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Supply Function Equilibria and Nonprofit-Maximizing Objectives^{*}

Junichi Haraguchi [†] and Yuta Yasui [‡]

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Abstract

We examine the *supply function equilibrium* (SFE), which is often used in the analysis of multi-unit auctions such as wholesale electricity markets, among (partially) public firms. In a general model, we characterize the SFE of such firms and examine the properties of symmetric SFE. We show, analyzing an asymmetric SFE in a duopoly model with linear demand and quadratic cost functions, that, when a partially public firm weighs more on the social welfare, the supply functions of not only the partially public firm but also a profit maximizing firm are flatter at the equilibrium. We also confirm that in a linear-quadratic model, the SFE converges to the (inverse) marginal cost function when the firms' social concerns increase symmetrically in the industry.

Key words: supply function equilibrium, electricity markets, partial privatization, corporate social responsibility, mixed oligopoly

JEL code: H42, L13, L33

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1 Introduction

Since Green and Newbery (1992) applied the *supply function equilibrium* (henceforth, SFE) to the analysis of wholesale electricity markets, a bunch of applications to the electricity or treasury bills market have appeared in the literature.¹ In this paper, we assume that firms have social welfare concerns; that is, they can be regarded as (partially) public firms or firms with corporate social responsibility (CSR). When a model with linear demand and quadratic cost functions is at an equilibrium, the supply functions of not only a partially public firm (when it weighs more on the social welfare) but also a private firm are closer to their marginal cost functions. Thus, in contrast to Matsumura (1998), the society benefits from the existence of a perfectly public firm. We also confirmed that in the linear-quadratic model, the supply function equilibrium converges to the (inverse) marginal cost function when the publicity of firms or the extent of social concern is improved symmetrically in the industry, while it is not guaranteed in the general model.

1.1 SFE and its applications

The SFE, introduced by Klemperer and Meyer (1989), is an equilibrium in a game where firms choose their own supply function flexibly. Firms offer their own supply schedule simultaneously, and the market then clears such that the total supply matches the demand at a certain price. In a model with demand uncertainty, the market clears after the uncertainty is realized. The SFE is defined as a (pure strategy) Nash equilibrium in this game. A remarkable feature of the SFE is that it is characterized as the locus of *ex post* optimal price-quantity pairs given the other firms' supply functions. The logic is as follows. Each firm guesses the others' (fixed) supply schedules. After the realization of demand uncertainty, combined with the other firms' supply functions, a residual demand function is determined. Suppose that a firm chooses its *ex post* optimal price-quantity pair along with the residual demand. Since the *ex post* optimal points vary according to the uncertainties realized even

¹See Holmberg and Newbery (2010) for recent SFE applications to the electricity markets.

though the firm assumes that the others' supply functions are fixed, the locus of ex post optimal points becomes a function from price to quantity. Since the firm can obtain ex post optimized profit through this supply function, it has no incentive to take other supply functions in the first stage, given others' supply functions. Thus, the locus is a best response to the others' supply functions. By considering such best responses for each firm, we obtain Nash Equilibrium in this game, in other words, SFE.

The most famous application of the SFE would be wholesale electricity markets. In deregulated wholesale electricity markets, generating companies offer their own supply schedules and retailers bid to supply their own customers such as consumers or other companies. If there are many retailers, the setting of supply function competition fits well with the structure in those markets because the retail demand is subject to exogenous shocks such as weather, holidays, and major sporting events. SFE has been extensively used as a tool to analyze electricity markets, both theoretically and empirically, ever since Green and Newbery (1992) employed it to analyze wholesale electricity markets.²

Another important, but less focused, interpretation would be the equilibria in the *conjectural variation* model with appropriately specified strategies. In general, the conjectural variation first introduced by Bowley (1924) implies that each firm chooses its quantity while believing that other firms change their quantities by $r_j dq_i$ in response to a change in firm i 's quantity dq_i . It has been used in empirical literature to capture various market structures in a reduced form.³ A major critique on this model is that it contradicts the equilibrium concept in the field of economics, that is, firms are *not* assumed to choose quantity, *given others' quantity*.⁴ However, in supply function competitions, strategies are (usually posi-

²Vives (2011) introduces uncertainty into the cost function rather than the demand function, treating it as asymmetric information. Holmberg et al. (2013) show that the SFE in step functions, such as actual offers in the electricity markets, converge to a continuous SFE as the steps becomes finer. Since it is difficult to obtain an analytical solution of the SFE, computation methods to calculate it are also developed in the operations research literature (see Holmberg (2009)).

³Iwata (1974) and Gollop and Roberts (1979) propose empirical methods to estimate or test the conjectural variation. They also apply it to Japanese plate glass industry and US coffee industry. Brander and Zhang (1993) estimate the conjectural variation in the US airline industry with dynamic settings. See also Bresnahan (1989) for review and discussion on the conjectural variation.

⁴Farrell and Shapiro (1990) explain that we can consider it as "the equilibrium of an (unmodeled) dynamic

tively sloped) supply functions. Therefore, other firms change their quantity in response to a quantity change by one firm. In other words, at the SFE, the conjectural parameter r_j is determined by the equilibrium strategy of firm j . Of course, there might be countless equilibria if there is no uncertainty in the market because the shape of each supply function only affects the off-equilibrium outcomes and works as an empty threat. However, in the market with demand uncertainty, the SFE restricts the set of possible outcomes much more narrowly and helps us to understand the market.

1.2 Mixed Oligopoly, Partial Privatization, and CSR

In this paper, we introduce (partially) public firms into the SFE.⁵ Oligopoly markets with public firms, called mixed oligopolies, were first examined by Merrill and Schneider (1966) and are extensively discussed in the literature.⁶ Matsumura (1998) generalizes it to the model of partial privatization by introducing a partially public firm that maximizes a weighted average of its own profit and the social welfare. In mixed oligopoly markets, the effects of the public firms are not straightforward because of strategic interactions.⁷ For instance, De Fraja and Delbono (1989) show that welfare may be higher when a public firm is a profit-maximizer rather than a welfare-maximizer, and Matsumura (1998) finds that the optimal level of partial privatization is neither full privatization nor full nationalization. Matsumura and Sunada (2013) examine a mixed oligopoly with misleading advertising competition and find that a public firm engages in rather than cancels out the misleading advertisement. Even if firms are fully symmetric but concerned about social welfare, the economic implication could be completely different from standard models with profit-maximizing firms. For

oligopolistic game,” and Cabral (1995) provides an explicit example of the repeated game of which the CV solution is an exact reduced form of the equilibrium.

⁵Mixed oligopolies occur in various industries, such as the airline, steel, automobile, railway, natural gas, electricity, postal service, education, hospital, home loan, and banking industries.

⁶See, De Fraja and Delbono (1989), Fjell and Pal (1996), Matsumura and Kanda (2005), and Ghosh and Mitra (2010).

⁷Assuming that public firms can commit to a certain strategy, we can apply general analysis tools to the effects of “exogenous competition” discussed by Weyl and Fabinger (2013). However, assuming that the public firm cannot make such a commitment and responds to others’ strategies, we would need to employ mixed oligopoly analysis.

instance, Ghosh and Mitra (2014) examine an oligopoly market where every firm maximizes a weighted average of its own profit and social welfare. Competition among firms with social welfare concerns can be understood as that among firms in transition and developing economies, where the extent of private ownership is restricted, or that among firms with CSR. They compare Cournot and Bertrand competition in a symmetrically differentiated market and show that Bertrand competition yields higher profit and lower social welfare than Cournot competition when profit is weighted sufficiently low. Therefore, we must be careful in considering the existence of (partially) public firms. Fortunately, it turns out that in supply function competition, a public firm encourages private firms to take socially better action through strategic complementarities in terms of the slopes of each supply function.

The remainder of the paper is organized as follows. Section 2 presents the general model. Section 3 analyzes the linear-quadratic model. Section 4 concludes the paper. The proofs of propositions can be found in the appendix.

2 The General Model

With symmetric and private firms, Klemperer and Meyer (1989) characterize the SFE under demand uncertainty as solutions of a differential equation and examine some of its general properties. In this subsection, we characterize the SFE with partially public firms analogously and examine the general properties. In the next subsection, we specify a model with a linear demand function and quadratic cost functions in order to further clarify the implication of the effects of the public firm.

Let us denote the demand function as $Q = D(p) + \epsilon$, where p is the price of the product and ϵ is a scalar random variable with strictly positive density everywhere on the support $[\underline{\epsilon}, \bar{\epsilon}]$. We assume that $-\infty < D'(p) < 0$, and $D''(p) \leq 0$. The firms have identical cost functions C s.t. $C'(q) \geq 0$ and $0 < C''(q) < \infty \forall q \in [0, \infty)$. Without loss of generality, let $C'(0) = 0$.⁸ A strategy for firm i ($i = 1, 2$) is defined as a function mapping from price

⁸If $C'(0) \neq 0$, we can normalize it by considering $\tilde{p} = p - C'(0)$ as in Klemperer and Meyer (1989).

into quantity: $S_i : [0, p] \rightarrow (-\infty, \infty)$. Here, we focus on pure strategy Nash equilibria, in which S_i maximizes i 's payoff given that j chooses S_j ($i, j = 1, 2, j \neq i$).

Given S_j , firm i 's ex post objective function is as follows:

$$\begin{aligned} v_i(p) &= (1 - \theta) [p(D(p, \epsilon) - S_j(p)) - C(D(p, \epsilon) - S_j(p))] \\ &+ \theta \left[\int_p^{\bar{p}} D(\dot{p}, \epsilon) d\dot{p} + pD(p, \epsilon) - C(D(p, \epsilon) - S_j(p)) - C(S_j(p)) \right]. \end{aligned} \quad (1)$$

Therefore, the first order condition is derived as follows:

$$\begin{aligned} \frac{\partial v_i}{\partial p} &= (1 - \theta) [(D(p) + \epsilon - S_j(p)) + (p - C'(D(p) + \epsilon - S_j(p))) (D'(p) - S'_j(p))] \\ &+ \theta [(p - C'(D(p) + \epsilon - S_j(p))) D'(p) + (C'(D(p) + \epsilon - S_j(p)) - C'(S_j(p))) S'_j(p)] = 0. \end{aligned} \quad (2)$$

Since S_i is determined by the locus of the optimal price $p^*(\epsilon)$ and the corresponding quantity $D(p^*(\epsilon) + \epsilon - S_j(p^*(\epsilon)))$, we can replace $D(p) + \epsilon - S_j(p)$ by S_i . Then, the above equation becomes a differential equation:

$$S'_j(p) = \frac{(1 - \theta_i) [S_i + (p - C'(S_i)) D'(p)] + \theta_i [(p - C'(S_i)) D'(p)]}{(1 - \theta_i) (p - C'(S_i)) + \theta_i [(p - C'(S_i)) - (p - C'(S_j))]} \equiv f_i(p, S_i, S_j). \quad (3)$$

Since a set of functions (S_1, S_2) that solves a system of differential equations $S'_j(p) = f_i(p, S_i, S_j)$ for $i = 1, 2, j \neq i$ satisfy the first order condition given the others' strategies, it is an SFE if the payoff function, given the others' strategies, satisfies the second order conditions. In a model with only private firms, it is known that the symmetric SFE strategy $S = S_i$ ($i = 1, 2$) must satisfy $0 < S'(p) < \infty$ for all p if ϵ has a full support ($\underline{\epsilon} = -D(0), \bar{\epsilon} = \infty$). In the competition among partially public firms, this property still holds, and the set of SFE is bounded by a function that shifts upwards (or becomes flatter if we take quantity on the horizontal axis and price on the vertical axis) as θ increases.

Proposition 1 (Necessity of positive slope) *If ϵ has full support ($\underline{\epsilon} = -D(0)$, $\bar{\epsilon} = \infty$) and S is symmetric SFE tracing through ex post optimal points, then $\forall p \geq 0$, S satisfies (3) and $0 < S'(p) < \infty$. Furthermore, S is bounded from below by a function $S^0(p, \theta)$ with its derivative*

$$S^{0'}(p) = -\frac{D_p(p) + D_{pp}(p)(p - C'(S^0(p)))}{(1 - \theta) - D_p(p)C''(S^0(p))}. \quad (4)$$

Proof. See the appendix.

As can be seen from eq. (4), $S^{0'}(p, \theta)$ is increasing in θ for all p , and thus, the lower bound $S^0(p, \theta)$ approaches the (inverse) marginal cost function when θ goes up. It is also worth noting that, even if θ approaches 1, $S^0(p, \theta)$ generally does not converge to the (inverse) marginal cost function, $C'^{-1}(p)$. Thus, we cannot guarantee that firms behave (almost) in a socially optimal way even if they have social welfare concerns and hardly care about profits.

In the following section, we specify the demand and cost functions to analyze the asymmetric SFE and the role of a public firm.

3 Linear Demand and Quadratic Cost Function

We specify the cost functions and a demand function for an analytical solution. The identical cost functions are defined as $C(S) = \frac{\epsilon}{2}S^2$ and the total demand function is defined as $Q = D(p, \epsilon) = \alpha + \epsilon - mp$. That is, $U(Q) = \left(\frac{\alpha + \epsilon}{m}\right)Q - \frac{1}{m}Q^2/2$, where U is a surplus function. Suppose player j 's strategy to be $q_j = S_j(p) = a + bp$. Then, the residual demand is written as

$$q_i(p, \epsilon) = D(p, \epsilon) - s_j(p) = \alpha + \epsilon - mp - a - bp. \quad (5)$$

Therefore, the *ex post* profit maximization problem for firm i , given the others' strategies, is written as follows:

$$\begin{aligned} \max_p \quad & (1 - \theta_i) [pq_i(p, \epsilon) - C(q_i(p, \epsilon))] \\ & + \theta_i [U(q_i(p, \epsilon) + S_j(p)) - C(q_i(p, \epsilon)) - C(S_j(p))]. \end{aligned} \quad (6)$$

We can consider the *ex post* optimal price for each ϵ by taking the first order condition with respect to p . Since the corresponding q_i for each ϵ is determined along the residual demand function $q_i(p, \epsilon)$, we can obtain the following locus of optimal points by canceling out ϵ in the FOC and (5):

$$q_i = \frac{(m + b) + \theta_i(bc - 1)b}{(1 - \theta_i) + c(m + b)}p + \frac{\theta_i abc}{(1 - \theta_i) + c(m + b)}.$$

Then, the SFE is constructed from the following supply functions:

$$S_i(p) = a_i^* + b_i^*p \text{ for all } i = 1, 2,$$

where

$$a_i^* = \frac{\theta_i a_j^* b_j^* c}{(1 - \theta_i) + c(m + b_j^*)}, \text{ and} \quad (7)$$

$$b_i^* = \frac{(m + b_j^*) + \theta_i(b_j^* c - 1)b_j^*}{(1 - \theta_i) + c(m + b_j^*)} \equiv B_i(b_j^*; \theta_i) \text{ (} i = 1, 2, j \neq i\text{)}. \quad (8)$$

Suppose $b_j^* \neq 0$. Then, since $a_1^* = a_2^* = 0$ must hold to satisfy eq. (7), we can obtain the SFE by solving eq. (8).

Even though eq. (8) is difficult to solve analytically, we can obtain some properties of the equilibrium without solving it explicitly. If we take b_i^* as a function of b_j^* , b_i^* shifts upward when θ_i increases. On the other hand, b_i^* is increasing in b_j^* if firm i is a private firm ($\theta_i = 0$).

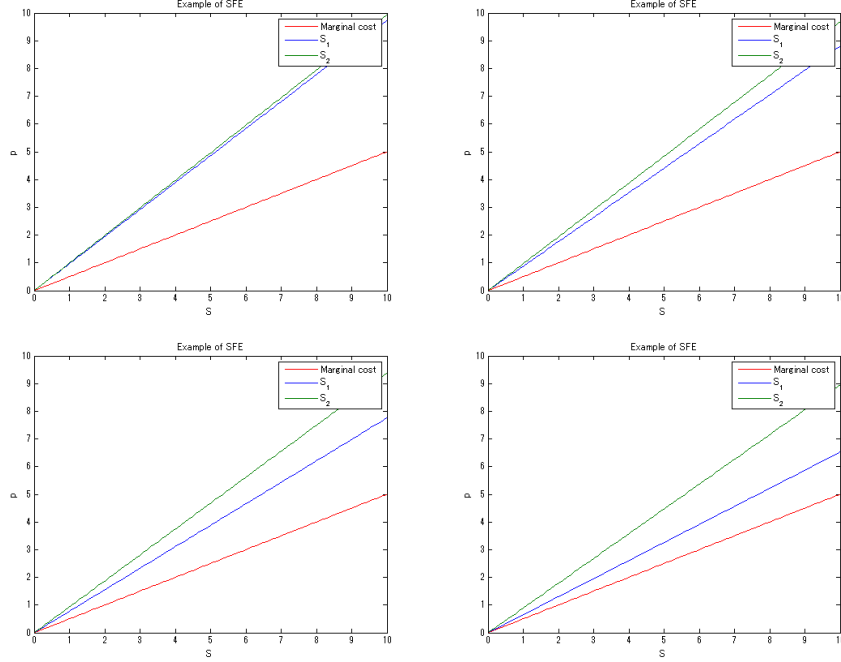


Figure 1: Numerical examples of SFE where $m = 1$, $c = 0.5$, and $\theta_2 = 0$. $\theta_1 = 0.1$: upper left, $\theta_1 = 0.4$: upper right, $\theta_1 = 0.7$: lower left, and $\theta_1 = 0.999$: lower right

Here, suppose that firm 1 is partially privatized and firm 2 is completely private. If we increase θ_1 , b_1^* shifts upward. Therefore, not only b_1^* but also b_2^* would increase since b_2^* is increasing in b_1^* . (Numerical examples are illustrated in Fig. 1.)

Proposition 2: (Effect of partial privatization) *If we consider a linear demand function and quadratic cost functions, an SFE is characterized as $S_i(p) = b_i^*p \ \forall i = 1, 2$, where $b_i^* = [(m + b_j^*) + \theta_i (b_j^*c - 1) b_j^*] / [(1 - \theta_i) + c(m + b_j^*)]$ ($j \neq i$). Furthermore, if firm 2 is a private firm ($\theta_2 = 0$), then both $S_1'(p)$ and $S_2'(p)$ increase when θ_1 increases.*

Proof. See Appendix.

Since both firms' supply functions become closer to the (inverse) marginal cost function for larger θ_1 , the social welfare increases with θ_1 . Thus, we can state the following corollary.

Corollary of Proposition 2: (Mixed oligopoly) *Suppose firm 2 is a private firm ($\theta_2 = 0$). Then, $\theta_1 = 1$ (completely public) is the optimal level of partial privatization.*

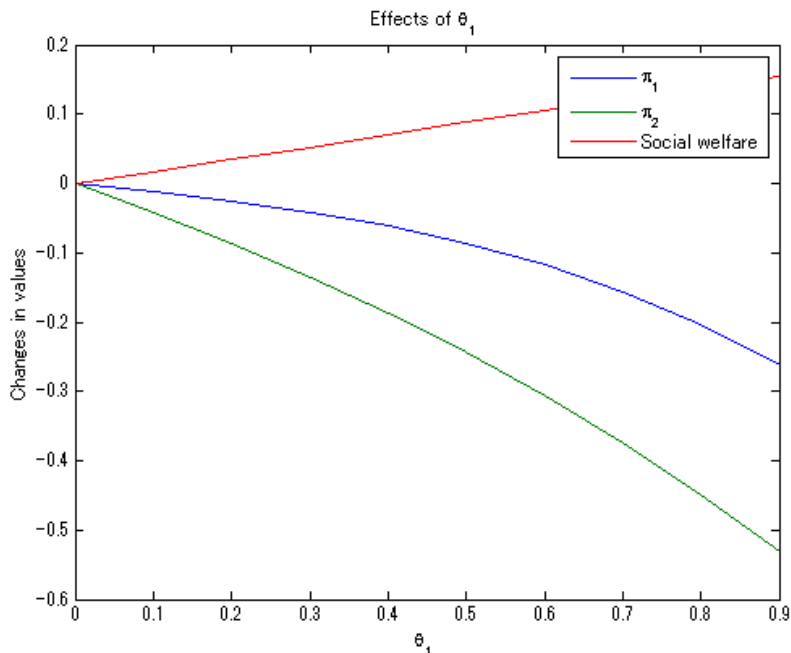


Figure 2: Effects on social welfare and profits. ($\alpha = 5$, $m = 1$, $c = 0.5$, $E[\epsilon] = 0$, $E[\epsilon^2] = 2$, and $\theta_2 = 0$.)

Thus, existence of a public firm increases social welfare in supply function competition, in contrast to the Cournot competition examined by Matsumura (1998). (Welfare improvement is illustrated in Fig. 2.)

Although we do not have any analytical solutions of eq. (8) for arbitrary θ_1 and θ_2 , we can solve it for symmetric social concern: $\theta_1 = \theta_2 = \theta$. In addition, it turns out that the solution of eq. (8), which is derived from the assumption that the supply function is linear, coincides with the unique solution of the differential equation (3), which satisfies SOC of the payoff function for each firm. Therefore, the uniqueness of the equilibrium is also proven for symmetric θ . Since we have an analytical formula for SFE, we can confirm that the supply function converges to the (inverse) marginal cost function when θ approaches 1 under the symmetry assumption of θ .

Proposition 3: (Symmetric SFE in a linear-quadratic model) *In a symmetric setting ($\theta_1 = \theta_2 = \theta$), we can obtain the unique symmetric SFE:*

$$S_i(p) = \frac{-m + \sqrt{m^2 + 4m(1-\theta)/c}}{2(1-\theta)}p \quad \forall i = 1, 2.$$

Furthermore, it converges to the (inverse) marginal cost function as $\theta \rightarrow 1$.

Proof. See Appendix.

We can interpret the symmetric θ as the extent of corporate social responsibility and an increase in θ as the improvement in the industry's social concern. The model guarantees that the supply function shifts upward monotonically as the industry's social concern improves and that it converges to the socially optimal level as $\theta \rightarrow 1$.

4 Conclusion

In this paper, we examine the supply function equilibrium introduced by Klemperer and Meyer (1989) for (partially) public firms. First, we generally characterize the SFEs in partially privatized markets. We show the necessity of a positive slope for a symmetric SFE. Second, we specify the demand and cost functions and show that the supply functions of not only a partially public firm (when it weighs more on the social welfare) but also a private firm is flatter. Thus, existence of a public firm increases social welfare in supply function competition in contrast to the Cournot competition examined by Matsumura (1998). Finally, we confirmed that the supply function equilibrium converges to the (inverse) marginal cost function when the firm generates more publicity or social concern in the industry increases symmetrically.

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Appendix

A Proofs

We characterize the differential equation (3) by the following series of lemmas.

Lemma 1 *The locus of points satisfying $f(p, S) = 0$ is a continuous, differentiable function $S = S^0(p)$, satisfying*

- (i) $S^0(0) = 0$,
- (ii) $S^0(p) < (C')^{-1}(p)$, $\forall p > 0$,
- (iii) $S^{0'}(p)$ is positive and increasing in θ , $\forall p \geq 0$, and
- (iv) $S^{0'}(0) < \frac{1}{C''(0)}$.

Proof of Lemma 1: Differentiating (3) w.r.t. S yields

$$f_S(p, S) = \frac{p - C'(S) + SC''(S)}{(p - C'(S))^2}$$

so for all $(\bar{p}, \bar{S}) \neq (0, 0)$ such that $f(\bar{p}, \bar{S}) = 0$,

$$\begin{aligned} f_S(\bar{p}, \bar{S}) &= \frac{1}{\bar{p} - C'(\bar{S})} + \frac{\bar{S}}{\bar{p} - C'(\bar{S})} \frac{C''(\bar{S})}{\bar{p} - C'(\bar{S})} \\ &= \frac{1 - \frac{1}{(1-\theta)} D_p(\bar{p}) C''(\bar{S})}{\bar{p} - C'(\bar{S})} \neq 0. \end{aligned}$$

Therefore, by the implicit function theorem, $f(p, S) = 0$ implicitly defines, in the neighborhood of any such (\bar{p}, \bar{S}) , a unique function $S = S^0(p)$, which is continuous and differentiable.

To prove (i) and (ii), observe that $\forall \theta \in [0, 1)$, $S^0(p)$ and $p - C'(S^0(p))$ are both positive

or both zero since $-\infty < D_p(p) < 0$ and $f(p, S^0(p)) = 0$. Hence, $p > C'(S^0(p))$ whenever $S^0(p) > 0$. Furthermore, $S^0(0) = 0$ is the unique solution to $f(0, S) = 0$. For all $p > 0$, $S^0(p) > 0$ (otherwise, $S^0(p) = 0$ and $p - C'(S^0(p)) > 0$) and $p > C'(S^0(p))$. Since $C'' > 0$, we can take the inverse function of C' and obtain $(C')^{-1}(p) > S^0(p)$ for all $p > 0$. Here, as $p \rightarrow 0$, the upper bound of $S^0(p)$ converges to zero and $S^0(p) > 0$ for all $p > 0$. Then, $S^0(p) \rightarrow 0$ as $p \rightarrow 0$. Thus, $S^0(p)$ is continuous at $p = 0$.

To prove (iii) and (iv), differentiate $f(p, S^0(p)) = 0$ totally with respect to p and substitute using this equation to get

$$S^{0'}(p) = -\frac{D_p(p) + D_{pp}(p)(p - C'(S^0(p)))}{(1 - \theta) - D_p(p)C''(S^0(p))}.$$

Now $\lim_{p \rightarrow 0} S^{0'}(p)$ exists and equals

$$-\frac{D_p(0)}{(1 - \theta) - D_p(0)C''(0)} \equiv S^{0'}(p),$$

where $0 < S^{0'}(p) < \frac{1}{C''(0)}$, and thus $S^0(p)$ is continuous and differentiable at $p = 0$. **Q.E.D.**

Lemma 2 *The locus of the points satisfying $f(p, S) = \infty$ is a continuous, differentiable function, $S = S^\infty(p) \equiv (C')^{-1}(p)$. Hence, $S^\infty(0) = 0$ and $0 < S^{0'}(p) < \infty \forall p \geq 0$.*

Proof of Lemma 2 (same as the proof of claim 2 in KM): From (3), $S^\infty(p)$ solves $f(p, S^\infty(p)) = \infty$ implies that $S^\infty(p)$ solves $p - C'(S^\infty(p)) = 0$. Hence, since $C'' > 0$, $S^\infty(p) = (C')^{-1}(p) \forall p$. The stated properties of $S^\infty(p)$ follow from the assumptions for $C'(S)$. **Q.E.D.**

Lemma 3 *For all points (p, S) between the $f = 0$ and $f = \infty$ loci, $0 < f(p, S) < \infty$. For all points in the first quadrant above the $f = 0$ locus or below the $f = \infty$ locus, $0 > f(p, S) > -\infty$.*

Proof of Lemma 3 (same as the proof of claim 3 in KM): Since $\frac{S}{p-C'(S)}$ is finite and increasing in S as long as $p > C'(S)$, for a given \bar{p} , $f(\bar{p}, S)$ is finite and monotonically increasing in S for $S \in [0, (C')^{-1}(p)]$. Below the $f = \infty$ locus, $0 > \frac{S}{p-C'(S)} > -\infty$. Hence, since $0 > D_p > -\infty$, $0 > f(p, S) > -\infty$. **Q.E.D.**

Lemma 4 *If $S(p)$ solves (3) and the other firm takes $S(p)$, the second derivative of i 's payoff with respect to p for a given ϵ evaluated at an intersection of $S(p)$ and residual demand function $D(p) + \epsilon - S(p)$ is written as*

$$\begin{aligned} \frac{\partial^2 v_i(p, \epsilon; S(p))}{\partial p^2} \Big|_{p=p^*} &= (D_p(p^*) - S'(p^*)) ((1 - \theta) + C''(D(p^*) + \epsilon - S(p^*))) \\ &\quad - C''(D(p^*) + \epsilon - S(p^*)) (D_p(p^*) - S'(p^*))^2 - (1 - \theta) S'(p^*), \end{aligned} \quad (9)$$

where p^* is the price that solves $D(p^*) + \epsilon - 2S(p^*) = 0$.

Proof of Lemma 4: Given that j chooses $S(p)$, the second order derivative of i 's payoff with respect to p for a given ϵ is

$$\begin{aligned} \frac{\partial^2 v_i(p, \epsilon; S(p))}{\partial p^2} &= (2 - \theta) \{D_p(p) - S'(p)\} - C''(D(p) + \epsilon - S(p)) (D_p(p) - S'(p))^2 \\ &\quad + (p - C'(D(p) + \epsilon - S(p))) (D_{pp}(p) - S''(p)) \\ &\quad + \theta S'(p) - \theta C''(S(p)) (S'(p))^2 + \theta (p - C'(S(p))) S''(p). \end{aligned} \quad (10)$$

If $S(p)$ solves (3), we can differentiate (3) totally with respect to p to obtain an expression for $S''(p)$:

$$S''(p) = \frac{X_1}{((1 - \theta)(p - C'(S(p))))^2}, \quad (11)$$

where

$$X_1 \equiv [(1 - \theta) S'(p) + (1 - C''(S(p)) S'(p)) D_p(p) + (p - C'(S(p))) D_{pp}(p)] [(1 - \theta)(p - C'(S(p)))] \\ - [(1 - \theta) S(p) + (p - C'(S(p))) D_p(p)] [(1 - \theta)(1 - C''(S(p)) S'(p))].$$

Substituting (3) for $S(p)$ into (11) gives

$$S''(p) = \frac{(1 - \theta) S'(p) + (1 - C''(S(p)) S'(p)) (D_p(p) - (1 - \theta) S'(p)) + (p - C'(S(p))) D_{pp}(p)}{(1 - \theta)(p - C'(S(p)))}, \quad (12)$$

and thus, when $S(p)$ solves (3), $S''(p)$ in (10) is replaced by (12). Moreover, if we evaluate at $p = p^*$ where p^* solves $D(p^*) + \epsilon - 2S(p) = 0$, (10) becomes (9). **Q.E.D.**

By these lemmas, we can prove Proposition 1.

Proof of Proposition 1 Satisfaction of (3) $\forall p \geq 0$ is a necessary condition for a supply function defined for all $p \geq 0$ to trace through ex post optimal points when the other firm commits to the same supply function. To show that $0 < S'(p) < \infty \forall p \geq 0$ is also a necessary condition, we show that if, for some p , S ever crosses either $f = 0$ from below or $f = \infty$ from the left, then S must eventually violate the global optimality⁹.

Once trajectory S crosses $f = 0$ from below, S' become and stays negative and, from (A2), S'' also becomes and stays negative. Therefore, the trajectory would eventually intersect the $S = 0$ axis at a point $(p_0, 0)$ with $p_0 > C'(0)$, where $S'(p_0) = f(p_0, 0) = \frac{1}{1-\theta} D_p(p_0)$. Therefore, for $\epsilon = e(0, p_0)$, $Q = D(p_0, \epsilon) = 0$ by definition and then, residual demand $D(p_0, \epsilon) - S(p_0) = 0$. Then, given firm j takes S , p_0 satisfies the first order condition but that results in $q_i = q_j = 0$ and $SW = \pi_i = v_i = 0$. On the other hand, since $S'(p_0) = \frac{1}{1-\theta} D_p(p_0)$, $S(p)$ and the residual demand $D(p, \epsilon) - S(p)$ for the same $\epsilon = e(0, p_0)$ cross each other

⁹Such a part of S represents ones of multiple intersections for certain ϵ which results in a smaller profit than that of another intersection for the same ϵ .

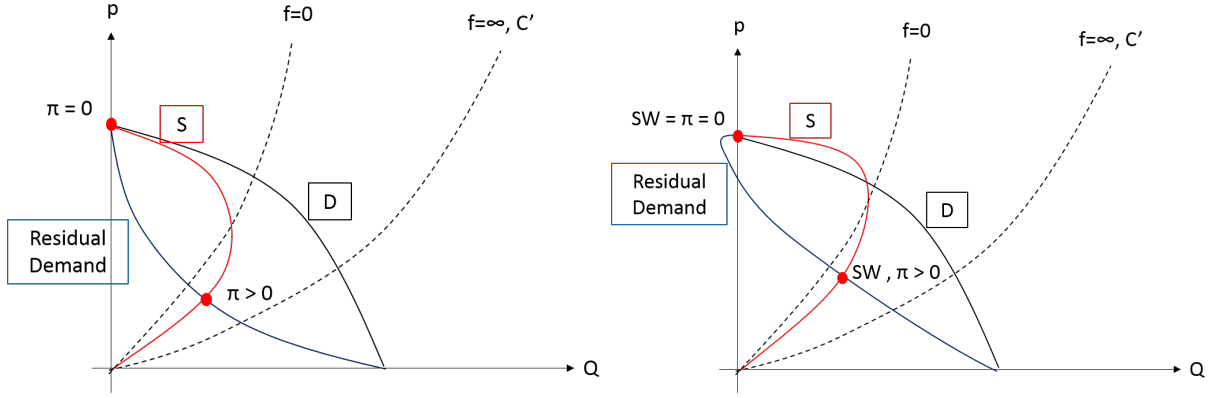


Figure 3: A supply function (satisfying FOC and symmetry) violating global optimality. (Left: $\theta = 0$, Right: $0 < \theta < 1$)

at another point (p_1, q_1) where $q_1 > 0$ and $p_1 > C'(q_1)$ (Fig.3). Since $SW, \pi_i, v_i > 0$ at (p_1, q_1) , firm i has an incentive to adjust from p_0 to p_1 . Thus, S eventually violates the global optimality. **Q.E.D.**

Proof of Proposition 2 WLOG, let assume that firm 1 is a partially public firm ($\theta_1 \in [0, 1]$) and firm 2 is a private firm ($\theta_2 = 0$). Then, $B_i(b_j; \theta_i)$ is rewritten as follows:

$$B_1(b_2; \theta_1) = \frac{(m + b_2) + \theta_1 (b_2 c - 1) b_2}{(1 - \theta_1) + c(m + b_2)},$$

and

$$B_2(b_1) \equiv B_2(b_1; 0) = \frac{(m + b_1)}{1 + c(m + b_1)}.$$

At the equilibrium, b_1^* and b_2^* satisfy the following:

$$F(b_1^*, b_2^*; \theta) \equiv \begin{bmatrix} b_1^* - B_1(b_2^*; \theta_1) \\ b_2^* - B_2(b_1^*) \end{bmatrix} = 0$$

Then, the Jacobian of $F(b_1^*, b_2^*; \theta)$ w.r.t. (b_1^*, b_2^*) is written as $J_{(b_1^*, b_2^*)} = \begin{bmatrix} 1 & -\frac{\partial B_1(b_2^*, \theta_1)}{\partial b_2^*} \\ -\frac{\partial B_2(b_1^*)}{\partial b_1^*} & 1 \end{bmatrix}$

, and then, by Implicit Function Theorem,

$$\begin{aligned} \begin{bmatrix} \frac{db_1^*}{d\theta_1} \\ \frac{db_2^*}{d\theta_1} \end{bmatrix} &= -J_b^{-1} \begin{bmatrix} -\frac{dB_1(b_2^*, \theta_1)}{d\theta_1} \\ 0 \end{bmatrix} \\ &= \frac{1}{1 - \frac{\partial B_1(b_2^*; \theta_1)}{\partial b_2^*} \frac{\partial B_2(b_1^*)}{\partial b_1^*}} \begin{bmatrix} 1 & \frac{\partial B_1(b_2^*; \theta_1)}{\partial b_2^*} \\ \frac{\partial B_2(b_1^*)}{\partial b_1^*} & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial B_1(b_2^*; \theta_1)}{\partial \theta_1} \\ 0 \end{bmatrix} \end{aligned}$$

Then, what we need to show is that $\frac{\partial B_1(b_2^*; \theta_1)}{\partial \theta_1} > 0$, $\frac{\partial B_1(b_2^*; \theta_1)}{\partial b_2^*} < 1$, and $\frac{\partial B_2(b_1^*)}{\partial b_1^*} \in (0, 1)$.

In general, we have

$$\frac{\partial B_i(b_j; \theta_i)}{\partial b_j} = \frac{\{1 + \theta_i (2b_j c - 1)\} \{(1 - \theta_i) + c(m + b_j)\} - c \{(m + b_j) + \theta_i (b_j c - 1) b_j\}}{\{(1 - \theta_i) + c(m + b_j)\}^2}$$

Thus, for firm 2,

$$\frac{\partial B_2(b_1)}{\partial b_1} = \frac{1}{\{1 + c(m + b_1)\}^2} > 0.$$

Furthermore, we can rewrite $\frac{\partial B_i(b_j; \theta_i)}{\partial b_j}$ as the following:

$$\frac{\partial B_i(b_j; \theta_i)}{\partial b_j} = 1 + \frac{\{-(1 - \theta) cb_j\} \{1 - \theta_i + cm + cb_j\} - cb_j (1 - \theta)^2 - (1 - \theta_i) cmb_j c - cm \{2 - \theta_i + cm\}}{\{(1 - \theta_i) + c(m + b_j)\}^2}.$$

Since the second term is less than 0, we have $\frac{\partial B_i(b_j; \theta_i)}{\partial b_j} < 1$. Lastly, by taking the derivative w.r.t. θ_1 ,

$$\begin{aligned} \frac{\partial B_1(b_2^*; \theta_1)}{\partial \theta_1} &= \frac{(1 - \theta_1) (b_2 c - 1) b_2 + c(m + b_2) (b_2 c - 1) b_2 + (m + b_2) + \theta_1 (b_2 c - 1) b_2}{\{(1 - \theta_1) + c(m + b_2)\}^2} \\ &= \frac{b_2^3 c^2 + (b_2^2 c^2 - b_2 c) m + m}{\{(1 - \theta_1) + c(m + b_2)\}^2} \\ &= \frac{b_2^3 c^2 + (b_2 c - \frac{1}{2})^2 m + \frac{3}{4} m}{\{(1 - \theta_1) + c(m + b_2)\}^2} > 0 \end{aligned}$$

Thus, if we suppose that $\theta_2 = 0$, then $\frac{db_1^*}{d\theta_1} > 0$ and $\frac{db_2^*}{d\theta_1} > 0$ **Q.E.D.**

Proof of Proposition 3 By proposition 1, for S to be a symmetric SFE, S must satisfy (3) and $0 < S'(p) < \infty$. In the linear case, (3) is rewritten as follows:

$$\begin{aligned} S'(p) &= \frac{(1 - \theta) S + \{p - C'(S)\} \cdot D_p(p)}{(1 - \theta) \{p - C'(S)\}} \\ &= \frac{(1 - \theta) S + \{p - cS\} \cdot (-m)}{(1 - \theta) \{p - cS\}}. \end{aligned}$$

In an autonomous form,

$$\begin{aligned} \frac{dS}{dt} &= (1 - \theta) S + \{p - cS\} \cdot (-m) \\ \frac{dp}{dt} &= (1 - \theta) \{p - cS\}. \end{aligned}$$

Then,

$$\begin{aligned} \begin{bmatrix} \frac{dS}{dt} \\ \frac{dp}{dt} \end{bmatrix} &= \begin{bmatrix} (1 - \theta + mc) S - mp \\ -c(1 - \theta) S + (1 - \theta) p \end{bmatrix} \\ &= \begin{bmatrix} (1 - \theta + mc) & -m \\ -c(1 - \theta) & (1 - \theta) \end{bmatrix} \begin{bmatrix} S \\ p \end{bmatrix}. \end{aligned}$$

For any eigenvalue r , the following equation must be satisfied:

$$\begin{aligned} \det \left(\begin{bmatrix} (1 - \theta + mc) & -m \\ -c(1 - \theta) & (1 - \theta) \end{bmatrix} - rI \right) &= 0 \\ \Leftrightarrow r &= \frac{2(1 - \theta) + mc \pm \sqrt{m^2 c^2 + 4mc(1 - \theta)}}{2}. \end{aligned}$$

We have different and unequal eigenvalues. For each eigenvalue r_1, r_2 , the eigenvectors are defined as follows:

$$\begin{aligned} & \left\{ \begin{bmatrix} (1-\theta+mc) & -m \\ -c(1-\theta) & (1-\theta) \end{bmatrix} - r_i I \right\} \begin{bmatrix} u_i \\ w_i \end{bmatrix} = 0 \\ & \Leftrightarrow \begin{bmatrix} ((1-\theta+mc) - r_i) u_i - m w_i \\ -c(1-\theta) u_i + ((1-\theta) - r_i) w_i \end{bmatrix} = 0. \end{aligned}$$

Then,

$$\begin{aligned} \frac{u_i}{w_i} &= \frac{(1-\theta) - r_i}{c(1-\theta)} \\ &= \frac{(1-\theta) - (1-\theta) - \frac{mb \pm \sqrt{m^2 c^2 + 4mc(1-\theta)}}{2}}{c(1-\theta)} \\ &= \frac{-mb \mp \sqrt{m^2 c^2 + 4mc(1-\theta)}}{2c(1-\theta)} \\ &= \frac{-m \mp \sqrt{m^2 + \frac{4m(1-\theta)}{c}}}{2(1-\theta)}. \end{aligned}$$

Let larger eigenvalue be r_1 . Then, $\frac{u_1}{w_1} < 0$ and $\frac{u_2}{w_2} > 0$. Since the eigenvalues are real and unequal, the solution to the differential equation is written as

$$\begin{pmatrix} S \\ p \end{pmatrix} = A_1 e^{\lambda_1 t} \begin{pmatrix} u_1 \\ w_1 \end{pmatrix} + A_2 e^{\lambda_2 t} \begin{pmatrix} u_2 \\ w_2 \end{pmatrix} \quad (13)$$

where A_1 and A_2 are arbitrary constants. Here, if $A_1 \neq 0$,

$$\begin{aligned} \frac{S}{p} &= \frac{A_1 e^{\lambda_1 t} u_1 + A_2 e^{\lambda_2 t} u_2}{A_1 e^{\lambda_1 t} w_1 + A_2 e^{\lambda_2 t} w_2} \\ &= \frac{A_1 u_1 + A_2 (e^{(\lambda_2 - \lambda_1)t}) u_2}{A_1 w_1 + A_2 (e^{(\lambda_2 - \lambda_1)t}) w_2} \\ &\rightarrow \frac{u_1}{w_1} < 0 \text{ as } t \rightarrow \infty, \end{aligned}$$

and thus, all trajectories eventually leave the region between $f = 0$ and $f = \infty$, and their slope becomes negative. Therefore, the only remaining S that satisfies the necessary conditions is (13) with $A_1 = 0$:

$$S(p) = \frac{-m + \sqrt{m^2 + \frac{4m(1-\theta)}{c}}}{2(1-\theta)} p \equiv \frac{g(\theta)}{h(\theta)}. \quad (14)$$

Suppose that the other firm takes this linear supply function. Then, the local optimality for firm i 's payoff function is satisfied along $S(p)$ by lemma 5, and the residual demand is also linear since the demand function is defined as linear. Since both the residual demand and marginal cost are linear in p , given ϵ , firm i 's profit function π_i is written as a function quadratic in p . On the other hand, since the demand function and industrial marginal costs are linear in p , given ϵ , SW is written as a function quadratic in p . Therefore, the payoff for firm i , which is the weighted average of i 's profit and SW , is written as a quadratic function. Therefore, the local optimal point given ϵ is actually a unique global maximizer given ϵ . Thus, (14) is a symmetric SFE.

We check the effect of θ . Since $g(1) = h(1) = 0$ by l'Hopital's Rule we have

$$\lim_{\theta \rightarrow 1} S(p) = \lim_{\theta \rightarrow 1} \frac{g(\theta)}{h(\theta)} = \lim_{\theta \rightarrow 1} \frac{g'(\theta)}{h'(\theta)} = \lim_{\theta \rightarrow 1} \frac{m}{c} \frac{p}{\sqrt{m^2 + \frac{4m(1-\theta)}{c}}} = \frac{p}{c}$$

Thus, θ converges to 1 and the supply function converges to the marginal production cost.

Q.E.D.