On the system-theoretical foundations of non-economic parameter constancy assumptions in economic growth modeling

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14 November 2017

On the System-Theoretical Foundations of Non-Economic Parameter Constancy Assumptions in Economic Growth Modeling

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14th November 2017

Abstract. In general, positive/quantitative growth models assume that (some of) the model parameters that are determined in non-economic systems are exogenous and constant. Such non-economic parameter constancy assumptions (abbr. 'NEPCAs') are not necessarily consistent with the empirical evidence on significant cross-system interactions and, in particular, long-run interactions between the economic system and the non-economic systems (e.g. socio-cultural, political, and ecological system). We derive the system-theoretical/mathematical conditions under which NEPCAs are good approximations of cross-system interactions in economic growth models: we (a) discuss the standard types of dynamic equilibrium and the problems that arise when using them to justify NEPCAs in economic long-run models (in presence of cross-system interactions), (b) formulate an equilibrium type (a 'stable partial dynamic equilibrium') that solves these problems, and (c) demonstrate the applicability of this equilibrium type as a foundation of the NEPCAs used in the AK growth model. Finally, we discuss some topics for further research.

Keywords. Economic growth, long run, parameter conditions, cross-system interactions, economic system, socio-cultural system, political system, ecological system, dynamic systems theory, dynamic equilibrium, AK model, population growth.

JEL Codes. O40, A12.

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1. Introduction

Over the last decades, economic growth theory has reoriented towards quantitative, positive, and predictive models that can ‘reproduce’ the observed quantitative characteristics of the long-run dynamics of economic variables while assuming that some ‘non-economic parameters’ are constant. These ‘non-economic parameters’ are the exogenous model parameters (e.g., time-preference/savings rate, population growth rate, and depreciation rate) that are primarily determined in non-economic systems (e.g., in the socio-cultural, political, and ecological system). Major examples of such a model are (the positive interpretations of) the Solow (1956) model and the Ramsey-(1928)-Cass-(1965)-Koopmans-(1967) model, which are the basis for numerous (positive/quantitative) growth and development models.

The ‘non-economic parameter constancy assumptions’ (abbr. ‘NEPCAs’) described above are associated with a major problem. According to the empirical evidence, there are interactions between the economic system and the non-economic systems.[1] Such cross-system interactions imply that although the non-economic parameters are determined in non-economic systems, they are not necessarily independent of economic system dynamics, since economic system dynamics may have an effect on non-economic systems that leads to a change of the non-economic parameters of the economic model/system.[2] [3] That is, the assumption that the non-economic parameters of the positive/quantitative economic growth models are exogenous or constant is not necessarily consistent with the empirical evidence on cross-system interactions; for example, the explanatory or predictive validity of the

[1] In particular, the literature has studied the interactions between specific economic variables/systems and specific non-economic variables/systems. See, e.g., (a) Bourguignon (2005) on the impact of economic development on social structures, (b) Alesina and Giuliano (2015) on the impact of culture on economic development (via institutions), (c) Acemoglu et al. (2001) on the effect of institutions on per-capita income, (d) Acemoglu et al. (2001), Alesina and Fuchs-Schündeln (2007), and Fuchs-Schündeln and Paolo (2016) on the effect of the political system (colonial origin, socialism) on the economic system (via impacts on institutions, preferences, and education), and (e) the literature on the impact of economic development on democracy and vice versa (e.g., Huber et al. (1993) and Persson and Tabellini (2006)). Similar literature can be found on the linkages between the economic and the ecological system via, e.g., pollution and resource depletion (see, e.g., Brock and Taylor (2005) for an overview and Kollenbach (2015, 2017) for a recent theoretical contribution to this topic).

[2] For example, the positive interpretation of the Ramsey-(1928)-Cass-(1965)-Koopmans-(1967) model seeks to explain, among others, the observable long-run economic dynamics (among others Kaldor’s stylized facts of economic growth) by relying on capital accumulation and exogenous technological progress while assuming that several parameters (e.g., time-preference rate, depreciation rate, intertemporal elasticity of substitution, and elasticity of substitution between capital and labor) are constant. In general, it can be assumed that, e.g., the time-preference rate depends, among others, on the socio-cultural characteristics of the society/economy and that in the long run, technological progress and income growth have an impact on the socio-cultural system (cf. Section 4).

[3] This problem is less important in the case of normative growth models, which, simply speaking, generate conditional statements of the following kind: “Given a preference and technology structure of the type…, the optimal growth path is characterized by…”.

positive/quantitative economic long-run models that are based on NEPCAs may be restricted/biased in presence of cross-system interactions.

This discussion challenges the common practice of basing economic growth models on NEPCAs and questions whether we need to change the approach to long-run economic dynamics modeling by focusing on *large-scale (interdisciplinary) models that endogenize 'all' NEPCAs* or by searching for empirical evidence on the various NEPCAs that are used in standard economic growth models ('*empirical foundations of NEPCAs*').[4] Before shifting towards such time-intensive, complex, and model-specific research methods, it makes sense to analyze more exactly the implications of cross-system interactions for NEPCAs in general. As we will see, cross-system interactions may but need not necessarily imply that NEPCAs are inadequate. In particular, in some types of dynamical system, NEPCAs are a good approximation of cross-system linkages (even if there are interactions between the systems). Thus, it seems interesting to discuss under which (system-theoretical) conditions are NEPCAs good approximations of cross-system interactions in economic growth modeling.

We approach this question as follows. First, we set-up a general dynamical system representing the evolution of the economic and non-economic system and the interactions between the two systems. Then, we apply some standard types of dynamic equilibrium (e.g., structurally stable equilibriums and homeostasis) to this dynamical system and discuss whether the dynamics arising in these equilibriums are consistent with NEPCAs. As we show, two major problems arise when applying the standard equilibrium types: the standard equilibrium types either do not allow for cross-system interactions or are 'unreliable' foundations of NEPCAs since they bear the possibility that NEPCAs are violated at some future point of (system) time. Therefore, we formulate an equilibrium type (which we name 'partial dynamic equilibrium') that does not give rise to these problems and, thus, may serve as a system-theoretical foundation of NEPCAs. Moreover, we provide a version of the AK growth model with interactions between the non-economic and economic system (via socio-

[4] It is always possible to provide an *empirical foundation of NEPCAs* used in a specific economic model by showing that the specific economic model's NEPCAs are supported by empirical evidence. However, many parameters of economic models are highly theoretical and, thus, difficult to estimate by using empirical data. Moreover, as discussed in Sections 3.3 and 5, such empirical NEPCAs foundations are restricted in validity unless they are supported by interdisciplinary (or non-economic) theoretical models. However, in general, even *interdisciplinary theoretical (large-scale) models* (that seek to endogenize NEPCAs by incorporating empirically proven inter-system linkages) cannot cover all thinkable cross-system linkages, i.e., in general, they rely on some sort of NEPCAs. Thus, NEPCAs seem inevitable in economic modeling and a discussion of their (mathematical) foundations seems to be a valuable task and a good complement to empirical NEPCA foundations and theoretical large-scale modeling.

cultural development and population growth) demonstrating the applicability of the partial dynamic equilibrium as a foundation of the NEPCAs used in the AK model.

Overall, we identify the conditions under which the NEPCAs used in economic growth modeling are consistent with cross-system interactions. As discussed in Section 5, these conditions can be (a) used for further methodological discussion of economic growth modeling with respect to the necessity of interdisciplinarity, (b) applied in future modeling of cross-system interactions (as demonstrated in Section 4), and (c) used for identifying the real-world non-economic (sub-)systems that are *not* modelable by NEPCAS and, thus, elaborating an interdisciplinary research program on endogenization of NEPCAs in economic growth modeling.

The rest of the paper is structured as follows. In the next section, we introduce the mathematical description of the economic and non-economic system (which relies on differential equation systems). Section 3 is devoted to the discussion of the standard types of dynamical system equilibrium in the context of NEPCAs and the derivation of the concept of the partial dynamic equilibrium. In Section 4, the latter concept is applied for modeling the interaction between the economic and non-economic system based on the AK growth model. Concluding remarks are provided in Section 5.

2. A Mathematical Description of the Systems

While there are different mathematical notational conventions, we choose the following notation for reasons of simplicity: small letters (e.g., x) or small Greek letters (e.g., χ) denote scalars; bold small letters (e.g., $\mathbf{x}$) denote vectors or vector functions; capital Greek letters (e.g., Φ) denote vector functions; capital letters (e.g., X) denote sets; R is the set of real numbers; and a dot indicates a derivative with respect to time (e.g., $\dot{x}$ is the derivative of x with respect to time).

Let $\mathbf{e}(t) \equiv (e_1(t), e_2(t), \ldots, e_\varepsilon(t)) \in E \subseteq R^\varepsilon$ denote the ε-dimensional vector of variables describing the state of the economic system at time $t \in [0, \infty)$, where E is the set of all feasible or meaningful states of the economic system. Moreover, let $\mathbf{n}(t) \equiv (n_1(t), n_2(t), \ldots, n_\eta(t)) \in N \subseteq R^\eta$ be the η-dimensional vector of variables describing the state of a non-economic system at time $t \in [0, \infty)$, where N is the set of all feasible or meaningful states of the non-economic system. As discussed in Section 1, we assume that the economic system is dependent on the parameter vector $\mathbf{p}(t) \equiv (p_1(t), p_2(t), \ldots, p_\pi(t)) \in P \subseteq R^\pi$ and that the parameter vector is dependent on the non-economic system, i.e.,

(1) $\mathbf{p}(t) = \Phi^{p}(\mathbf{n}(t))$

where Φ^{p} is a vector function of the type $\Phi^{p}: N \rightarrow P$.

Without loss of generality, we rely on differential equations for modeling the dynamics of the systems. In particular, we assume that the economic system dynamics are determined as follows (cf. Section 1):

(2) $\dot{\mathbf{e}}(t) = \Gamma^{e}(\mathbf{e}(t), \mathbf{p}(t))$

(3) $\mathbf{e}(0) = \mathbf{e}_0 \in E$

where Γ^{e} is a vector function of the type $\Gamma^{e}: E \times P \rightarrow R^{\varepsilon}$ and $\mathbf{e}_0$ is the initial state of the economic system. If we define the function $\Phi^{e}: E \times N \rightarrow R^{\varepsilon}$, $\Phi^{e}(\mathbf{e}(t), \mathbf{n}(t)) := \Gamma^{e}(\mathbf{e}(t), \Phi^{p}(\mathbf{n}(t)))$, then, we can transform (2) as follows:

(4) $\dot{\mathbf{e}}(t) = \Gamma^{e}(\mathbf{e}(t), \Phi^{p}(\mathbf{n}(t))) = \Phi^{e}(\mathbf{e}(t), \mathbf{n}(t))$

In line with the previous discussion, we model the non-economic system by using differential equations and assume that the non-economic system is dependent on the economic system (cf. Section 1), i.e.,

(5) $\dot{\mathbf{n}}(t) = \Phi^{n}(\mathbf{e}(t), \mathbf{n}(t))$

(6) $\mathbf{n}(0) = \mathbf{n}_0 \in N$

where Φ^{n} is a vector function of the type $\Phi^{n}: E \times N \rightarrow R^{\eta}$ and $\mathbf{n}_0$ is the initial state of the non-economic system.

Overall, this discussion implies that (a) the dynamics of the economic system **e** depend on the state of the economic and non-economic system (cf. (4)) and (b) the dynamics of the non-economic system **n** depend on the state of the economic and non-economic system (cf. (5)). Moreover, as we can see, we use *autonomous* differential equations for describing the dynamics of the economic and non-economic system. In general, this is not a problem (in long-run growth modeling), since we can always define the terms of the differential equations that are explicitly dependent on time as auxiliary variables and assign them to the non-economic system **n**, as demonstrated in Section 4.

Moreover, we define the vector $\mathbf{s}(t) \equiv (e_1(t), e_2(t), \ldots, e_\varepsilon(t), n_1(t), n_2(t), \ldots, n_\eta(t))$, which represents the state of the overall system at time t, such that the economic system **e** and the non-economic system **n** can be interpreted as subsystems of the system **s**. Our discussion implies that $\mathbf{s}(t) \in E \times N$ and

(7) $\dot{\mathbf{s}}(t) = (\Phi^{e}(\mathbf{s}(t)), \Phi^{n}(\mathbf{s}(t))) =: \Phi^{s}(\mathbf{s}(t))$

where $\Phi^{s}: E \times N \rightarrow R^{\varepsilon+\eta}$ is a vector function.

In terms of the notation introduced in this section, the 'non-economic parameters constancy assumptions' discussed in Section 1 can be represented as follows:

(8) $\forall t \in T \;\; \dot{\mathbf{p}}(t) = 0$ 'NEPCAs'

where $T \subseteq R$ is the model application period representing the past and future time to which the (economic) model is assumed to apply.

3. Types of Dynamical System Equilibrium and their Implications for NEPCAs

In this section, we discuss different types of dynamical system equilibrium known from mathematical dynamical systems theory, systems theory, and economics. As we will see, many of the dynamical system equilibrium types seem valuable for modeling the dynamics of the economic and non-economic system. However, only one type (namely, the partial dynamic equilibrium) seems to support NEPCAs when there are interactions between the systems.

3.1 Structural Stability of the Non-Economic System and Bifurcations

Structural stability is a very important concept in dynamics modeling (see, e.g., Andronov et al. (1987) and Guckenheimer and Holmes (1989) for a discussion). A dynamic system is regarded as structurally stable if marginal variations in model parameters do not change the qualitative behavior of the system. Obviously, the structural stability of the dynamic system used to model economic dynamics is essential, since if even marginal parameter variations change the qualitative predictions of the model, the model is not a very reliable explanation of economic dynamics in the light of measurement problems/inadequacies regarding model parameters and variables (cf., e.g., Andronov et al. (1987, p.374 and p.405)). Thus, the structural stability of the economic system **e** (with respect to the changes in the parameters **p** determined by the non-economic system **n**) is a premise for economic modeling and, henceforth, we assume that the economic system is structurally stable in this sense.

For the following discussion, the structural stability of the non-economic system **n** is much more interesting. When studying the interactions between the economic system **e** and the non-economic system **n**, the structural stability of the non-economic system **n** and the structural stability of the economic system **e** can be understood as antipodal concepts: while the structural stability of the *economic system* refers to the reaction of the economic system **e** in response to a change in non-economic variables **n** (cf. (4)), the structural stability of the *non-economic system* refers to the reaction of the non-economic system **n** in response to a change in economic variables **e** (cf. (5)). We can formalize this concept of the structural stability of the non-economic system $\mathbf{n}$ by using the model introduced in Section 2 as follows. First, assume that the economic-variables vector $\mathbf{e}$ is given/constant, e.g., $\mathbf{e}(t) = \tilde{\mathbf{e}} \equiv (\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_\varepsilon)$, where $\tilde{e}_1$ = const., $\tilde{e}_2$ = const., …, $\tilde{e}_\varepsilon$ = const., and $\tilde{\mathbf{e}} \in E$. Then, (5) implies that the dynamics of the non-economic system $\mathbf{n}$ are given by the following equation.

(5') $\quad \dot{\mathbf{n}}(t) = \Phi^{\mathrm{n}}(\tilde{\mathbf{e}}, \mathbf{n}(t)), \quad \mathbf{n}(0) = \mathbf{n}_0 \in N$

Based on these assumptions, we can postulate the following definition of the structural stability of the non-economic system $\mathbf{n}$, which is restricted to stable fixed points, yet sufficient for our purposes.

Definition 1. *Let the dynamics of the vector* $\mathbf{n}(t)$ *be determined by (5). Moreover, assume that (5) is such that for all* $\mathbf{n}_0 \in N$*, the vector* $\mathbf{n}(t)$ *converges to the stable fixed point* $\mathbf{n}^*(\tilde{\mathbf{e}}) \in N$ *if* $\forall t \in T\ \mathbf{e}(t) = \tilde{\mathbf{e}} \in E$*. In other words, given the parameter vector* $\tilde{\mathbf{e}} \in E$*, the dynamic behavior of the non-economic system (5') is (per assumption) characterized by a stable steady state* $\mathbf{n}^*(\tilde{\mathbf{e}}) \in N$*. In this case, the non-economic system* $\mathbf{n}$ *is structurally stable on the (connected) set* $\tilde{E} \subseteq E$ *(where* $\tilde{\mathbf{e}} \in \tilde{E}$*) if for each* $\mathbf{e} \in \tilde{E}$*, there exists a (stable) fixed point* $\mathbf{n}^*(\mathbf{e}) \in N$*.*

If the non-economic system $\mathbf{n}$ changes its qualitative behavior when $\mathbf{e}$ leaves $\tilde{E}$ (e.g., if the omega limit-set changes from a fixed point to a limit-cycle), then the boundary $\mathrm{bd}(\tilde{E}) := \mathrm{cl}(\tilde{E})\backslash\mathrm{int}(\tilde{E})$ represents the set of points of bifurcation.

Obviously, on its own, the concept of structural stability of the non-economic system $\mathbf{n}$ (cf. Definition 1) does not allow us justify NEPCAs in economic modeling. If the non-economic system (5') has a (globally) stable fixed point $\mathbf{n}^*$, as stated by Definition 1, then changes in the initial conditions $\mathbf{n}_0$ of the non-economic system $\mathbf{n}$ are accompanied by the convergence to one and the same fixed point $\mathbf{n}^*$. This does not, however, imply that the fixed point of the non-economic system $\mathbf{n}^*$ does not change if the economic variables $\mathbf{e}$ change. In general, if the non-economic system $\mathbf{n}$ is (a) dependent on the economic system $\mathbf{e}$, as stated in (5'), and (b) structurally stable with respect to the economic system $\mathbf{e}$ (cf. Definition 1), then the focus $\mathbf{n}^*$ is a function of the economic system $\mathbf{e}$, i.e., $\mathbf{n}^*(\mathbf{e})$. That is, a change in $\mathbf{e}$ leads to a change in $\mathbf{n}^*$. Thus, in presence of *continuous* economic dynamics and cross-system interactions, the structurally stable non-economic system $\mathbf{n}$ described by Definition 1 does not necessarily generate a constant/static 'fixed point' $\mathbf{n}^*$, i.e., $\mathbf{n}$ is not necessarily constant in the limit. Therefore, $\mathbf{p}$ is not necessarily constant in the limit (cf. (1)), i.e., NEPCAs (8) can be violated even in the limit. We turn now to special types of structurally stable system that can be consistent with (8).

3.2 A Stable and Independent Fixed Point of the Non-Economic System

As explained in Section 3.1, the fixed point type ($\mathbf{n}^*$) described by Definition 1 is not necessarily consistent with the NEPCAs (8), since, in general, $\mathbf{n}^*$ is dependent on $\mathbf{e}$. A special case arises if the non-economic system $\mathbf{n}$ is dependent on the economic system $\mathbf{e}$, as stated in (5'), but the fixed point $\mathbf{n}^*$ is not. In this case, each time a change in $\mathbf{e}$ occurs, the economy seeks to converge to one and the same $\mathbf{n}^*$. However, the economic system $\mathbf{e}$ is, in general, not describable by one or several discrete (non-systematic/erratic) changes; rather, the economic system $\mathbf{e}$ changes continuously, perpetually, and systematically (according to the economic theory). Thus, even if $\mathbf{n}^*$ is independent of $\mathbf{e}$, a steadily changing economic system $\mathbf{e}$ leads to steady or even increasing deviation of the non-economic system $\mathbf{n}$ from the fixed point $\mathbf{n}^*$, i.e., $\mathbf{n}$ is not necessarily constant in the limit. Therefore, the NEPCAs (8) may be violated in the limit (cf. (1)). Overall, even in the case of independency described in this section, the fixed-point type ($\mathbf{n}^*$) described by Definition 1 is not necessarily consistent with the NEPCAs (8) in the limit, and we turn now to a special case named homeostasis.

3.3 Homeostasis of the Non-Economic System

If we assume that not only the fixed point $\mathbf{n}^*$ of the non-economic system $\mathbf{n}$ is independent of the economic system $\mathbf{e}$ (cf. Section 3.2) but also that there are no (significant) transitional dynamics of the non-economic system $\mathbf{n}$ for a given set ($\bar{E}$) of economic variables $\mathbf{e}$, we obtain the concept of 'homeostasis of the non-economic system', which we define as follows.

Definition 2. *For a given vector* $\bar{\mathbf{e}} \in E$*, the non-economic system* $\mathbf{n}$ *(cf. (5)) is in the state of homeostasis if (a) there exist open and connected sets* $\bar{E}$ *and* $\hat{E}$ *such that*

(5'') $\quad \bar{E} \subset \hat{E} \subseteq E \wedge (\dot{\mathbf{n}}(t) = 0 \text{ } if \text{ } \mathbf{e} \in \bar{E}) \wedge (\dot{\mathbf{n}}(t) \neq 0 \text{ } if \text{ } \mathbf{e} \in \hat{E})$

and (b) $\bar{\mathbf{e}} \in \bar{E}$*.*

The concept of homeostasis seems adequate for modeling of, e.g., switching of policy/political regimes and ecological changes. Homeostasis of the non-economic system $\mathbf{n}$ represents a problem for the application of NEPCAs in economic modeling and their empirical validation: (5'') states that the non-economic system $\mathbf{n}$ is stable/static as long as the economic system **e** is within the set $\bar{E}$. If the economic system **e** leaves the set $\bar{E}$, the non-economic system **n** may start to change over time. Thus, for example, if the economic system **e** is *initially* within the set $\bar{E}$ and we measure the empirical indexes representing the non-economic system **n**, we may come to the wrong conclusion that the non-economic system **n** is stable or does not react to economic dynamics and, thus, the NEPCAs (8) are satisfied (cf. (1)). However, if *subsequently*, the economic system **e** develops such that it leaves the set $\bar{E}$ (which may be plausible in long-run modeling), the non-economic system **n** may start to change and to react to economic system changes (cf. (5'')). In this case, the assumption '$\dot{\mathbf{n}}(t) = 0$' and, thus, '$\dot{\mathbf{p}} = 0$' (cf. (1)) is not adequate for modeling the long-run dynamics, i.e., (1), (8), and (5'') may contradict each other in long-run modeling.

Overall, if some of the non-economic subsystems may behave according to the concept of homeostasis, neither can *long-run* predictions of economic dynamics rely on the NEPCAs (8) nor can these NEPCAs be validated by empirical evidence on the constancy of non-economic parameters **p**. In particular, a theory of the development of the non-economic system **n** that excludes the possibility of homeostasis of the non-economic system **n** is required (in addition to the empirically measured stability of the non-economic system **n**) to justify the NEPCAs in economic models. In general, this requires interdisciplinary or non-economic theoretical research.

Obviously, if the non-economic system **n** is in the state of homeostasis, it does not react to economic system dynamics (cf. Definition 2). This contradicts the evidence showing that the economic and non-economic systems interact (cf. Section 1). Nevertheless, it can be attempted to merge the concept of homeostasis of the non-economic system **n** with the empirical evidence on cross-system interactions by assuming that the cross-system interactions have been observed while the economic system **e** has been outside the set $\bar{E}$. Then, however, for applying the concept of homeostasis (and, thus, the NEPCAs (8)) in the modeling of *future* non-economic system dynamics (over the period T), it is necessary to show (theoretically) that even though the non-economic system **n** has not been in the state of homeostasis in the past, it will be in the state of homeostasis in future (over T). Again, this requires interdisciplinary or non-economic theoretical research. Moreover, the concept of homeostasis, as used in this paragraph, can be substituted by the concept discussed in Section 3.6.

Note that if homeostasis is defined such that $\forall \mathbf{e} \in E \;\; \dot{\mathbf{n}}(t) = 0$ (i.e., $\bar{E} = E$, cf. Section 2 and (5'')), the non-economic system **n** is independent of the economic system **e**. Therefore, this definition of homeostasis does not serve our purposes, since empirical evidence implies that economic dynamics have impacts on non-economic dynamics (cf. Section 1).

3.4 Slow/Weak Dynamics/Reaction of the Non-Economic System

One of the conventional wisdoms about institutional and socio-cultural change is that it is relatively slow or that it reacts weakly to the changes in the economic system (see, e.g., Roland (2004), Streeck and Thelen (2005), and Acemoglu and Robinson (2010) for a discussion of institutional development). Thus, it may be argued that the parameter changes reflecting the institutional and socio-cultural change or, in general, change of the non-economic system **n** could be neglected (provided that the economic system **e** does not 'overreact' to small changes in the parameters **p** determined in the non-economic system **n**, as discussed in Section 3.5).

The problems associated with this argument are manifold. First, even if the non-economic system **n** changes slowly, the changes may accumulate over time such that the cumulative change may become significant or measurable over the long periods to which long-run economic models refer (cf., e.g., Streeck and Thelen (2005), p.8). Second, not only cumulation across time but also across systems is relevant: since, in general, the parameters **p** of an economic model do not only depend on one non-economic subsystem but on many different non-economic subsystems, the case may arise that the impacts of each non-economic subsystem are neglectable while the overall impact of all non-economic systems is significant. Third, even small (cumulative) changes in the non-economic (sub)system(s) may have strong impacts on the economic system **e** and, thus, may be not neglectable in economic modeling if (a) the elasticity of the economic system **e** with respect to the non-economic system parameters **p** is great ('overreaction to small changes in **p**'), (b) the economic system **e** is not structurally stable with respect to the changes in the non-economic parameters **p**, or (c) the economic system **e** is close to some point of bifurcation, such that even a relatively small change in the non-economic parameters **p** leads to a large change in the economic system **e** or to a change in the qualitative properties of economic dynamics (cf. Section 3.1).[5] Fourth, even if it can be empirically shown that the non-economic system **n** has changed slowly or reacted weakly to economic dynamics in the past, we cannot exclude that due to (quasi-)homeostatic nature of the non-economic system **n** (cf. Section 3.3), **n** may start to

[5] The occurrence of characteristics (a) to (c) can be tested by studying the economic model solely, i.e., interdisciplinary research is not necessary to analyze whether the (model of the) economic system is overly elastic, structurally unstable, or close to bifurcation points.

change/react much more quickly/strongly at some future point in time (when the economic system **e** leaves the set $\bar{E}$). Thus, empirical validation of slow/weak non-economic system dynamics/reaction does not provide a firm foundation of NEPCAs and either interdisciplinary or non-economic theoretical research is necessary to do so, as discussed in Section 1. Fifth, in general, the statement that the dynamics of a non-economic system **n** are weak/slow such that they can be neglected seems to be imprecise or vague. (When is an impact channel weak enough such that it can be neglected?)

We can conclude this discussion as follows: due to accumulation over time and over systems, even slow/weak dynamics of the non-economic (sub)system(s) become sooner or later measurable or significant and, thus, must be accounted by a change in (a) the non-economic variables **n** and (b) the non-economic parameters **p** (cf. (1)) at some point in time *t'*, which can be mathematically expressed as follows:

(5''') $(\mathbf{p}(t) = \mathbf{p}^1 \in P \text{ for } t \leq t')$ and $(\mathbf{p}(t) = \mathbf{p}^2 \in P \text{ for } t > t')$

where *t'* is the point in time at which the cumulative change in the non-economic parameters **p** becomes measurable or significant.

3.5 Weak Reaction of the Economic System to Non-Economic System Changes

Another interesting case arises when the reaction of the economic model to the changes in its non-economic parameters **p** is relatively weak. At the first look, the study of this case does not require interdisciplinary research but only the study of the effects of parameter changes in the corresponding economic model. For example, we may ask, what happens in the Ramsey-(1928)-Cass-(1965)-Koopmans-(1967) model if the time-preference changes over time. In particular, we can analyze the effects of non-economic parameter changes on the *quantitative and qualitative* results of the economic model.

As discussed in Section 3.1, for studying the *qualitative* reaction of an economic model to the changes in its non-economic parameters **p**, the concept of structural stability can be used. For example, the Ramsey-(1928)-Cass-(1965)-Koopmans-(1967) model is, in general, structurally stable (with respect to time-preference rate changes).

Regarding the *quantitative* effects of non-economic parameter changes in an economic model, arguments similar to the arguments discussed in Section 3.4 can be developed. First, even if we can show that for a given velocity of non-economic parameter change, the economic model can neglect these changes at the present (since they have a relatively weak impact on the economic variables **e**), we cannot exclude that due to quasi-homeostatic nature of the non-economic system **n**, the non-economic system dynamics accelerate in future such that the changes in **n** become relevant for the economic system **e**. Thus, the argument that (for a certain empirically observed velocity-range of non-economic system dynamics) the effects of the non-economic parameter changes are neglectable in an economic model does provide a firm foundation of NEPCAs and either interdisciplinary or non-economic theoretical research is necessary to do so, as discussed in Section 1. Second, the statement that the effects of the non-economic parameter changes are weak and, thus, neglectable is imprecise (cf. Section 3.4).

3.6 Partial Dynamic Equilibrium

The concepts discussed in Sections 3.1-3.5 allow only for limited cross-system interactions (cf., e.g., Section 3.3), are vague/imprecise (cf., e.g., Sections 3.4 and 3.5), or imply that there is the possibility that the NEPCAs (8) become inadequate if the economic system **e** develops beyond some scope (cf., e.g., $\bar{E}$ or t') and, thus, bear some uncertainty regarding the applicability of the NEPCAs (8). The concept of the 'partial dynamic equilibrium' suggested in this section is a derivate of the previously discussed concepts and tries to solve the problems associated with them. In particular, we try to show in this section that it is possible to model two interacting systems (the economic and non-economic system) by applying the NEPCAs (8) while reducing the vagueness and uncertainty associated with the concepts discussed in Sections 3.1-3.5.

3.6.1 Partial Dynamic Equilibrium and NEPCAs

In economic models, two types of dynamic equilibria arise: (standard) dynamic equilibria and asymptotic dynamic equilibria. Analogously, we distinguish between a (standard) partial dynamic equilibrium and an asymptotic partial dynamic equilibrium. As formulated by Definition 3a, a partial dynamic equilibrium can be achieved in finite time (for example, if the initial conditions are such that the economy is in partial dynamic equilibrium at $t = 0$). In contrast, an asymptotic partial dynamic equilibrium refers only to the limit dynamics (cf. Definition 3b).

Definition 3. *Assume that the dynamics of the system* **s** *are governed by the differential equation system (1)/(2)/(5), where the initial conditions* $\mathbf{e}_0$ *and* $\mathbf{n}_0$ *are given by (3) and (6). We say that (given the initial conditions (3) and (6))*

(a) the system **s** *is in partial dynamic equilibrium over the period* $\underline{T} \subseteq T$ *if* $\forall t \in \underline{T}\ \ \dot{\mathbf{e}}(t) \neq 0\ \wedge$ $\mathbf{p}(t) = \Phi^{p}(\mathbf{n}(t)) = \mathbf{p}^{*} \in P$ *(cf. (1)) and*

(b) the system **s** *is characterized by an asymptotic partial dynamic equilibrium if* $\lim_{t\to\infty} \dot{\mathbf{e}}(t) \neq 0$ *and* $\lim_{t\to\infty} \mathbf{p}(t) = \lim_{t\to\infty} \Phi^{\mathrm{p}}(\mathbf{n}(t)) = \mathbf{p}^* \in P$ *(cf. (1)).*

Now, we discuss Definition 3 rather abstractly, while in Section 4, we provide examples of the concepts discussed here.

Definition 3a states that in partial dynamic equilibrium, the economic system **e** is non-static while the non-economic system **n** behaves such that the parameter vector **p** is constant (and, thus, NEPCAs (8) are satisfied). **p** can be constant in two cases. First, obviously, $\mathbf{p}(t) = \Phi^{\mathrm{p}}(\mathbf{n}(t))$ is constant if $\mathbf{n}(t)$ is constant. This case is not very interesting, since it presumes that the non-economic system **n** is static, which contradicts the premise made in Section 1 that economic dynamics lead to non-economic system dynamics. Second, the parameter vector **p** may be constant ($\dot{\mathbf{p}}(t) = 0$, $\mathbf{p}(t) = \mathbf{p}^*$) even if the non-economic system is non-static ($\dot{\mathbf{n}}(t) \neq 0$) provided that the parameter equation system $\Phi^{\mathrm{p}}(\mathbf{n}(t)) = \mathbf{p}^*$ is underdetermined (cf. Meckl (2002)). For example, if $\Phi^{\mathrm{p}}(\mathbf{n}(t)) = \mathbf{p}^*$ is a system of *linear* equations with $\pi < \eta$, then there may exist a one- or higher-dimensional subspace/set N^* (e.g., a line, a plane, etc.) of vectors **n** satisfying $\Phi^{\mathrm{p}}(\mathbf{n}) = \mathbf{p}^*$; thus, if $\mathbf{n}(t)$ is non-constant and its law of motion satisfies $\mathbf{n}(t) \in N^*\ \forall t$, then **p** is constant for all t.[6] Alternatively, if $\Phi^{\mathrm{p}}(\mathbf{n}(t)) = \mathbf{p}^*$ is a smooth (underdetermined) *non-linear* equation system, then $\Phi^{\mathrm{p}}(\mathbf{n}) = \mathbf{p}^*$ may define a (smooth) (hyper)surface/manifold; as long as the non-economic system **n** moves along this (hyper)surface/manifold, **p** can be constant while $\mathbf{n}(t)$ is non-constant.

Overall, when the system **s** is in partial dynamic equilibrium (cf. Definition 3a), (a) the NEPCAs (8) are satisfied, (b) the economic system **e** is non-static, and (c) the non-economic system **n** can be non-static in the case that the parameter equation system $\Phi^{\mathrm{p}}(\mathbf{n}(t)) = \mathbf{p}^*$ is underdetermined.[7] Moreover, if the system **s** is right from the beginning (i.e., for all $t \geq 0$) in partial dynamic equilibrium, then the non-economic system **n** does not have any impacts on the economic system **e**, since these impacts are transmitted via **p** in our model (cf. (1) and

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[6] For example, assume, first, that $\Phi^{\mathrm{p}}(\mathbf{n}(t)) = \mathbf{A}\mathbf{n}(t) = \mathbf{p}^*$, where $\mathbf{A} = \{b_{ij}\}$ is a rank-3 3×3-Matrix with given constant elements b_{ij}, i.e., $\pi = 3$, $\eta = 3$, and $\mathbf{p}^* \equiv (p_1^*, p_2^*, p_3^*)$ and $\mathbf{n}(t) \equiv (n_1(t), n_2(t), n_3(t))$ are column vectors. In this case, there can exist only one solution ($\mathbf{n}^*$) of the linear equation system $\mathbf{A}\mathbf{n}(t) = \mathbf{p}^*$ and, thus, $\forall t\ \mathbf{n}(t) = \mathbf{n}^*$, i.e., **n** must be constant. However, if **A** has rank 2 (i.e., the equation system $\mathbf{A}\mathbf{n}(t) = \mathbf{p}^*$ is underdetermined), then there exists a solution of the following form: $n_1(t) = c_1 + c_2 n_3(t) \wedge n_2(t) = c_3 + c_4 n_3(t)$, where c_1-c_4 are functions of b_{ij} and of some p_i^*. If we assume that $n_1(t)$ and $n_2(t)$ satisfy these equations for all t, then we can choose an arbitrary law of motion (which may be derived from a non-economic theory) for $n_3(t)$ and, nevertheless, the equation system $\mathbf{A}\mathbf{n}(t) = \mathbf{p}^*$ is satisfied for all t. The set N^* is then given as follows: $N^* = \{\mathbf{n} \equiv (n_1, n_2, n_3) \in R^3: n_1 = c_1 + c_2 n_3 \wedge n_2 = c_3 + c_4 n_3\}$.

[7] If such a system (i.e., a system that (a) is characterized by an underdetermined parameter equation system $\Phi^{\mathrm{p}}(\mathbf{n}(t)) = \mathbf{p}^*$ and (b) is in partial dynamic equilibrium while the economic and non-economic variables are non-static) existed in reality, empirical investigations could identify correlations between some economic and some non-economic variables (cf. the empirical studies listed in Footnote 1).

(2)) and **p** is constant in partial dynamic equilibrium (cf. Definition 3a). Thus, the partial dynamic equilibrium (cf. Definition 3a) itself, i.e., the assumption that the system **s** is in partial dynamic equilibrium for all $t \geq 0$ does not serve our purposes.[8] (Nevertheless, the partial dynamic equilibrium can be very useful if we assume that the system **s** is not in partial dynamic equilibrium at $t = 0$ but later; cf. Section 3.6.2.)

In contrast, the *asymptotic* partial dynamic equilibrium (cf. Definition 3b) seems to be useful in general. According to Definition 3b, the parameter vector $\mathbf{p}(t)$ converges to the fixed point $\mathbf{p}^*$. Thus, the NEPCAs (8) are satisfied asymptotically (cf. Figure 1). Moreover, Definition 3b allows for interactions between the economic and non-economic system during the transition period (i.e., before the limit); in particular, $\mathbf{e}(t)$, $\mathbf{n}(t)$, and $\mathbf{p}(t)$ are not necessarily constant during this transition period ($\mathbf{p}(t)$ must be constant only in the limit). In general, a system **s** that is characterized by an asymptotic partial dynamic equilibrium seems to be useful for founding NEPCAs (in presence of cross-system interactions). In particular, when the system has converged sufficiently close to $\mathbf{p}^*$ (at time t^*) such that the future changes in **p** are relatively small (i.e., $\forall t \geq t^*\ \mathbf{p}(t) \approx \mathbf{p}^*$) while $\mathbf{e}(t)$ and $\mathbf{n}(t)$ are (still) not constant, the NEPCAs (8) are approximately satisfied while cross-system interactions exist.

Overall, the asymptotic partial dynamic equilibrium has a major advantage in comparison to the equilibrium types discussed in Sections 3.3-3.5. While the NEPCAs (8) are satisfied asymptotically in the case of an asymptotic partial dynamic equilibrium, the equilibrium types discussed in Sections 3.3-3.5 allow for a violation of (8) at some future point of (system) time; exactly speaking, in the case of the equilibria described in Sections 3.3-3.5, a future violation of (8) is possible or even asymptotically inevitable.

[8] However, there are two interesting special cases where cross-system interactions (of a very special type) exist despite constant parameters **p**. These special/knife-edge cases may not be of practical interest or may require a sound interdisciplinary theoretical foundation. First, assume that the non-economic system **n** has an autonomous component, i.e., it is non-static even if the economic system **e** is static. Moreover, assume that the dynamics of the economic system **e** are such that they offset the autonomous dynamics of the non-economic system **n**. In other words, the relations between the systems are such that the impact of the economic system **e** on the non-economic system **n** leads to a steady state of the non-economic system **n** (and, thus, constant **p**). Second, assume that the non-economic system **n** represents many different non-economic subsystems. Moreover, assume that these subsystems are in a type of equilibrium where their changes offset each other with respect to **p**. For example, assume that **p** is determined by many different non-economic subsystems (e.g., socio-cultural, political, ecological,…) and that (while interacting with the economic system **e**) the dynamics of some of these subsystems (e.g., n_1, n_2, …, n_m) have a positive (increasing) effect on **p** while the dynamics of the others (e.g., n_{m+1}, n_{m+2}, …, n_η) have a negative (decreasing) effect on **p**, such that the increases offset the decreases and **p** is constant (trend-wise). In this case, empirical investigations (investigating only the interactions between one specific non-economic subsystem and the economic system) could identify correlations between some economic variables and some non-economic variables representing some specific non-economic subsystems. Yet, there would not be any interactions between the economic system **e** and the (overall) non-economic system **n**.

3.6.2 Stability of the Partial Dynamic Equilibrium, Transitional Dynamics, and NEPCAs

Now, we turn to the definition and discussion of the stability of the partial dynamic equilibrium, the transitional dynamics, and their relevance for NEPCAs.

Definition 4. *Let the dynamics of the system* $\mathbf{s}$ *be governed by the differential equation system (1)/(2)/(5), and assume that (1)/(2)/(5) has solutions on the initial conditions set* $S° \equiv E°\times N° \subseteq S \equiv E\times N$. *In particular, assume that for each* $\mathbf{s}_0 \equiv (\mathbf{e}_0, \mathbf{n}_0) \in S°$, *there exists a function* $\mathbf{s}(t, \mathbf{s}_0) \equiv (\mathbf{e}(t, \mathbf{s}_0), \mathbf{n}(t, \mathbf{s}_0))$ *that is consistent with (1)/(2)/(5) for all* $t \in T = [0, \infty)$ *and satisfies* $\mathbf{s}(0, \mathbf{s}_0) \equiv (\mathbf{e}(0, \mathbf{s}_0), \mathbf{n}(0, \mathbf{s}_0)) = \mathbf{s}_0$.

a) Assume that the system $\mathbf{s}$ *is in partial dynamic equilibrium at* $\underline{t} \in T$ *(cf. Definition 3a) given the initial state* $\underline{\mathbf{s}}_0 \in \underline{S} \equiv \underline{E}\times\underline{N} \subseteq S°$, *i.e.,* $\Phi^{\mathrm{p}}(\mathbf{n}(\underline{t}, \underline{\mathbf{s}}_0)) = \mathbf{p}^* \in P$ *and* $\dot{\mathbf{e}}(\underline{t}, \underline{\mathbf{s}}_0) \neq 0$. *This partial dynamic equilibrium is (asymptotically) stable on the set* $\underline{S}$ *if (9) and (10) are satisfied, where*

(9) $\quad \forall t > \underline{t} \;\; \Phi^{\mathrm{p}}(\mathbf{n}(t, \underline{\mathbf{s}}_0)) = \mathbf{p}^* \;\wedge\; \dot{\mathbf{e}}(t, \underline{\mathbf{s}}_0) \neq 0$

(10) $\quad \forall \mathbf{s}_0 \in \underline{S} \;\; lim_{t\to\infty}\, \Phi^{\mathrm{p}}(\mathbf{n}(t, \mathbf{s}_0)) = \mathbf{p}^* \;\wedge\; lim_{t\to\infty}\, \dot{\mathbf{e}}(t, \mathbf{s}_0) \neq 0.$

b) Assume that the system $\mathbf{s}$ *is characterized by an asymptotic partial dynamic equilibrium (cf. Definition 3b) given the initial state* $\underline{\underline{\mathbf{s}}}_0 \in \underline{\underline{S}} \equiv \underline{\underline{E}}\times\underline{\underline{N}} \subseteq S°$, *i.e.,* $\lim_{t\to\infty} \Phi^{\mathrm{p}}(\mathbf{n}(t, \underline{\underline{\mathbf{s}}}_0)) = \mathbf{p}^* \in P$ *and* $\lim_{t\to\infty} \dot{\mathbf{e}}(t, \underline{\underline{\mathbf{s}}}_0) \neq 0$. *This asymptotic partial dynamic equilibrium is stable on the set* $\underline{\underline{S}}$ *if*

(11) $\quad \forall \mathbf{s}_0 \in \underline{\underline{S}} \;\; \lim_{t\to\infty} \Phi^{\mathrm{p}}(\mathbf{n}(t, \mathbf{s}_0)) = \mathbf{p}^* \in P \;\wedge\; \lim_{t\to\infty} \dot{\mathbf{e}}(t, \mathbf{s}_0) \neq 0.$

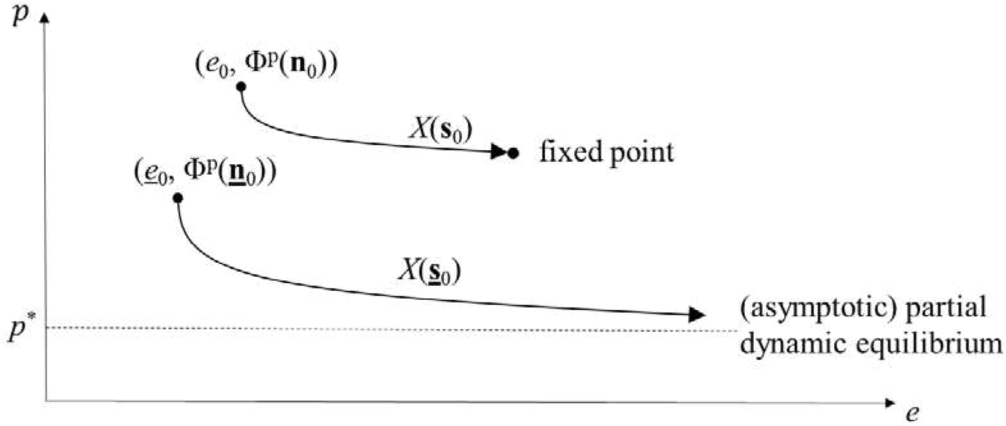
Definition 5. *If all the statements of Definition 4 hold for* $\underline{S} = S°$ $(\underline{\underline{S}} = S°)$, *the partial dynamic equilibrium (the asymptotic partial dynamic equilibrium) is globally stable.*

The stability criterions used in Definition 4 are standard. Stability condition (9) states that if the system $\mathbf{s}$ is in partial dynamic equilibrium (at the time point $\underline{t}$), then it stays in partial dynamic equilibrium (for all $t \geq \underline{t}$); condition (10) ensures that if (a) the system $\mathbf{s}$ is not in partial dynamic equilibrium and (b) the initial conditions are within some stability set ($\underline{S}$), then $\mathbf{s}$ converges to the partial dynamic equilibrium. Moreover, an asymptotic partial dynamic equilibrium is stable on the set $\underline{\underline{S}}$ if the system $\mathbf{s}$ converges to the asymptotic partial dynamic equilibrium for all initial conditions belonging to the set $\underline{\underline{S}}$. Only the requirement that the economic system $\mathbf{e}$ is non-static all the time (cf. (9)–(11)) may be regarded as a deviation from the standard stability definition.

For discussing the geometrical properties of a *stable* partial dynamic equilibrium, let $X(\mathbf{s}_0) := \{(\mathbf{e}(t, \mathbf{s}_0), \Phi^{\mathrm{p}}(\mathbf{n}(t, \mathbf{s}_0)) \in E\times P\!: t \in [0, \infty)\}$ be a trajectory in $\mathbf{e}$-$\mathbf{p}$ space, where $\mathbf{s}_0 \in \underline{S}$ or $\mathbf{s}_0 \in \underline{\underline{S}}$

(cf. Definition 4). Figure 1 depicts an example of the convergence to a stable partial dynamic equilibrium (cf. Definition 4a) in the case of a one-dimensional economic system ($\varepsilon = 1$) and a one-dimensional parameter system ($\pi = 1$). Alternatively, this depiction of the transition to a stable partial dynamic equilibrium (cf. Definition 4a) can be interpreted as an asymptotic partial dynamic equilibrium (cf. Definition 3b). Moreover, Figure 1 depicts an example of a standard dynamic equilibrium (stable fixed point) for reasons of comparison.

Figure 1. Examples: (asymptotic) (partial) dynamic equilibrium (ε = π = 1).



If we do not only analyze the dynamics in partial dynamic equilibrium as done in Section 3.6.1, but also consider the transitional dynamics (i.e., the convergence to the partial dynamic equilibrium) as implied by Definitions 4 and 5, the (transition phase to the) partial dynamic equilibrium can be an interesting foundation of NEPCAs. Assume that (a) the dynamical system **s** satisfies all the assumptions postulated in Definitions 3a and 4a and (b) initially (i.e., at $t = 0$), the system **s** is not in partial dynamic equilibrium but converges to it according to Definition 4a (i.e., the system **s** is in the transition phase). Under these assumptions, the dynamics of the system **s** can be consistent with the NEPCAs (8) despite interactions between the economic system **e** and the non-economic system **n**. The proof is straight forward. First, since the system is not in partial dynamic equilibrium over the transition phase, interactions between the economic and non-economic system are not ruled out in general; in particular, $\mathbf{e}(t)$, $\mathbf{n}(t)$, and $\mathbf{p}(t)$ are not necessarily static over the transition phase (cf. Definition 4a). Second, since the parameter vector $\mathbf{p}(t)$ converges to the steady state $\mathbf{p}^*$ (cf. Definition 4a), the NEPCAs (8) are satisfied asymptotically. In particular, if the system has converged sufficiently close to the stable partial dynamic equilibrium (at time t^*) such that the future changes in **p** are relatively small (i.e., $\forall t \geq t^* \ \mathbf{p}(t) \approx \mathbf{p}^*$) while $\mathbf{e}(t)$ and $\mathbf{n}(t)$ are (still) not constant, the NEPCAs (8) are approximately satisfied while cross-system interactions exist.

As we can see, the concept of the transition to a stable partial dynamic equilibrium has a major advantage in comparison to the equilibrium types discussed in Sections 3.3-3.5: while the NEPCAs (8) are satisfied asymptotically in the case of a stable partial dynamic equilibrium, the equilibrium types discussed in Sections 3.3-3.5 allow for a violation of (8) at some future point of (system) time (cf. Section 3.6.1).

3.6.3 Summary and Discussion

The discussion in Sections 3.6.1 and 3.6.2 implies that we can provide a foundation of NEPCAs in presence of cross-system interactions if we assume that the system (1)–(7) is either characterized by a (stable) asymptotic partial dynamic equilibrium (cf. Definition 3b) or in the transition phase to a stable partial dynamic equilibrium (cf. Definition 4a). In both cases, the system **s** has the following properties:

(a) the economic and non-economic system are dependent upon each other (cf. (1)–(7)), i.e., there are cross-system linkages;

(b) the cross-system interactions (i.e., the interactions between **e** and **n**) are measurable over the transitional phase (i.e., during the convergence to the equilibrium) and, thus, can be consistent with the empirical evidence on cross-system interactions (cf. Footnote 1);

(c) the NEPCAs (8) are satisfied asymptotically; moreover, they are approximately satisfied in finite time;

(d) the economic system **e** is even asymptotically non-static, which is consistent with the empirical evidence on long-run economic dynamics; and

(e) the NEPCAs (8) may be satisfied (asymptotically) even if the non-economic system **n** is (asymptotically) non-static.

Note that the definition of the partial dynamic equilibrium can be reformulated such that it covers the case of an asymptotically static economic system **e**. In this case, both, the economic system **e** and the parameter vector **p** are static in the limit. Thus, additional conditions become necessary to ensure that **p** converges more quickly to its equilibrium than **e** does, such that for some relatively large t, there are significant economic dynamics ($\dot{\mathbf{e}} \neq 0$) while the NEPCAs (8) are approximately satisfied ($\mathbf{p}(t) \approx \mathbf{p}^*$). These 'additional conditions' can be formulated in terms of limit tangential vector angles associated with the trajectory $X(\mathbf{s}_0)$ describing the dynamics of the system **e-p**, as will be discussed in a separate paper.

For a discussion of the application of the concept of partial dynamic equilibrium in the context of structural change in multi-sector frameworks and for a comparison of this concept to the related concepts (e.g., 'generalized balanced growth', 'aggregate balanced growth', and 'asymptotically constant growth path') used in structural change modeling, see Stijepic (2011). Note, however, that in some sense, the concepts applied in the structural change literature are antipodal to the concepts discussed in our paper: the structural change theories search for a growth path that allows for some sort of *dynamic equilibrium* ('balanced growth') *of the aggregate economic system* while another system (namely, the economic *sector* system) is not in dynamic equilibrium; in contrast, we search for a trajectory (of the system **s**) along which the *(aggregate) economic system is not in dynamic equilibrium*, while another system (namely, the non-economic system) is in a dynamic equilibrium.

4. The Partial Dynamic Equilibrium as a Foundation of the AK Model NEPCAs

In this section, we suggest a simple model of the long-run interaction between the economic and non-economic system based on the textbook AK model (see, e.g., Barro and Sala-i-Martin (2004, p.63ff.) for a description of the latter). The AK model assumes that the productivity parameter (a), the savings rate (σ), the population growth rate (λ), and the depreciation rate (δ) are constant and exogenous. While there are papers that show that these parameters can be endogenized in economic models (e.g., the Ramsey-(1928)-Cass-(1965)-Koopmans-(1967) model endogenizes the savings rate), it makes sense to assume that these parameters are determined (at least to some extent) in non-economic systems (e.g., in the socio-cultural, ecological/climate, and political system; cf. Footnote 1 for references/evidence). In fact, the models that endogenize the AK model parameters are dependent upon other constant/exogenous parameters that are, in general, determined in the non-economic system (e.g., the Ramsey-(1928)-Cass-(1965)-Koopmans-(1967) model assumes that the savings rate is determined by an exogenous and constant time preference rate among others). Thus, even such models are based on NEPCAs. Since this Section 4 is devoted to a demonstration of the application of the partial dynamic equilibrium in the context of NEPCAs and not to a full theoretical foundation of the AK model, we simplify the discussion and the mathematical derivations significantly by assuming that all the AK model parameters are determined in the non-economic system. The reader may, however, keep in mind that the AK model parameters are partially determined in the economic system and partially determined in the non-economic system, i.e., they are 'partial' NEPCAs. Thus, further research may deal with the precise distinction between non-economic and economic determinants of these parameters.

Following the standard AK model, we assume that per-capita capital (k) is accumulated according to the following equation.

(12) $\dot{k}(t) = \sigma(t)a(t)k(t) - [\lambda(t) + \delta(t)]k(t),\ \ k(0) = k_0$ is given

Per-capita output (y) and per-capita consumption (c) are determined by (13) and (14).

(13) $y(t) = a(t)k(t)$

(14) $c(t) = [1 - \sigma(t)]\,y(t)$

Thus, according to the terminology introduced in Section 2, the economic system $\mathbf{e}$ encompasses the three variables k, y, and c and the parameter vector $\mathbf{p}$ consists of the parameters λ, σ, a, and δ, as stated by (15) and (16).

(15) $\mathbf{e}(t) := (k(t), y(t), c(t)),\ \ \varepsilon = 3$

(16) $\mathbf{p}(t) := (\lambda(t), \sigma(t), a(t), \delta(t)),\ \ \pi = 4$

The textbook AK model assumes that the parameters $\mathbf{p}$ are constant, i.e.,

(17) $\mathbf{p}(t) = \mathbf{p}^* := (\lambda^*, \sigma^*, a^*, \delta^*)$.

As discussed at the beginning of Section 4, (17) can be interpreted as a NEPCA. In contrast, to the textbook AK model, we assume that (a) the parameter vector $\mathbf{p}(t)$ is endogenously determined in the non-economic system $\mathbf{n}$ and (b) the non-economic system $\mathbf{n}$ and the economic system $\mathbf{e}$ depend upon each other. Without loss of generality, we implement these assumptions as follows. First, we assume that the non-economic variable $n_1(t)$ is determined by (18)–(20).

(18) $n_1(t) = \eta_1/[\eta_2(t) + y(t)]^{\eta_3}$

(19) $\eta_2(t) = \eta_0\exp(\eta_4 t)$

(20) $\eta_0, \eta_1, \eta_2, \eta_3, \eta_4 > 0$

For example, n_1 may be interpreted as an inverse index of socio-cultural development, where the index value is within the range $(0, \eta_1/\eta_0^{\eta_3}]$. The lower n_1, the higher the socio-cultural development level. For example, a high n_1 indicates that the society is relatively patriarchic, ruled by religious and family institutions, hierarchic, etc., while a relatively low n_1 indicates that the society is relatively emancipated, liberal, government ruled, etc. (cf., e.g., Bourguignon (2005)).

As we can see, (18) implies that economic development supports socio-cultural development, since n_1 decreases with per-capita income y. This assumption is supported by the literature on the positive effect of industrialization (which is closely related to per-capita income growth in early stages of development) on socio-cultural development (cf., e.g., Bourguignon (2005)).

Moreover, (18) and (19) imply that (in the economy being considered) there is some autonomous socio-cultural development indicated by η_2. That is, there is socio-cultural development even without economic development. This assumption reducing the relevance of economic development for socio-cultural development seems to make sense. It is, however, not crucial for any of our results.

We assume that the population growth rate λ is dependent on socio-cultural development (n_1) and, via the function φ_1, on some other non-economic variables (n_2, n_3, and n_4), as stated by (21) and (22).

(21) $\lambda(t) = \underline{\lambda} + n_1(t) + \varphi_1(n_2, n_3, n_4)$

(22) $\underline{\lambda} > 0, \ \varphi_1 > 0$

Equation (21) states that socio-cultural development has a negative impact on the population growth rate. This may make sense, since, e.g., emancipation and decreasing role of religious institutions decrease the fertility rate.

To simplify the discussion, we assume, without loss of generality, that (23) and (24) are true.

(23) $\forall t \quad \sigma(t) = \varphi_2(n_2, n_3, n_4) \equiv \sigma^* > 0 \ \wedge \ a(t) = \varphi_3(n_2, n_3, n_4) \equiv a^* > 0 \ \wedge \ \delta(t) = \varphi_4(n_2, n_3, n_4) \equiv \delta^* > 0$

(24) n_2, n_3, and n_4 are given and constant.

That is, the parameters σ, a, and δ are functions (φ_2, φ_3, and φ_4) of constant non-economic parameters (n_2, n_3, and n_4) and are, thus, constant. Even with this restriction, we can demonstrate all the relevant aspects of the partial dynamic equilibrium and NEPCAs.

According to the terminology introduced in Section 2, we can define the vector **n** as follows.

(25) $\mathbf{n}(t) := (n_1(t), n_2, n_3, n_4), \ \eta = 4$

Overall, we can see that in this model, the economic system **e**, the parameter vector **p**, and the non-economic system **n** are defined by (15), (16), and (25), respectively. The functions $\Gamma^e(\mathbf{e}(t), \mathbf{p}(t))$ (cf. (2)), $\Phi^p(\mathbf{n}(t))$ (cf. (1)), and $\Phi^n(\mathbf{e}(t), \mathbf{n}(t))$ (cf. (5)), which relate **e**, **p**, and **n**, are implied by (12)–(14), (21)/(23), and (18)/(24). We can see that our model, which is determined by (12)–(16) and (18)–(25), has the following characteristics:

1.) The three-dimensional economic system **e** (cf. (15)) depends on the four-dimensional parameter vector **p** (cf. (16)) via (12)–(14).

2.) The parameter vector **p** (cf. (16)) depends on the four-dimensional non-economic system **n** (cf. (25)) via (21) and (23).

3.) The non-economic system **n** (cf. (25)) depends on the economic system **e** (cf. (15)) via (18).

In particular, we can see that, in accordance with Sections 1 and 2, economic system dynamics have an impact on non-economic system dynamics (cf. n_1 and y in (18)) and vice versa (cf. k, λ, and n_1 in (12) and (21)), i.e., there are cross-system interactions. Moreover, the model assumptions (12)–(16) and (18)–(25) imply that there exists a (locally) stable asymptotic partial dynamic equilibrium (cf. Definition 3b) that is consistent with the AK model NEPCAs (17) in the limit, which can be proven as follows. (12), (21), and (23) imply (26). (13), (18), (19), (21), and (23) imply (27).

(26) $\dot{k}(t)/k(t) = \sigma^*a^* - [\lambda(t) + \delta^*]$

(27) $\lambda(t) = \underline{\lambda} + \eta_1/[\eta_0\exp(\eta_4 t) + a^*k(t)]^{\eta_3} + \varphi_1(n_2, n_3, n_4)$

Analogous to the standard AK model, (23) and (26) imply that $\dot{k}(0)/k(0) > 0$ if the product of savings rate and productivity parameter (σ^*a^*) is greater than the sum of population growth rate and depreciation rate ($\lambda(0) + \delta^*$), which is a standard assumption in the AK model. If we assume that the model parameters are such that $\dot{k}(0)/k(0) > 0$, then (20), (26), and (27) imply that (a) $\forall t \geq 0$ $\dot{k}(t)/k(t) \geq 0$ and (b) $\lambda(t)$ decreases strictly monotonously over time and converges to $\underline{\lambda} + \varphi_1(n_2, n_3, n_4)$, i.e., $\lim_{t\to\infty} \lambda(t) = \underline{\lambda} + \varphi_1(n_2, n_3, n_4) =: \lambda^*$. This result, (16), (23), and (25) imply that while the non-economic system **n** and the parameter vector **p** are non-static over time (since $\lambda(t)$ is non-static), they are constant in the limit, i.e., $\lim_{t\to\infty} \mathbf{n}(t) = \mathbf{n}^* \equiv (0, n_2, n_3, n_4)$ and $\lim_{t\to\infty} \mathbf{p}(t) = \mathbf{p}^* \equiv (\lambda^*, \sigma^*, a^*, \delta^*)$. Moreover, (13)–(15) and the fact that $\forall t \geq 0$ $\dot{k}(t)/k(t) > 0$ imply that the economic system **e** is non-static (even in the limit). This completes the proof of the existence of an asymptotic partial dynamic equilibrium in our model (cf. Definition 3b). The local stability of this asymptotic partial dynamic equilibrium is implied by the fact that the equilibrium exists for a non-empty and connected set of initial states **e**(0) and **n**(0). In particular, k_0, $y(0)$, $c(0)$, and $n_1(0)$ can be varied (within some ranges) without changing (a) the limit value ($\lim_{t\to\infty} \mathbf{p}(t) = \mathbf{p}^*$) of the parameter vector **p** and (b) the qualitative limit dynamics ($\lim_{t\to\infty} \dot{\mathbf{e}}(t) > 0$) of the economic vector **e** (cf. Definition 4b).

Overall, the AK model version presented in this section provides a foundation of the standard AK model's NEPCAs (17) while allowing for cross-system interactions. In particular, (a) the NEPCAs (17) are satisfied in the limit (and approximately satisfied when the parameter vector **p** is close to its limit state $\mathbf{p}^*$), (b) there are interactions between the economic system **e** and the non-economic system **n** while the parameter vector **p** converges to its limit state $\mathbf{p}^*$ (cf. the discussion of Definition 3b in Section 3.6.1), and (c) all the limit-predictions of our model are identical to the limit-predictions of the standard AK model despite cross-system interactions. Moreover, our model adds a transitional phase to the textbook AK model and, thus, increases the consistency of the AK model with the empirical evidence on the existence of transitional phases. In particular, the GDP growth rate ($\dot{y}/y$) increases and the population growth rate ($\dot{\lambda}/\lambda$) decreases over the transitional phase of our model because of interactions between socio-cultural and economic development. Thus, our model may serve as a joint socio-cultural and economic explanation of the transition from the pre-industrial 'Malthusian development phase' (which is characterized by slow GDP growth and fast population growth) to the modern industrial development phase (which is characterized by relatively fast GDP growth and relatively slow population growth). See also Galor (2011) for a detailed discussion of unified growth theory.

5. Concluding Remarks

As discussed in Section 1, (a) over the last decades, economic growth theory has reoriented towards quantitative, positive, and predictive models that are based on non-economic parameter constancy assumptions (abbr. NEPCAs), (b) NEPCAs seem to be very useful if not inevitable in long-run economic dynamics modeling, and (c) in the light of the empirical evidence on the interactions between the economic and non-economic system and the limits to the inclusion/study of all the specific interactions between all the economic and all the non-economic subsystems, it seems important to discuss the system-theoretical foundations of NEPCAs in presence of cross-system interactions. Devoting our paper to the latter, we approached as follows.

First, in Sections 3.1-3.5, we discussed the known types of dynamic equilibrium (among others structurally stable systems, homeostasis of the non-economic system, and slowly developing non-economic systems) that seem to be standard candidates for generating the dynamics that are consistent with NEPCAs in economic models. This discussion yields two mayor results:

(a) There are two major arguments against the validation of NEPCAs via empirical evidence on the constancy of the corresponding parameters or via the study of the interactions between the economic system and *specific* non-economic (sub)systems:

> (i) non-economic systems may be homeostatic, i.e., they may be stable over long periods of time and, nevertheless, change significantly or even drastically when the economic variables surpass certain threshold levels, such that the present/past

empirical information on the constancy of non-economic variables/parameters does not imply that these variables/parameters will be constant in the near future, and

(ii) even if it can be shown (theoretically or empirically) that each non-economic (sub)system's interaction with the economic system is marginal, the cumulative magnitude of the interactions between the economic system and the group of all relevant non-economic (sub)systems may be significant.

(b) There are several problems when using the standard types of dynamic equilibrium (cf. Sections 3.1-3.5) for justifying NEPCAs in economic modeling. Most importantly, the standard equilibrium types

(i) do not allow for cross-system interactions,

(ii) bear the possibility that NEPCAs are violated at some future point of (system) time and, thus, are not reliable foundations of NEPCAs,

(iii) are not consistent with NEPCAs, or

(iv) require interdisciplinary theoretical research[9] for justifying their application in modeling of cross-system interactions.

Second, we formulated a dynamic equilibrium type (which we name partial dynamic equilibrium) that solves these problems. In this sense, the concept of the (asymptotic) partial dynamic equilibrium represents the conditions that ensure that NEPCAs are consistent with cross-system interactions, i.e., it reveals the mathematical/system-theoretical nature of NEPCAs in presence of cross-system interactions.

Finally, we provided a simple theoretical model of interactions between the economic and non-economic (in particular, socio-cultural) system to demonstrate the application of the partial dynamic equilibrium in the context of NEPCAs used in economic modeling (and, in particular, in the AK model). In our model, cross-system interactions arise, where (a) the socio-cultural development (e.g., emancipation) affects the population growth rate and, thus, the economic system/growth and (b) economic development has a positive impact on socio-cultural development (cf., e.g., Bourguignon (2005)). These interactions generate a transition phase (from the Malthusian stage to the modern industrial stage), while the NEPCAs of the standard AK model are satisfied in the limit, i.e., our model converges to the standard AK model in the limit.

While we do not seek to support or oppose the usage of NEPCAs in long-run economic dynamics modeling, our results can be understood as a support of NEPCAs, since we show

[9] e.g., theoretical research trying to exclude the possibility that the non-economic system is (quasi-)homeostatic (cf. Sections 3.3 and 3.4).

that NEPCAs can be consistent with cross-system interactions while avoiding the problems that arise when standard dynamic equilibrium types (cf. Sections 3.1-3.5) are used to justify NEPCAs, as discussed above.

The concept of the partial dynamic equilibrium and, in particular, the conditions under which NEPCAs can be consistent with cross-system interactions (see Section 3.6.3 for a summary) can be used in further research as follows. First, the methodological implications of the partial dynamic equilibrium for economic growth modeling could be studied and further types of mathematical foundations of NEPCAs in presence of cross-system interactions could be elaborated. Second, our results can be applied in future theoretical modeling of cross-system interactions, as demonstrated in Section 4. In particular, the application of partial dynamic equilibriums (and dynamic equilibriums in general) simplifies the analysis of cross-system interactions and system dynamics. Thus, a modeling approach focusing on modeling partial dynamic equilibria could help to cope with the complex dynamics arising in the analysis of cross-system interactions. Third, theoretical and empirical research could try to identify non-economic (sub-)systems that are *not* modelable by partial dynamic equilibria and, therefore, should be incorporated/endogenized in economic growth modeling. In this way, the weaknesses of the models that rely on NEPCAs (and, in particular, the system interactions that are neglected by NEPCAs) could be identified more clearly and an (interdisciplinary) research program on endogenization of NEPCAs in economic growth modeling could be elaborated.

6. References

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