Comparing Incomparable Alternatives when certain intergenerational choice axiom are to be satisfied

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Abstract: Since it has been shown that completeness and constructiveness is an unattainable goal for paretian and anonymous social welfare relations on infinite utility streams, recent contributions generally propose criteria displaying some incomparability. I suggest an interpretation of incomparability which sets a limit to the recourse to the judgment of incomparability when certain intergenerational choice axioms are to be satisfied. It leads to see as incompatible the axioms *fixed step relative anonymity* and *stationarity.* This conclusion is obtained thanks to a generalization to infinite dimension of a lemma of [d'Aspremont-Gevers 1977] on the equivalence between *relative anonymity* and *anonymity.* It contrasts with [Asheim-d'Aspremont-Banerjee 2008] who build a preorder satisfying both *fixed step relative anonymity* and *stationarity.* The reason of this disagreement is that their result is obtained at the cost of additional judgments of incomparability which are not justified according to the interpretation of incomparability suggested here.

Keywords: Intergenerational choice; Comparability; Anonymity; Infinite utility streams; Completeness.

JEL Classification Numbers: D63; D71.

1 Introduction

Intergenerational choice models (or intertemporal choice models with infinite horizon) have shown that tensions may exist between different desirable properties. For example, it is difficult to reconcile *continuity, strong Pareto* and *finite anonymity* [Diamond 1965]. Tensions also exist between the properties of consistency (*transitivity, completeness, precision*) of a preorder (i.e. a transitive and reflexive binary relation, possibly incomplete) satisfying *strong Pareto* and *finite anonymity* in the context of intergenerational choice. [Fleurbaey-Michel 2003] provide an example where, when seeking to complete such a preorder, we can lose in *precision* (i.e. some strict preferences are converted into indifference). If we want to keep the same degree of precision, we can lose *transitivity.* Szpilrajn's theorem [Szpilrajn 1930] seemingly solves this dilemma between *precision, completeness* and *transitivity.* However, the problem of *incalculability* appears. This problem means that, after having shown the existence of an order (i.e. a complete preorder) having the desired properties, we cannot find a really applicable method of implementation [Lauwers 2007], [Zame 2007].

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This is why some authors choose to weaken the requirement of *completeness*, rather than weakening *strong Pareto* and *finite anonymity.* Among others, [Basu-Mitra 2007] or [Asheim-d'Aspremont-Banerjee 2008]. To weaken the requirement of *completeness* amounts adding the value "is incomparable with" to the set of values a judgment can give. This set then becomes:

{"is better than", "is worse than", "is indifferent to", "is incomparable with"}

It is however legitimate to ask about the practical meaning of the value "is incomparable with". If it is true that adding this value formally solves some incompatibilities between axioms, what intuitive meaning could be given to it, particularly in the context of social or intergenerational choice? This question is important if it is considered that a social welfare relation does not only represent a cold mechanism of aggregation of the preferences, but also expresses the tastes of the "ethical observer".

In the following, I suggest an interpretation of incomparability which leads to the concept of forced judgment. Forced judgments set a limit to the recourse to the judgment of incomparability when axioms consisting in equivalence relations on the space of pairs of alternatives are to be satisfied. The concept is applied to the axiom *relative anonymity* defined on the space of pairs of infinite utility streams. It leads to see as incompatible the axioms *fixed step relative anonymity* and *stationarity.* This result of incompatibility is obtained thanks to a generalization to infinite dimension of a lemma of [d'Aspremont-Gevers 1977] on the equivalence between *relative anonymity* and *anonymity.* It contrasts with [Asheim-d'Aspremont-Banerjee 2008] who build a preorder satisfying both *fixed step relative anonymity* and *stationarity.* But their preorder displays judgments of incomparability which are not justified according to the interpretation of incomparability suggested here.

Section 2 exposes the interpretation of incomparability and the resulting concept of forced judgments. Section 3 introduces the notation. Section 4 explores some general questions related to forced judgments. Section 5 presents a finite case aiming at illustrating the concepts introduced in section 4: forced judgments, forced preorder and primary judgments. Section 6 studies the relation between the axioms *relative anonymity* and *anonymity.* It gives the generalization to infinite dimension of the lemma of [d'Aspremont-Gevers 1977] on the equivalence between *relative anonymity* and *anonymity.* It is shown that, for any order on infinite streams, *relative anonymity* with respect to any periodic permutation is equivalent to *anonymity* with respect to that permutation (theorem *15*). Periodic permutations are a set of permutations contained in the set of cyclic permutations and containing the set of fixed step permutations. Cyclic and fixed step permutations are well-known [Mitra-Basu 2007] or [Lauwers 2007] whereas periodic permutations are introduced in the present paper. Moreover, theorem *15* characterizes completely periodic permutations. So that this generalization be accessible without necessarily passing by the definitions related to forced judgments, results and proofs of subsection 6.2 are reformulated without reference to these definitions (namely theorem *15*, lemma *17* and corollary *18*).

Section 7 discusses the question of incompatibility between *fixed step relative anonymity* and *stationarity* in the light of the concepts and results of previous sections.

2 Incomparability and forced judgments

2.1 Practically, what does incomparability mean?

The intuitive interpretation of a judgment of incomparability (i.e. a judgment taking the value "is incomparable with", section 3 gives formal definitions of judgments and values) does not seem as direct and clear as that of the other judgments. As an illustration, one never meets *explicitly* the choice "is incomparable with" in vote procedures to choose political leaders. Although to abstain from voting may be interpreted as a judgment of incomparability, some people could also interpret abstention as a judgment of indifference. Actually, it is not certain that people who are not accustomed to mathematical formalism could distinguish between incomparability and indifference.

Because of that, one can be tempted to assimilate the judgment of incomparability with that of indifference, since the selection of an alternative among two which are incomparable, is an arbitrary choice exactly as if the two alternatives were indifferent. But the problem is that the order obtained by replacing incomparability with indifference is highly likely to be nontransitive.

One can also be tempted to prohibit such a value, which amounts requiring completeness. But on the practical level, a preorder can be a tool of decision as useful as an order, and easier to implement since it is authorized to remain silent on certain pairs of alternatives. For example, if the two incomparable alternatives are dominated by a third one, incomparability does not prevent from selecting the optimal alternative, i.e. the third one. In this case, the exact knowledge of the comparison between the first and the second alternative is not of any help.

These remarks on incomparability are rather technical. It is also useful to examine the significance of jugments compared to the final goal of the choice, supposing that such a final goal exists, which should be the case if the considered choice problem is correctly defined. For example, the significance of the judgment "the alternative x is better than the alternative y" is that x is more appropriate than y to achieve the final goal of the choice. As for the judgment of incomparability, it seems to me that the only case where the recourse to that judgment may be legitimate, is when we do not know the impact of the substitution of an alternative by the other on the final goal of the choice. In other words, to judge that x is incomparable with y, in the context of a well-specified choice problem with a clear final goal, is equivalent to not knowing which of the values "is better than", "is worse than" or "is indifferent to" to assign to the pair (x, y).

But not to know the consequence of a choice does not mean that the consequence does not exist. Consider the case of the political vote. If you don't know which candidate will be better, this does not prevent that if elected, each candidate will take a succession of decisions which will allow you to reach some utility level. If you could travel in time, you would see the utility level each candidate would allow you to reach. You would return then to present and compare the candidates in a precise way. The fact of not being able to travel in time and not to know the ranking of the candidates, does not prevent that this ranking exists objectively. It is just not known. It is as if, behind the preorder representing the known preferences, there is an *underlying order* able to compare all the candidates and of which the considered preorder is a subrelation.

In short, the preorder representing the known preferences is the apparent side of the underlying order representing the real preferences.

I am not trying to say that it is possible to prefer a thing to another without being aware of this preference. Rather, I argue that if one does not know which alternative is better, then, with more attention and information and with a thorough analysis of the consequence of exchanging an alternative by the other on the final goal of the choice, one should be able to know. If not, if it is absolutely and definitively impossible to decide, even by devoting all the necessary effort and time, one should agree to declare the two alternatives indifferent.

Note that this interpretation of incomparability as not knowing the underlying judgment, is weaker than the well-known axiom 1 of [Arrow 1950] imposing completeness. [Arrow 1950] justifies his axiom as follows (page 331):

> ...the chooser considers in turn all pairs of alternatives, say x and y, and for each pair he makes one and only one of three decisions: x is preferred to y, x is indifferent to y, or y is preferred to x.

That the value “is incomparable with” does not appear in this enumeration testifies to its not very intuitive nature.

Another possible interpretation of the judgment of incomparability is to give to it a value in itself, autonomous with respect to the values "is better than", "is worse than" or "is indifferent to". This amounts to give the following answer to the title of the section: "Incomparability means incomparability". But such an interpretation seems to me purely formal. I tend to suppose that it would make the mathematical modeling of preferences less intuitive. Consequently, the model would deviate from the object to model. Therefore I stick, in this paper, to the first interpretation.

2.2 If it can neither be worse nor similar, it is surely better...

If you know the reason why you are making a choice, that is, if you have a clear idea of the final goal of the choice, and if you know that in reference to that final goal, a given alternative can neither be worse than nor similar to another one, then you have good reasons to believe that it is better.

Let’s formalize that a little more. The interpretation of the judgment of incomparability (as not knowing the consequences of the choice) does not invite to reject the concept of preorder, but rather to be interested in the family of orders of which the considered preorder constitutes a subrelation. For a preorder R on a set of alternatives E, let us denote $\Re(R)$ this family of orders. All the orders of $\Re(R)$ agree on all judgments established by R, but some can be in dissension on a pair of alternatives (x, y) when R considers (x, y) to be incomparable. Moreover, if an axiom Π (or a set of axioms) is considered as evident or ethically desirable, it is of interest to consider the subset $\Re(R, \Pi)$ of $\Re(R)$, made up of the orders of $\Re(R)$ satisfying Π.

Suppose that R considers (x, y) to be incomparable and that all the orders of $\Re(R, \Pi)$ establish the same judgment on a pair of alternatives (x, y), for instance "x is better than y". If one adheres to the suggested interpretation of the judgment of incomparability and insists on satisfying Π, it becomes inevitable, I believe, to grant some plausibility to the judgment "x is better than y". Indeed, on the one hand, the incomparability of (x, y) is understood as not knowing which of the three values "is better than", "is worse than" or "is indifferent to" to assign to the pair (x, y), on the other hand judgments "x is worse than y" or "x is indifferent to y" would violate the axiom Π.

Notice that what is of interest here is not merely the preorder R, but the preferences R is supposed to model. If satisfying the axiom Π implies that we know that the only possible underlying judgment is "x is better than y", then the position which consists in saying "We do not know which value to assign to (x, y)" becomes difficult to defend. At least, as long as the suggested interpretation of incomparability is seen as acceptable, it should be recognized that the judgement "x is incomparable with y" does not have the same strength as if there was two orders in $\Re(R, \Pi)$ giving two different judgments on (x, y).

In the case where all the orders of $\Re(R, \Pi)$ establish the same judgment on (x, y), I suggest the terminology *forced judgment* under R and Π. Formal definition is given further (section 4).

The terminology and the idea of forced judgment was inspired to me by [d'Aspremont 2007] where it is question of forced adoption of a criterion for a population of a given size when this criterion is adopted for populations of lower sizes. A sequence of criteria applying to populations of increasing sizes and checking this condition of forced adoption is said to be a proliferating sequence [d'Aspremont 2007]. In an obvious way, forced judgments on some pairs of alternatives result from the forced adoption of a criterion. Here, I do not investigate the forced judgments which could result from the increase in population, as in [d'Aspremont 2007], but that which could result from the adoption of certain axioms, considered as evident or ethically desirable and accompanying the social preorder.

3 Notation

The (non empty) set of alternatives is denoted E. A preorder on E is a reflexive and transitive binary relation. An order on E is a complete preorder.

For a preorder R and two alternatives x and y, $x \succsim_R y$ means "x is preferred or indifferent to y", $x \succ_R y$ means "x is preferred to y" and $x \sim_R y$ means "x is indifferent to y". The graph of a preorder R (which is a subset of $E \times E$) is $G(R) = \{(x, y) \in E \times E / x \succsim_R y\}$. A preorder expressing social choice may be associated to one or more axioms which are considered evident or ethically desirable. Axioms used in intergenerational choice theory can be preorders on E. For example: *Hammond equity* axiom. In that case, the preorder expressing intergenerational choice, say R, is required to admit the axiom as subrelation. Axioms can also be equivalence relations on E. In that case, $\sim_R$is required to admit the axiom as subrelation. For example: axioms of *anonymity.* Lastly, axioms can also be equivalence relations on $E \times E$. It is this case which concerns us here. Let Π be an equivalence relation on $E \times E$. The preorder R is said to satisfy Π if:

$$\forall (x, y) \text{ and } (x', y') \text{ in } E \times E,\ \ [(x, y)\Pi(x', y')] \Longrightarrow [x \succsim_R y \Longrightarrow x' \succsim_R y'] \quad (1)$$

For example: the axiom *relative anonymity* ([d'Aspremont-Gevers 1977] or [Asheim-d'Aspremont-Banerjee 2008]), or the axiom *invariance with respect to individual change of origin,* denoted $inv(a_i + u_i)$ in [d'Aspremont-Gevers 2002]. More details on *relative anonymity* and *anonymity* axioms will be provided hereafter. Many other axioms of intergenerational choice are expressed in this form of equivalence relation on $E \times E$ (see among others [d'Aspremont-Gevers 2002]).

We can associate to an equivalence relation Π on $E \times E$, a correspondence π in the following way:

$$\begin{aligned} \pi \quad &: \quad E \times E \to P(E \times E) \\ (x, y) \quad &\to \quad \{(x', y') \in E \times E / (x, y)\Pi(x', y')\} \end{aligned} \quad (2)$$

where $P(E \times E)$ is the set of subsets of $E \times E$. For a subset A of $E \times E$, denote $\pi(A) = \cup_{\alpha \in A} \pi(\alpha)$. The relation between Π and π is bi-univocal. So, we can refer indifferently to the relation Π or the correspondence π.

The properties inherited by π are:

· *reflexivity*: $\forall \alpha \in E \times E,\ \alpha \in \pi(\alpha)$

· *symmetry*: $\forall \alpha, \beta \in E \times E,\ \beta \in \pi(\alpha) \Longrightarrow \alpha \in \pi(\beta)$

· *transitivity*: $\forall \alpha \in E \times E,\ \pi(\pi(\alpha)) \subset \pi(\alpha)$

Of course, *transitivity* and *symmetry* involve *reflexivity.*

The condition (1) writes $\pi(G(R)) \subset G(R)$.

The following notations will also be convenient for the sequel: $\Re(R)$ is the set of orders of which the preorder R is a subrelation, i.e. for all S in $\Re(R)$ and for all x, y in E, $x \succsim_R y$ implies $x \succsim_S y$ and $x \succ_R y$ implies $x \succ_S y$; $\Re(R, \Pi)$ (or $\Re(R, \pi)$) is the set of orders on E which satisfy Π (condition (1)) and admit R as a subrelation; V is the set of values $\{\succ, \prec, \sim\}$; a judgment is an element of the set $E \times E \times V$, for example $(x, y, \succ)$ means that the value $\succ$ is assigned to the pair (x, y) ,i.e. $x \succ y$; $J(R)$ is the list of judgments established by the preorder R, it is a subset of $E \times E \times V$.

For example, with these notations, $x \succ_R y$ is equivalent to $(x, y, \succ) \in J(R)$ and $S \in \Re(R)$ is equivalent to $J(R) \subset J(S)$.

I would request from the reader a little patience as for the introduction of notations which may appear unusual such as judgments and correpondences[2].

4 Some general considerations on forced judgments

4.1 Definitions

Definition 1 *A preorder R is said to be compatible with an axiom Π iff $\Re(R, \Pi)$ is not empty.*

Remark 2 *R compatible with Π does not entail that condition (1) is checked. However, theorem 6 shows that there exist a preorder, of which R is a subrelation, checking condition (1). The issue of knowing to what extent condition (1) implies that R is compatible with Π, is not tackled here.*

Definition 3 *Let $x, y \in E$. If (and only if) there exists $v \in V = \{\prec, \succ, \sim\}$ such that, for all S in $\Re(R, \Pi)$, $(x, y, v) \in J(S)$, then (x, y, v) is a forced judgment on the pair (x, y) under the preorder R and the axiom Π. Moreover, if (x, y, v) is a forced judgment on the pair (x, y) under the preorder R and the axiom Π for every preorder R on E, then (x, y, v) is a forced judgment on the pair (x, y) under the axiom Π.*

Remark 4 *It is equivalent to say that (x, y, v) is a forced judgment under the axiom Π or to say that every order which satisfies Π establishes the judgment (x, y, v). We will use sometimes the expression "forced indifference" which indicates a forced judgment taking the value "is indifferent to".*

[2] Concerning judgments, it was necessary to define the forced preorder as being the list of forced judgments. This requires to give a formal definition to the concept of forced judgment. Therefore it was necessary as a preliminary to define what a judgment is.

Concerning correspondences of equivalence, the study of the compatibility between a preorder R on E and a relation of equivalence on $E \times E$, amounts to see to what point one can " project " a judgment established by R on a given pair of alternatives, on a subset of $E \times E$. This subset of $E \times E$ consists of the list of pairs of alternatives equivalent to the initial pair. In this diagram, one mentally associates an element of $E \times E$ to a subset of $E \times E$. It is the concept of correspondence which expresses most naturally and directly this mental association. For example, if this notation is avoided, it would be necessary each time to replace the sentence "for all β in $\pi(\alpha)$" by "for all (x', y') such that (x', y') is equivalent to (x, y) according to Π". This would certainly encumber the text. Especially for the proof of proposition 7, this notation seems inevitable. Notice that, to avoid confusion, one cannot use the symbol $\sim$ for a relation of equivalence on $E \times E$ because it is already used for equivalence on E.

It is possible to weaken this definition by replacing the unanimity of the orders of $\Re(R,\Pi)$ by a more flexible decision rule. That requires to equip the set $\Re(R,\Pi)$ with an adapted structure. As first approach, I will stick here to the rule of the unanimity.

4.2 Forced preorder

Suppose that R is compatible with Π. Denote $F(R,\Pi)$ (or $F(R,\pi)$) the set of forced judgments under R and Π. We may like to know if there exists a preorder which list of judgments corresponds exactly to $F(R,\Pi)$. If so, it would be the preorder forced by R and Π. To know that, it is necessary to determine what conditions have to be imposed on a subset of $E \times E \times V$ so that this subset corresponds to a preorder.

Lemma 5 *Let $J \subset E \times E \times V$. J is the list of judgments of a preorder on E iff:*

C1) Uniqueness: for any pair (x,y) of $E \times E$, there exists at most one $v \in V$ such that $(x,y,v) \in J$.

C2) Symmetry between $\prec$ and $\succ$: for any pair (x,y) of $E \times E$, $(x,y,\prec) \in J$ implies $(y,x,\succ) \in J$.

C3) Symmetry of $\sim$: for any pair (x,y) of $E \times E$, $(x,y,\sim) \in J$ implies $(y,x,\sim) \in J$.

C4) Reflexivity: for any $x \in E$, $(x,x,\sim) \in J$.

C5) Transitivity: for any (x,y,z) of $E\times E\times E$, $[(x,y,\succ) \in J$ or $(x,y,\sim) \in J]$ and $[(y,z,\succ) \in J$ or $(y,z,\sim) \in J]$implies $[(x,z,\succ) \in J$ or $(x,z,\sim) \in J]$.

Proof. If R is a preorder on E, then $J(R)$ obviously satisfies conditions *C1* to *C5*. Conversely, denote R_J the binary relation on E defined by: $x \succsim_{R_J} y$ iff11 $[(x,y,\succ) \in J$ or $(x,y,\sim) \in J]$. From conditions *C4* and *C5*, R_J is reflexive and transitive. We must check that the list of judgments of R_J is J, i.e. $J(R_J) = J$. That is, we must check the following equivalences for all x, y in E:

$$x \succsim_{R_J} y \text{ and } y \succsim_{R_J} x \Leftrightarrow (x,y,\sim) \in J \tag{3}$$

$$x \succsim_{R_J} y \text{ and non } y \succsim_{R_J} x \Leftrightarrow (x,y,\succ) \in J \tag{4}$$

$$x \precsim_{R_J} y \text{ and non } y \precsim_{R_J} x \Leftrightarrow (x,y,\prec) \in J \tag{5}$$

(3) According to the definition of $x \succsim_{R_J} y$, it is easily seen that $(x,y,\sim) \in J$ implies $x \succsim_{R_J} y$ and $y \succsim_{R_J} x$. Conversely, having $x \succsim_{R_J} y$ and $y \succsim_{R_J} x$ amounts to having $[(x,y,\succ) \in J$ or $(x,y,\sim) \in J]$ and $[(y,x,\succ) \in J$ or $(y,x,\sim) \in J]$. It is not possible to have $(x,y,\succ) \in J$ and $(y,x,\succ) \in J$, if not, by transitivity, it would yield $(x,x,\succ) \in J$, what contradicts conditions *C4* and *C1*. It is neither possible to have $(x,y,\succ) \in J$ and $(y,x,\sim) \in J$. By condition *C3*, we would have $(x,y,\sim) \in J$. Together with $(x,y,\succ) \in J$, this contradicts conditions

C1. Likewise for $(y,x,\succ) \in J$ and $(x,y,\sim) \in J$. Thus, we have necessarily $(x,y,\sim) \in J$ and $(y,x,\sim) \in J$.

(4) According to the definition of $x \succsim_{R_J} y$, we see that $(x,y,\succ) \in J$ implies $x \succsim_{R_J} y$. It also implies, by condition *C1*, that we have neither $(x,y,\prec) \in J$ nor $(x,y,\sim) \in J$. So, by conditions *C2* and *C3*, we have neither $(y,x,\succ) \in J$ nor $(y,x,\sim) \in J$. Thus we have not $y \succsim_{R_J} x$.

(5) Same proof as (4) ∎

Theorem 6 *If the preorder R is compatible with Π, then there exists a unique preorder $F_{R,\Pi}$ which list of judgments corresponds exactly to the set of forced judgments under R and Π . Moreover, $F_{R,\Pi}$ satisfies Π in the sense of condition (1).*

Proof. Thanks to lemma *5*, it is enough now to prove that $F(R,\Pi)$ checks conditions *1* to *5*. Suppose there exist x,y in E and v_1, v_2 in V such that $(x,y,v_1) \in F(R,\Pi)$ and $(x,y,v_2) \in F(R,\Pi)$.For all S in $\Re(R,\Pi)$, $(x,y,v_1) \in J(S)$ and $(x,y,v_2) \in J(S)$. But a given S have exactly one judgment on a given pair (x,y). Hence $v_1 = v_2$.Thus, condition *1* is checked. In the same manner, conditions *2*, *3*, *4* and *5* are easily checked.

Now let (x,y) and (x',y') be two pairs of alternatives such that $(x,y)\Pi(x',y')$ and $x \succsim_{F_{R,\Pi}} y$. For all S in $\Re(R,\Pi)$, $F_{R,\Pi}$ being a subrelation to S, we have $x \succsim_S y$. Together with $(x,y)\Pi(x',y')$, and since S satisfies Π, we deduce $x' \succsim_S y'$. Thus $x' \succsim y'$ is a forced judgment, i.e. $x' \succsim_{F_{R,\Pi}} y'$. This shows that the preorder $F_{R,\Pi}$ satisfies Π. ∎

Notice that $F_{R,\Pi}$ is not in general complete, as the example in section 5.

4.3 Primary judgments: a draft

It can be interesting to study the relation between on the one hand R and Π, on the other hand $F_{R,\Pi}$. More precisely, it is of interest to know to what extent it is possible to reduce R to a subrelation R' without changing the preorder $F_{R,\Pi}$, i.e. with $F_{R,\Pi} = F_{R',\Pi}$? This "minimization problem" would lead to decompose the set of forced judgments under R and Π into two components: on the one hand the judgments of a minimal relation R', that could be interpreted as primary judgments of R, freed from the influence of Π; on the other hand the judgments obtained from primary judgments by application of Π. An example of a set of primary judgments is provided in section 5.

As first step, it would be of interest to answer to another question. Let r be a set of representatives of the equivalence classes of Π. In the sequel, r is called a reduction of Π. Using the correspondence π instead of Π (for the definition of π,see (2), page 6), a reduction r of π is a subset of $E \times E$, maximal for inclusion, for which there does not exist α and β in r checking $\alpha \neq \beta$ and $\beta \in \pi(\alpha)$. Suppose that R and Π check condition (1). If we know the set

$r \cap G(R)$, we can deduce the remaining forced judgments under R and π.The question is: can we reduce the "size" of the set $r \cap G(R)$ by choosing another reduction of π? Formally: does it exist a reduction r' such that $r' \cap G(R)$ is strictly included in $r \cap G(R)$?

Proposition 7 answers this question negatively, what suggests that changing the reduction cannot contribute to the determination of primary judgments.

The proof of proposition 7 is based on a partitioning of $G(R)$ in two parts, one part being $r \cap G(R)$ and the other part the image of the former by the correspondence $\widetilde{\pi} = \pi - Identity$ (see figure 1). It follows that reducing the set $r \cap G(R)$ to a smaller reduction, say $r' \cap G(R)$, would also reduce its image. Thus, we would not obtain a new partition of $G(R)$, what would have been the case if r' was a reduction of π. A contradiction.

Proposition 7 *If R and Π check condition (1), for any reduction r of π, $r \cap G(R)$ is maximal. I.e. there does not exist another reduction r' such as $r' \cap G(R)$ strictly contains $r \cap G(R)$.*

Proof. Let r be a reduction of π. Denote $\widetilde{\pi}$ the correspondence from $E \times E$ to $P(E \times E)$ defined by $\forall \alpha \in E \times E$, $\widetilde{\pi}(\alpha) = \pi(\alpha) - \{\alpha\}$. Let's show first that $r \cap G(R')$ and $\widetilde{\pi}(r \cap G(R'))$ make a partition of $G(R')$ (see figure 1).

We check easily that r and $\widetilde{\pi}(r)$ make a partition of $E \times E$. Thus $r \cap G(R)$ and $\widetilde{\pi}(r) \cap G(R)$ make a partition of $G(R)$. Let's show that $\widetilde{\pi}(r \cap G(R)) = \widetilde{\pi}(r) \cap G(R)$. By definition, $\widetilde{\pi}(r \cap G(R)) = \cup_{\alpha \in r \cap G(R)} \widetilde{\pi}(\alpha)$. Thus $\beta \in \widetilde{\pi}(r \cap G(R))$ means that there exists α in $r \cap G(R)$ such that $\beta \in \widetilde{\pi}(\alpha)$. As a result $\beta \in \widetilde{\pi}(r)$ and $\beta \in \widetilde{\pi}(G(R))$. Thus $\beta \in \widetilde{\pi}(r) \cap G(R)$. Conversely, let $\beta \in \widetilde{\pi}(r) \cap G(R)$.Thus there is $\alpha \in r$ such that $\beta \in \widetilde{\pi}(\alpha)$. π being symmetrical, we deduce that $\alpha \in \widetilde{\pi}(\beta)$. Since $\beta \in G(R)$, we have $\alpha \in \widetilde{\pi}(G(R))$. Since condition (1) on R and π implies $\widetilde{\pi}(G(R)) \subset G(R)$, we have $\alpha \in G(R)$. Thus we have $\alpha \in r \cap G(R)$ and since $\beta \in \widetilde{\pi}(\alpha)$, $\beta \in \widetilde{\pi}(r \cap G(R))$.

Let's suppose now that there exist two reductions r_1 and r_2 such that $r_2 \cap G(R)$ strictly contains $r_1 \cap G(R)$. There would exist α in $r_2 \cap G(R)$ and not in $r_1 \cap G(R)$. Since $r_1 \cap G(R)$ and $\widetilde{\pi}(r_1 \cap G(R))$ make a partition of $G(R)$, α should be in $\widetilde{\pi}(r_1 \cap G(R))$. But $r_1 \cap G(R) \subset r_2 \cap G(R)$ implies $\widetilde{\pi}(r_1 \cap G(R)) \subset \widetilde{\pi}(r_2 \cap G(R))$. Consequently α should also be in $\widetilde{\pi}(r_2 \cap G(R))$, what would be contradictory with $\alpha \in r_2 \cap G(R)$, since $r_2 \cap G(R)$ and $\widetilde{\pi}(r_2 \cap G(R))$ make a partition of $G(R)$. ■

Remark 8 *Proposition 7 requires condition (1) on R and Π. If R and Π only check the following weaker condition*

$$\forall (x,y) \text{ and } (x',y') \text{ in } E \times E, \;\; [(x,y)\Pi(x',y')] \Longrightarrow [x \succsim_R y \Longrightarrow non(x' \prec_R y')] \tag{6}$$

one could apply proposition 7 to the preorder R' defined by: $x \succsim_{R'} y$ iff there is (x',y') in $\pi(x,y)$ such that $x' \succsim_R y'$. Indeed, R' and Π obviously check

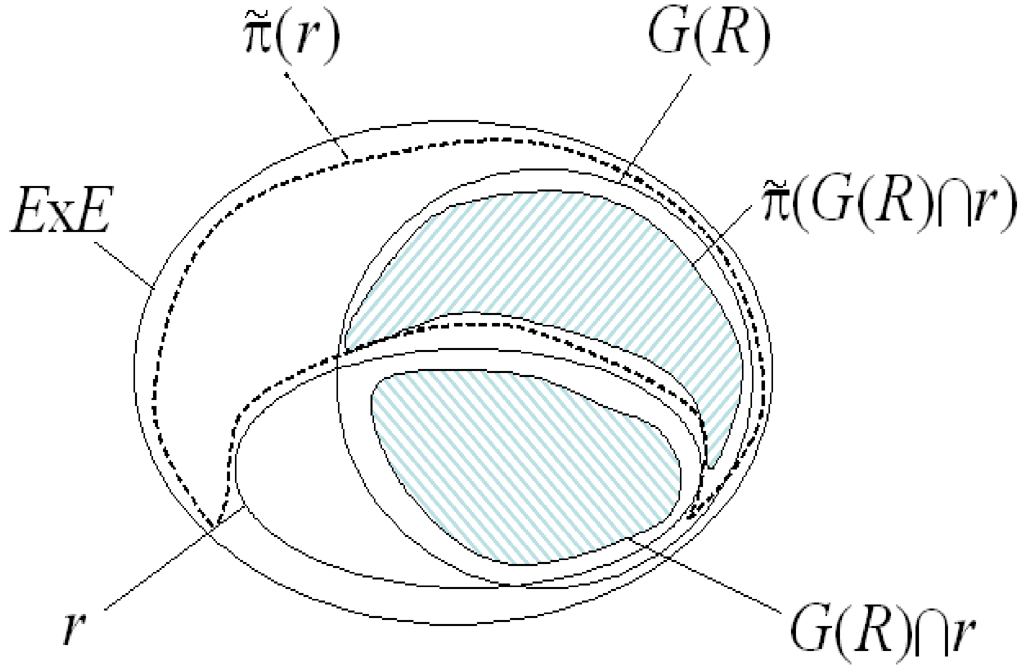

Figure 1: Partition of $G(R)$ according to $\widetilde{\pi}$

condition (1) and primary judgments are the same for R and R'. It is easily seen that condition (6) is weaker than the condition "R and Π are compatible".

4.4 Is there forced judgments without imposing an axiom ?

Section 6 deals with a situation where, even if the graph of the preorder is empty , i.e. no preorder is imposed, the correspondence induces forced judgments. On the contrary, the following proposition, which is a corollary of Szpilrajn's theorem, shows that a preorder alone never presents forced judgments apart from its own judgments.

Proposition *9* should be useful for the determination of forced preorders. Indeed, by ensuring the existence of orders having different judgments on a pair of alternatives, say (x, y), one proves that x and y are incomparable with respect to the forced preorder considered.

Proposition 9 *If R is a preorder on a set E, and if there exists $x \in E$ and $y \in E$ such hat x and y are incomparable according to R, then there exist three orders R_1, R_2 and R_3 of which R is a subrelation, such that $x \succ_{R_1} y$, $x \prec_{R_2} y$ and $x \sim_{R_3} y$.*

Proof. 1- Existence of $R_1$and R_2 : It is enough to prove the existence of R_1. The existence of R_2 would result by symmetry.

Let's consider the relation R' defined by

$$\begin{aligned} u & \sim_{R'} v \Longleftrightarrow u \sim_R v \\ u & \succ_{R'} v \Longleftrightarrow [u \succ_R v \text{ or } (u \succsim_R x \text{ and } y \succsim_R v)] \end{aligned}$$

The proof showing that R' is a preorder which completes R can be found (with the help of some minor adjustments) in the proof of Szpilrajn's theorem ([Szpilrajn 1930], lemma 1). Moreover, we have $x \succ_{R'} y$. According to Szpilrajn's theorem, there exists an order R_1 completing R'. Consequently, R_1 completes R and checks $x \succ_{R_1} y$.

2- Existence of R_3 : Let's consider the relation R"defined by

$$u \succsim_{R"} v \Longleftrightarrow [u \succsim_R v \text{ or } [(u \succsim_R x \text{ or } u \succsim_R y) \text{ and } (x \succsim_R v \text{ or } y \succsim_R v)]]$$

First let's check that R" completes R. If $v \succ_R u$, the condition

$$[(u \succsim_R x \text{ or } u \succsim_R y) \text{ and } (x \succsim_R v \text{ or } y \succsim_R v)]$$

would imply

$$[v \succ_R u \text{ and } (u \succsim_R x \text{ or } u \succsim_R y)]$$

But, by transitivity of R, ($v \succ_R u$ and $u \succsim_R x$) would yield $v \succ_R x$ and ($v \succ_R u$ and $u \succsim_R x$) would yield $v \succ_R y$. Suppose $v \succ_R x$. Let's consider the condition $(x \succsim_R v$ or $y \succsim_R v)$. With $v \succ_R x$, it would yield (by transitivity) $y \succsim_R v$. We would then have $y \succsim_R v \succ_R x$, thus $y \succ_R x$. This would contradict the assumption of incomparability between x and y. Thus, the condition $(x \succsim_R v$ or $y \succsim_R v)$ cannot be true. Symmetrically, if we suppose $v \succ_R y$, we would end in the same way to *non* $(x \succsim_R v$ or $y \succsim_R v)$. Consequently, the condition

$$[(u \succsim_R x \text{ or } u \succsim_R y) \text{ and } (x \succsim_R v \text{ or } y \succsim_R v)]$$

is not consistent with $v \succ_R u$. This shows that R" completes R.

The reflexivity of R" results from the reflexivity of R. For the transitivity, let u, v and w be such that $u \succsim_{R"} v$ and $v \succsim_{R"} w$. The 4 following cases must be considered:

a) $u \succsim_R v$ and $v \succsim_R w$: In this case, $u \succsim_R w$ thus $u \succsim_{R"} w$.

b) $u \succsim_R v$ and $[(v \succsim_R x \text{ or } v \succsim_R y) \text{ and } (x \succsim_R w \text{ or } y \succsim_R w)]$: This implies, by transitivity of R, $(u \succsim_R x$ or $u \succsim_R y)$, thus

$$[(u \succsim_R x \text{ or } u \succsim_R y) \text{ and } (x \succsim_R w \text{ or } y \succsim_R w)]$$

thus $u \succsim_{R"} w$.

c) $[(u \succsim_R x \text{ or } u \succsim_R y) \text{ and } (x \succsim_R v \text{ or } y \succsim_R v)]$ and $v \succsim_R w$: This implies, by transitivity of R, $(x \succsim_R w \text{ or } y \succsim_R w)$, thus

$$[(u \succsim_R x \text{ or } u \succsim_R y) \text{ and } (x \succsim_R w \text{ or } y \succsim_R w)]$$

thus $u \succsim_{R"} w$.

d)

$$[(u \succsim_R x \text{ or } u \succsim_R y) \text{ and } (x \succsim_R v \text{ or } y \succsim_R v)]$$

and

$$[(v \succsim_R x \text{ or } v \succsim_R y) \text{ and } (x \succsim_R w \text{ or } y \succsim_R w)]$$

This directly yields

$$[(u \succsim_R x \text{ or } u \succsim_R y) \text{ and } (x \succsim_R w \text{ or } y \succsim_R w)]$$

thus $u \succsim_{R"} w$.

Same manner as for R_1 and R_2, it is enough to complete R" by an order R_3.

■

5 A finite case

5.1 The axiom and the preorder

Consider the following set of alternatives

$$E = \{A; B; C; TA; TB\}$$

where TA and TB are respectively alternatives obtained by applying a transformation T to the two alternatives A and B. For example, this model can represent the choice among the seaside resorts A= Hammamet, B= Soussa and C= Kerkena. T is the transformation "70 km towards the south". Thus, TA indicates Akouda, TB indicates Mahdia. It is that $TTA = B$, since Soussa is located 140 km south of Hammamet. There is no seaside resorts 70 km south of Mahdia. Neither is there seaside resorts 70 km north and south of Kerkena (which is an island). Hence, TB has not an image by T and C has neither an image nor an antecedent by T.

Consider the correspondence $\widehat{\pi}$ defined on $E \times E$ by

$$\widehat{\pi}(x,y) = \left\{ \begin{array}{c} (x',y') \in E \times E / (x',y') = (Tx,Ty) \text{ or} \\ (x,y) = (Tx',Ty') \text{ or } (x,y) = (x',y') \end{array} \right\}$$

$\widehat{\pi}$ is reflexive and symmetric, but not transitive. Let π be the transitive closure of $\widehat{\pi}$,i.e.:

$$\pi(x,y) = \left\{ \begin{array}{c} (x',y') \in E \times E / \exists \text{ a sequence } (x_1,y_1) \ldots (x_n,y_n) \\ \text{such that } (x_1,y_1) = (x,y), (x_n,y_n) = (x',y') \\ \text{and } (x_i,y_i) \in \widehat{\pi}(x_{i-1},y_{i-1}), \ i = 2, \ldots n \end{array} \right\}$$

Let's calculate the correspondence π for any couple of $E \times E$. Notice that $(x', y') \in \pi(x, y) \Longleftrightarrow (y', x') \in \pi(y, x)$, so that following calculations determine π completely:

$\pi(A, B) = \{(A, B); (TA, TB)\}$
$\pi(A, TB) = \{(A, TB)\}$
$\pi(A, TA) = \{(A, TA); (TA, B); (B, TB)\}$
$\pi(A, A) = \{(A, A); (TA, TA); (B, B); (TB, TB)\}$
$\pi(A, C) = \{(A, C)\}$
$\pi(B, C) = \{(B, C)\}$
$\pi(TA, C) = \{(TA, C)\}$
$\pi(TB, C) = \{(TB, C)\}$
$\pi(C, C) = \{(C, C)\}$

We can see that a reduction (i.e. a set of representatives of the equivalence classes of π, see 3.c) of π is

$$r = r_1 \cup r_2 \cup r_3$$

where

$$\begin{aligned} r_1 &= \{(A, B); (A, TB); (A, TA); (A, C); (B, C); (TA, C); (TB, C)\} \\ r_2 &= \{(B, A); (TB, A); (TA, A); (C, A); (C, B); (C, TA); (C, TB)\} \\ r_3 &= \{(A, A); (C, C)\} \end{aligned}$$

Suppose that it is considered desirable that π is satisfied, what amounts saying that if one prefers x to y, one must also prefer Tx to Ty.

Consider the following preorder R defined by his graph .

$$G(R) = \{(A, A); (B, B); (TB, TB); (TA, TA); (C, C); (A, TA); (TA, B); (C, TA)\}$$

We exhibit the forced judgments under R and π, the sets $\Re(R, \pi)$ and $F(R, \Pi)$, the preorder $F_{R,\Pi}$ and the primary judgments.

5.2 Forced judgments

Observe that R does not satisfy π since, for example, $(B, TB) \in \pi(A, TA)$ though the judgments of R on (B, TB) and (A, TA) are not similar.

Any order S in $\Re(R, \pi)$ judges necessarily : $A \succ TA$, $TA \succ B$ and $B \succ TB$. Indeed, $(A, TA, \succ) \in J(R)$ implies $(A, TA, \succ) \in J(S)$. Since S satisfies π, for all $(x, y) \in \pi(A, TA)$ we have $(x, y, \succ) \in J(S)$. Consequently, $(A, TA, \succ)$, $(TA, B, \succ)$ and $(B, TB, \succ)$ are in $F(R, \pi)$. These judgments can be written

$$A \succ TA \succ B \succ TB$$

Consequently, the transitive and reflexive closure of the set of judgments $A \succ TA \succ B \succ TB$ is a subrelation to every order S in $\Re(R, \pi)$. Obviously, $\Re(R, \pi)$ is not empty. For example, the transitive and reflexive closure of the set of judgments $C \succ A \succ TA \succ B \succ TB$ is an element of $\Re(R, \pi)$ .Thus R and

π are compatible. According to theorem *6*, the set $F(R,\pi)$ defines a preorder $F_{R,\pi}$ that satisfies π. Hence, the transitive and reflexive closure of the set of judgments $A \succ TA \succ B \succ TB$ is a subrelation to $F_{R,\pi}$. It is easily checked that it corresponds in fact exactly to $F_{R,\pi}$. $\Re(R,\pi)$ is then the set of orders on E admitting $F_{R,\pi}$ as subrelation and satisfying π.

As it can be seen, $F_{R,\pi}$ is not complete. It judges C incomparable with the other alternatives. Hence, arguing for the interpretation of incomparability suggested in the introduction does not involve requiring completeness. However, $F_{R,\pi}$ displays less incomparability than R. For instance, $F_{R,\pi}$ judges $A \succ B$ whereas A and B are incomparable according to R. Observe that if C is removed from E, all pairs of alternatives would become comparable according to $F_{R,\pi}$. Thus, $F_{R,\pi}$ would become an order and $\Re(R,\pi) = \{F_{R,\pi}\}$. In this case, there would be no more room for incomparability.

5.3 Primary judgments

We have $r \cap G(R) = \{(A,A)\,;(C,C)\,;(A,TA)\,;(C,TA)\}$. Consider the extension of R by the preorder R' as in remark 8. It can be checked that it is not possible to decrease the set $r \cap G(R')$ by choosing another reduction. Proposition 7 shows that this holds in the general case. Primary judgments are to be sought among judgments $(A,TA,\succ)$,$(TA,A,\prec)$, $(C,TA,\succ)$, $(TA,C,\prec)$, $(A,A,\sim)$ and $(C,C,\sim)$, since the other forced judgments under R and π can be obtained by applying π. In fact we can check that these judgments form a set of primary judgments. That is, if we remove one of them, the set of forced judgments under the relation thus defined and π, would be different from the set of forced judgments under R and π.

6 The *relative anonymity* axiom

6.1 The finite dimension case

I seek to study the forced judgments emanating from the axiom *relative anonymity* (see definition further) in the context of intergenerational choice. The consequences of this axiom were studied in [d'Aspremont-Gevers 1977] in the context of a social welfare functional and with a finite number of individuals. The authors established (lemma 4) the implication *relative anonymity*[3] $\Longrightarrow$ *finite anonymity.*

First, I present the assumptions of the lemma followed by the lemma. Then, within the framework of formal welfarism, which is a usual assumption in intergenerational choice theory, I give the translation of the lemma in term of social order instead of social welfare functional. In the next subsection, I extend the

[3] This property is called anonymity in [d'Aspremont-Gevers 1977].

lemma to infinite populations. That will be used to highlight forced judgments induced by relative anonymity in the case of infinite population.

In [d'Aspremont-Gevers 1977], the set E of the alternatives is unspecified. The population is composed of n individuals. U is the set of bounded functions from E to $\mathbb{R}^n$ ($\mathbb{R}$ denotes the real line) and f a social welfare functional which associates to each $u \in U$ an order on E denoted R_u. $u(x)_i$ is the n^{th} component of $u(x)$. f may satisfy the following properties. Recall that $x \succsim_R y$ means "x is preferred or indifferent to y according to a relation R".

independence of irrelevant alternatives: For all u_1, u_2 in U, and x, y in E

$$\begin{aligned} & [u_1(x) = u_2(x) \text{ and } u_1(y) = u_2(y)] \\ \Longrightarrow \quad & R_{u_1} \text{ and } R_{u_2} \text{ have the same judgment on } (x, y) \end{aligned}$$

strong Pareto: $\forall x, y$ in E, $\forall u$ in U, $x \succsim_{R_u} y$ if $\forall i$ in $\{1,..,n\}$, $u(x)_i \geq u(y)_i$. If moreover $\exists j$ in $\{1,..,n\}$ such that $u(x)_j > u(y)_j$, then $x \succ_{R_u} y$.

relative anonymity: For all permutation σ on $\{1,..,n\}$, if u_1 and u_2 in U are such that $\forall i$ in $\{1,..,n\}$ and $\forall x$ in E we have $u_1(x)_i = u_2(x)_{\sigma(i)}$, then $R_{u_1} = R_{u_2}$.

Proposition 10 *([d'Aspremont-Gevers 1977], lemma 4): If f satisfies to independence of irrelevant alternatives, strong Pareto and relative anonymity, then $\forall u$ in U, $\forall x, y$ in E, if x and y are such that there exists a permutation σ on $\{1,..,n\}$ such that $\forall i$ in $\{1,..,n\}$ we have $u(x)_i = u(y)_{\sigma(i)}$, then $x \sim_{R_u} y$.*

The assumptions *independence of irrelevant alternatives* and *strong Pareto* are only used to guarantee what of [d'Aspremont-Gevers 2002] call *formal welfarism.* That means that "*the goodness of a state of affairs can be judged entirely by the goodness of the utilities in that state*", according to [Sen 1980] quoted by [d'Aspremont-Gevers 2002]. In the context of this section, we are from the start within the framework of *formal welfarism.* This makes it possible to regard utility streams as alternatives. E, the set of alternatives, becomes $\mathbb{R}^n$. The properties *strong Pareto* and *relative anonymity* of the *social welfare functional* are inherited by the image-order R on $\mathbb{R}^n$ as it will be specified. The assumption *independence of irrelevant alternatives* is automatically checked. Here is the translation of the properties and the lemma in this context[4].

strong Pareto: $\forall x, y$ in $\mathbb{R}^n$, $x \succsim_R y$ if $\forall i$ in $\{1,..,n\}$, $x_i \geq y_i$. if moreover $\exists j$ in $\{1,..,n\}$ such that $x_j > y_j$, then $x \succ_R y$.

relative anonymity: For all permutation σ on $\{1,..,n\}$, $\forall i$ in $\{1,..,n\}$ and $\forall x, y$ in $\mathbb{R}^n$ we have $x \succsim_R y \Longrightarrow \sigma(x) \succsim_R \sigma(y)$, where $\sigma(x)$ is obtained by permuting the components of x according to σ.

The lemma's translation is:

[4] To arrive at this translation starting from R, it is enough to consider the social welfare functional which associates to a bounded function from $\mathbb{R}^n$ to $\mathbb{R}^n$, denoted u, the order R_u defined by $x \succsim_{R_u} y$ iff $u(x) \succsim_R u(y)$.

Proposition 11 *If R is an order on $\mathbb{R}^n$ satisfying strong Pareto and relative anonymity, for all permutation σ on $\{1,..,n\}$ and $\forall x$ in $\mathbb{R}^n$ we have $x \sim_R \sigma(x)$.*

Condition "$\forall x$ in $\mathbb{R}^n$: $x \sim_R \sigma(x)$" is called the *anonymity* condition. If x has an infinity of components and if σ permutes only a finite number of components, the condition takes the name of *finite anonymity.*

6.2 The infinite dimension case

I now seek to extend proposition *11* to a set of alternatives made up of all infinite real sequences, or streams. This is carried out by theorem *15*.

In order to facilitate the reading for those who are not interested in the concept of forced judgments, statements having recourse to this concept (namely theorem *15*, lemma *17* and corollary *18*) will be followed of a translation free from this concept.

Before stating theorem *15*, it is necessary to give definition *12*, definition *13* and proposition *14* which connects these definitions.

Denote the set of all infinite real sequences $\mathbb{R}^{\mathbb{N}^*}$, where $\mathbb{N}^*$ is the set of positive integers. Denote σ a permutation on $\mathbb{N}^*$. In what follows permutations are on $\mathbb{N}^*$. To reduce the notations, write $\sigma(k)$ for the image of an integer k by σ and also $\sigma(x)$ for the vector of $\mathbb{R}^{\mathbb{N}^*}$ obtained by permuting the components of x according to σ.

Definition 12 *Periodic permutation: A permutation σ is periodic iff there exists an integer $n \geq 1$ such that $\sigma^n = identity$.*

Definition 13 *Fixed step permutation: A permutation σ on $\mathbb{N}^*$ is said to be fixed step iff there exists a partition of $\mathbb{N}^*$: N_1, N_2...such that $\forall i, j, |N_i| = |N_j|$ and σ can be written as a composition of permutations $\sigma_1 \circ \sigma_2 \circ$ ...where for all i and j such that $j \neq i$, σ_i leaves invariant all the elements of N_j.*

Definition *12* is more demanding than the condition of cyclicity met in the literature ([Mitra-Basu 2007] or [Lauwers 2007]). In other words, any periodic permutation is cyclic, but not the converse. Definition *13* results directly from the definitions found, among others, in [Asheim-d'Aspremont-Banerjee 2008]. All the σ_i are in fact finite permutations on $\{1,..,p\}$ where $p = |N_i|$.

Proposition 14 *Every fixed step permutation σ is periodic.*

Proof. For all finite permutation σ_i on $\{1,..,p\}$ and for all k in $\{1,..,p\}$, there exists an integer p_i such that $\sigma_i^{p_i}(k) = k$. Moreover, it is necessary to have $p_i \leq p$, otherwise $\sigma_i^n(k)$ would go out of $\{1,..,p\}$ for some n. It is deduced that $\sigma_i^{p!} =$ identity, $\forall \sigma_i$. Since composition between σ_i and σ_j is commutative because they operate on disjoint subsets of $\mathbb{N}^*$, we have

$$\begin{aligned} \sigma^{p!} &= (\sigma_1 \circ \sigma_2 \circ ...)^{p!} \\ &= \sigma_1^{p!} \circ \sigma_2^{p!} \circ ... = \text{identity} \end{aligned}$$

■

The set of fixed step permutations, equipped with composition, is a group ([Mitra-Basu 2007], [Lauwers 2007]). But the set of periodic permutations is not a group. For example let us consider the following permutations

$$\pi_1 : (1,2)\,(3,4)\,...\,(2p-1,2p)\,...$$

and

$$\pi_2 : (1)\,(2,3)\,(4,5)\,...\,(2p,2p+1)\,...$$

π_1 and π_2 are periodic. But $\pi_3 = \pi_1 \circ \pi_2$ is not (see [Lauwers 2007] for more details on π_1, π_2 and π_3).

Denote $P(\mathbb{R}^{\mathbb{N}^*})$ the set of subsets of $\mathbb{R}^{\mathbb{N}^*}$ and $P(\mathbb{R}^{\mathbb{N}^*} \times \mathbb{R}^{\mathbb{N}^*})$ the set of subsets of $\mathbb{R}^{\mathbb{N}^*} \times \mathbb{R}^{\mathbb{N}^*}$.

Let's consider now the correspondence π_{fsra} :

$$\begin{aligned} \mathbb{R}^{\mathbb{N}^*} \times \mathbb{R}^{\mathbb{N}^*} &\rightarrow P\left(\mathbb{R}^{\mathbb{N}^*} \times \mathbb{R}^{\mathbb{N}^*}\right) \\ (x,y) &\rightarrow \{(x',y')\,/\exists \sigma \text{ fixed step permutation}, x' = \sigma(x) \text{ and } y' = \sigma(y)\} \end{aligned} \tag{7}$$

π_{fsra} is an equivalence correspondence because the set of fixed step permutations, equipped with composition, is a group.

Let σ be a permutation. Denote π_σ the correspondence

$$\begin{aligned} \mathbb{R}^{\mathbb{N}^*} \times \mathbb{R}^{\mathbb{N}^*} &\rightarrow P\left(\mathbb{R}^{\mathbb{N}^*} \times \mathbb{R}^{\mathbb{N}^*}\right) \\ (x,y) &\rightarrow \{(x',y')\,/\exists n \in \mathbb{Z},\, x' = \sigma^n(x) \text{ and } y' = \sigma^n(y)\} \\ & \quad\ \text{where } \mathbb{Z} \text{ is the set of integers} \end{aligned}$$

Not to weigh down the notations, denote also π_σ the correspondence:

$$\begin{aligned} \mathbb{R}^{\mathbb{N}^*} &\rightarrow P\left(\mathbb{R}^{\mathbb{N}^*}\right) \\ x &\rightarrow \{x'/\exists n \in \mathbb{Z},\, x' = \sigma^n(x)\} \end{aligned}$$

We can check that it is an equivalence correspondence.

We are now ready to state the generalization of proposition *11*:

Theorem 15 *σ is periodic iff for all x in $\mathbb{R}^{\mathbb{N}^*}$, the judgment on $(x, \sigma(x))$ is forced indifference under π_σ.*

Theorem 15 without forced judgments: σ is periodic iff for all x in $\mathbb{R}^{\mathbb{N}^*}$and for all orders on $\mathbb{R}^{\mathbb{N}^*}$ satisfying π_σ, x is indifferent to $\sigma(x)$.

Theorem *15* offers to some extent a characterization of periodic permutations. To establish theorem *15*, I show initially that a necessary and sufficient condition for a permutation σ to be periodic, is that the sequence of successive images of any x in $\mathbb{R}^{\mathbb{N}^*}$ has only a finite number of terms (lemma *16*). Then I show that this last condition is also necessary and sufficient so that the judgment between any element of $\mathbb{R}^{\mathbb{N}^*}$ and its image by σ is forced indifference under π_σ (lemma *17*).

Lemma 16 *A permutation σ is periodic iff $\forall x \in \mathbb{R}^{\mathbb{N}^*}$, the sequence*

$$x, \sigma(x), \sigma^2(x), \sigma^3(x) ..., \sigma^n(x) ...$$

takes a finite number of values.

Proof. Sufficiency is evident. Let's suppose that σ is not periodic and show that the condition

$$\forall x, \text{ the sequence } x, \sigma(x), \sigma^2(x), \sigma^3(x) ..., \sigma^n(x) ... \text{is finite}$$

does not hold.

σ non periodic writes : $\forall n \geq 1$, $\exists k \in \mathbb{N}^*$ such that $\sigma^n(k) \neq k$. Denote one of these k, $k(n)$. Take $x = (1, 2, 3, 4...)$. Let i and j be two nonnegative integers such that $i > j$. We have $\sigma^i(x) = \sigma^{i-j}(\sigma^j(x))$. Denote $i - j = n$. For all y in $\mathbb{R}^{\mathbb{N}^*}$, the $k(n)^{th}$ component of y does not return in its place in $\sigma^n(y)$ since $\sigma^n(k(n)) \neq k(n)$. Take $y = \sigma^j(x)$. As all the components of y are pairwise different, we have necessarily $\sigma^n(y) \neq y$. Thus $\sigma^i(x) \neq \sigma^j(x)$.The sequence $x, \sigma(x), \sigma^2(x), \sigma^3(x) ..., \sigma^n(x)$ ...is thus infinite. ■

Lemma 17 *Let $x \in \mathbb{R}^{\mathbb{N}^*}$ and σ be a permutation. Then, the judgment on $(x, \sigma(x))$ is forced indifference under π_σ iff the sequence*

$$x, \sigma(x), \sigma^2(x), \sigma^3(x) ..., \sigma^n(x) ...$$

takes a finite number of values.

Lemma 17 without forced judgments: Let $x \in \mathbb{R}^{\mathbb{N}^*}$ and σ be a permutation. Then, x is indifferent to $\sigma(x)$ for any order R on $\mathbb{R}^{\mathbb{N}^*}$ which satisfies π_σ, iff the sequence

$$x, \sigma(x), \sigma^2(x), \sigma^3(x) ..., \sigma^n(x) ...$$

takes a finite number of values.

Proof. If the sequence takes a finite number of values, there exist two different nonnegative integers n, m such that $\sigma^n(x) = \sigma^m(x)$. Let us suppose $n > m$. Let R be an order which satisfies π_σ. Let j be the judgment of R on $(x, \sigma(x))$. j is equal either to $\succ$ or $\prec$ or $\sim$. Since R satisfies π_σ, R will have the same judgment on $\left(\sigma(x), \sigma^2(x)\right)$ and on

$$\left(\sigma^2(x), \sigma^3(x)\right) ... \left(\sigma^m(x), \sigma^{m+1}(x)\right) ... \left(\sigma^{n-1}(x), \sigma^n(x)\right) = \left(\sigma^{n-1}(x), \sigma^m(x)\right)$$

The transitivity of R imposes that this judgment can only be indifference.

Conversely, Suppose that the terms of the sequence

$$x, \sigma(x), \sigma^2(x), \sigma^3(x) ..., \sigma^n(x) ...$$

are pairwise different. Denote

$$X = \left\{x, \sigma(x), \sigma^2(x), \sigma^3(x) ..., \sigma^n(x) ...\right\}$$

and

$$\pi_\sigma(X) = \cup_{y \in X} \pi_\sigma(y)$$

Notice first that, because of the transitivity of π_σ, $\pi_\sigma(X)$ and $\overline{\pi_\sigma(X)}$ (the complementary of $\pi_\sigma(X)$ in $\mathbb{R}^{\mathbb{N}^*}$) are stable by π_σ. Moreover, for each z in $\pi_\sigma(X)$, there exists a unique i in $\mathbb{Z}$ such that $z = \sigma^i(x)$. Denote $i(z)$ that integer.

Then, define the following order R_1:

· For (y, z) in $\pi_\sigma(X) \times \pi_\sigma(X)$, $y \succsim_{R_1} z$ iff $i(y) \leq i(z)$.
· For (y, z) in $\overline{\pi_\sigma(X)} \times \pi_\sigma(X)$, $y \succ_{R_1} z$.
· For (y, z) in $\overline{\pi_\sigma(X)} \times \overline{\pi_\sigma(X)}$, $y \sim_{R_1} z$.

We see that R_1 is complete, reflexive, transitive and that it satisfies π_σ. Moreover, we have $x \succ_{R_1} \sigma(x)$. Symmetrically, build R_2 in such a way that it coincide with R_1 on $\overline{\pi_\sigma(X)} \times \pi_\sigma(X)$ and $\overline{\pi_\sigma(X)} \times \overline{\pi_\sigma(X)}$ with, for all (y, z) in $\pi_\sigma(X) \times \pi_\sigma(X)$, $y \succsim_{R_2} z$ iff $i(z) \leq i(y)$. We see that R_2 establishes the judgment: $x \prec_{R_2} \sigma(x)$. Since there are two possible different judgments on $(x, \sigma(x))$ by orders which satisfy π_σ, there is no forced judgment on $(x, \sigma(x))$ under π_σ. ■

Proof. *(of theorem 15)*: It suffices to join lemma *16* and lemma *17*. ■

Since any fixed step permutation is periodic (proposition *14*), we deduce from theorem *15*:

Corollary 18 *Under π_{fsra} (for the definition of π_{fsra}, see (7), page 18), for all x in $\mathbb{R}^{\mathbb{N}^*}$ and σ fixed step permutation, the judgment on $(x, \sigma(x))$ is forced indifference.*

Corollary 18 without forced judgments: For any order on $\mathbb{R}^{\mathbb{N}^*}$ satisfying π_{fsra} , any x in $\mathbb{R}^{\mathbb{N}^*}$ and any fixed step permutation σ, x is indifferent to $\sigma(x)$.

A preorder R which satisfies π_{fsra} is said to be *fixed step relatively anonymous.* The indifference between x and a *fixed step permuted* of x is known as *fixed step anonymity.*

Corollary *18* generalizes proposition *11* on the one hand by extending it to infinite dimension, on the other hand by not requiring the condition *strong Pareto.* Also let's notice that it requires the preorder to satisfy only *fixed step relative anonymity*, weaker than *strong relative anonymity* (which means satisfying π_σ for any permutation σ). It offers the property of *fixed step anonymity*, stronger than *finite anonymity.* In the finite setting, there is no difference between *fixed step relative anonymity* and *strong relative anonymity* nor between *fixed step anonymity* and *finite anonymity.*

7 The incompatibility between *fixed step relative anonymity* and *stationarity*

In this section, we need to use the axioms *finite Pareto* and *stationarity. Finite Pareto* is weaker than *strong Pareto.* It states that if there is a finite number of generations that are better off in x compared to y, and if the remaining generations have the same situations in both streams, then x should be preferred to y. *Stationarity* stipulates that adding the same first component to two streams should not change the judgment on these two streams. I refer to [Asheim-d'Aspremont-Banerjee 2008] for formal definitions of these two axioms (section 2.3).

In [Asheim-d'Aspremont-Banerjee 2008], it is noticed that the axioms *fixed step anonymity, finite Pareto* and *stationarity* are incompatible. For example, both following streams are indifferent under the terms of *fixed step anonymity*:

$$\begin{array}{l} x = 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad .. \\ y = 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad .. \end{array}$$

According to *stationarity* and starting from the previous example, one can deduce that both following streams should be also indifferent:

$$\begin{array}{l} y = 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad .. \\ z = 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad .. \end{array}$$

But, according to *finite Pareto*,

$$x = 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad ..$$

strictly dominate

$$z = 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad ..$$

Consequently, in the present circumstance, the judgment established by *fixed step anonymity* and *stationarity* is inconsistent with the judgment established by *finite Pareto.* Since *finite Pareto* is generally seen as the most uncontroversial among these three axioms, the example demonstrates the incompatibility of *fixed step anonymity* and *stationarity.*

Moreover, [Asheim-d'Aspremont-Banerjee 2008] show that it is possible to build a preorder they call the *generalized time-invariant overtaking* so that it checks *Finite Pareto, stationarity* and *fixed step relative anonymity*

([Asheim-d'Aspremont-Banerjee 2008], definition 2 and theorem 1). Hence, loosing *fixed step anonymity* to the weaker axiom *fixed step relative anonymity* may appear as a mean to solve the incompatibility between *fixed step anonymity* and *stationarity.*

However, the cost of satisfying simultaneously *fixed step anonymity* and *stationarity* is additional incomparability, compared with more traditional preorders such as the *catching up* or the *fixed-step catching up* (definitions in [Asheim-d'Aspremont-Banerjee 2008], section 4). For example, the *generalized time-invariant overtaking* declares the streams x and y to be incomparable whereas for the *catching up* x is better than y and for the *fixed-step catching up* x and y are indifferent.

If we stick to the interpretation of incomparability suggested in the introduction, such a judgment of incomparability is not possible while claiming to satisfy *fixed step relative anonymity.* Indeed, corollary *18* affirms that there is not a real difference between *fixed step relative anonymity* and *fixed step anonymity.* In other words, to affirm that the preferences satisfy *fixed step relative anonymity* amounts to affirm that they satisfy *fixed step anonymity.* Recall that the suggested interpretation of incomparability implies that the preferences are represented by the forced preorder, not merely by the preorder we chose to start with. One might restate that in the form of an axiom of comparability:

Axiom of comparability: If the preferences are to satisfy a preorder R and an axiom Π, then they satisfy $F_{R,\Pi}$.

Under this axiom, it cannot exist preferences satisfying *finite Pareto* and satisfying at the same time *fixed step relative anonymity* and *stationarity.* It is thus necessary to choose one of these two axioms.

If we choose *fixed step relative anonymity*, preferences are then represented by $F_{R,\textit{fixed step relative anonymity}}$. Assume that the preorder R and *fixed step* *relative anonymity* are compatible[5], that is,

$$\Re\left(R, \textit{fixed step relative anonymity}\right) \neq \varnothing$$

and that R checks *finite Pareto.*

We then have

$$x \sim_{F_{R,\textit{fixed step relative anonymity}}} y \tag{8}$$

Finite Pareto implies

$$x \succ_R z$$

thus, we also have

$$x \succ_{F_{R,\textit{fixed step relative anonymity}}} z \tag{9}$$

(8) and (9) imply

$$y \succ_{F_{R,\textit{fixed step relative anonymity}}} z \tag{10}$$

(8) and (10) imply that $F_{R,\textit{fixed step relative anonymity}}$ does not check *stationarity.*

Now let's choose *stationarity.* Assume that the preorder R and *stationarity* are compatible and that R checks *finite Pareto.* The previous example with $F_{R,\textit{fixed step relative anonymity}}$ shows that there cannot be indifference between x and y.

Suppose we had

$$y \succ_{F_{R,stationarity}} x$$

By *stationarity*, this would give

$$z \succ_{F_{R,stationarity}} y$$

Thus, by transitivity

$$z \succ_{F_{R,stationarity}} x$$

what would violate *finite Pareto.* Therefore, x could only be either better than or incomparable with y with respect to $F_{R,stationarity}$. But saying this amounts to say that "x is better than y" is a forced judgment under R and *stationarity.* Consequently, we have necessarily

$$x \succ_{F_{R,stationarity}} y \tag{11}$$

Thus, $F_{R,stationarity}$ does not check *fixed step relative anonymity.* Observe that (11) shows that *finite Pareto* and *stationarity* entail a certain form of impatience (suggested by [van Liedekerke-Lauwers 1997]).

Hence, even if the preorder R satisfies simultaneously *fixed step relative anonymity* and *stationarity,* as it is the case for the *generalized time-invariant*

[5] As it is not central to the argument, the problem of existence of a preorder R *finite Pareto* and compatible with either *fixed step relative anonymity* or *stationarity,* is not adressed here. However, it can be proved that the *generalized time-invariant overtaking* is *finite Pareto* and compatible with *fixed step relative anonymity* and a discounted-sum with constant discount-rate is an example of an order *finite Pareto* and compatible with *stationarity.*

overtaking, this should not be interpreted (if we stick to the interpretation of incomparability suggested in the introduction) as if the preferences induced by R were satisfying these two axioms simultaneously.

Now, suppose that R satisfies simultaneously *fixed step relative anonymity* and *stationarity* and that $F_{R,stationarity}$ is seen as more plausible to represent preferences. This amounts to choose *stationarity* over *fixed step relative anonymity* . We saw that in this case the preferences do not satisfy *fixed step relative anonymity.* How then to interpret the fact that R satisfies *fixed step relative anonymity*? Can't one deduce that the preferences represented by $F_{R,stationarity}$ recover some virtue from that fact?

Actually, the preferences represented by $F_{R,stationarity}$ satisfy *fixed step relative anonymity* only on part of the set of pairs of alternatives (to which (x, y) does not belong, although (x, y) is in $G(F_{R,stationarity})$). One could call that *partial fixed step relative anonymity.* Thus, the question may be reformulated as follows: Does *partial fixed step relative anonymity* entails, for preferences, some valuable kind of equity?

Let's consider a simpler example. Consider the following set of alternatives

$$E = \{(4, 2); (2, 4); (3, 2); (2, 3)\}$$

and the social welfare function defined by $f(x) = 3x_1 + x_2$.

f satisfies *relative anonymity* on the following subset of $E \times E$:

$$\{[(4, 2), (3, 2)]; [(2, 4), (2, 3)]; [(2, 3), (2, 4)]; [(3, 2), (4, 2)]\}$$

Although this kind of equity is undoubtedly better than no equity at all, we cannot however deduce from it that f shows a great sense of equity. Indeed, it is seen for example that f prefers $(3, 2)$ over $(2, 4)$. Thus, like $F_{R,stationarity}$, f displays impatience, although, admittedly, the impatience of f is much sharper than that of $F_{R,stationarity}$.

To sum up: The preferences induced by R and *stationarity* display impatience. Assigning to (x, y) the value "is incomparable with" instead of "is preferred to", whereas the only underlying judgment consistent with R and *stationarity* is "x is preferred to y", cannot, I suppose, make preferences more equitable.

Nevertheless, preferences induced by R and *stationarity* check *fixed step relative anonymity* on some pairs of alternatives. This is valuable insofar as it can be considered valuable to check an axiom on part of the domain.

8 Concluding remarks

The concept of forced judgment developed in this paper could facilitate the selection of the desirable preorder. It guarantees a certain intuitive value for the judgments if certain axioms consisting of equivalence relations on the Cartesian product of the set of alternatives by itself, are adopted. This concept proved useful in showing the proximity between *relative anonymity* and *anonymity*.

As future research, it would seem interesting to apply this concept for the axiom *invariance with respect to individual change of origin*, asserting the invariance of the ranking of two utility streams if one applies the same translation to both. Indeed, [d'Aspremont-Gevers 2002] give a theorem (theorem 17) which can be interpreted as stating the existence of forced judgments when alternatives are "separated" by a given hyperplane. However, there would remain some steps to be crossed. Indeed, whereas the associated correspondence is $\pi_{inv(a_i+u_i)}$:

$$\begin{array}{rcl} \mathbb{R}^n \times \mathbb{R}^n & \rightarrow & P\left(\mathbb{R}^n \times \mathbb{R}^n\right) \\ (x, y) & \rightarrow & \{(x', y') \,/ \exists a \text{ in } \mathbb{R}^n \text{ such } x' = x + a \text{ and } \ y' = y + a\} \end{array}$$

it remains to determine a set of primary judgments, i.e. a minimal preorder R on $\mathbb{R}^n$ that would generate, joined with $\pi_{inv(a_i+u_i)}$, the set of orders characterized by theorem 17 of [d'Aspremont-Gevers 2002].

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