On the diversification benefit of reinsurance portfolios

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Abstract

In this paper we compare the diversification benefit of portfolios containing excess-of-loss treaties and portfolios containing quota-share treaties, when the risk measure is the (excess) Value-at-Risk or the (excess) Expected Shortfall. In a first section we introduce the set-up under which we perform our investigations. Then we show that when the losses are continuous, independent, bounded, the cover unlimited and when the risk measure is the Expected Shortfall at a level κ close to 1, a portfolio of n excess-of-loss treaties diversifies better than a comparable portfolio of n quota-share treaties. This result extends to the other risk measures under additional assumptions. We further provide evidence that the boundedness assumption is not crucial by deriving analytical formulas in the case of treaties with i.i.d. exponentially distributed original losses. Finally we perform the comparison in the more general setting of arbitrary continuous joint loss distributions and observe in that case that a finite cover leads to opposite results, i.e. a portfolio of n quota-share treaties diversifies better than a comparable portfolio of n excess-of-loss treaties at high quantile levels.

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1 Introduction

A proper diversification provides a powerful mechanism for a financial institution to manage its risks and offer good value to its costumers. The business proposition of reinsurers is to offer a more diversified portfolio to its customers in order to reduce their capital cost of carrying it (see for instance, Dacorogna and Hummel [2005]). Thus it is fundamental for such business to understand the extent of diversification that it can achieve with various forms of contracts. In this paper, we aim at contributing to a better understanding of the diversification benefit of certain reinsurance portfolios. More specifically, we establish a framework in which we can compare, from the perspective of a reinsurer, the diversification benefit of portfolios containing quota-share contracts to the diversification benefit of portfolios containing excess-of-loss contracts. In a quota-share contract, the issuer (reinsurer) assumes a share $\alpha \in (0,1)$ of the client's loss L to which we refer as the *original loss*. These contracts are aggregated into portfolios to which we refer as "proportional portfolios". On the other hand, the set-up of an excess-of-loss treaty is slightly more involved. In this case, the issuer covers the loss of the client that exceeds a pre-agreed amount D and pays the client up to a pre-determined amount C. These treaties are commonly referred to as "C excess D ", or more simply C xs D. We refer to D as the *deductible* of the contract and to C as the *cover (limit)* of the contract. These treaties are pooled into portfolios to which we refer as "non-proportional portfolios". The *diversification benefit* of a portfolio covering n losses $L_1,\ldots,L_n$ is defined as in Bürgi et al. [2008] by

$$\mathrm{DB}(S,\rho) = 1 - \frac{\rho(\sum_{i=1}^n L_i)}{\sum_{i=1}^n \rho(L_i)}, \tag{1}$$

with $S = \sum_{i=1}^n L_i$ and ρ a given risk measure. For subadditive risk measures (as defined in Artzner et al. [1999]), the function $\mathrm{DB}(\cdot)$ takes values in [0,1]. A non-proportional portfolio and its proportional counterpart, which form the subject of our comparison, are chosen such that they cover via their respective contracts the same original losses $L_1,\ldots,L_n$ (client's losses). A thorough comparison allows then to identify conditions under which one portfolio outperforms the other in terms of diversification benefit. We mainly focus on assessing the diversification benefit with the Expected Shortfall. However, for completeness discussions are also provided for $\mathrm{DB}(\cdot)$ measured with the Value-at-Risk and the corresponding capital (see Busse et al. [2013]), which is defined as the deviation to the expectation.[1]

Our paper is organized as follows: we present in Section 2 the modelling of the quota-share and the excess-of-loss contracts, together with the corresponding portfolios. Then we proceed in Section 3 by comparing the diversification benefit between these portfolios for excess-of-loss treaties with an infinite cover. The original losses in Section 3.1 are assumed to be independent bounded and in Section 3.2 to be independent exponential. The comparison is further performed in Section 4 for excess-of-loss treaties with a finite cover, where we can remove the independence and boundedness assumptions of the original losses. Conclusions are drawn in Section 5.

[1] In this paper we discuss the deviation of Value-at-Risk and Expected Shortfall to the expectation (xVaR and xES).

2 Modelling of reinsurance portfolios

Let L be a non-negative, (absolutely) continuous and integrable random variable (rv) representing the loss distribution of a client (original loss), e.g. of an insurance company. The reinsurer offers a contract that covers a part of this loss L in return for a premium. Let us observe what happens in the cases of quota-share and excess-of-loss contracts.

Quota-share

The loss that a reinsurer bears in a quota-share contract with a pre-agreed percentage $\alpha \in (0,1)$ is

$$L^{\mathrm{P}} = \alpha L, \tag{2}$$

where L denotes the original loss of the client. We keep the notation L^{P} to refer to the reinsurer's loss arising from a single quota-share contract (proportional business).

Excess-of-loss

Having issued an excess-of-loss treaty with a deductible D and a cover C, the reinsurer bears the following loss,

$$L^{\mathrm{NP}} = \begin{cases} 0, & \text{if } L \le D; \\ L - D, & \text{if } D < L \le D + C; \\ C, & \text{if } L > D + C. \end{cases} \tag{3}$$

The random variable L^{NP}, modelling the loss arising from an excess-of-loss treaty, is a mixed random variable with positive mass at zero and at the cover C. We first compute in Lemma 2.1 the distribution function (df) of an excess-of-loss treaty L^{NP} (as defined in (3)), and then derive in Corollary 2.2 analytical formulas for the risk measures of L^{NP}. In all what follows we will denote the positive part of a real number x by

$$(x)^+ = \max(x, 0).$$

Lemma 2.1. *(Distribution function of an excess-of-loss treaty.) Let L be a non-negative, continuous and integrable random variable on $[0,\infty)$ with df F_L and let $D, C \in (0,+\infty)$. Let $L^{\mathrm{NP}} = (L-D)^+ - (L-D-C)^+$ be the loss borne by the reinsurer as defined in (3). Then, $\forall x \in \mathbb{R}$ the distribution function of L^{NP} satisfies*

$$F_{L^{\mathrm{NP}}}(x) = \begin{cases} 0, & \text{if } x < 0; \\ F_L(x+D), & \text{if } 0 \le x < C; \\ 1, & \text{if } x \ge C. \end{cases} \tag{4}$$

The proof is given in Appendix A.1. It is important to note that $F_{L^{\mathrm{NP}}}$ is not differentiable, hence L^{NP} does not have a continuous probability density function. Figure 1 illustrates the results of Lemma

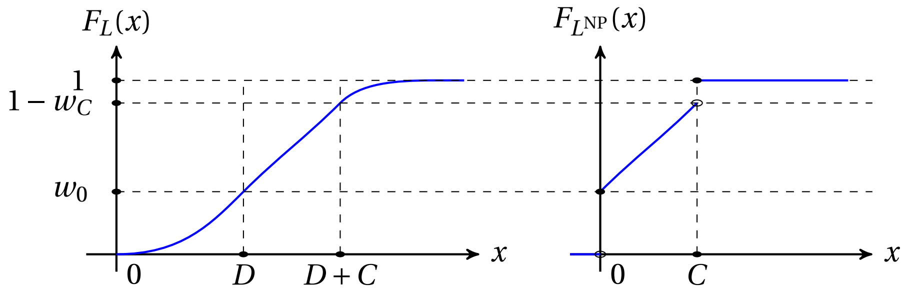


Figure 1: On the left, generic df F_L of the original loss L; on the right, df $F_{L^{NP}}$ of the corresponding excess-of-loss treaty with deductible D and cover C.

2.1 by showing the shape of the df's of L and L^{NP}. It can be seen from the graph that $F_{L^{NP}}$ has two jumps: one at zero with height w_0 equal to P$(L \leq D)$, and one at the cover C with height w_C equal to $P(L \geq D+C)$. Keeping Figure 1 in mind, we continue further, by deriving the Value-at-Risk and Expected Shortfall of L^{NP}. The *Value-at-Risk* of the rv (modelling losses) X with df F_X at probability level $\kappa \in (0,1)$ is defined as

$$\mathrm{VaR}_\kappa(X) = \inf\{x \in \mathbb{R}, F_X(x) \geq \kappa\}.$$

Intuitively, the statement "The Value-at-Risk of L at 95% is 371" ($\mathrm{VaR}_{95\%}(L) = 371$) means that with at least 95% probability, the loss L will not exceed 371. It is important to note that VaR_κ is not always a subadditive risk measure and hence not always coherent; this topic is extensively studied in the literature, see e.g. Embrechts et al. [2002]. The *Expected Shortfall* of X with $\mathbb{E}[|X|] < \infty$ at a probability level $\kappa \in (0,1)$ is,

$$\mathrm{ES}_\kappa(X) = \frac{1}{1-\kappa}\int_\kappa^1 \mathrm{VaR}_u(X)\,du.$$

Expected Shortfall ES_κ is a coherent measure of risk (see Embrechts and Wang [2015]). It can be interpreted as the average loss in the right tail of the distribution. By definition, it provides more information than Value-at-Risk, since it takes into consideration not only the frequency but also the severity of potential extreme losses. Estimating these risk measures contributes to a deeper understanding of the risk faced when holding portfolios and thus to a better management of this risk. The results are presented in the following corollary.

Corollary 2.2. (VaR_κ *and* ES_κ *of an excess-of-loss treaty.)* *Let D and C be two positive real numbers, L^{NP} be defined as in Lemma 2.1 and $\kappa \in (0,1)$. Then,*

$$\mathrm{VaR}_\kappa(L^{NP}) = \min\{\max\{\mathrm{VaR}_\kappa(L) - D, 0\}, C\},$$

$$\mathrm{ES}_\kappa(L^{NP}) = \begin{cases} \frac{\mathbb{E}[L^{NP}]}{1-\kappa}, & \text{if } \kappa \leq w_0; \\ \frac{1}{1-\kappa}\int_{\mathrm{VaR}_\kappa(L)}^{C+D} u f_L(u)\,du + (C+D)\frac{w_C}{1-\kappa} - D, & \text{if } w_0 < \kappa < 1-w_C; \\ C, & \text{if } \kappa \geq 1-w_C, \end{cases}$$

where $w_0 = F_L(D)$ and $w_C = 1 - F_L(D+C)$.

The proof is given in Appendix A.2. We present in Figure 2 an illustration of the Value-at-Risk for a generic original loss L and for the corresponding excess-of-loss treaty L^{NP} defined as in Eq. (3). It is better visualized from this example how the Value-at-Risk of an excess-of-loss contract compares to the Value-at-Risk of the underlying original loss. Figure 2 is also a graphical representation of the first result of Corollary 2.2. It can be seen that for $\kappa \in [w_0, 1-w_C]$, the Value-at-Risk of L^{NP} equals the Value-at-Risk of L shifted down by D units and that for $\kappa \geq (1-w_C)$, the Value-at-Risk and the Expected Shortfall of L^{NP} are both equal to the cover C. These results will be further used in Sections 3 and 4 when studying the diversification benefit of portfolios containing n excess-of-loss treaties.

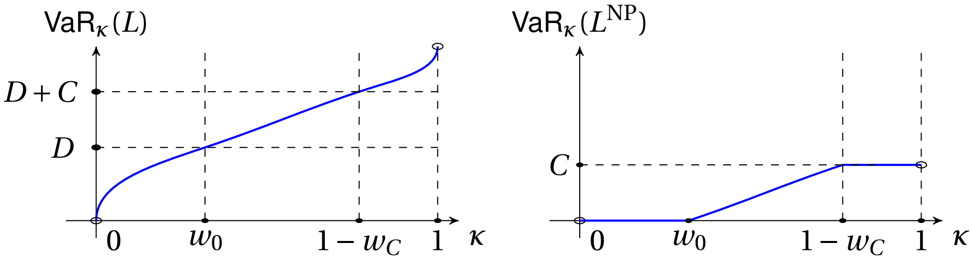


Figure 2: Value-at-Risk of the original loss L (left) and of the corresponding excess-of-loss contract with finite cover C and finite deductible D (right).

Let us also introduce the *excess Expected Shortfall* and *excess Value-at-Risk* at level $\kappa \in (0,1)$ for an integrable random variable X as follows:

$$\mathrm{xES}_\kappa(X) = \mathrm{ES}_\kappa(X) - \mathbb{E}[X], \qquad \text{and} \qquad \mathrm{xVaR}_\kappa(X) = \mathrm{VaR}_\kappa(X) - \mathbb{E}[X], \tag{5}$$

respectively, which represents the capital as defined in Busse et al. [2013].

Aggregation into portfolios

For a given set of original losses $(L_1, L_2, \ldots, L_n)$, we model the loss of the underlying portfolio by

$$S_n = \sum_{i=1}^{n} L_i.$$

We consider non-proportional and proportional portfolios that have the same set of original losses. More precisely, the non-proportional portfolio is modelled by

$$S_n^{\mathrm{NP}} = \sum_{i=1}^{n} L_i^{\mathrm{NP}},$$

where each L_i^{NP} is defined as in Eq. (3) with a deductible D_i, cover C_i and original loss L_i. The corresponding proportional portfolio is modelled by

$$S_n^{\mathrm{P}} = \sum_{i=1}^{n} L_i^{\mathrm{P}},$$

where each L_i^{P} is applied with a fixed share α on the original losses L_i (see Eq. (2)). Next we compare under various assumptions the diversification benefit of S_n^{P} and S_n^{NP}.

3 Comparison for independent treaties and unlimited covers

So far we have modelled original losses by non-negative, continuous and integrable rv's. In this section we also assume that the original losses are independent and that the excess-of-loss treaties have an unlimited cover. The *right endpoint* of a random variable X is defined as

$$r_X = \sup\{x \in \mathbb{R},\ F_X(x) < 1\}. \tag{6}$$

In Section 3.1 we assume additionally that original losses have a finite right endpoint and compare in this setting the diversification benefit of the proportional and non-proportional portfolios (Theorem 3.3). This comparison is further complemented in Section 3.2 by the case where the original losses have an infinite right endpoint, with the example of identical exponential distributions. This leads to analytical formulas.

3.1 Original losses having a finite right endpoint

For $n \in \mathbb{N}\,(n \geq 2)$, let the original losses L_i, $i = 1, \ldots, n$, be independent with df F_i and a finite right endpoint r_i, and let us assume w.l.o.g. that

$$r_n - D_n \leq \sum_{i=1}^{n-1} r_i - D_i. \tag{7}$$

Let the non-proportional portfolio of n excess-of-loss treaties $L_i^{\mathrm{NP}} = (L_i - D_i)^+$, $i = 1, \ldots, n$, be denoted by $S_n^{\mathrm{NP}} = \sum_{i=1}^n L_i^{\mathrm{NP}}$, where the deductible D_i is finite, positive and smaller than r_i. Then, $L_1^{\mathrm{NP}}, \ldots, L_n^{\mathrm{NP}}$ are also independently distributed with df $F_{L_i^{\mathrm{NP}}}$. Note that we now consider excess-of-loss contracts with an unlimited cover[2], i.e. every L_i^{NP} is a mixed random variable with mass only at zero. From Lemma 2.1, it follows that the distribution function of every L_i^{NP} satisfies $\forall x \in \mathbb{R}$,

$$F_{L_i^{\mathrm{NP}}}(x) = \begin{cases} 0, & \text{if } x < 0; \\ F_i(x + D_i), & \text{if } 0 \leq x < r_i - D_i; \\ 1, & \text{if } x \geq\ r_i - D_i. \end{cases} \tag{8}$$

On Figure 3 we compare the df of the excess-of-loss contract with the one of the original loss, where it can be seen that each L_i^{NP} has a jump at zero of height $w_{0,i} = F_i(D_i)$ (right-hand side). Then, we show on Figure 4 the Value-at-Risk of the original loss L and of the corresponding excess-of-loss L^{NP}. Both graphs simply represent the generalized inverse of the corresponding functions in Figure 3. Having set the ground, we can now assess the risk of a non-proportional portfolio and use this result to compare its diversification benefit to the one of its proportional counterpart, when the risk measure is evaluated at a level κ close to 1.

[2] It is sufficient that the cover is sufficiently high, precisely that $\forall i = 1, \ldots, n, C_i > r_i - D_i$.

Figure 3: Distribution function F_L of the original loss L (left) and of the corresponding excess-of-loss contract L^{NP} with unlimited cover and deductible D (right).

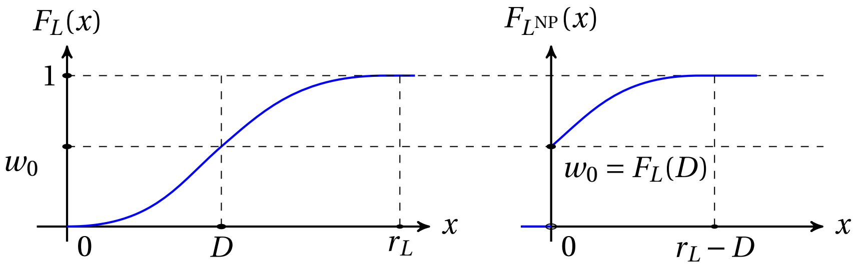
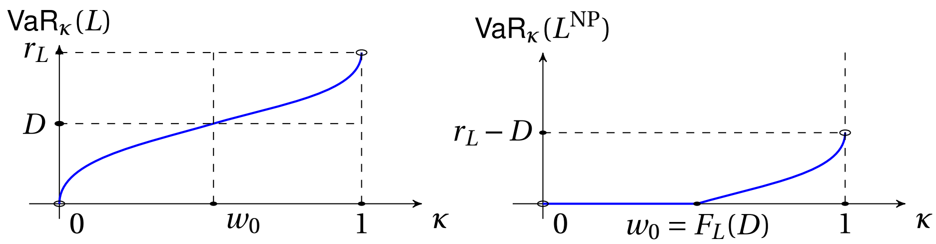


Figure 4: Value-at-Risk of the original loss (left), and of the corresponding excess-of-loss treaty with deductible D (right).



3.1.1 Measuring the risk of non-proportional portfolios

By means of induction, we can reveal the relationship between the distribution function of S_n^{NP} and S_n, for values $s \geq \sum_{i=1}^{n-1}(r_i - D_i)$, in which an analogous formula to Eq. (8) is derived. This is accomplished in Proposition 3.1.

Proposition 3.1. *For a fixed $n \in \mathbb{N}$, let L_i, $i = 1, \dots, n$, be independent, non-negative, continuous, integrable rv's with df F_i, with a finite right endpoint $r_i < \infty$ and such that Eq.* (7) *is satisfied. Let $S_n = \sum_{i=1}^n L_i$ and $S_n^{\text{NP}} = \sum_{i=1}^n L_i^{\text{NP}}$ where $L_i^{\text{NP}} = (L_i - D_i)^+$, $i = \{1, \dots, n\}$ with $D_i \in (0, r_i)$. Then, $\forall s \in [\sum_{i=1}^{n-1}(r_i - D_i), \infty)$,*

$$F_{S_n^{\text{NP}}}(s) = F_{S_n}\Big(s + \sum_{i=1}^{n} D_i\Big).$$

The proof is given in Appendix A.3. For the purpose of risk quantification, one can derive the relation between the risk measures of S_n^{NP} and S_n when the level κ is close to 1. This is done for the Value-at-Risk and the Expected Shortfall in the next corollary.

Corollary 3.2. (VaR_κ *and* ES_κ *for a portfolio of n independent excess-of-loss treaties.*) *Let $n \in \mathbb{N}$*

and S_n^{NP}, S_n be defined as in Proposition 3.1. Then, $\forall \kappa \in \left[F_{S_n}\left(\sum_{i=1}^{n-1} r_i + D_n\right), 1\right)$,

$$\mathrm{VaR}_\kappa(S_n^{\mathrm{NP}}) = \mathrm{VaR}_\kappa(S_n) - \sum_{i=1}^{n} D_i,$$

which yields also

$$\mathrm{ES}_\kappa(S_n^{\mathrm{NP}}) = \mathrm{ES}_\kappa(S_n) - \sum_{i=1}^{n} D_i.$$

The proof follows similar steps as the one given in Appendix A.2. This corollary enables us to compare the diversification benefit of a portfolio of n independent excess-of-loss contracts to the one of its proportional counterpart. This is the main result of this section and is presented in Theorem 3.3.

3.1.2 Main result: comparison of the diversification benefits

The main result of Theorem 3.3 is the comparison of the diversification benefit (1) between the non-proportional portfolio S_n^{NP} and the proportional portfolio S_n^{P}. Recall that the *proportional* portfolio is defined by $S_n^{\mathrm{P}} = \alpha \sum_{i=1}^{n} L_i$, where $\alpha \in (0,1)$.

Theorem 3.3. *(Comparison of the diversification benefits, independent bounded losses and infinite cover case.) For a fixed $n \in \mathbb{N}$, let S_n^{P} and S_n^{NP} denote a proportional portfolio and its non-proportional counterpart defined as in Proposition 3.1, respectively. Then*

$$\forall \kappa \in \left[F_{S_n^{\mathrm{NP}}}\left(\sum_{i=1}^{n-1}(r_i - D_i)\right), 1\right) \cap (\kappa^*, 1),$$

where $\kappa^* = \max_i F_i(D_i)$, we have the following results (the function $\mathrm{DB}(\cdot)$ is defined in (1)).

- *In the case of the Expected Shortfall,*

$$\mathrm{DB}(S_n^{\mathrm{NP}}, \mathrm{ES}_\kappa) \geq \mathrm{DB}(S_n^{\mathrm{P}}, \mathrm{ES}_\kappa).$$

- *In the case of the excess Expected Shortfall,*

$$\mathrm{DB}(S_n^{\mathrm{NP}}, \mathrm{xES}_\kappa) \geq \mathrm{DB}(S_n^{\mathrm{P}}, \mathrm{xES}_\kappa).$$

- *If the Value-at-Risk is subadditive for the given κ and the joint df of the original losses, i.e $\mathrm{VaR}_\kappa\left(\sum_{i=1}^{n} L_i\right) \leq \sum_{i=1}^{n} \mathrm{VaR}_\kappa(L_i)$, then*

$$\mathrm{DB}(S_n^{\mathrm{NP}}, \mathrm{VaR}_\kappa) \geq \mathrm{DB}(S_n^{\mathrm{P}}, \mathrm{VaR}_\kappa).$$

- *In the case of the excess Value-at-Risk:*

- *if the Value-at-Risk is subadditive for the given* κ *and the joint df of the original losses, and additionally either* $\sum_{i=1}^n \mathrm{VaR}_\kappa(L_i) > \sum_{i=1}^n (\mathbb{E}[L_i^{\mathrm{NP}}]+D_i)$ *or* $\sum_{i=1}^n \mathrm{VaR}_\kappa(L_i) < \sum_{i=1}^n \mathbb{E}[L_i]$*, or*
- *if the Value-at-Risk is superadditive for the given* κ *and the joint df of the original losses and additionally* $\sum_{i=1}^n \mathbb{E}[L_i] < \sum_{i=1}^n \mathrm{VaR}_\kappa(L_i) < \sum_{i=1}^n (\mathbb{E}[L_i^{\mathrm{NP}}] + D_i)$*,*

then

$$\mathrm{DB}(S_n^{\mathrm{NP}}, \mathrm{xVaR}_\kappa) \geq \mathrm{DB}(S_n^{\mathrm{P}}, \mathrm{xVaR}_\kappa).$$

Proof. Let $n \in \mathbb{N}$ and $\kappa \in \left[F_{S_n^{\mathrm{NP}}}\left(\sum_{i=1}^{n-1}(r_i - D_i)\right), 1\right) \cap (\kappa^*, 1)$. Let $S_n = \sum_{i=1}^n L_i$, then from the positive homogeneity of the Value-at-Risk and Expected Shortfall it follows

$$\mathrm{VaR}_\kappa\left(S_n^{\mathrm{P}}\right) = \alpha \mathrm{VaR}_\kappa\left(S_n\right) \quad \text{and} \quad \mathrm{ES}_\kappa\left(S_n^{\mathrm{P}}\right) = \alpha \mathrm{ES}_\kappa\left(S_n\right).$$

Corollary 3.2 then implies that

$$\mathrm{VaR}_\kappa(S_n^{\mathrm{NP}}) = \frac{\mathrm{VaR}_\kappa(S_n^{\mathrm{P}}) - \alpha \sum_{i=1}^n D_i}{\alpha} \qquad \text{and} \qquad \mathrm{ES}_\kappa(S_n^{\mathrm{NP}}) = \frac{\mathrm{ES}_\kappa(S_n^{\mathrm{P}}) - \alpha \sum_{i=1}^n D_i}{\alpha}.$$

Finally, applying Corollary 3.2 with $n = 1$ (recall that $\kappa \geq \kappa^*$) and plugging into the above equations, we obtain

$$\mathrm{DB}(S_n^{\mathrm{NP}}, \mathrm{VaR}_\kappa) = 1 - \frac{\mathrm{VaR}_\kappa(S_n^{\mathrm{NP}})}{\sum_{i=1}^n \mathrm{VaR}_\kappa(L_i^{\mathrm{NP}})} = 1 - \frac{\mathrm{VaR}_\kappa(S_n^{\mathrm{P}}) - \alpha \sum_{i=1}^n D_i}{\sum_{i=1}^n \mathrm{VaR}_\kappa(\alpha L_i) - \alpha \sum_{i=1}^n D_i} \geq \mathrm{DB}(S_n^{\mathrm{P}}, \mathrm{VaR}_\kappa),$$

and

$$\mathrm{DB}(S_n^{\mathrm{NP}}, \mathrm{ES}_\kappa) = 1 - \frac{\mathrm{ES}_\kappa(S_n^{\mathrm{NP}})}{\sum_{i=1}^n \mathrm{ES}_\kappa(L_i^{\mathrm{NP}})} = 1 - \frac{\mathrm{ES}_\kappa(S_n^{\mathrm{P}}) - \alpha \sum_{i=1}^n D_i}{\sum_{i=1}^n \mathrm{ES}_\kappa(\alpha L_i) - \alpha \sum_{i=1}^n D_i} \geq \mathrm{DB}(S_n^{\mathrm{P}}, \mathrm{ES}_\kappa),$$

where the last inequality comes in both cases from the fact that the function $x \mapsto \frac{a-x}{b-x}$ is non-increasing on $[0, b)$ if $b \geq a$ (here $x < b$ because we assumed $\kappa > \kappa^*$). Recall in the VaR case the additional assumption of subadditivity.
In the xES case, the previous argument still holds since

$$\sum_{i=1}^n (\mathbb{E}[L_i] - D_i) \leq \sum_{i=1}^n \mathbb{E}[L_i^{\mathrm{NP}}] \leq \sum_{i=1}^n (\mathrm{ES}_\kappa[L_i] - D_i),$$

where the first inequality follows the definition of L_i^{NP} and the second comes from the fact that $\kappa \geq \kappa^*$. The result for the xVaR is obtained by direct comparison. □

To sum things up, we have proved that the diversification benefit of a portfolio of n independent excess-of-loss treaties is higher or equal than the diversification benefit of a portfolio of n independent quota-share treaties, under some conditions given in Theorem 3.3.

We are aware that the assumption of independent contracts is restrictive and hence does not describe a vast amount of portfolios held in practice. To the best of our knowledge, however, Theorem 3.3 contributes to the actuarial literature as being the first analytical result on the comparison between the diversification benefit of non-proportional and proportional contracts. A potential path for future studies is the introduction of dependence between the original losses, or the restriction to particular distribution functions. Extensions of this framework are further discussed in Section 3.2 where we consider explicit distribution functions for the original losses. Contracts with a finite cover are considered in Section 4, since those are the types of contracts that are encountered most often in practice. In the next section we derive closed-form formulas for S_2^{NP} in the case of independent, identical and uniformly distributed original losses. This enables us to obtain a graphical illustration of our main result.

3.1.3 Example of uniform losses

In this section we derive closed-form formulas for the comparison between the diversification benefit of a non-proportional and a proportional portfolio with two i.i.d. uniformly distributed original losses L. First, we compute the diversification benefit of the proportional portfolio. Second, we derive the diversification benefit of the non-proportional portfolio and last, we compare the diversification benefit of these two portfolios when the proportional and non-proportional treaties have the same expectation. Note, this condition on the mean is also imposed in Rees and Wambach [2008], Section 2.4, in order to compare proportional and non-proportional contracts in the framework of utility theory.

Portfolio of 2 quota-share contracts

Let the original losses L_1, L_2, be independent uniformly distributed on the interval $[a,b]$ with $a,b>0$. The quota-share contracts are modelled by $L_i^{\mathrm{P}} = \alpha L_i$, $i=1,2$, and the proportional portfolio by $S_2^{\mathrm{P}} = L_1^{\mathrm{P}} + L_2^{\mathrm{P}}$. Then, the diversification benefit of S_2^{P}, measured with the Expected Shortfall and with the Value-at-Risk, can be expressed by the formulas below (the derivations are included in Appendix A.4).

$$\mathrm{DB}(S_2^{\mathrm{P}}, \mathrm{ES}_\kappa) = \begin{cases} \frac{(b-a)\kappa^{\frac{3}{2}}(2\sqrt{2}-3\sqrt{\kappa})}{3(\kappa-1)(a(\kappa-1)-b(\kappa+1))}, & \text{if } 0<\kappa<\frac{1}{2}; \\ \frac{(b-a)(3(1-\kappa)-\sqrt{8(1-\kappa)})}{3(a(\kappa-1)-b(\kappa+1))}, & \text{if } \frac{1}{2}\le\kappa<1. \end{cases} \tag{9}$$

$$\mathrm{DB}(S_2^{\mathrm{P}}, \mathrm{VaR}_\kappa) = \begin{cases} \frac{(b-a)(2\kappa-\sqrt{2\kappa})}{2(a+(b-a)\kappa)}, & \text{if } 0<\kappa<\frac{1}{2}; \\ \frac{(b-a)(2\kappa-2+\sqrt{2(1-\kappa)})}{2(a+(b-a)\kappa)}, & \text{if } \frac{1}{2}\le\kappa<1. \end{cases} \tag{10}$$

The diversification benefit of S_2^{P} measured with xES_κ and xVaR_κ writes

$$\mathrm{DB}(S_2^{\mathrm{P}}, \mathrm{xES}_\kappa) = \begin{cases} \frac{\sqrt{8\kappa}-3\kappa}{3(1-\kappa)}, & \text{if } 0<\kappa<\frac{1}{2}; \\ 1-\frac{1}{\kappa}+\frac{\sqrt{8(1-\kappa)}}{3\kappa}, & \text{if } \frac{1}{2}\le\kappa<1. \end{cases} \tag{11}$$

$$\mathrm{DB}(S_2^{\mathrm{P}}, \mathrm{xVaR}_\kappa) = \begin{cases} \frac{2\sqrt{\kappa}}{\sqrt{2}+2\sqrt{\kappa}}, & \text{if } 0<\kappa<\frac{1}{2}; \\ \frac{2(\kappa-1)+\sqrt{2-2\kappa}}{2\kappa-1}, & \text{if } \frac{1}{2}\le\kappa<1. \end{cases} \tag{12}$$

It is worth noting that the simplicity of the expressions is due to the set-up that we consider, namely two independent identical uniformly distributed original losses.

Portfolio of 2 excess-of-loss contracts

The excess-of-loss treaties are modelled by L_1^{NP} and L_2^{NP}, such that $L_i^{\mathrm{NP}} = (L_i - D)^+$, $i = 1,2$, where the original losses are independent uniformly distributed on (a,b). The portfolio containing these two excess-of-loss contracts is modelled by $S_2^{\mathrm{NP}} = L_1^{\mathrm{NP}} + L_2^{\mathrm{NP}}$. We take $a,b > 0$ and assume that $D \in (a,b)$, to avoid the trivial cases. The rv's L_1^{NP}, L_2^{NP} have each a positive mass w_0 at 0, where

$$w_0 = P(L_1^{\mathrm{NP}} = 0) = P(L_1 < D) = \frac{D-a}{b-a} > 0.$$

Then, we assess the diversification benefit as in Eq. (1) using VaR_κ, ES_κ, xES_κ and xVaR_κ. For example, the diversification benefit of S_2^{NP} measured with VaR_κ writes (see Appendix A.5 for the proof and the result for all four risk measures),

$$\mathrm{DB}(S_2^{\mathrm{NP}}, \mathrm{VaR}_\kappa) = \begin{cases} \frac{2b\kappa - 2a\kappa - \sqrt{2}\sqrt{(a-D)^2 + (a-b)^2\kappa}}{2(\kappa(b-a)-(D-a))}, & \text{if } w_0 < \kappa \le F_{S_2^{\mathrm{NP}}}(b-D); \\ \frac{(b-a)\left(2(\kappa-1)+\sqrt{2}\sqrt{1-\kappa}\right)}{2(\kappa(b-a)-(D-a))}, & \text{if } F_{S_2^{\mathrm{NP}}}(b-D) < \kappa < 1. \end{cases} \tag{13}$$

We note that for $\kappa < w_0$, $\mathrm{DB}(S_2^{\mathrm{NP}}, \mathrm{VaR}_\kappa)$ is not defined since in this case $\mathrm{VaR}_\kappa(L_1^{\mathrm{NP}}) = 0$ (division by zero). Next we compare the diversification benefits of the two portfolios.

Comparison of the diversification benefits for S_2^{NP} and S_2^{P}

Finally, we compare the diversification benefit of S_2^{NP} and S_2^{P} for quota-share and excess-of-loss contracts having the same expectation. This is accomplished by choosing an appropriate deductible D such that $\mathbb{E}[L_i^{\mathrm{NP}}] = \mathbb{E}[L_i^{\mathrm{P}}]$, $i = 1,2$, i.e.

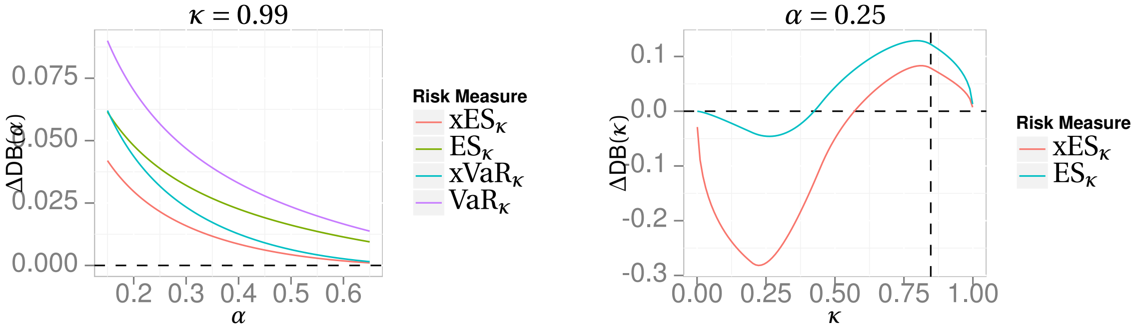
$$\frac{\alpha(a+b)}{2} = \frac{(b-D)^2}{2(b-a)} \quad \Rightarrow D = b - \sqrt{\alpha(b^2-a^2)}, \tag{14}$$

where α is the parameter of the quota-share contract. The diversification benefit of S_2^{P} is derived in equations (9), (10), (11) and (12), and the diversification benefit of S_2^{NP} is derived in equations (13) and (30). For a given interval (a,b) of the original losses, the difference in diversification benefit

$$\Delta\mathrm{DB}(\rho) = \mathrm{DB}(S_2^{\mathrm{NP}}, \rho) - \mathrm{DB}(S_2^{\mathrm{P}}, \rho)$$

is then evaluated. On Figure 5 we plot $\Delta\mathrm{DB}$ versus α (on the left) and versus κ (on the right), for given fixed values of a and b. From the plot on the left it can be seen that for $\kappa = 0.99$ the difference in diversification benefit is positive for all values of α, which implies that the non-proportional portfolio offers a higher diversification benefit. On the right, we plot the diversification benefit measured with xES_κ and ES_κ versus κ. For κ close to zero, the proportional contracts give a higher diversification and for κ close to 1 the non-proportional portfolio diversifies better. The graph confirms the results of Theorem 3.3, namely that $\Delta\mathrm{DB}(\mathrm{ES}_\kappa) \ge 0$ for κ on the right of the vertical dashed line which corresponds to $\kappa \ge F_{S_2^{\mathrm{NP}}}(b-D)$. In the next chapter we provide an extension of this comparison to original losses with infinite right endpoints.

Figure 5: Difference in diversification benefit between the non-proportional and the proportional portfolios plotted against the parameter of the proportional contracts α for fixed $a = 1$, $b = 10$ and $\kappa = 0.99$ (left), and difference in diversification benefit between the non-proportional and the proportional portfolios plotted against the level of the risk measures κ for fixed $a = 1$, $b = 10$ and $\alpha = 0.25$ (right). The vertical dashed line corresponds to $F_{S_2^{\mathrm{NP}}}(b - D)$. Each portfolio contains two i.i.d. treaties with uniformly distributed original losses where D is calibrated as in (14). A positive value indicates that the non-proportional contracts diversify better.



3.2 Original losses with an infinite right endpoint: case of n i.i.d. contracts with exponentially distributed original losses

Theorem 3.3 states that the diversification benefit, measured at κ close to 1, is higher for non-proportional than for proportional portfolios, under the assumption that these portfolios originate from losses which are independent and continuous, with finite right-endpoints. The aim of the current section is to carry out this comparison when the assumption of the original losses having a finite-right endpoint is challenged. For this we consider identical exponentially distributed original losses. First we assess the risk of the non-proportional portfolio, from which the risk of the proportional portfolio follows directly (by taking the limit as D goes to zero and multiplying by α). For the comparison we assume the "fair" calibration criterion (i.e., that the underlying proportional and non-proportional treaties have the same expectation).

3.2.1 Derivation of the risk measures

Let the original losses $L_1, L_2, \ldots, L_n$ ($n \in \mathbb{N}$) be i.i.d. random variables following the exponential distribution with df $F_L(x) = 1 - e^{-\lambda x}$, $\forall x \in [0, +\infty)$. Moreover, let $L_1^{\mathrm{NP}}, L_2^{\mathrm{NP}}, \ldots, L_n^{\mathrm{NP}} \sim F_{L^{\mathrm{NP}}}$ be the n i.i.d. random variables modelling the excess-of-loss contracts given by $L_i^{\mathrm{NP}} = (L_i - D)^+$, with a finite deductible $D \in (0, +\infty)$. From Eq. (8) it follows that the distribution function $F_{L^{\mathrm{NP}}}$ is given by

$$F_{L^{\mathrm{NP}}}(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1 - e^{-\lambda(x+D)}, & \text{if } 0 \le x < \infty. \end{cases} \tag{15}$$

We aim at assessing the risk of the non-proportional portfolio S_n^{NP}. First, we derive the df of S_n^{NP}, for which we apply the theory of Laplace transforms. For some non-negative random variable X with df F_X, the *Laplace-Stieltjes transform* of its distribution function F_X is given for every $t \in [0, +\infty)$, by

$$\mathfrak{L}\{F_X\}(t) = \int_0^{+\infty} e^{-tx} dF_X(x) = \mathbb{E}\left[e^{-tX}\right]. \tag{16}$$

Recall that the *Laplace transform* of a function f defined on $[0, \infty)$ is given, $\forall\, t \in [0, +\infty)$ by

$$f^*(t) = \int_0^{+\infty} e^{-tx} f(x) dx. \tag{17}$$

The link between the Laplace-Stieltjes transform in Eq. (16) and the Laplace transform in Eq. (17) is useful to retrieve the distribution of the portfolio from its Laplace-Stieltjes transform. Concretely, for a random variable X with df F_X,

$$\mathfrak{L}\{F_X\}(s) = sF_X^*(s), \quad \forall\, s \in [0, \infty),$$

where F_X^* is the Laplace transform of the df F_X and $\mathfrak{L}\{F_X\}$ its Laplace-Stieltjes transform (see Pfeiffer [1990, pp. 423-425]). Hence, having obtained $\mathfrak{L}\{F_{S_n^{\mathrm{NP}}}\}$, one can retrieve $F_{S_n^{\mathrm{NP}}}$ by applying the inverse Laplace transform on $s^{-1}\mathfrak{L}\{F_{S_n^{\mathrm{NP}}}(s)\}$, since the Laplace-Stieltjes transform characterizes the distribution of a random variable [Feller, 1971, pp. 430-431]. We do not provide an explicit definition for the inverse Laplace transform since it is not directly needed for the next steps. For a function f with a Laplace transform f^* (see Eq. (17)), the inverse Laplace transform is implicitly defined by

$$\mathfrak{I}\left(f^*\right) = f.$$

Inverse transform tables or symbolic software can be used for a closed-form derivation of the inverse Laplace transforms. Alternatively, numerical methods can be applied.
The Laplace-Stieltjes transform of the random variable S_n^{NP}, corresponding to the sum of n i.i.d. excess-of-loss treaties is given by (see Limani [2015])

$$\mathfrak{L}\left\{F_{S_n^{\mathrm{NP}}}\right\}(s) = \left(w_0 + \frac{\lambda}{\lambda + s} e^{-\lambda D}\right)^n, \quad \forall\, s \in [0, \infty),$$

where $w_0 = P(L^{\mathrm{NP}} = 0) = 1 - e^{-\lambda D}$. Applying the binomial theorem, we obtain

$$\mathfrak{L}\left\{F_{S_n^{\mathrm{NP}}}\right\}(s) = \sum_{k=0}^{n} \frac{\binom{n}{k} w_0^{n-k} \lambda^k e^{-\lambda D k}}{(\lambda + s)^k}, \quad \forall\, s \in [0, \infty).$$

This completes the derivation of the Laplace-Stieltjes transform of $F_{S_n^{\mathrm{NP}}}$. In this particular case a closed-form solution of the inverse Laplace transform $\mathfrak{I}\left(s^{-1}\mathfrak{L}\{F_{S_n^{\mathrm{NP}}}(s)\}\right)$ is attainable using the upper and lower *incomplete Gamma functions*, defined respectively by

$$\Gamma(a, x) = \int_x^{+\infty} r^{a-1} e^{-r} dr, \quad \text{and} \quad \gamma(\alpha, x) = \int_0^x r^{\alpha-1} e^{-r} dr, \tag{18}$$

for $a \geq 0$ and $\alpha > 0$ (see Gradshteyn and Ryzhik [2007, pp. 899-902]). Then, the distribution function of S_n^{NP} is derived $\forall t \in [0, +\infty)$ as follows:

$$
\begin{aligned}
F_{S_n^{\mathrm{NP}}}(t) &= \mathfrak{I}\left\{\frac{\mathfrak{L}\left\{F_{S_n^{\mathrm{NP}}}\right\}(s)}{s}\right\}(t) = \mathfrak{I}\left\{\sum_{k=0}^{n} \frac{\binom{n}{k} w_0^{n-k} \lambda^k e^{-\lambda D k}}{s(\lambda+s)^k}\right\}(t) \\
&= w_0^n + \sum_{k=1}^{n} \binom{n}{k} w_0^{n-k} \lambda^k \frac{e^{-\lambda D k}}{(k-1)!} \left(\Gamma(k,0) - \Gamma(k,\lambda t)\right) \lambda^{-k} \\
&= w_0^n + \sum_{k=1}^{n} \binom{n}{k} w_0^{n-k} \frac{e^{-\lambda D k}}{(k-1)!} \gamma(k,\lambda t).
\end{aligned}
\tag{19}
$$

In the third equality we use the linearity of the inverse Laplace transform and the formulas given in Abramowitz and Stegun [1972, p. 1020]. The derivation of the df of S_n^{NP} is hence accomplished. We note that as D goes to zero, the limit of (19) equals $\gamma(n,\lambda s)/(n-1)!$, the df of an Erlang$(n,\lambda)$ distributed random variable corresponding to the sum of n i.i.d. exponential random variables with rate λ (see Lakatos et al. [2013, p. 41]).
The Expected Shortfall of S_n^{NP} follows directly. Let $\kappa \in (0,1)$, then using (19) the Expected Shortfall of S_n^{NP} is given by

$$
\begin{aligned}
\mathrm{ES}_\kappa(S_n^{\mathrm{NP}}) &= \frac{1}{1-\kappa} \int_{\mathrm{VaR}_\kappa(S_n^{\mathrm{NP}})}^{+\infty} x\, dF_{S_n^{\mathrm{NP}}}(x) = \frac{1}{1-\kappa} \sum_{k=1}^{n} \binom{n}{k} w_0^{n-k} \lambda^k \frac{e^{-\lambda D k}}{(k-1)!} \int_{\mathrm{VaR}_\kappa(S_n^{\mathrm{NP}})}^{+\infty} e^{-\lambda x} x^k\, dx \\
&= \frac{1}{1-\kappa} \sum_{k=1}^{n} \binom{n}{k} w_0^{n-k} \frac{e^{-\lambda D k}}{(k-1)!} \frac{\Gamma(k+1, \lambda \mathrm{VaR}_\kappa(S_n^{\mathrm{NP}}))}{\lambda}.
\end{aligned}
$$

Equipped with this formula, we are able to derive and compare the diversification benefits between proportional and non-proportional portfolios, for all of the four risk measures of interest.

3.2.2 Comparison of the diversification benefits

Finally, we investigate in the current set-up how the diversification benefit of proportional and non-proportional portfolios differ. Recall that the formulas for the proportional portfolio S_n^{P} are obtained directly by taking the limit of the formulas for the non-proportional one S_n^{NP}, letting the deductible D tend to zero. In order to guarantee a "fair" comparison, the deductible D of the excess-of-loss treaties is again chosen such that the means of proportional and non-proportional treaties are equal, i.e. $\mathbb{E}[L_i^{\mathrm{NP}}] = \mathbb{E}[L_i^{\mathrm{P}}], i = 1,2$, which gives

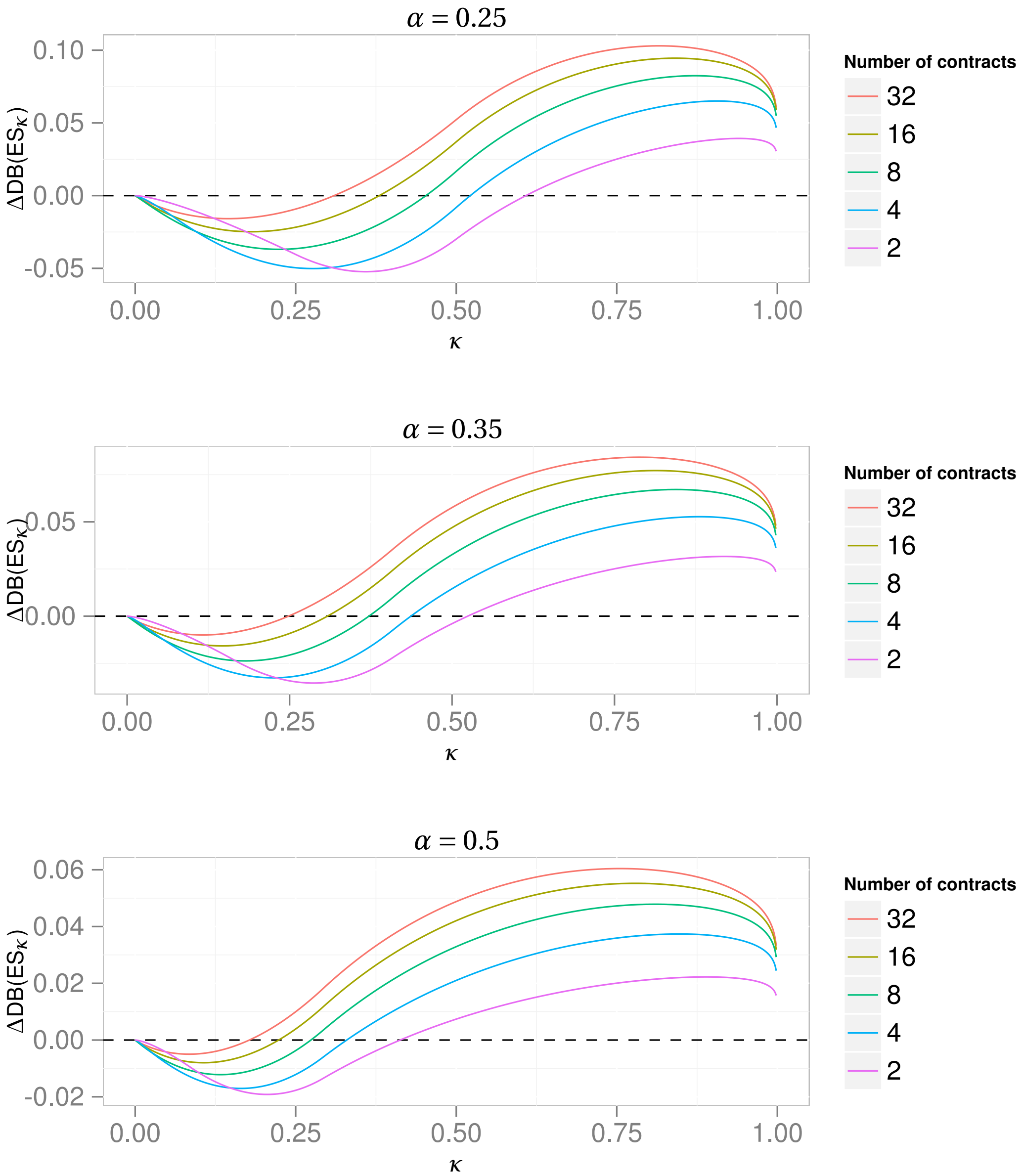
$$
\frac{\alpha}{\lambda} = \frac{e^{-\lambda D}}{\lambda} \quad \Rightarrow \quad D = -\frac{\log \alpha}{\lambda}. \tag{20}
$$

As a side effect, this causes the diversification benefit to depend only on κ and α. Note that the parameter of the exponential distribution λ does not appear in the final formulas, since the term λD in Eq. (19) becomes $-\log(\alpha)$ after applying (20). The difference in the diversification benefit measured with ES_κ between the non-proportional and proportional portfolios,

$$
\Delta \mathrm{DB}(\mathrm{ES}_\kappa) = \mathrm{DB}(S_n^{\mathrm{NP}}, \mathrm{ES}_\kappa) - \mathrm{DB}(S_n^{\mathrm{P}}, \mathrm{ES}_\kappa),
$$

is plotted against κ in Figure 6 for different values of α, where a positive value of the difference indicates that the non-proportional portfolio diversifies better.

Figure 6: $\Delta\mathrm{DB}$ measured with ES_κ is plotted against the level κ for portfolios of n i.i.d. exponentially distributed original losses and varying the number of contracts $n \in \{2,4,8,16,32\}$. The parameter of the original losses is $\lambda = \frac{1}{3}$ and the deductible is set to $D = -\frac{\log(\alpha)}{\lambda}$, where $\alpha = 0.25$ (top), $\alpha = 0.35$ (middle) and $\alpha = 0.5$ (bottom).

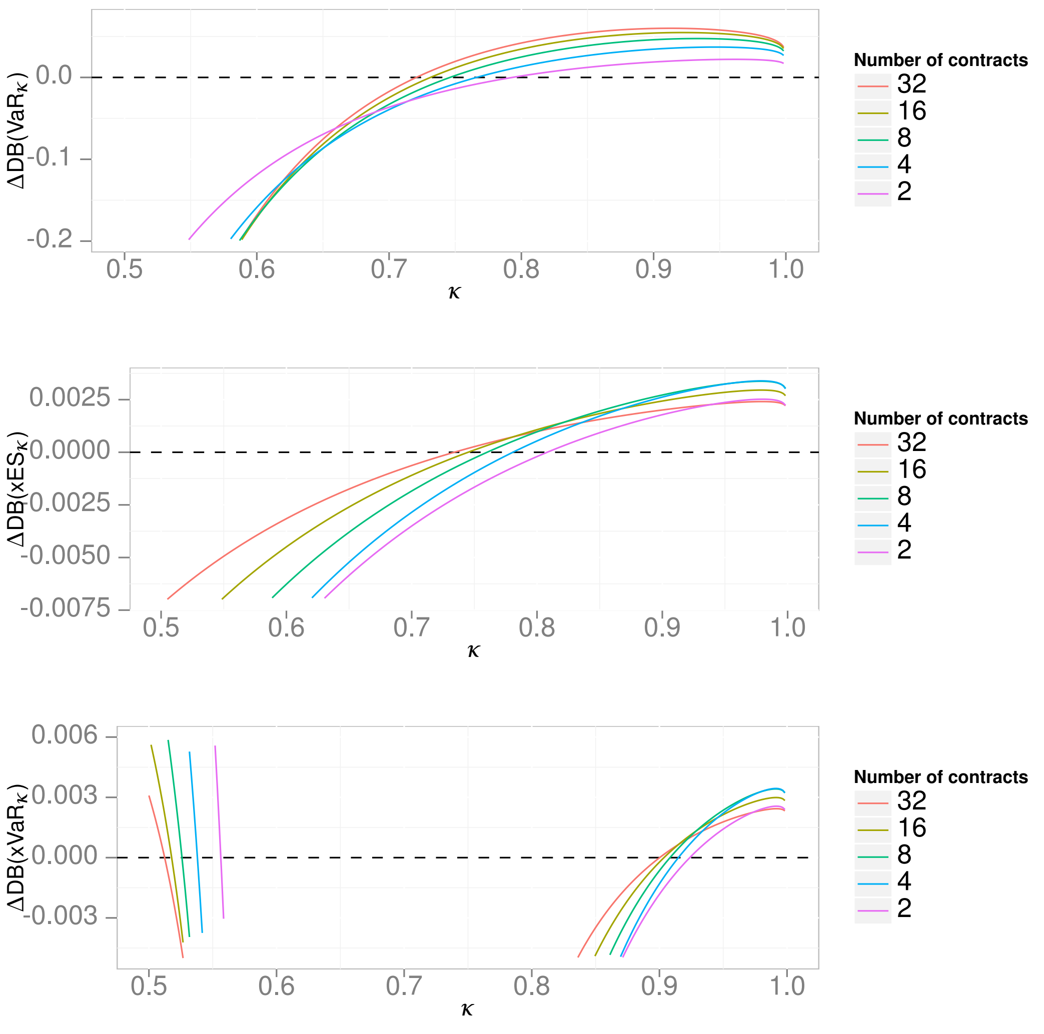


As mentioned earlier, this comparison is particularly important since it is not covered by Theorem 3.3, given that now the original losses have an infinite right endpoint. Moreover, obtaining a formula for $n \in \mathbb{N}$ allows us to study if the result of the comparison depends on the number of contracts and how.

It is observed that for κ close to 1, the non-proportional portfolio diversifies better than the proportional one. Furthermore, as the number of contracts increases (from purple to red lines), the difference between the diversification benefits increases, as well as the interval of κ values for which this difference is positive. In addition, as α increases, the absolute difference in the diversification benefits decreases. This is due to the fact that the contracts become more and more similar as D reaches 0, or equivalently as α reaches 1, see (20). Above all, it is interesting to observe that the shape of $\Delta\mathrm{DB}\,(\mathrm{ES}_\kappa)$ is identical to the plot on the right-hand side of Figure 5.

The difference in diversification benefit measured with VaR_κ, xES_κ and xVaR_κ is plotted against κ on Figure 7. It is observed that the non-proportional portfolio diversifies better for κ close to 1. For VaR_κ (and for ES_κ as seen on Figure 6) the difference becomes more pronounced as the number of contracts increases. However, for xES_κ and xVaR_κ the relationship is not that straightforward. For example, when κ is close to 1 the difference is higher for portfolios of 8 contracts than for portfolios of 16 contracts, and lower for portfolios of 2 contracts than for portfolios of 4 contracts. Note that the discontinuity observed in the case of xVaR_κ (bottom graph) corresponds to $\kappa = 1 - e^{-1} \approx 0.63$, for which $\mathrm{VaR}_\kappa(L) = \mathbb{E}[L]$.

Figure 7: ΔDB measured with VaR_κ (top), xES_κ (middle) and xVaR_κ (bottom), plotted against the level κ for portfolios of n i.i.d. exponentially distributed original losses and varying the number of contracts $n \in \{2,4,8,16,32\}$. The parameter of the original losses is $\lambda = \frac{1}{3}$ and the deductible is set to $D = -\frac{\log(\alpha)}{\lambda}$, where $\alpha = 0.5$.



To conclude, we have seen that the diversification benefit of a non-proportional portfolio is higher than the one of a proportional portfolio for κ close to 1, when the contracts have the same expectation and the original losses are i.i.d. exponentially distributed. So far, we have studied the difference in diversification benefit between proportional and non-proportional portfolios assuming excess-of-loss treaties with an unlimited cover. In the next section we challenge this last assumption and carry out the comparison for non-proportional treaties with a finite cover C.

4 Comparison for excess-of-loss treaties with a finite cover

Let us now consider excess-of-loss contracts with a finite cover C, which gives us the modified loss distribution $L^{\mathrm{NP}} = (L-D)^+ - (L-C-D)^+$. We study first a simplified example with two independent and identically distributed exponential original losses, and derive closed-form formulas. Then, we finalize the section by extending our discussion to a more general framework.

4.1 Case of 2 contracts with identically distributed independent exponential original losses

Let the original losses L_1, L_2 be i.i.d. with distribution function $F_L(s) = \left(1-e^{-\lambda s}\right)\mathbb{1}_{\{s\geq 0\}}$. We first assess the risk of the proportional and non-proportional portfolios and then compare their diversification benefits.

4.1.1 Derivation of the risk measures

Proportional Portfolio S_2^{P}

It can be easily shown (see e.g. Cruz et al. [2015, p. 119]) that the risk measures of the proportional contracts (2) are given by

$$\mathrm{VaR}_\kappa(L_1^{\mathrm{P}}) = \mathrm{VaR}_\kappa(L_2^{\mathrm{P}}) = -\frac{\alpha}{\lambda}\log(1-\kappa),$$
$$\mathrm{ES}_\kappa(L_1^{\mathrm{P}}) = \mathrm{ES}_\kappa(L_2^{\mathrm{P}}) = \alpha\left(\frac{1-\log(1-\kappa)}{\lambda}\right).$$

We can then derive the risk measures of the proportional portfolio $S_2^{\mathrm{P}} = \sum_{i=1}^2 L_i^{\mathrm{P}}$, given by

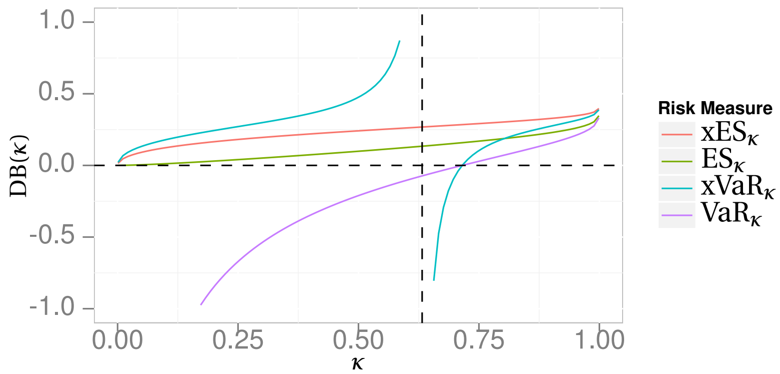
$$\mathrm{VaR}_\kappa(S_2^{\mathrm{P}}) = -\alpha\frac{1+W\left(\frac{\kappa-1}{e}\right)}{\lambda},$$
$$\mathrm{ES}_\kappa\left(S_2^{\mathrm{P}}\right) = \frac{\alpha}{\lambda(1-\kappa)}e^{1+W\left(\frac{\kappa-1}{e}\right)}\left(1+W^2\left(\frac{\kappa-1}{e}\right)\right),$$

where W is the Lambert function, which satisfies

$$W(x)e^{W(x)} = x, \quad x\in\mathbb{C}, \tag{21}$$

see Corless et al. [1996] for more details. The formulas of the corresponding diversification benefit measured with VaR_κ, xVaR_κ, ES_κ, and xES_κ are provided in Appendix A.6. We show on Figure 8 the diversification benefits plotted against the level κ. It is interesting to observe that in all these cases $\mathrm{DB}(S_2^{\mathrm{P}},\rho)$ is independent of λ and α. It is observed on Figure 8 that the diversification benefit measured with xVaR_κ has a discontinuity at $\kappa = 1-e^{-1}$ (vertical dashed line), for which the denominator of (33) becomes zero. Moreover, the diversification benefit measured with VaR_κ is negative for e.g. $\kappa = 0.7$ and positive for e.g. $\kappa = 0.95$, due to lack of subadditivity and subadditivity of VaR_κ for these values of κ, respectively. This observation is in line with the example in McNeil et al. [2015, pp. 297-299]. Next we calculate the diversification benefit of the non-proportional portfolio S_2^{NP}.

Figure 8: Diversification benefit of S_2^{P} for each risk measure indicated in the legend, plotted against κ. The original losses are exponentially distributed. The shapes of the graphs are not affected by λ or α.



Non-proportional portfolio S_2^{NP}

Applying Lemma 2.1 and performing a direct integration, we obtain the df of S_2^{NP} (see Limani [2015] for the details),

$$F_{S_2^{\mathrm{NP}}}(s) = \begin{cases} 0, & \text{if } s < 0; \\ F_L(s+D)\,w_0 + \int_D^{s+D} F_L(s+2D-u)\,f_L(u)\,du, & \text{if } s \in [0, C); \\ F_L(s+D-C)(1+w_C) + \int_{s+D-C}^{C+D} F_L(s+2D-u)\,f_L(u)\,du, & \text{if } s \in [C, 2C); \\ 1, & \text{if } s \geq 2C, \end{cases} \tag{22}$$

where $w_0 = F_L(D)$ and $w_C = 1 - F_L(D+C)$. Note that the distribution function of a portfolio of two i.i.d. excess-of-loss treaties S_2^{NP} has three jumps. It has a jump at zero of size w_0^2, a jump of size $2w_0 w_C$ at the cover C and of size w_C^2 at the "double cover" $2C$.
In the exponential case under consideration, the df of L^{NP} (modelling the loss arising from an excess-of-loss treaty with deductible D and cover C) is given by

$$F_{L^{\mathrm{NP}}}(s) = \begin{cases} 0, & \text{if } s < 0; \\ 1 - \mathrm{e}^{-\lambda(s+D)}, & \text{if } 0 \leq s \leq C;, \\ 1 & \text{if } s \geq C, \end{cases}$$

see Lemma 2.1, and a straightforward calculation shows that

$$\mathbb{E}\left[L^{\mathrm{NP}}\right] = \frac{\mathrm{e}^{-\lambda D}}{\lambda}\left(1 - \mathrm{e}^{-\lambda C}\right).$$

The corresponding risk measures are derived from Corollary 2.2:

$$\mathrm{VaR}_\kappa(L^{\mathrm{NP}}) = \min\left\{\max\left\{\frac{-\log(1-\kappa)}{\lambda} - D, 0\right\}, C\right\}, \tag{23}$$

$$
\mathrm{ES}_\kappa(L^{\mathrm{NP}}) = \begin{cases} \frac{\mathrm{e}^{-\lambda D}\left(1-\mathrm{e}^{-\lambda C}\right)}{\lambda(1-\kappa)}, & \text{if } \kappa \le w_0; \\ \frac{1}{\lambda}\left[1-\log(1-\kappa)-\frac{\mathrm{e}^{-\lambda(C+D)}}{1-\kappa}\right]-D, & \text{if } w_0 < \kappa < 1-w_C; \\ C, & \text{if } \kappa \ge 1-w_C, \end{cases} \tag{24}
$$

with $w_0 = P(L^{\mathrm{NP}} = 0) = 1 - e^{-\lambda D}$ and $w_C = P(L^{\mathrm{NP}} = C) = e^{-\lambda(C+D)}$. Now we can derive from (22) the df and the risk measures of the non-proportional portfolio S_2^{NP} in the exponential case,

$$
F_{S_2^{\mathrm{NP}}}(s) = \begin{cases} 0, & \text{if } s < 0 \\ 1 - e^{-\lambda(s+2D)}\left(\lambda s - 1 + 2e^{\lambda D}\right), & \text{if } 0 \le s < C \\ 1 - e^{-\lambda(s+2D)} + \lambda e^{-\lambda(s+2D)}(s-2C), & \text{if } C \le s < 2C \\ 1, & \text{if } s \ge 2C, \end{cases} \tag{25}
$$

(see Limani [2015] for the details of the derivation). Closed-form expressions can then be derived for the (excess) Value-at-Risk and (excess) Expected Shortfall of S_2^{NP} (see Appendix A.7), and consequently for the corresponding diversification benefits. Hence we are ready to compare the diversification benefit of S_2^{NP} and of S_2^{P}, which is the subject of the next section.

4.1.2 Comparison of the diversification benefits

Finally, we can compare the diversification benefit of S_2^{NP} and S_2^{P} using the formulas given in Appendix A.6 and A.7. For this purpose we adopt the "fair" calibration condition, imposing the same mean of the proportional and non-proportional contracts. The current set-up is slightly more involved than in the case of an unlimited cover since now we have to calibrate two parameters, D and C, relative to α. We fix α and set $D = \frac{p}{\lambda}$ for some $p > 1$.[3] Then the mean condition writes

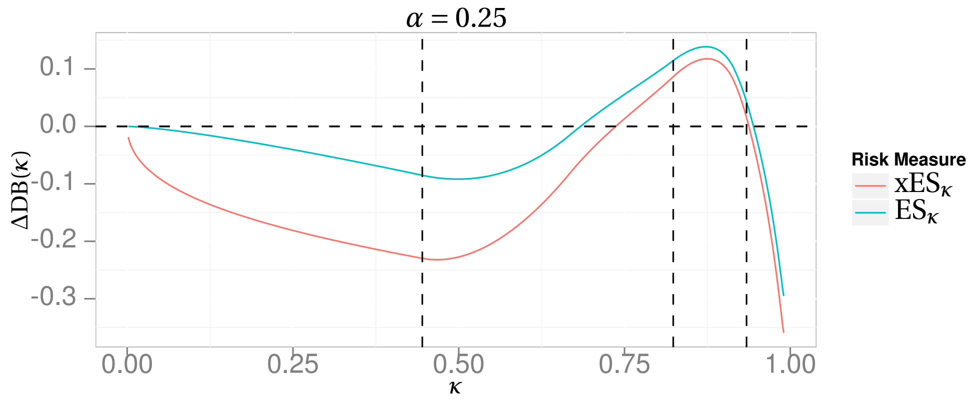
$$
\frac{\alpha}{\lambda} = \frac{e^{-\lambda D}}{\lambda}\left(1-\mathrm{e}^{-\lambda C}\right) \quad \Rightarrow \quad C = -\frac{\log\left(1-\alpha \mathrm{e}^{\lambda D}\right)}{\lambda}, \tag{26}
$$

provided that $e^{-p} > \alpha$. In Figure 9 we show the plot of $\Delta\mathrm{DB}(\mathrm{xES}_\kappa)$ and $\Delta\mathrm{DB}(\mathrm{ES}_\kappa)$ against κ for $\alpha = 0.25, p = 1.1$ and $\lambda = 0.5$. Starting from the left, the first vertical dashed line corresponds to $F_{S_2^{\mathrm{NP}}}(2D)$. In the region before this line the proportional portfolio diversifies better. The second vertical dashed line corresponds to $\kappa = F_{S_2^{\mathrm{NP}}}(C-)$, the probability that the loss of the portfolio is less than the amount of the cover C. In other words, none of the contracts exhausts their cover. The third vertical dashed line corresponds to $\kappa = F_{S_2^{\mathrm{NP}}}(C)$, the probability that the loss of the portfolio reaches at most the amount of the cover C. For κ in the interval $(F_{S_2^{\mathrm{NP}}}(C-), F_{S_2^{\mathrm{NP}}}(C))$, the difference is positive, hence the non-proportional portfolio diversifies better. Finally, the most important result is that for κ close to 1 we observe negative values, which indicates that the proportional portfolio S_2^{P} diversifies better.

We have seen in Section 3 that for portfolios of independent bounded contracts and unlimited covers the non-proportional portfolio offers a better diversification for κ close to 1. However, for κ close to 1, when considering an example of contracts with finite cover C, our results are reversed and we observe that the proportional portfolio offers a higher diversification. The reason for this is relatively simple and is discussed in the next section.

[3] A reinsurer tries to set the deductible D such that it is higher than the expected loss, hence we choose $p > 1$.

Figure 9: Difference in diversification benefit measured with xES_κ and ES_κ plotted against κ for fixed $\alpha = 0.25$, $D = 2.2$ and C defined as in Eq. (26). Each portfolio contains two treaties from i.i.d. exponential original losses with parameter $\lambda = 0.5$. The vertical dashed lines correspond to $\kappa = w_0^2, \kappa = F_{S_2^{\mathrm{NP}}}(C-)$ and $\kappa = F_{S_2^{\mathrm{NP}}}(C)$, in this order. A positive difference indicates that the non-proportional portfolio diversifies better.



4.2 Extension and Discussion

We consider in this section n original losses modelled by random variables $L_1, \ldots, L_n$ with corresponding continuous distribution functions $F_1, F_2, \ldots, F_n$. Moreover, we assume that the multivariate random vector $(L_1, \ldots, L_n)$ is distributed according to some continuous multivariate df F with the above mentioned margins. We emphasize that so far we have studied portfolios of independent original losses, whereas now we take a step that widely generalizes our investigative framework by considering non-identical contracts with some continuous multivariate distribution function F. Recall that each original loss is modelled by a continuous, integrable and non-negative rv.
The non-proportional portfolio S_n^{NP} is given by $S_n^{\mathrm{NP}} = \sum_{i=1}^n L_i^{\mathrm{NP}}$, where

$$L_i^{\mathrm{NP}} = (L_i - D_i)^+ - (L_i - D_i - C_i)^+,$$

with positive finite deductible D_i and finite cover C_i, $i = 1, \ldots, n$.
First, we derive the diversification benefit (1) for the non-proportional portfolio S_n^{NP} and κ close to 1, and then provide a discussion on how this compares to a portfolio of proportional contracts. We note that $F_{S_n^{\mathrm{NP}}}$ has a jump at zero with height $w_0 = F(D_1, \ldots, D_n)$, which equals the probability that none of the original losses exceeds the corresponding deductible D_i. If the original losses are independent, the mass at zero w_0 can be written

$$w_0 = \prod_{i=1}^n F_i(D_i).$$

The highest value where $F_{S_n^{\mathrm{NP}}}$ has a jump occurs at $C := \sum_{i=1}^n C_i$ with a height

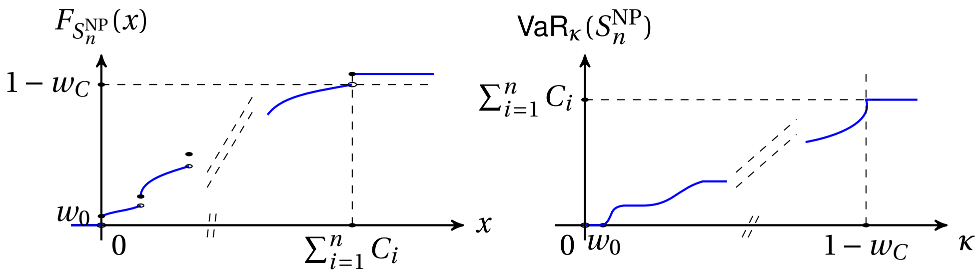
$$w_C = P(L_1 \geq D_1 + C_1, \ldots, L_n \geq D_n + C_n).$$

Again, for independent original losses w_C simply equals

$$w_C = \prod_{i=1}^{n} (1 - F_i(D_i + C_i)).$$

An illustration of the df and Value-at-Risk of S_n^{NP} are shown on Figure 10, to help us better explain the implications of the shape of $F_{S_n^{\mathrm{NP}}}$ for risk measurement purposes. It is seen from the Value-at-

Figure 10: Distribution function of the non-proportional portfolio $F_{S_n^{\mathrm{NP}}}$ (left), and of the corresponding non-proportional one $\mathrm{VaR}_\kappa(S_n^{\mathrm{NP}})$ (right).



Risk plot (right-hand side) that for $\kappa \in [1 - w_C, 1)$,

$$\mathrm{VaR}_\kappa(S_n^{\mathrm{NP}}) = \sum_{i=1}^{n} C_i.$$

This implies also that the Expected Shortfall of S_n^{NP} equals $\sum_{i=1}^n C_i$, for $\kappa \in [1 - w_C, 1)$. Intuitively speaking, when measuring risk at the probability level $\kappa \in [1 - w_C, 1)$, we are in the region where all the covers are exhausted. Hence, for a sufficiently high probability level, for example for an extreme event, both Value-at-Risk and Expected Shortfall deliver the sum of all the individual covers of the treaties in the portfolio. Note that the higher the tail dependence between the underlying original losses, the larger the interval $[1 - w_C, 1)$.
Let us denote by $\tilde{\kappa}^* = \max_i F_i(D_i + C_i)$ the lowest probability level at which the Value-at-Risk and Expected Shortfall of each individual contract equals their cover. The following theorem summarizes the results in the case of a finite cover and is the counterpart of Theorem 3.3 where the losses are independent, bounded and the cover infinite. The inequalities are reverted.

Theorem 4.1. *(Comparison of the diversification benefits, finite cover case.) Under the assumptions and notations from Section 4.2,* $\forall \kappa \in [\max(\tilde{\kappa}^*, 1 - w_C), 1)$ *we have the following results (the function* $\mathrm{DB}(\cdot)$ *is defined in* (1)*).*

- *In the case of the Expected Shortfall,*

$$\mathrm{DB}(S_n^{\mathrm{P}}, \mathrm{ES}_\kappa) \geq \mathrm{DB}(S_n^{\mathrm{NP}}, \mathrm{ES}_\kappa).$$

- *In the case of the excess Expected Shortfall,*

$$\mathrm{DB}(S_n^{\mathrm{P}}, \mathrm{xES}_\kappa) \geq \mathrm{DB}(S_n^{\mathrm{NP}}, \mathrm{xES}_\kappa).$$

- *If the Value-at-Risk is subadditive for the given κ and the joint df of the original losses, i.e* $\mathrm{VaR}_\kappa\left(\sum_{i=1}^n L_i\right) \leq \sum_{i=1}^n \mathrm{VaR}_\kappa(L_i)$, *then*

$$\mathrm{DB}(S_n^{\mathrm{P}}, \mathrm{VaR}_\kappa) \geq \mathrm{DB}(S_n^{\mathrm{NP}}, \mathrm{VaR}_\kappa).$$

- *In the case of the excess Value-at-Risk:*
  - *if the Value-at-Risk is subadditive for the given κ and the joint df of the original losses and additionally $\sum_{i=1}^n \mathbb{E}[L_i] < \sum_{i=1}^n \mathrm{VaR}_\kappa(L_i)$, or*
  - *if the Value-at-Risk is superadditive for the given κ and the joint df of the original losses and additionally $\sum_{i=1}^n \mathbb{E}[L_i] > \sum_{i=1}^n \mathrm{VaR}_\kappa(L_i)$,*

  then

$$\mathrm{DB}(S_n^{\mathrm{P}}, \mathrm{xVaR}_\kappa) \geq \mathrm{DB}(S_n^{\mathrm{NP}}, \mathrm{xVaR}_\kappa).$$

Proof. By definition of $\tilde{\kappa}^*$, $\forall \kappa \in [\tilde{\kappa}^*, 1)$ and $\forall i \in \{1, \ldots, n\}$, we have

$$\mathrm{VaR}_\kappa(L_i^{\mathrm{NP}}) = C_i \text{ and } \mathrm{ES}_\kappa(L_i^{\mathrm{NP}}) = C_i.$$

Thus for $\kappa \in [\max(\tilde{\kappa}^*, 1 - w_C), 1)$,

$$\mathrm{DB}(S_n^{\mathrm{NP}}, \mathrm{ES}_\kappa) = 1 - \frac{\sum_{i=1}^n C_i}{\sum_{i=1}^n C_i} = 0. \tag{27}$$

Similarly in the case of the other risk measures, for $\kappa \in [\max(\kappa^*, 1 - w_C), 1)$,

$$\mathrm{DB}\left(S_n^{\mathrm{NP}}, \mathrm{xES}_\kappa\right) = \mathrm{DB}\left(S_n^{\mathrm{NP}}, \mathrm{VaR}_\kappa\right) = \mathrm{DB}\left(S_n^{\mathrm{NP}}, \mathrm{xVaR}_\kappa\right) = 0.$$

For the proportional portfolio, we obtain from the subadditivity of the Expected Shortfall that for all $\kappa \in (0, 1)$,

$$\mathrm{DB}(S_n^{\mathrm{P}}, \mathrm{ES}_\kappa) \geq 0. \tag{28}$$

Hence (27) and (28) imply that $\forall \kappa \in [\max(\tilde{\kappa}^*, 1 - w_C), 1)$,

$$\mathrm{DB}\left(S_n^{\mathrm{P}}, \mathrm{ES}_\kappa\right) \geq \mathrm{DB}\left(S_n^{\mathrm{NP}}, \mathrm{ES}_\kappa\right). \tag{29}$$

The results for the other risk measures are easily checked as well. □

To summarize, we note that the Value-at-Risk and Expected Shortfall are additive at any level $\kappa \in [\max(\tilde{\kappa}^*, 1-w_C), 1)$ for a portfolio of non-proportional contracts S_n^{NP}. The length of this interval increases for a random vector of original losses with a high tail dependence. This implies that for κ sufficiently high, the diversification benefit of a non-proportional portfolio is not affected by the subadditivity features of the Value-at-Risk of the random vector modelling the original losses. Second, for κ sufficiently high both the Value-at-Risk and Expected Shortfall deliver the same number. Third, it was observed that for κ close to 1 the diversification benefit of the proportional portfolio measured with ES_κ and xES_κ is greater or equal than the diversification benefit of the non-proportional portfolio. When measured with the Value-at-Risk, the comparison depends on the subadditivity of VaR_κ.

5 Conclusion

A financial institution can mitigate its exposure to the various downside risks by holding well-diversified portfolios. Diversification is important not only at the individual level, but also for the overall stability of the financial system. Our study in the first part of this paper was motivated by the following question: *"How does the diversification gain of a portfolio of quota-share treaties compare to the diversification gain of a portfolio of excess-of-loss treaties?"*

To the best of our knowledge, this question was first tackled via a simulation study in Bettinger et al. [2015]. In the present paper, we establish through a theoretical analysis a framework where we can compare, from the point of view of a reinsurer, the diversification benefit between portfolios of quota-share and excess-of-loss treaties. For this comparison we take excess-of-loss and quota-share treaties which have the same underlying original loss L (modelled as a continuous, integrable and non-negative rv). Our first major result is presented in Theorem 3.3, where we show that the diversification benefit of a portfolio of n excess-of-loss treaties with unlimited cover, measured with ES_κ and xES_κ at κ close to 1 is higher than the diversification benefit of a portfolio of n quota-share treaties, when the underlying losses are independent and bounded. This result holds for the Value-at-Risk as well if it is in addition subadditive at the probability level κ. For the xVaR additional conditions are needed. Note that the independence assumption is restrictive and it would be interesting to study the effect of a dependency between losses.

As an application, we derive formulas for the diversification benefit of a toy model with two i.i.d. uniformly distributed original losses, where we observe that for values of κ close to 1, the non-proportional portfolio offers a better diversification, confirming the result of Theorem 3.3 (see Fig. 5). Departing from the assumption of bounded original losses we derive in Section 3.2 an analytical formula for the diversification benefit of portfolios of i.i.d. treaties with exponentially distributed original losses. For κ close to 1, we find that the non-proportional portfolio diversifies better as well (see Fig. 6). In these two examples we assumed that the proportional and non-proportional contracts have the same expectation, in order to ensure a fair comparison.

The second major finding of this paper is obtained by extending the investigation to non-proportional portfolios of excess-of-loss treaties with a finite cover. Contrary to our expectations, in Section 4 we observe that the results are in opposition to those given in Theorem 3.3 and that the diversification benefit measured with ES_κ at κ close to 1 is higher for proportional treaties than for their non-proportional counterpart. The same finding is valid for the Value-at-Risk provided that it is sub-additive at the level κ. These results are summarized in Theorem 4.1. We emphasize that these findings are obtained by relaxing the assumption of independent original losses (see Section 4.2). Future research could be directed towards considering portfolios of both quota-share and excess-of-loss contracts, which is a more realistic representation of the portfolios held in practice, and studying how to determine the weights between these non-proportional and proportional treaties such that an optimal diversification is attained.

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A Technical Appendix

A.1 Proof of Lemma 2.1

It is clear given the excess-of-loss distribution (3) that

$$F_{L^{\mathrm{NP}}}(x) = 0, \forall x < 0,$$

$$F_{L^{\mathrm{NP}}}(x) = 1, \forall x \geq C,$$

and

$$F_{L^{\mathrm{NP}}}(0) = F_L(D).$$

For the remaining case $x \in (0, C)$,

$$\begin{aligned}
F_{L^{\mathrm{NP}}}(x) &= P(L^{\mathrm{NP}} \leq x) \\
&= P(L^{\mathrm{NP}} = 0) + P(0 < L^{\mathrm{NP}} \leq x) \\
&= F_L(D) + P(0 < L - D \leq x) \\
&= F_L(D) + P(D < L \leq x + D) \\
&= F_L(x + D).
\end{aligned}$$

A.2 Proof of Corollary 2.2

Recall the notations $w_0 = F_L(D)$ and $w_C = 1 - F_L(C + D)$. From Eq. (3) it is clear that

$$\mathrm{VaR}_\kappa(L^{\mathrm{NP}}) = 0, \forall \kappa \leq w_0,$$

and

$$\mathrm{VaR}_\kappa(L^{\mathrm{NP}}) = C, \forall \kappa \geq 1 - w_C.$$

For the remaining case $\kappa \in (w_0, 1 - w_C)$,

$$\begin{aligned}
\mathrm{VaR}_\kappa(L^{\mathrm{NP}}) &= \inf\{x \in \mathbb{R}, F_{L^{\mathrm{NP}}}(x) \geq \kappa\} \\
&= \inf\{x \in \mathbb{R}, F_{L-D}(x) \geq \kappa\} \\
&= \inf\{x \in \mathbb{R}, F_L(x) \geq \kappa\} - D \\
&= \mathrm{VaR}_\kappa(L) - D,
\end{aligned}$$

which is increasing in κ, worth 0 when $\kappa = w_0$ and C when $\kappa = 1 - w_C$. In summary,

$$\mathrm{VaR}_\kappa(L^{\mathrm{NP}}) = \min\{\max\{\mathrm{VaR}_\kappa(L) - D, 0\}, C\}.$$

Turning now to the Expected Shortfall, it is clear using the above derivations that

$$\mathrm{ES}_\kappa(L^{\mathrm{NP}}) = \frac{\mathbb{E}[L^{\mathrm{NP}}]}{1 - \kappa}, \forall \kappa \leq w_0,$$

and

$$\mathrm{ES}_\kappa(L^{\mathrm{NP}}) = C, \forall \kappa \geq 1 - w_C.$$

For the remaining case $\kappa \in (w_0, 1-w_C)$,

$$\begin{aligned}
\mathrm{ES}_\kappa(L^{\mathrm{NP}}) &= \frac{1}{1-\kappa}\int_\kappa^1 \mathrm{VaR}_u(L^{\mathrm{NP}})du \\
&= \frac{1}{1-\kappa}\left(\int_\kappa^{1-w_C} \mathrm{VaR}_u(L^{\mathrm{NP}})du + \int_{1-w_C}^1 \mathrm{VaR}_u(L^{\mathrm{NP}})du\right) \\
&= \frac{1}{1-\kappa}\left(\int_\kappa^{1-w_C} (\mathrm{VaR}_u(L) - D)\,du + \int_{1-w_C}^1 C du\right) \\
&= \frac{1}{1-\kappa}\left(\int_\kappa^{1-w_C} \mathrm{VaR}_u(L)du - D(1-w_C-\kappa) + w_C C\right) \\
&= \frac{1}{1-\kappa}\int_{\mathrm{VaR}_\kappa(L)}^{C+D} x dF_L(x) + \frac{1}{1-\kappa}[w_C C - D(1-w_C-\kappa)] \\
&= \frac{1}{1-\kappa}\int_{\mathrm{VaR}_\kappa(L)}^{C+D} u f_L(u)du + (C+D)\frac{w_C}{1-\kappa} - D.
\end{aligned}$$

A.3 Proof of Proposition 3.1

Under the assumptions of Proposition 3.1, denote by $\mathscr{P}_k$ the proposition:

$$F_{S_k^{\mathrm{NP}}}(s) = F_{S_k}\left(s + \sum_{i=1}^k D_i\right), \ \forall s \in [\sum_{i=1}^{k-1}(r_i - D_i), +\infty).$$

The fact that $\mathscr{P}_1$ is true follows directly from (8). Now let us assume that $\mathscr{P}_k$ is true for a given $k < n$, and recall that $w_{0,k+1} = F_{k+1}(D_{k+1})$. Then, for $s \geq \sum_{i=1}^k (r_i - D_i)$, we obtain

$$\begin{aligned}
&F_{S_{k+1}^{\mathrm{NP}}}(s) = \int_0^\infty F_{S_k^{\mathrm{NP}}}(s-x)dF_{L_{k+1}^{\mathrm{NP}}}(x) = \int_0^{r_{k+1}-D_{k+1}} F_{S_k^{\mathrm{NP}}}(s-x)dF_{L_{k+1}^{\mathrm{NP}}}(x) \\
&= \int_0^{r_{k+1}-D_{k+1}} F_{S_k^{\mathrm{NP}}}(s-x)\,w_{0,k+1}d\delta_0(x) + \int_0^{r_{k+1}-D_{k+1}} F_{S_k^{\mathrm{NP}}}(s-x)f_{k+1}(x+D_{k+1})dx \\
&= w_{0,k+1}\underbrace{F_{S_k^{\mathrm{NP}}}(s)}_{=1} + \int_0^{\min\{s, r_{k+1}-D_{k+1}\}} F_{S_k^{\mathrm{NP}}}(s-x)f_{k+1}(x+D_{k+1})dx \\
&\stackrel{(7)}{=} w_{0,k+1} + \int_0^{r_{k+1}-D_{k+1}} F_{S_k^{\mathrm{NP}}}(s-x)f_{k+1}(x+D_{k+1})dx \\
&= w_{0,k+1} + \int_{D_{k+1}}^{r_{k+1}} F_{S_k^{\mathrm{NP}}}(s+D_{k+1}-u)f_{k+1}(u)du \\
&= w_{0,k+1} + \int_{D_{k+1}}^0 F_{S_k^{\mathrm{NP}}}(s+D_{k+1}-u)f_{k+1}(u)du + \int_0^{r_{k+1}} F_{S_k^{\mathrm{NP}}}(s+D_{k+1}-u)f_{k+1}(u)du
\end{aligned}$$

$$
\begin{aligned}
&= w_{0,k+1} + \int_{D_{k+1}}^{0} f_{k+1}(u)du + \int_0^{r_{k+1}} \underbrace{F_{S_k}(s + \sum_{i=1}^{k+1} D_i - u)}_{\mathscr{P}_k} f_{k+1}(u)du \\
&= w_{0,k+1} - w_{0,k+1} + F_{S_{k+1}}\left(s + \sum_{i=1}^{k+1} D_i\right) \\
&= F_{S_{k+1}}\left(s + \sum_{i=1}^{k+1} D_i\right),
\end{aligned}
$$

hence $\mathscr{P}_{k+1}$ is also true. The result follows by induction.

A.4 Diversification benefit of a proportional portfolio with two independent identically distributed uniform original losses

We use here the same notation as in Section 3.1.3. The Value-at-Risk is directly obtained from the inverse of $F_{L_1^{\mathrm{P}}}$, namely $\forall \kappa \in [0,1]$,

$$
\mathrm{VaR}_\kappa(L_1^{\mathrm{P}}) = F_{L_1^{\mathrm{P}}}^{-1}(\kappa) = \alpha\,(a + (b-a)\kappa).
$$

Similarly, the Expected Shortfall is obtained via

$$
\begin{aligned}
\mathrm{ES}_\kappa(L_1^{\mathrm{P}}) &= \alpha \mathrm{ES}_\kappa(L_1) \\
&= \frac{\alpha}{1-\kappa} \int_{\mathrm{VaR}_\kappa(L_1)}^{b} \frac{x}{b-a} dx \\
&= \frac{\alpha\left(b^2 - \mathrm{VaR}_\kappa(L_1)^2\right)}{2(1-\kappa)(b-a)}.
\end{aligned}
$$

The df of $L_1 + L_2$ is given, for every $z \in \mathbb{R}$ by

$$
F_{L_1+L_2}(z) = \mathbb{1}_{[2a,a+b]}(z)\frac{(z-2a)^2}{2(b-a)^2} + \mathbb{1}_{(a+b,2b)}(z)\left(1 - \frac{(z-2b)^2}{2(b-a)^2}\right) + \mathbb{1}_{[2b,+\infty)}(z)
$$

(this can be seen by integrating the density function given by

$$
f_{L_1+L_2}(z) = \mathbb{1}_{[2a,a+b]}(z)\frac{z-2a}{(b-a)^2} - \mathbb{1}_{(a+b,2b)}(z)\frac{z-2b}{(b-a)^2},
$$

see e.g. Theorem 1 of Bradley and Gupta [2002]). The formula of the Value-at-Risk is obtained by solving $\mathrm{VaR}_\kappa(L_1+L_2) = F_{L_1+L_2}^{-1}(\kappa)$, $\forall \kappa \in (0,1)$, which yields

$$
\mathrm{VaR}_\kappa(L_1+L_2) = \begin{cases} 2a + \sqrt{2\kappa(b-a)^2}, & \text{if } \kappa < \frac{1}{2}; \\ 2b - \sqrt{(1-\kappa)2(b-a)^2}, & \text{if } \kappa \geq \frac{1}{2}. \end{cases}
$$

Then, for the Expected Shortfall we obtain by direct integration

$$
\mathrm{ES}_\kappa(L_1+L_2) = \begin{cases} \frac{1}{(1-\kappa)(b-a)^2}\left(\frac{b^3 - 2a^3 - \mathrm{VaR}_\kappa^3(L_1+L_2)}{3} + a\left(\mathrm{VaR}_\kappa^2(L_1+L_2) - ab\right)\right) + \frac{a+2b}{3(1-\kappa)} & \text{if } \kappa < \frac{1}{2}; \\ \frac{1}{(1-\kappa)(b-a)^2}\left(\frac{4}{3}b^3 - \mathrm{VaR}_\kappa^2(L_1+L_2)\left(b - \frac{\mathrm{VaR}_\kappa(L_1+L_2)}{3}\right)\right) & \text{if } \kappa \geq \frac{1}{2}. \end{cases}
$$

After plugging the above formulas into (1), we finally obtain Equations (9)-(10)-(11)-(12).

A.5 Diversification benefit of a non-proportional portfolio with two independent identically distributed uniform original losses

We use here the same notations as in Section 3.1.3. The mathematical expectation of L_1^{NP} satisfies

$$\mathbb{E}[L_1^{\mathrm{NP}}] = \frac{1}{b-a}\int_0^{b-D} s\,ds = \frac{(b-D)^2}{2(b-a)}.$$

Recall that $w_0 = \frac{D-a}{b-a}$, then from Corollary 2.2 (with $C=\infty$) we obtain

$$\mathrm{VaR}_\kappa(L_1^{\mathrm{NP}}) = \begin{cases} 0, & \text{if } \kappa \le w_0; \\ \kappa(b-a)-(D-a), & \text{if } \kappa > w_0, \end{cases}$$

for the Value-at-Risk, and

$$\mathrm{ES}_\kappa(L_1^{\mathrm{NP}}) = \begin{cases} \frac{(b-D)^2}{2(1-\kappa)(b-a)}, & \text{if } \kappa \le w_0; \\ \frac{(b-D)+\mathrm{VaR}_\kappa(L_1^{\mathrm{NP}})}{2}, & \text{if } \kappa > w_0, \end{cases}$$

for the Expected Shortfall. For $s \in [0, b-D)$, the distribution of the portfolio S_2^{NP} is given by

$$F_{S_2^{\mathrm{NP}}}(s) = w_0^2 + 2w_0\frac{s}{b-a} + \frac{s^2}{2(b-a)^2}.$$

From the above equation and Corollary 3.2 we derive the Value-at-Risk of S_2^{NP},

$$\mathrm{VaR}_\kappa(S_2^{\mathrm{NP}}) = \begin{cases} 0, & \text{if } 0<\kappa \le w_0^2; \\ 2(a-D)+\sqrt{2}\sqrt{(D-a)^2+\kappa(b-a)^2}, & \text{if } w_0^2<\kappa \le F_{S_2^{\mathrm{NP}}}(b-D); \\ 2(b-D)-(b-a)\sqrt{2(1-\kappa)}, & \text{if } F_{S_2^{\mathrm{NP}}}(b-D)<\kappa<1. \end{cases}$$

For the Expected Shortfall we then obtain

$$\mathrm{ES}_\kappa(S_2^{\mathrm{NP}}) = \begin{cases} \frac{(b-D)^2}{(1-\kappa)(b-a)}, & \text{if } 0<\kappa \le w_0^2; \\ \frac{w_0}{(1-\kappa)(b-a)}\left((b-D)^2-\mathrm{VaR}_\kappa(S_2^{\mathrm{NP}})^2\right) - \frac{\mathrm{VaR}_\kappa(S_2^{\mathrm{NP}})^3}{3(1-\kappa)(b-a)^2} + \frac{(b-D)^3}{(1-\kappa)(b-a)^2}, & \text{if } w_0^2<\kappa \le F_{S_2^{\mathrm{NP}}}(b-D); \\ \frac{(b-D)(4(b-D)^2-\mathrm{VaR}_\kappa(S_2^{\mathrm{NP}})^2)}{(1-\kappa)(b-a)^2} - \frac{8(b-D)^3-\mathrm{VaR}_\kappa(S_2^{\mathrm{NP}})^3}{3(1-\kappa)(b-a)^2}, & \text{if } F_{S_2^{\mathrm{NP}}}(b-D)<\kappa<1. \end{cases}$$

The diversification benefit (1) is then computed with the risk measures VaR_κ, xVaR_κ, ES_κ and xES_κ. When $\rho = \mathrm{xVaR}_\kappa$, we have

$$\mathrm{DB}(S_2^{\mathrm{NP}}, \mathrm{xVaR}_\kappa) = \begin{cases} 1-\frac{\mathbb{E}[S_2^{\mathrm{NP}}]}{\mathbb{E}[S_2^{\mathrm{NP}}]} = 0, & \text{if } 0<\kappa \le w_0^2; \\ \frac{(b-a)\left(2(a-D)+\sqrt{2}\sqrt{(a-D)^2+(a-b)^2\kappa}\right)}{(b-D)^2}, & \text{if } w_0^2<\kappa \le w_0; \\ \frac{(a-b)\sqrt{2}\sqrt{(a-D)^2+(a-b)^2\kappa}+2\kappa(a-b)^2}{2(b-a)(\kappa(b-a)-(D-a))-(b-D)^2}, & \text{if } w_0<\kappa \le F_{S_2^{\mathrm{NP}}}(b-D); \\ \frac{(b-a)^2\left(2\kappa-2+\sqrt{2-2\kappa}\right)}{2(b-a)(\kappa(b-a)-(D-a))-(b-D)^2}; & \text{if } F_{S_2^{\mathrm{NP}}}(b-D)<\kappa<1. \end{cases} \tag{30}$$

Note that $\mathrm{DB}(S_2^{\mathrm{NP}}, \mathrm{xVaR}_\kappa)$ is not defined for $\kappa = \frac{(b-D)^2}{2(b-a)^2} + \frac{D-a}{b-a}$ due to a division by zero. The expressions of $\mathrm{DB}(S_2^{\mathrm{NP}}, \mathrm{VaR}_\kappa)$ (13), $\mathrm{DB}(S_2^{\mathrm{NP}}, \mathrm{ES}_\kappa)$ and $\mathrm{DB}(S_2^{\mathrm{NP}}, \mathrm{xES}_\kappa)$ are obtained in a similar way.

A.6 Diversification benefit of a proportional portfolio with two independent identically distributed exponential original losses

We present here the derivation of $\mathrm{DB}(S_2^{\mathrm{P}},\rho)$ for the four risk measures ρ of interest. Recall that W is the Lambert function (21).

$$\mathrm{DB}(S_2^{\mathrm{P}},\mathrm{VaR}_\kappa(\cdot)) = 1-\frac{1+W\left(\frac{\kappa-1}{e}\right)}{2(\log(1-\kappa))}; \tag{31}$$

$$\mathrm{DB}(S_2^{\mathrm{P}},\mathrm{ES}_\kappa) = 1-\frac{e^{1+W\left(\frac{\kappa-1}{e}\right)}\left(1+W^2\left(\frac{\kappa-1}{e}\right)\right)}{2(1-\kappa)(1-\log(1-\kappa))}; \tag{32}$$

$$\mathrm{DB}(S_2^{\mathrm{P}},\mathrm{xVaR}_\kappa) = 1-\frac{3+W\left(\frac{\kappa-1}{e}\right)}{2(\log(1-\kappa)+1)}; \tag{33}$$

$$\mathrm{DB}(S_2^{\mathrm{P}},\mathrm{xES}_\kappa) = 1-\frac{1}{\log(1-\kappa)}\left[1-\frac{e^{1+W\left(\frac{\kappa-1}{e}\right)}\left(1+W^2\left(\frac{\kappa-1}{e}\right)\right)}{2(1-\kappa)}\right]. \tag{34}$$

These four functions of κ (only!) are plotted on Figure 8.

A.7 Value-at-Risk and Expected Shortfall of a non-proportional portfolio with two independent identically distributed exponential original losses.

Recall that $w_0 = 1-e^{-\lambda D}$ and W is the Lambert function (21). Then from (25) we obtain, for $\kappa \in (0,1)$,

$$\mathrm{VaR}_\kappa(S_2^{\mathrm{NP}}) = \begin{cases} 0, & \text{if } 0<\kappa\le w_0^2; \\ \frac{1-2e^{\lambda D}-W\left(e^{1+2D\lambda-2e^{\lambda D}}(\kappa-1)\right)}{\lambda}, & \text{if } w_0^2<\kappa<1-e^{-\lambda(C+2D)}(C\lambda-1+2e^{\lambda D}); \\ C, & \text{if } 1-e^{-\lambda(C+2D)}(C\lambda-1+2e^{\lambda D})\le\kappa<F_{S_2^{\mathrm{NP}}}(C); \\ \frac{1+2\lambda C-W\left(-e^{1+2\lambda(C+D)}(\kappa-1)\right)}{\lambda}, & \text{if } F_{S_2^{\mathrm{NP}}}(C)\le\kappa<1-e^{-2\lambda(C+D)}; \\ 2C, & \text{if } 1-e^{-2\lambda(C+D)}\le\kappa\le 1, \end{cases}$$

and, using the notation $q_\kappa = \mathrm{VaR}_\kappa(S_2^{\mathrm{NP}})$,

$$\mathrm{ES}_\kappa(S_2^{\mathrm{NP}}) = \begin{cases} \frac{2\left(e^{-\lambda D}-e^{-\lambda(D+C)}\right)}{\lambda(1-\kappa)}, & \text{if } 0<\kappa\le w_0^2; \\ \frac{e^{-\lambda(C+2D+q_\kappa)}\left(2(\lambda q_\kappa+1)e^{\lambda(C+D)}+\lambda^2q_\kappa^2e^{\lambda C}-2e^{\lambda(D+q_\kappa)}\right)}{\lambda(1-\kappa)}, & \text{if } w_0^2<\kappa<1-e^{-\lambda(C+2D)}(\lambda C-1+2e^{\lambda D}); \\ C+\frac{Ce^{-\lambda(C+2D)}}{1-\kappa}, & \text{if } 1-e^{-\lambda(C+2D)}(\lambda C-1+2e^{\lambda D})\le\kappa<F_{S_2^{\mathrm{NP}}}(C); \\ \frac{e^{-\lambda(2D+q_\kappa)}\left(2C(1+\lambda q_\kappa)-\lambda q_\kappa^2\right)}{1-\kappa}, & \text{if } F_{S_2^{\mathrm{NP}}}(C)\le\kappa<1-e^{-2\lambda(C+D)}; \\ 2C, & \text{if } 1-e^{-2\lambda(C+D)}<\kappa<1. \end{cases}$$

From this on it is possible to deduce the formulas of the diversification benefits, which are more complex than in A.6 and not reported here.