstract

Barelli and De Breu Passôa proved that Inada conditions imply asymptotic Cobb-Douglas behavior of the production function. This was corrected by Litina and Palivos by putting that only the elasticity of substitution is equal to one. We will correct both proof and arguments and come up with a proof without limitations, leaving the conclusion of Barelli unaltered but with a less restrictive proof. In addition we show that the asymptotic Cobb-Douglas power α of capital can be estimated by $\alpha(0) = \lim_{k\to 0} \frac{kf'(k)}{f(k)}$ for $k = 0$ and by $\alpha(\infty) = \lim_{k\to\infty} \frac{kf'(k)}{f(k)}$ for $k = \infty$. Furthermore if Inada conditions apply then the elasticity of substitution is bounded.

Keywords: Inada condition, production function, elasticity of substitution, Cobb-Douglas

JEL Classification E13 · E23

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1. Introduction

It all started by Solow (1956) when he presented his growth model. Uzawa (1961,1962) proved in his two-sector growth model certain conditions were sufficient to ensure existence of equilibria.

In 1963, Inada (1963) noticed those conditions and made them more explicit.

Hahn (1965) and Solow (1961) made some comments on the findings of Uzawa.

Solow:
"Uzawa finds that his model economy is always stable... if the consumption-goods sector is more capital-intensive than the investment-goods sector. It seems paradoxical to me that such an important characteristic of the equilibrium path should depend on such a casual property of the technology. And since this stability property is the one respect in which Uzawa's results seem qualitatively different from those of my 1956 paper on a one-sector model, I am anxious to track down the source of this difference."

and Hahn:
"It is evident that in all these constructions the condition that the equilibrium at a moment in time be unique is crucial. The rest of the story is really concerned with ensuring that there is a steady state with positive factor prices. But the assumptions required to establish uniqueness of momentary equilibrium are all terrible assumptions."

The two sector model and the existence and uniqueness of the solution were at that time popular topics in discussions.

Later Barelli and De Breu Passôa (2003) proved that the limiting cases when k approaches zero or infinity the production function will behave Cobb-Douglas, which was corrected by Litina and Palivos (2008). Their conclusion was that Barelli and De Breu Passôa had proved the elasticity of substitution σ is equal to 1, $\sigma = 1$. We could not share their insight.

In order to investigate the problem we will formulate the Inada conditions and explain how it is used together with a simple model. We analyze the proof of Barelli and De Breu Passôa and make some comments. The same is done for the paper of Litina and Palivos for which we will examine their counterexamples, we cannot agree on, in detail. Then we will present an alternative proof that Inada conditions imply asymptotic Cobb-Douglas behavior and that Inada conditions imply boundedness of the elasticity of substitution. Finally we will show how to estimate the capital power coefficient of the asymptotic Cobb-Douglas production function.

2. The Inada conditions

Production functions with two factors labor L and capital K of production that are homogeneous of degree one can be written as per capita production functions with only capital deepening k as variable.

The Inada conditions stated in a modern form read that a (per-capita) production function $f: \mathcal{R}^+ \to \mathcal{R}^+$ should satisfy

- $f(0) = 0$ (1)
- $\lim_{k\to\infty} f'(k) = 0$ (2)
- $\lim_{k\to 0} f'(k) = \infty$ (3)
- $\lim_{k\to\infty} f(k) = \infty$ (4)
  strictly increasing
- $f'(k) > 0 \; \forall k \in \mathcal{R}^+$ (5)
  strictly concave
- $f''(k) < 0 \; \forall k \in \mathcal{R}^+$ (6)

As economy we use a simple one sector model. For simplicity we put income $y = f(k)$ and this income is spent on consumption and investment. In a closed economy without government savings is equal to investment.

The fraction of income consumed is

$$c = c_1 y \tag{7}$$

$$y = c + i \tag{8}$$

$$y = w + (r + \delta)k \tag{9}$$

Capital is accumulating according

$$\dot{k} = i - \delta k \tag{10}$$

equation (10) with consumer behavior will transfer to

$$\dot{k} = (1 - c_1)y - \delta k \tag{11}$$

Because of all the discussions in the 60's and a lot of misunderstanding even today about how to use Inada conditions we will formulate the use of it as follows.

If in a closed economy without government as given above, with $c_1 = constant$ expressing consumer's behavior, for an arbitrary chosen $c_1 \in [0,1]$, for any given production function satisfying the Inada conditions and for any arbitrary model parameters then for that point c_1 there exists a solution under maximum profit conditions for which you can resolve *k*, *w* and *r* , it is unique and stable.

This holds for every economically relevant depreciation rate δ. In a slightly adapted form it holds for a two sector model and also when there is a growth rate g, although the sum of depreciation and growth should be less than one.

$$g + \delta < 1 \tag{12}$$

If you change consumer's behavior you have to reexamine the problem.

The strong point is that it is a simple statement, which is also its weak point, because even the most common CES functions used today do not satisfy the Inada conditions.

With a CES model we used it was easy to create consumer behavior in such a way that solutions are not unique nor stable.

$$f(k) = p[\alpha(k)^{\gamma} + (1-\alpha)]^{1/\gamma} \tag{13}$$

where

$$\sigma = \frac{1}{1-\gamma} \tag{14}$$

The CES model do not fulfil the Inada conditions in general, because e.g. for $\gamma < 0$ and k to infinity

$$f(k) = \lim_{k\to\infty} = p[\alpha(k)^{\gamma} + (1-\alpha)]^{1/\gamma} = p(1-\alpha)^{1/\gamma} \tag{15}$$

which is not equal to infinity. In case of Cobb-Douglas, $\sigma = 1$, the conditions are fulfilled, in which case the solution is unique and stable at any $c_1 = constant$ as we showed in De la Fonteijne (2012). We also like to point out that even if the Inada conditions do not hold we can claim for CES function in particular the same outcome for all σ and at all $c_1 = constant$ within a certain validity range.

3. Analysis

First Barelli and De Breu Passôa made explicit use of the fact that σ is bounded, so he added another condition. His conclusion, therefore, should read, if Inada conditions and σ is bounded then asymptotic behavior is Cobb-Douglas. Apart from this fact Litina and Palivos gave two counterexamples to demonstrate that there was a mistake in the proof of Barelli and De Breu Passôa, because the 2 counter examples they gave both not fulfil the conditions that σ is bounded or their example was true, either way, they cannot serve as a counter example.

In their first counterexample

$$f(k) = b + \frac{ak}{1+k} \tag{16}$$

$$f(0) = b, \qquad f(\infty) = a + b \tag{17}$$

$$f'(0) = a, \qquad f'(\infty) = 0 \tag{18}$$

$$f''(0) = -2a, \quad f''(\infty) = 0 \tag{19}$$

using

$$\sigma = -\frac{f'(k)[f(k)-kf'(k)]}{kf(k)f''(k)} \tag{20}$$

gives

$$\sigma(0) = \infty \tag{21}$$

σ is not bounded and this examples cannot serve as a counterexample for the proof Barelli gave.

At infinity we have

$$\sigma(\infty) = \frac{1}{2}$$

which is bounded but here there is no contradiction because

$$\sigma(\infty) < 0, f(\infty) = a + b < \infty, f'(\infty) = 0 \text{ and } \lim_{k\to\infty} \frac{kf'(k)}{f(k)} = 0$$

And therefore it cannot serve as a counterexample. Its asymptotic behavior is a labor only production function.

In their second counterexample σ is not bounded for $k \to 0$ and for $k \to \infty$ and therefore cannot serve as a counterexample.

So far I could not understand their argument about $\lim_{k\to 0} \frac{f(k)}{k} = f'(0)$ because Barelli is explicitly using that $f(0) = 0$ and $f'(0) = \infty$ to draw his conclusion that $\sigma = 1$.

A second point is that Barelli bracketed this function by 2 CES functions using ε and δ technique to prove that $\sigma = 1$ asymptotically, which is not only highly suggesting that its character is Cobb-Douglas, but in fact proving it. A last comment on the paper of Litina is the last example where they used the following homogeneous of degree one production function satisfying the Inada conditions.

$$f(k) = A(1 + k^{-\rho})^{-\frac{1}{\rho}} - k - 1 \tag{22}$$

where $-1 < \rho < 0$

Of course Litina conclude it is $\sigma = 1$ asymptotically, which is true. And because of our statement and the proof of Barelli it is asymptotically Cobb-Douglas. To make it plausible look at 2 special cases.

First let $\rho = -.5$

$$f(k) = A((1 + k^{.5})^2 - k - 1) = 2Ak^{.5} \tag{23}$$

which is Cobb-Douglas for all levels of k.

In the second example we let $\rho = -.2$

$$f(k) = A((1 + k^{.2})^5 - k - 1) = A(5k^{.2} + 10k^{.4} + 10k^{.6} + 5k^{.8}) \qquad (24)$$

For k large $f(k) \approx 5Ak^{.8}$ which is also Cobb-Douglas. With the same examples Litina came to another conclusion because they misinterpreted a rule they used.

Interesting to see how Klump (2008) constructed the class of CES production functions out of the definition of the elasticity of substitution in two different ways, which also implies that if the production function has an elasticity of substitution $\sigma = 1$ on a certain interval then the function is Cobb-Douglas on that particular interval.

4. Alternative proof without using boundedness of the elasticity of substitution

Because f is strictly increasing $f'(k) > 0\ \forall k \in \mathcal{R}^+$ and strictly concave $f''(k) < 0\ \forall k \in \mathcal{R}^+$ it is easy to see that $0 < kf'(k) < f(k)$, which we rewrite as

$$0 < \frac{kf'(k)}{f(k)} < 1 \qquad (25)$$

Taking the limit for k to zero or infinite then clearly zero or 1 is not a solution. As the function k, $f(k)$ and $f'(k)$ are strictly monotone then below and beyond a certain point k the function $\frac{kf'(k)}{f(k)}$ is monotone. Following the monotone convergence theorem there exists a value α for which

$$\lim_{k\to 0} \frac{kf'(k)}{f(k)} = \alpha \qquad (26)$$

and a value β for which

$$\lim_{k\to \infty} \frac{kf'(k)}{f(k)} = \beta. \qquad (27)$$

It is then straightforward to see that the solution for f can be calculated as

$$f(k) = ak^{\alpha}\ ,\ 0 < \alpha < 1 \qquad \text{for } k \to 0 \qquad (28)$$

and

$$f(k) = bk^{\beta}\ ,\ 0 < \beta < 1 \qquad \text{for } k \to \infty \qquad (29)$$

which concludes the proof: if Inada conditions apply then the asymptotic behavior of f is Cobb-Douglas with of course $\sigma = 1$.

We have the feeling that the proof is on the razor's edge in the mathematical sense that there is few or none space left, it is or it isn't, and therefore at some point the proof can be made more mathematical precise. Readers are invited to do so. It all boils down to the limiting value of $\frac{kf'(k)}{f(k)}$.

We return to the boundedness of the elasticity of substitution and the proof of Barelli.

Recall that

$$\sigma(k) = -\frac{f'(k)[f(k)-kf'(k)]}{kf(k)f''(k)} \tag{30}$$

which we rewrite as

$$\sigma(k) = -\frac{f'(k)}{kf''(k)}\left[1-\frac{kf'(k)}{f(k)}\right] \tag{31}$$

which is bounded for every value of $k \in (0,\infty)$ and only the points at zero and infinity have to be examined more carefully by taking the limit using the rule of De l'Hôpital

$$\lim_{k\to 0}\frac{kf'(k)}{f(k)} = \lim_{k\to 0}\frac{kf''(k)}{f'(k)} + 1 = \alpha \tag{32}$$

or

$$\lim_{k\to 0}\frac{kf''(k)}{f'(k)} = \alpha - 1 \tag{33}$$

Use the results from (32) and (33) to evaluate σ from (31)

$$\sigma(0) = -\lim_{k\to 0}\frac{f'(k)}{kf''(k)}\left[1-\frac{kf'(k)}{f(k)}\right] = -\frac{1-\alpha}{\alpha-1} = 1 \tag{34}$$

For k to infinity you have to follow the same procedure.

Combining these results proves that if Inada conditions apply σ is bounded.

The boundedness of σ together with the proof of Barelli proves his claim of asymptotic Cobb-Douglas behavior.

As an example we take the last example of Litina

$$f(k) = A(1+k^{-\rho})^{-\frac{1}{\rho}} - k - 1 \tag{35}$$

$$\alpha(0) = \lim_{k\to 0}\frac{kf'(k)}{f(k)} = -\rho \tag{36}$$

$$\alpha(\infty) = \lim_{k\to \infty}\frac{kf'(k)}{f(k)} = 1+\rho \tag{37}$$

For $\rho = -.5$ the result is $\alpha(\infty) = .5$ and for $\rho = -.2$ the result is $\alpha(\infty) = .8$ as we have already seen.

Notice the difference in using $\lim_{k\to\infty}\frac{kf'(k)}{f(k)} = 0$ in the proof of Barelli.

Another example is

$$f(k) = ak^{\alpha} + bk^{\beta} \qquad 0 < \alpha < \beta < 1 \tag{38}$$

The power coefficients for asymptotic Cobb-Douglas behavior follow from

$$\alpha(0) = \lim_{k\to 0}\frac{kf'(k)}{f(k)} = \alpha \tag{39}$$

and

$$\alpha(\infty) = \lim_{k\to\infty}\frac{kf'(k)}{f(k)} = \beta \tag{40}$$

Notice that the production function itself is not Cobb-Douglas.

5. Conclusion

The proof of Barelli was limited to a bounded elasticity of substitution, so strictly speaking he could not claim that Inada conditions implies asymptotic Cobb-Douglas behavior.

With a new proof we have shown that the conclusion of Barelli is still valid and the Inada conditions imply not only that the elasticity of substitution is bounded but also that the behavior of the production function considered is asymptotic Cobb-Douglas.

We have proved that if Inada conditions apply *f* is asymptotically Cobb-Douglas and therefore we can estimate the capital power coefficient α by

$\alpha(0) = \lim_{k\to 0}\frac{kf'(k)}{f(k)}$ for $k = 0$ and $\alpha(\infty) = \lim_{k\to\infty}\frac{kf'(k)}{f(k)}$ for $k = \infty$. The limits are not necessarily the same.

The general purpose of Inada conditions is its strong point and at the same time its weakness in excluding one of the most common used production function, i.e. the CES production function. Due to the fact that Inada conditions are very restrictive its practical purpose is limited.

6. Acknowledgement

This paper is part of a study to reduce unemployment in a sustainable way whilst keeping governmental debt within sustainable limits and improve prosperity. In our opinion this is one of the most important things to achieve in society from a macro economic and from a participation society point of view. To put it simple, the right to every member of society to have a job in order to have the possibility to contribute in a positive and meaningful way to society.

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