Multiplier effect and comparative statics in global games of regime change

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Abstract

This paper provides a general analysis of comparative statics results in global games. I show that the effect of a change in any parameter of the global game model can be decomposed into the direct effect, which captures the effect of a change in parameters when agents' beliefs are held constant, and the multiplier effect, which captures the role of adjustments in agents' beliefs. I characterize conditions under which the multiplier effect is strong and relate it to the strength of strategic complementarities and the publicity multiplier emphasized in earlier work. Finally, I use the above insights to identify when comparative statics can be deduced from the model's primitives, when they do not depend on the information structure, and when they coincide with predictions of the complete information model.

Key words: global games, comparative statics, multiplier effect, strategic complementarities, publicity multiplier

JEL codes: D83, D84

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1 Introduction

Global games are coordination games with incomplete information where agents' payoffs depend on whether the status quo is preserved or abandoned. This class of games was first introduced to the literature by Carlsson and Damme (1993) and popularized by Morris and Shin (1998, 2003). Since then, global games have been fruitfully used to study economic phenomena that feature coordination motives such as currency crises, sovereign debt crises, bank runs, business cycle fluctuations, or political revolts, among others.[1]

The popularity of global games stems from the fact that, in contrast to coordination games with complete information, global games tend to have a unique equilibrium. This uniqueness of equilibrium allows one to obtain unambiguous comparative statics results and policy prescriptions. Indeed, in applications of global games a significant effort is typically devoted to establishing comparative statics results. However, these results are derived on a case-by-case basis, and there exist few general results that could be invoked to simplify such analysis. Furthermore, there have been few attempts to understand how the presence of incomplete information structure and heterogeneous beliefs affect comparative statics results.

The goal of this paper is to fill in this gap in the literature. In particular, I focus on the following questions: What is the role of heterogeneous beliefs and the information structure in determining comparative statics results? When are the beliefs important drivers of comparative statics results? How different, qualitatively and quantitatively, are predictions of global games than predictions derived from the underlying complete information models?[2] To answer the above questions, I consider a general global game model. Using standard techniques, I compute the unique equilibrium of the model, which, as usual, is characterized by a regime change threshold θ^* (i.e., the value of fundamentals below which the status quo collapses and above which the status quo prevails). I then turn my attention to the comparative statics analysis of the regime change threshold, which is the focus of the paper.

The main result of the paper, which all other results in the paper build on, is that the change in the threshold θ^*, following a change in *any parameter* of the model, can be decomposed into a product of a "direct effect" and a "multiplier effect." The direct effect captures how a change in a parameter of the model affects the regime change threshold when agents' beliefs are held constant. Thus, the direct effect captures the fundamental (i.e., "non-

[1] For example, currency crises have been considered in Morris and Shin (1998), Hellwig, Mukherji, and Tsyvinski (2006), Angeletos, Hellwig, and Pavan (2006, 2007); debt crises have been addressed in Szkup (2017) and Zabai (2014); political revolts have been treated in Edmond (2013); and business cycles have been delt with in Schaal and Taschereau-Dumouchel (2016). For applications to banking and the design of banking regulation, see Eisenbach (2016), Goldstein and Pauzner (2005), Rochet and Vives (2004), or Vives (2014).

[2] With every global game model, we can associate a complete information model which is identical to the global game model but where agents observe the underlying fundamentals.

belief") channels through which a change in a parameter of the model affects the equilibrium. The multiplier effect, on the other hand, captures the effect of the adjustment in agents' beliefs about the likelihood of a regime change. I show that the multiplier effect is always greater than 1, and the same for each parameter of the model.

The above decomposition of comparative statics has three immediate consequences. First, it indicates that in order to determine whether a given change in a policy parameter decreases the likelihood of regime change, one can focus on the direct effect and abstract from adjustments in beliefs (i.e., hold beliefs constant). Second, it indicates that adjustments in beliefs act like an amplification mechanism that always magnifies the initial effect of the parameter change. Third, since the multiplier effect associated with a change in each parameter is the same, to identify which parameters have the strongest effect on the equilibrium it suffices to compare their direct effects. Thus, the above decomposition not only clarifies the role of beliefs in the model but also can be used to simplify comparative statics analysis.

In the remainder of the paper, I further investigate the properties of the multiplier and direct effects. I first relate the multiplier effect to the "publicity multiplier" and to the strength of strategic complementarities in the model. Morris and Shin (2003, 2004) showed that the impact of public information on the agents' equilibrium threshold signals is stronger than justified by its information content, and referred to this effect as the "publicity multiplier." I show that the publicity multiplier is a special case of the multiplier effect identified above, and that a similar effect is associated with other parameters of the model. I also find that the multiplier effect is large precisely when best-response functions are steep at the equilibrium threshold.[3] I then use this observation to characterize when the multiplier effect is strong, which allows me to identify conditions when a small shock to the model can have large equilibrium consequences.

Next, I turn my attention to the direct effect. Since the direct effect determines the sign of comparative statics results, I use it to answer three related questions: (1) When can the comparative statics results be deduced from the model's primitives?, (2) When are comparative statics results independent of the assumed information structure? (3) When do predictions of the global game model coincide with predictions based on analysis of the extremal equilibria of the complete information model? I provide a simple condition on the model's primitives under which comparative statics results can be deduced without solving the model and under which they are robust to changes in the information structure. I also provide two examples where this condition is violated and show that in such a case the information structure may affect the results. Finally, I provide conditions under which predictions of the

[3] This result identifies the relevant measure of strategic complementarities for comparative statics analysis of global games. The importance of accounting for the strength of strategic complementarities in global games when performing comparative statics analysis has recently been emphasized by Vives (2014).

global game model and the underlying complete information model coincide.

Throughout the paper, I show how the results established herein can be used to derive new results, improve understanding of the existing results, or extend existing results.

Related Literature — This paper contributes to the ever-growing literature on global games. Global games were introduced by Carlsson and Damme (1993), and extended by Frankel, Morris, and Pauzner (2003) and Oury (2013).[4] While global games have been extensively studied, there have been few attempts to derive general comparative statics results for global games or to understand the role that heterogeneous beliefs play in those results. The notable exceptions are Iachan and Nenov (2015), who study the effects of changes in the precision in private information quality on the regime change threshold, and Guimaraes and Morris (2007), who compare the predictions of global game model with those of a complete information framework in a context of a currency crisis model. Similar, to Iachan and Nenov (2015), I consider a general global game model, but do not limit myself to changes in information structure.

The analysis in the paper builds on insights from Cooper and John (1988) and Vives (2014). Cooper and John (1988) were the first to emphasize that models with strategic complementarities tend to feature a multiplier effect, although their analysis was limited to a complete information framework. Vives (2014) stresses the importance of taking into account the strength of strategic complementarities when performing comparative statics analysis in global games. The direct motivation for this work, however, comes from the applied literature and the difficulty of deriving (and interpreting) comparative statics results in complex global game models such as those in Eisenbach (2016), Szkup (2017), or Zabai (2014). Indeed, in Szkup (2017) I apply the results presented in this paper to analyze the effects of various government policies aimed at preventing self-fulfilling debt crises.[5]

From a broader perspective, this paper is also related to the work on monotone comparative statics and supermodular games (see, for example, Milgrom and Roberts, 1990; Topkis, 1998; Van Zandt and Vives, 2007; Vives, 1990; and Vives, 2004). One of the goals of these papers is to characterize a condition where a change in a parameter leads to a monotone adjustment either in the agent's choice (in a single-agent decision problem) or in the agent's

[4] See also Angeletos, Hellwig, and Pavan (2007), Mathevet and Steiner (2013), and Steiner (2008) for analysis of dynamic global games; Hellwig (2002), Morris and Shin (2004), Angeletos and Werning (2006), Hellwig, Mukherji, and Tsyvinski (2006), and Tarashev (2007) for analysis of the impact of exogenous and endogenous public information on the global games equilibrium; Dasgupta (2007) for analysis of an option to delay decisions; Szkup and Trevino (2015) for the analysis of information acquisition; Angeletos, Hellwig, and Pavan (2006) and Angeletos and Pavan (2013) for analysis of signalling; or Edmond (2013) for analysis of information manipulation.

[5] While the model in Szkup (2017) does not fit directly into the framework considered in this paper I show there that similar results can be extended to a more complicated micro-founded environment.

best-response function (in strategic environments). These papers are also helpful in the analysis of global games, however, by relying on specific properties of global games, I am able to derive more detailed results and to uncover properties that are related to the structure of global games in particular.

2 The Model

In this section, I describe the general model within which I perform my analysis. In Section 2.2 I provide several well-known examples that fit my setup. I also briefly characterize the unique equilibrium of the model, which will serve as the starting point for the comparative statics analysis performed in the remainder of the paper.

2.1 Setup

There is a continuum of players indexed by i, $i \in I$, where without loss of generality I is normalized to $[0,1]$. The set of players I is partitioned into a finite set $\mathcal{S}$ of types of players, $\mathcal{S}= \{s_1, ..., s_N\}$. For every $n \in \{1, ..., N\}$, s_n contains a continuum of identical players of measure λ_n, with $\Sigma_{n=1}^{N} \lambda_n = 1$. The type of player i is denoted by $s(i)$. All agents, regardless of their type, have the same action set $A = \{0,1\}$ and choose $a_i \in \{0,1\}$, where $a_i = 1$ corresponds to attacking the regime and $a_i = 0$ corresponds to not attacking the regime (i.e., supporting the status quo).[6] Let $m = \Sigma_{n=1}^{N} \left(\int_{i \in s_n} a_i di \right)$ denote the proportion of agents choosing to attack the status quo.

The economy is characterized by a state variable $\theta \in \mathbb{R}$, referred to as the strength of the regime, and by the regime status $\mathcal{R} \in \{0,1\}$, where $\mathcal{R} = 1$ denotes a change of the regime while $\mathcal{R} = 0$ means that the status quo is preserved. Initially, the economy is in the status quo. The regime changes, that is, $\mathcal{R} = 1$, if and only if

$$R(\theta, m; \boldsymbol{\psi}) < 0,$$

where $\boldsymbol{\psi}$ is a vector that contains all the parameters of the model.[7,8] The function R measures the resilience of the regime and is assumed to be continuously differentiable in all its arguments, with $R_1 > 0$ and $R_2 < 0$. That is, the resilience of the regime increases with

[6] I focus on binary global games since not only such games are more tractable but also because most of the literature focused on such games. It is also worth pointing out that the main results can be extended to the case of continuum of actions, but at the cost of substantially complicating the analysis.

[7] For example, in Morris and Shin (1998) the status quo is a currency peg, while the alternative regime is floating exchange rate regime; in Dasgupta (2007) the status quo is unprofitable (or unsuccessful) investment, while the alternative regime is the state where investment is profitable (successful); in Goldstein and Pauzner (2005) the status quo is a bank being solvent, while the alternative state is the bank becoming insolvent, etc.

[8] I provide specific examples of the vector $\boldsymbol{\psi}$ in Section 2.2.

θ, the intrinsic strength of the regime, and decreases with m, the proportion of agents that decide to challenge the status quo. Finally, I assume that for sufficiently low enough θ the regime will change even if no agent attacks it, while for sufficiently high θ the regime will survive even if all agents decide to challenge it. In other words, there exist $\underline{\theta}$ and $\overline{\theta}$ such that

$$R\left(\underline{\theta}, 0; \boldsymbol{\psi}\right) = 0 \quad \text{and} \quad R\left(\overline{\theta}, 1; \boldsymbol{\psi}\right) = 0,$$

and the regime collapses for all $\theta < \underline{\theta}$ while it survives for all $\theta > \overline{\theta}$ irrespective of the proportion of the agents that decides to attack it.[9]

The types of players differ in respect to their payoff functions. Since the action space is binary, when making their decisions, agents care only about the differential payoff between attacking the status quo and not attacking it. Thus, it is enough to specify the payoff differential functions rather than the payoff functions themselves. Let $\pi^n\left(\theta, m; \boldsymbol{\psi}\right)$ denote the payoff gain from choosing $a_i = 1$ rather than $a_i = 0$ for an agent of type s_n (the superscript on the function π denotes the type of the agent). Then

$$\pi^n\left(\theta, m, \boldsymbol{\psi}\right) = \begin{cases} H^n\left(\theta; \boldsymbol{\psi}\right) & \text{if } \mathcal{R} = 1 \\ L^n\left(\theta; \boldsymbol{\psi}\right) & \text{if } \mathcal{R} = 0 \end{cases},$$

where $H^n\left(\theta; \boldsymbol{\psi}\right) > 0$ is the payoff differential between attacking the status quo and not attacking it for an agent of type s_n when the regime changes and $L^n\left(\theta; \boldsymbol{\psi}\right) < 0$ is the corresponding when the status quo is preserved. For every $n \in \{1, ..., N\}$, H^n and L^n are differentiable in all their arguments, bounded, and non-increasing in θ.

The strength of the regime, θ, is distributed uniformly over the real line and is initially unobserved.[10] Agent i observes a private signal

$$x_i = \theta + \tau_{s(i)}^{-1/2} \varepsilon_i,$$

where ε_i is distributed according to an absolutely continuous distribution $F_{s(i)}$ with mean 0 and continuously differentiable density $f_{s(i)}$. The ε_i are identically distributed across agents of the same type, independent across all agents, and independent of θ. The parameter $\tau_{s(i)}$ measures the precision of agent i's signal, with a higher $\tau_{s(i)}$ implying a more precise signal. Note that all agents of the same type observe identically distributed signals with the same precision. However, the distribution of the signals and their precisions may vary across types.

As stated above, $\boldsymbol{\psi} = \{\psi_1, ..., \psi_M\} \in \mathbb{R}^M$ is the vector of all the parameters of the model, with ψ_m denoting a specific parameter. The vector $\boldsymbol{\psi}$ includes both the parameters of the

[9] Following the literature, I refer to $\left[\underline{\theta}, \overline{\theta}\right]$ as the coordination region.

[10] The assumption of a uniform improper prior is made for simplicity. Nevertheless, in Sections 4.1 and 5.2 I consider a model with a proper prior (though in those cases I limit myself to the Gaussian information structure).

information structure as well as parameters that directly affect the regime change function R, or the payoff differential functions H^n and L^n, $n = 1, ..., N$. Examples of this vector are given in the next section, in the context of specific global game models used in earlier papers.

2.2 Examples

The environment described above is general and encompasses a large number of models used in the applied global game literature. Below I show how several well-known models map into the setup described above. Later, I will use these particular models to illustrate applications of the results developed in the paper.

2.2.1 Morris and Shin (1998)

In their pioneering work, Morris and Shin (1998) used global games to study self-fulfilling currency crises. In their setup, there is a continuum of ex-ante identical speculators, indexed by i with $i \in [0, 1]$, who are deciding whether to attack a currency peg. The payoff from attacking the peg is $e^* - s(\theta) - t$ if the peg collapses following the attack, and $-t$ if the peg survives. Here, t captures a transaction cost associated with attacking the currency, e^* is the prevailing exchange rate (at which the currency is fixed), and $s(\theta)$ is the shadow exchange rate (i.e., the exchange rate that would materialize if the currency were allowed to float). Finally, θ captures the strength of the fundamentals, so that a higher θ is associated with a higher shadow exchange rate (i.e., $s'(\theta) > 0$). The speculators do not observe θ but only noisy signals $x_i = \theta + \sigma\varepsilon_i$, $\varepsilon_i \sim F$, with zero mean, and with ε_i that are *i.i.d.* across agents and independent of θ. The currency peg is maintained by a central bank with an objective function $v - c(\theta, m)$, where m is the proportion of speculators that attack the currency; v captures the benefit of maintaining the currency peg, while $c(\theta, m)$ is the cost of defending the peg, with $c_\theta < 0$ and $c_m > 0$.

The model described above fits into the framework of Section 2.1. To see this, let $\boldsymbol{\psi} = (v, e^*, t, \sigma)$ be the vector of the parameters of the model, assume that there is a single type of agent (or speculator), and set

$$R(\theta, m; \boldsymbol{\psi}) = v - c(\theta, m), \ H(\theta; \boldsymbol{\psi}) = e^* - s(\theta) - t, \text{ and } L(\theta; \boldsymbol{\psi}) = -t$$

2.2.2 Sakovics and Steiner (2012)

Sakovics and Steiner (2012) ask who should be subsidized in an investment game with strategic complementarities in order to maximize the probability of successful investment. They model the investment game as a global game.

In their setup, there is a continuum of investors divided into N groups, where λ_n is the measure of agents in group n, $n = 1, ..., N$, with $\Sigma_{n=1}^{N} \lambda_n = 1$. Investors simply choose whether to invest or not. For an investor that belongs to group n, the cost of investment is c_n. If he invests and the investment is successful, then he earns a benefit b; otherwise his benefit is 0. The other option is to refrain from investing, with a return of 0. The investment is successful if and only if enough agents invest, that is, if

$$m \geq 1 - \theta,$$

where m is the total mass of agents that invest. Agents do not observe θ, but each agent in group n observes a private signal $x_i = \theta + \tau^{-1/2}\varepsilon_i$, where $\varepsilon_i \sim F_n$ (so that agents in different groups have different signal distributions), with 0 mean.

Again, it is easy to see that this setup fits into the general setup described in Section 2.1. Define $\widehat{\theta} \equiv 1 - \theta$, let $\boldsymbol{\psi} = \{b_1, c_1, ..., b_N, c_N, \tau\}$, and set

$$R(\theta, m; \boldsymbol{\psi}) = \widehat{\theta} - m, \; H^n = b_n - c_n, \text{ and } L^n = -c_n$$

Here, I redefine the state of the economy as $\widehat{\theta} \equiv 1 - \theta$ so that R is increasing in the state of the economy, as was assumed in Section 2.1.

2.2.3 Bebchuk and Goldstein (2011)

Bebchuk and Goldstein (2011) use global games to model inefficient credit market freezes and to investigate policies that could prevent such undesirable outcomes. In their model, there is a continuum $[0, 1]$ of risk-neutral banks, each with a net worth of \$1, and which decide whether to invest in a risk-free asset or provide a loan to non-financial corporations. The return on the risk-free investment is equal to $1 + r$. The return on a corporate loan is equal to $1 + R$ (with $R > r$) if the economic fundamentals are strong and a sufficient number of corporations obtain credit, and 0 otherwise. In particular, a corporate loan pays net return R if and only if $\theta + zm > b$, where θ captures the strength of the economy, m is the mass of firms that received funding from the banks, z captures the strength of aggregate investment complementarities in the economy, and b is a threshold level for the loans to be profitable. As usual, banks do not observe θ, but each bank observes a private noisy signal $x_i = \theta + \tau^{-1/2}\varepsilon_i$, $\varepsilon_i \sim N(0, 1)$, with the ε_i *i.i.d.* across agents and independent of θ.

Again, it is easy to see that this setup is included in the model of Section 2 with $\boldsymbol{\psi} = \{r, R, z, b, \tau\}$, and

$$R(\theta, m; \boldsymbol{\psi}) = \widehat{\theta} - zm, \; H^k = R - r, \text{ and } L^k = -(1 + r),$$

where again I redefine the state of the economy as $\widehat{\theta} \equiv b - \theta$ so that R is increasing in the state of the economy.

2.3 The Unique Equilibrium

In this section, I characterize the unique equilibrium of the model described in Section 2.1. While this result is standard, the equilibrium conditions described below will be the starting point for the subsequent analysis given in paper. But first, I make standard assumptions pertaining to the regime change function R and the payoff functions which will be maintained throughout the paper.

Assumption 1 *1. If $R(\theta, m, \psi) = 0$, then $\mathcal{R} = 1$.*

2. If indifferent, agent i attacks the regime.

3. For each $n \in \{1, .., N\}$, $H^n(\theta; \boldsymbol{\psi})$ and $L^n(\theta; \boldsymbol{\psi})$, as well as $\partial H^n(\theta; \boldsymbol{\psi}) / \partial\theta$ and $\partial L^n(\theta; \boldsymbol{\psi}) / \partial\theta$, are bounded and integrable with respect to the measure induced by the CDF F^n.

The first two parts of Assumption 1 are commonly assumed tie-breaking assumptions. The last part imposes boundedness and integrability conditions on the payoff functions and their derivatives. These technical assumptions are maintained throughout the paper.

Let $\alpha_i : \mathbb{R} \to \{0, 1\}$ denote agent i's strategy. As usual in the literature, I focus on monotone strategies (also referred to as threshold strategies), that is, strategies where there is some $x_i^* \in \mathbb{R}$ such that $\alpha_i(x_i) = 1$ if and only if $x_i \le x_i^*$ and $\alpha_i(x_i) = 0$ if $x_i > x_i^*$. The threshold x_i^* is referred to as the threshold signal, that is, the value of the signal at which agent i switches from attacking the regime to not attacking it. An equilibrium in which all agents follow monotone strategies is called a monotone equilibrium.

The next result states that the model has a unique equilibrium and provides the equilibrium conditions.[11]

Proposition 1 *There exists a unique equilibrium where the regime changes if and only if $\theta \le \theta^*$ and where all of the following hold:*

1. *All agents of type $s_n \in \mathcal{S}$ use a monotone strategy with threshold x_n^*, where x_n^* is the unique solution to*

$$\int_{-\infty}^{\theta^*} H^n(\theta; \boldsymbol{\psi}) f_n(\theta | x_n^*) \, d\theta + \int_{\theta^*}^{\infty} L^n(\theta; \boldsymbol{\psi}) f_n(\theta | x_n^*) \, d\theta = 0 \tag{1}$$

2. *The regime change threshold θ^* is the unique solution to*

$$R\left(\theta^*, \sum_{n=1}^{N} \lambda_n F_n\left(\frac{x_n^* - \theta^*}{\tau_n^{-1/2}}\right); \boldsymbol{\psi}\right) = 0 \tag{2}$$

[11] Proofs of this and other results can be found in the Appendix.

3. In the limit as $\tau_n \to \infty$ for all $n = 1, ..., N$, the regime change threshold converges to

$$R\left(\theta^*, \sum_{n=1}^{N} \lambda_n \frac{H^n(\theta^*; \boldsymbol{\psi})}{H^n(\theta^*; \boldsymbol{\psi}) - L^n(\theta^*; \boldsymbol{\psi})}; \boldsymbol{\psi}\right) = 0$$

The above result establishes that the model has a unique equilibrium which is in monotone strategies and characterized by $N+1$ equations: the N payoff indifference equations (one for each type of agents, per Equation (1)), and the regime change condition (Equation (2)). It also establishes that in the limit the proportion of agents that attacks the regime when $\theta = \theta^*$ is given by $\sum_{n=1}^{N} \lambda_n \frac{H^n(\theta^*;\boldsymbol{\psi})}{H^n(\theta^*;\boldsymbol{\psi}) - L^n(\theta^*;\boldsymbol{\psi})}$.

Having established uniqueness of the equilibrium and derived the equilibrium conditions, I now turn my attention to the main focus of the paper, that is, the comparative statics results and the role played by the beliefs in their determination.

3 The Multiplier and Direct Effects

The starting point for my analysis is the simple observation that θ^* plays a dual role in the above system of equilibrium conditions (Equations (1) and (2)). First, θ^* is the actual equilibrium threshold such that if $\theta \leq \theta^*$ then the regime changes, while if $\theta > \theta^*$ the current regime stays in place (Equation (2)). Second, in the payoff indifference condition for type s_n, θ^* corresponds to the agent's belief about the regime change threshold (Equation(1)). Of course, in the equilibrium the actual and the expected regime change thresholds coincide, but away from equilibrium they might be different. As I show below, this simple observation leads to interesting insights into comparative statics predictions of global games.

Let $\psi_m \in \boldsymbol{\psi}$ be a parameter of interest, and suppose we are interested in understanding how a change in ψ_m affects the equilibrium thresholds θ^* and x^*. Motivated by the above discussion, I differentiate between "partial" and "total" changes in θ^* and x^* in response to a change in ψ_m. In particular, I denote by $\partial x_n^*/\partial \psi_m$ the effect that a change in ψ_m has on type s_n agents' threshold signal when agents' beliefs about θ^* are held constant. Similarly, I denote by $\partial \theta^*/\partial \psi_m$ the partial effect of a change in ψ_m on the regime change threshold when agents' strategies care held constant (i.e., with $\{x_n^*\}_{n=1}^N$ held constant).[12] Finally, I denote the total effects of a change in ψ_m on the equilibrium thresholds (including the effect through the change in beliefs) by $d\theta^*/d\psi_m$ and $dx_n^*/d\psi_m$. In other words, $d\theta^*/d\psi_m$ and $dx_n^*/d\psi_m$ correspond to the equilibrium effects induced by a change in ψ_m that one would typically compute when performing comparative statics analysis, while $\partial\theta^*/\partial\psi_m$ and

[12] Formally, $\partial x_n^*/\partial \psi_m$ is computed by applying the implicit function theorem to the payoff indifference condition for type s_n with θ^* treated as an exogenous constant, while $\partial\theta^*/\partial\psi_m$ is computed by applying the implicit function theorem to Equation (2) with $\{x_n^*\}_{n=1}^N$ held constant.

$\partial x_n^*/\partial\psi_m$ correspond to the partial effects implied by a change in ψ_m when ignoring the adjustments in endogenous variables.

Having introduced the above notation, I now state the main result of the paper, which all of the subsequent analysis is based on.

Theorem 1 *Fix $\boldsymbol{\psi}$. For any $\psi_m \in \boldsymbol{\psi}$, we have*

$$\frac{d\theta^*}{d\psi_m} = \underbrace{\frac{1}{1-\sum_{n=1}^N \frac{\partial\theta^*}{\partial x_n^*}\frac{\partial x_n^*}{\partial\theta^*}}}_{\text{The Multiplier Effect } (\mathcal{M}(\psi_m))} \underbrace{\left[\frac{\partial\theta^*}{\partial\psi_m}+\sum_{n=1}^N \frac{\partial\theta^*}{\partial x_n^*}\frac{\partial x_n^*}{\partial\psi_m}\right]}_{\text{The Direct Effect } (\mathcal{D}(\psi_m))}$$

Moreover,

1. $\mathcal{M}(\psi_m) \in (1,\infty)$ *if* $\tau_n < \infty$ *for all* $n \in \{1,...,N\}$.
2. *For any* $\psi_m, \psi_l \in \boldsymbol{\psi}$, *we have* $\mathcal{M}(\psi_m) = \mathcal{M}(\psi_l)$.
3. *If* $\tau_n = \tau$ *for all* $n \in \{1,...,N\}$, *then*

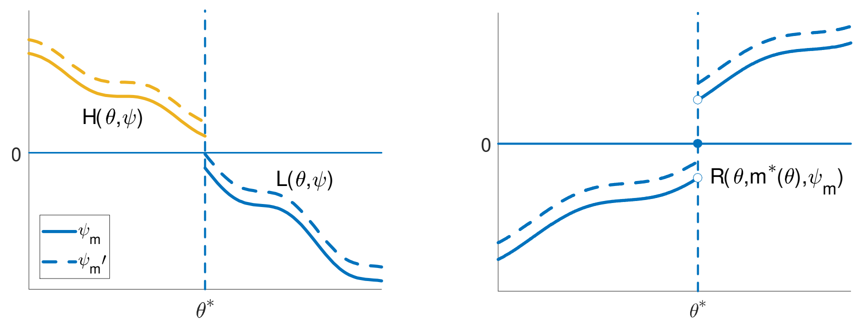
$$\lim_{\tau\to\infty}\mathcal{M} = \infty \quad \textit{and} \quad \lim_{\tau\to\infty}\mathcal{D}(\psi_m) = 0 \textit{ with } \lim_{\tau\to\infty}\mathcal{MD}\in\mathbb{R}$$

The first part of Theorem 1 establishes that a change in θ^* induced by a change in any parameter of the model can be decomposed into the "direct effect" and the "multiplier effect." The direct effect captures the effect that a change in ψ_m has on θ^* when all agents' beliefs about the regime change threshold are held constant (i.e., with θ^* constant in Equation (1)). In particular, a change in ψ_m can lead to a change in θ^* by directly affecting the regime change condition (as captured by $\partial\theta^*/\partial\psi_m$), or indirectly by affecting the payoff indifference conditions and leading to a change in individual threshold signals while holding agents beliefs about θ^* unchanged (as captured by $(\partial\theta^*/\partial x_n^*)(\partial x_n^*/\partial\psi_m)$). Both of these effects are captured by $\mathcal{D}(\psi_m)$. Thus, the "direct effect" captures the fundamental (i.e., "non-belief") channels through which a change in a parameter affects the equilibrium.

However, following a change in ψ_m, agents' beliefs are not constant. In particular, agents understand that a change in ψ_m leads to a change in θ^*, and hence adjust their beliefs and actions, inducing a further adjustment in θ^*. This leads to another round of adjustments in agents' beliefs, and hence in θ^*, and so on. These adjustments are captured by the "multiplier effect." Thus, the multiplier effect captures the role that adjustments in beliefs play in the change in θ^*.[13]

[13] The above discussion suggests that the decomposition of comparative statics stated in Theorem 1 can be obtained by analyzing equilibrium best-response dynamics (see, for example, Vives (2004)). Indeed, in the Appendix I show that the above result can be derived either by using the implicit function theorem or by computing the best-response dynamics. The latter has the advantage of providing an intuitive interpretation of this result.

The second part establishes several important properties of the direct and multiplier effects. First, it states that the multiplier effect is always positive and greater than 1, but finite as long as the precision of the information is finite. Second, a change in any element of $\boldsymbol{\psi}$ results in the same multiplier effect.[14] In other words, if ψ_m and ψ_l are two distinct parameters of the model, then the difference in the equilibrium effect of changes in ψ_m and ψ_l are fully attributed to the difference in their direct effects. As a consequence, we can simply denote the multiplier effect by $\mathcal{M}$. Finally, we see that as agents' signals become infinitely precise the multiplier effect tends to infinity, while the direct effect tends to 0, implying that in the limit all of the adjustments in θ^* are driven by the adjustments in beliefs.



Panel A: Expected payoff as $\tau \to \infty$ Panel B: Regime resistance as $\tau \to \infty$

Figure 1: Expected payoff and regime resistance as a function of θ in the limit as $\tau \to \infty$

Why do we have $\lim_{\tau\to\infty} \mathcal{D}(\psi_m) = 0$ and $\lim_{\tau\to\infty} \mathcal{M} = \infty$? Consider first the part of the direct effect that operates through the payoff indifference condition (i.e., $(\partial\theta^*/\partial x_n^*)(\partial x_n^*/\partial\psi_m)$). If agent i could observe θ, then he would always attack the regime if $\theta \geq \theta^*$ since in this case he would be certain to receive payoff $H^{s(i)}(\theta;\boldsymbol{\psi}) > 0$. Similarly, he would always refrain from attacking if $\theta < \theta^*$, since in this case he would be certain that he would receive $L^{s(i)}(\theta;\boldsymbol{\psi}) < 0$. Note that this is true regardless of the actual values that $H^{s(i)}(\theta;\boldsymbol{\psi})$ and $L^{s(i)}(\theta;\boldsymbol{\psi})$ take as long as $H^{s(i)}(\theta;\boldsymbol{\psi}) > 0$ and $L^{s(i)}(\theta;\boldsymbol{\psi}) < 0$. Thus, even if a change in ψ_m leads to changes in the payoff functions $H^{s(i)}(\theta;\boldsymbol{\psi})$ and $L^{s(i)}(\theta;\boldsymbol{\psi})$ this would have no impact on player i's behavior (as depicted on Panel A of Figure 1). But as $\tau \to \infty$ we converge to the case where agents can predict θ exactly, and thus in the limit $\partial x_n^*/\partial\psi_m \to 0$ for all $n \in \{1, ..., N\}$.

Note that the above discussion also implies that as $\tau \to \infty$ the equilibrium proportion of agents attacking the regime (which I denote by $m^*(\theta)$) converges to a step function with

[14] Note that this does not mean that multiplier effect is independent of parameters of the model. Indeed, in Section 5 I discuss how $\mathcal{M}$ varies with $\boldsymbol{\psi}$. Rather, it states that starting with a fixed $\boldsymbol{\psi}$ a small change in any element of $\boldsymbol{\psi}$ results in the same multiplier effect.

$m^* (\theta) = 1$ for all $\theta > \theta^*$ and $m^* (\theta) = 0$ for all $\theta < \theta^*$ with $m^* (\theta^*) \in (0, 1)$ as determined in Proposition 1. This, in turn, implies that in the limit the regime change function $R (\theta^*, m^* (\theta))$ is strictly smaller than 0 for $\theta < \theta^*$, takes the value 0 at $\theta = \theta^*$ and is strictly greater than 0 for all $\theta > \theta^*$. As such, any potential effect of a small change in ψ_m on θ^* is always dominated by the discontinuous jump in the proportion of agents attacking the regime (see Panel B of Figure 1). As a consequence, in the limit $\partial\theta^*/\partial\psi_m = 0$.

It is worth stressing that despite its simplicity, Theorem 1 proves to be a surprisingly useful tool for computing and understanding comparative statics results, as it clarifies the role of agents' beliefs and the channels through which a change in ψ_m affects θ^*. First, it tells us that in order to establish whether a change in ψ_m will increase, decrease, or left unchanged the regime change threshold θ^* it is enough to determine the sign of the direct effect. Thus, for the purpose of obtaining qualitative predictions, one can treat beliefs as a fixed object, which can substantially simplify the analysis.[15] Second, Theorem 1 implies that the adjustment in beliefs acts like an amplification mechanism that always magnifies the initial response of θ^* to a change in ψ_m. Finally, we see that in order to determine which parameter has the strongest effect on θ^*, it suffices to compare the direct effect induced by each parameter, the observation that I utilize in Section 3.1. I state the above observations as a corollary.

Corollary 1 *Consider the effect of a change in ψ_m on the equilibrium.*

1. *The direction of the change in θ^* is determined by the direct effect, that is*
$$sgn\left(\frac{\partial\theta^*}{\partial\psi_m}\right) = sgn\left(\mathcal{D}\left(\psi_m\right)\right)$$
2. *The adjustment in beliefs always amplifies the initial response of θ^*, that is,*
$$\left|\frac{d\theta^*}{d\psi_m}\right| \geq \left|\frac{\partial\theta^*}{\partial\psi_m}\right|,$$
with a strict inequality holding whenever $\frac{\partial\theta^*}{\partial\psi_m} \neq 0$.
3. *Suppose that $\psi_m \in \boldsymbol{\psi}$ leads to the strongest direct effect. Then*
$$\frac{d\theta^*}{d\psi_m} > \frac{d\theta^*}{d\psi_k} \quad \text{for all } \psi_k \in \boldsymbol{\psi},\ k \neq m$$

The remainder of the paper is devoted to investigating further properties of the multiplier effect (Sections 4 and 5) and understanding further implications of Theorem 1 for comparative statics analysis (Section 6). Finally, in Section 7 I discuss several extensions of Theorem 1.

[15] For the application of this approach see Szkup (2017).

3.1 Application: Design of Investment Subsidies (Sakovics and Steiner, 2012)

Part 3 of Corollary 1 implies that if we want to increase the regime change threshold, then we should change the parameter ψ_m which is associated with the largest direct effect. This observation can substantially simplify the analysis of optimal policy design. In this section, I show how this result can be used to extend the result of Sakovics and Steiner (2012) regarding the optimal design of investment subsidies. Another potential application of this result is in the design of optimal financial regulations.

Consider the model described in Section 2.2.2, where a continuum of investors decides whether to invest in a risky project.[16] Following Sakovics and Steiner (2012), suppose that a social planner wants to use investment subsidies in order to encourage investment. In particular, the planner wants to ensure that the threshold below which investment is successful is at least $\widehat{\theta}$, and he wants to achieve this in the least costly way. Let $\mathbf{v} = (v_1, ..., v_N)$ denote the vector of subsidies, with v_n denoting the subsidy granted to agents of type s_n. The planner's problem is then

$$\min \sum_{n=1}^{N} \lambda_n v_n$$
$$\text{s.t. } \theta^* (\mathbf{v}) \geq \widehat{\theta}$$
$$\overline{v} > v_n \geq 0 \text{ for all } n \in \{1, ..., N\}$$

where $\overline{v}$ is the maximum subsidy that can be given to agents (which ensures that the cost of investing is always non-negative) and where $\theta^* (\mathbf{v})$ is the threshold below which investment is successful when the vector of subsidies is $\mathbf{v}$.

The following result is a simple application of Theorem 1.

Proposition 2 *Suppose that for all feasible* $\mathbf{v}$ *we have*

$$\frac{\mathcal{D}(v_1; \mathbf{v})}{\lambda_1} > .... > \frac{\mathcal{D}(v_N; \mathbf{v})}{\lambda_N}$$

Then there exists $n^* \in \{1, ..., N\}$ *such that* $v_n = c_n$ *for all* $n < n^*$, $v_n = 0$ *for all* $n > n^*$, *and* $v_n \in [0, c_n]$ *for* $n = n^*$.

This result states that if the direct effect of a subsidy to agents in group n is uniformly greater than the direct effect to group l with $l > n$, then the planner should fully subsidize group n first, and only then subsidize group $n+1$, if further subsidies are still needed to reach

[16] In this case, the action of attacking the status quo is interpreted as investing, while refraining from challenging the status quo is interpreted as not investing, and θ is the inverse measure of the strength of the economic fundamentals, meaning that a lower value of θ is associated with a better state of the economy.

the desired threshold $\widehat{\theta}$. The above result might seem obvious, but it generalizes the results derived by Sakovics and Steiner (2012) to the case of non-linear payoff, non-linear regime change condition, and imperfectly informative signals.[17] This is the advantage of Theorem 1: It substantially simplifies analysis of problems that otherwise might seem difficult to solve.[18]

Proposition 2 also helps us to better understand the intuition behind the choice of subsidies. It implies that if the current level of subsidies is $\mathbf{v}$ but the planner would like to further increase the investment threshold θ^*, then on the margin he would subsidize the group with the largest $\mathcal{D}(v_n; \mathbf{v}) / \lambda_n$. Since $\mathcal{D}(v_n; \mathbf{v}) = [\partial\theta^* / \partial x_n^*] [\partial x_n^* / \partial v_n]$, it follows that on the margin the planner targets the group of agents which is (1) responsive to the subsidies (high $\partial x_n^* / \partial v_n$) and (2) which is influential at the aggregate level (high $\partial\theta^* / \partial x_n^*$) but (3) relatively small so the subsidy is not costly (low λ_n). Factors such as the strength of strategic complementarities between groups or the sensitivity of agents to changes in the aggregate threshold, which are captured by the multiplier effect, do not play any role in his choice.

4 Understanding the Multiplier Effect

In this section, I explore how the multiplier effect identified above is related to the publicity multiplier and to the strength of strategic complementarities. This analysis is motivated by the work of Morris and Shin (2003, 2004), who were the first to argue that in global games public news has a disproportionately strong impact relative to their informational content (and who referred to this property of public information as "publicity multiplier"), and by the work of Vives (2004, 2014) who stressed that the strength of strategic complementarities in global games is one of the key parameters of the model.

4.1 Publicity Multiplier

Morris and Shin (2003, 2004) drew attention to the role played by public information as a coordination device. They show that in global games a public news can have a disproportional impact on the behavior of players relative to its informational content, and refer to this effect as the "publicity multiplier."[19] Below, I investigate how the publicity multiplier is related to

[17] In particular, one can verify that in the specification considered by Sakovics and Steiner (2012) the condition of Proposition 2 is satisfied.

[18] Sakovics and Steiner (2012) uncover the fundamental property of global games, which intuitively states that the amount of optimism in the model is in fixed supply, and they base their investigation on this property. Each approach has its advantages and disadvantages, and they should be seen as complementary. In particular, their approach provides a useful restriction on the equilibrium which can be used to compute the equilibrium or its properties.

[19] The role played by public information in global games has been also investigated by Hellwig (2002), Bannier and Heinemann (2005), and Metz (2002). See also Morris and Shin (2002), Angeletos and Pavan

the multiplier effect derived above, and revisit the question of when the publicity multiplier is particularly strong that Morris and Shin (2003) originally explored.

For the purpose of this section, I introduce a proper prior belief into the model and restrict my attention to the Gaussian information structure and to a single type of agent. In particular, following Morris and Shin (2003) I assume that all agents are ex-ante identical, they share a common prior belief $\theta \sim N\left(\mu_\theta, \tau_\theta^{-1}\right)$ and each of them receives a private signal $x_i = \theta + \tau_x^{-1/2} \varepsilon_i$, with $\varepsilon_i \sim N(0,1)$, ε_i iid across agents, and independent of θ. Here, μ_θ can be interpreted as the public information available to the agents.[20] Finally, as in Morris and Shin (2003), I assume that μ_θ has no direct effect on either payoffs differential function $\pi(\theta; \boldsymbol{\psi})$ or the regime change condition $R(\theta, m; \boldsymbol{\psi})$ (i.e., $\partial \pi(\theta; \boldsymbol{\psi}) / \partial \mu_\theta = 0 = \partial R(\theta, m; \boldsymbol{\psi}) / \partial \mu_\theta$), but affects the equilibrium play only via its impact on agents' posterior beliefs. Otherwise, the setup is unchanged relative to Section 2.

Let θ^* be the unique equilibrium regime change threshold, and let x^* be the associated threshold signal.[21] Morris and Shin (2003) define the publicity multiplier as

$$\mathcal{P} \equiv \frac{\frac{dx^*}{d\mu_\theta}}{\frac{\partial x^*}{\partial \mu_\theta}},$$

where $dx^*/d\mu_\theta$ is the total change in x^* following a change in the mean of the prior (i.e., in public information) and $\partial x^* / \partial \mu_\theta$ measures the effect of a change in μ_θ through its impact on agents' posterior beliefs. In other words, $\frac{dx^*}{d\mu_\theta} = \mathcal{P} \frac{\partial x^*}{\partial \mu_\theta}$; here we differentiate between the partial and total effects of a change in μ_θ on x^* rather than on θ^*.

Next, note that μ_θ is just one of the parameters of the model, so that $\mu_\theta \in \boldsymbol{\psi}$. It follows that a similar multiplier effect can be derived for any $\psi_m \in \boldsymbol{\psi}$. Thus, we can define $\mathcal{P}(\psi_m)$ as the multiplier effect that a change in ψ_m has on x^*. Nevertheless, μ_θ does have a distinct property in the current setup: μ_θ affects only agents' payoff indifference condition (via its effect on agents' posterior beliefs) but has no effect on the regime change condition. With this last observation, I can state the following result.

(2007), and Ui and Yoshizawa (2015) for analysis of public information in closely related quadratic-Gaussian models.

[20] Equivalently, one can assume that agents have an improper uniform prior over $\mathbb{R}$ about θ, and observe a public signal $\mu_\theta = \theta + \tau_\theta^{-1/2} \eta$, $\eta \sim N(0,1)$.

[21] With public information, an equilibrium is unique if and only if private precision is precise enough relative to the precision of the public signal. In the current context, the required condition is

$$\tau_x^{1/2} / \tau_\theta > \frac{1}{\sqrt{2\pi}} \frac{-\overline{R}_2}{\underline{R}_1}$$

where $\underline{R}_1$ is the lower bound on $\partial R / \partial \theta$ and $\overline{R}_2$ is the upper bound on $\partial R / \partial m$. The derivations of this condition are standard and hence omitted from the paper.

Proposition 3 *Let ψ^P denote the vector of parameters that affect only the agents' payoff indifference conditions and let $\mathcal{P}(\psi_m)$ denote the multiplier effect associated with the change in $\psi_m \in \psi^P$.*[22] *Then*

$$\mathcal{P}(\psi_m) = \mathcal{M}, \text{ implying that } \frac{dx^*}{d\mu_\theta} = \mathcal{M}\frac{\partial x^*}{\partial \mu_\theta} \text{ for all } \psi_m \in \boldsymbol{\psi}^P$$

The above proposition has two implications. First, it implies that there is nothing special about the "publicity multiplier" and that such a multiplier applies to any parameter $\psi_m \in \boldsymbol{\psi}^P$. Second, we see that the effect of this multiplier, which is associated with the comparative statics of x^*, is the same as that which is associated with changes in θ^*.

The fact that $\mathcal{P}(\psi_m) = \mathcal{M}$ has important consequences: It implies that the publicity multiplier $\mathcal{P}(\mu_\theta)$ tends to tends to infinity as $\tau_x \to \infty$ and achieves its "maximum value" when public information is ignored by the agents. The latter observation seems counter-intuitive, as it seems to suggest that public information has the strongest impact when agents ignore it. The solution to this apparent contradiction is simple. As the precision of private information increases, agents attach less and less weight to the public information, and hence the direct effect of a higher μ_θ decreases. The decrease in the direct effect dominates the increase in the multiplier effect, and thus the total impact of public information on equilibrium threshold tends to 0 as $\tau \to \infty$. Thus, Morris and Shin (2003) are correct to point out that μ_θ has the strongest impact on the equilibrium when τ_θ is high, but this is driven to a large degree by the direct effect rather than the multiplier effect.

Following Morris and Shin (2003), in Proposition 3 I restricted my attention to parameters of the model that affects only the payoff indifference condition and considered a setup with a single type of agents. One may wonder whether these restrictions are important for the result. Not surprisingly, they do not, and a similar decomposition of the total change in $dx^*/d\psi_m$ into the multiplier effect and the partial effect can be derived (Section D.1 in the Appendix). Thus, I conclude that the model features a unique multiplier effect $\mathcal{M}$ irrespective of whether we focus on the change in x^* or the change in θ^*, or which parameter of the model we consider.

4.2 Relation to Strategic Complementarities

In this section, I investigate the relation between the magnitude of the multiplier effect and the strength of strategic complementarities. The analysis is motivated by findings of Vives (2014), who stressed that "the degree of strategic complementarity of investors' actions is the crucial parameter (...) for policy analysis" and used this insight to show that the effect of financial regulation depends on the strength of strategic complementarities. Since, as shown in Theorem 1, the multiplier effect determines the overall effect of a given parameter change

[22] In other words, $\partial R/\partial\psi_m = 0$ for all $\psi_m \in \psi^P$.

on the equilibrium, this suggests that there is a close connection between the magnitude of the multiplier effect and the degree of strategic complementarity of agents' actions. The main goal of this section is to understand this relation.

Typically, one measures the degree of strategic complementarity by the steepness of the agents' best-response functions. To compute best-response functions suppose that all agents use threshold strategies, and let $\widehat{x}_n$ be the threshold used by all the agents of types s_n, $n = 1, ..., N$. Let $\widehat{\boldsymbol{x}} = \{\widehat{x}_1, ..., \widehat{x}_N\}$ be the vector of these thresholds, and denote by $\widehat{\theta}\left(\widehat{\boldsymbol{x}}\right)$ the implied regime change threshold. Given that all other agents use monotone strategies, the best response of agent i is to use a monotone function with a threshold signal $\beta_{s(i)}\left(\widehat{\boldsymbol{x}}\right)$, where $\beta_{s(i)}\left(\widehat{\boldsymbol{x}}\right)$ is implicitly defined as the unique solution to agent i's indifference condition

$$\int_{-\infty}^{\widehat{\theta}(\widehat{\boldsymbol{x}})} H^{s(i)}\left(\theta;\boldsymbol{\psi}\right) f_{s(i)}\left(\theta|\beta_{s(i)}\left(\widehat{\boldsymbol{x}}\right)\right) d\theta + \int_{\widehat{\theta}(\widehat{\boldsymbol{x}})}^{\infty} L^{s(i)}\left(\theta;\boldsymbol{\psi}\right) f_{s(i)}\left(\theta|\beta_{s(i)}\left(\widehat{\boldsymbol{x}}\right)\right) d\theta = 0$$

The above equation defines implicitly the best-response function for agents of type $s\left(i\right)$, $\beta_{s(i)}\left(\widehat{\boldsymbol{x}}\right)$.

To measure the strength of strategic complementarities, one can ask how much $\beta_{s(i)}\left(\widehat{\boldsymbol{x}}\right)$ increases as all the $\widehat{x}_n$, $n = 1, ..., N$, increase by a small amount. This is equivalent to computing the directional derivative of $\beta_{s(i)}\left(\widehat{\boldsymbol{x}}\right)$ in the direction $\underline{\mathbf{1}} = (1, ..., 1)^T \in \mathbb{R}^N$. I denote this directional derivative by $\nabla_{\underline{\mathbf{1}}}\beta_{s(i)}\left(\widehat{\boldsymbol{x}}\right)$, where

$$\nabla_{\underline{\mathbf{1}}}\beta_{s(i)}\left(\widehat{\boldsymbol{x}}\right) = \sum_{n=1}^{N} \frac{\partial \beta_{s(i)}\left(\widehat{\boldsymbol{x}}\right)}{\partial\left(\widehat{x}_n\right)}$$

Using this definition, strategic complementarities are stronger when the best-response functions are steeper.

While natural, the above definition is cumbersome to use in practice. This is because in order to determine how the strength of strategic complementarities varies with the parameters of the model, we must compare the best-response functions on their entire domains, which is often challenging. It also suffers from the problem that in many cases a change in the setup will result in a best-response function becoming steeper at some $\widehat{\boldsymbol{x}}$ but flatter at others.[23]

Proposition 4 offers a solution to this problem. Specifically, it establishes that the magnitude of the multiplier effect, and hence the total effect of the change in θ^*, is determined by the slope of the best-response function evaluated at $\widehat{\boldsymbol{x}} = \mathbf{x}^*$, where $\mathbf{x}^* = (x_1^*, ..., x_N^*)$ is the vector of equilibrium signal thresholds. Thus, from the comparative statics point of view the relevant measure of the strategic complementarities in global games is the slope of the best-response function evaluated at $\widehat{\boldsymbol{x}} = \mathbf{x}^*$, where $\mathbf{x}^* = (x_1^*, ..., x_N^*)$ is the vector of equilibrium

[23] To circumvent this problem, Vives (2014) suggests using the maximal value of the slope of the best-response function as the measure of strategic complementarities.

signal thresholds, $\nabla_{\underline{\mathbf{1}}}\beta_n(\mathbf{x}^*)$. I refer to this measure as the "equilibrium degree of strategic complementarities" (as it involves computing the slope of the best-response function at the equilibrium signal thresholds).

Proposition 4 *Let β_n denote the best-response function for type s_n, $n = 1, ..., N$. Then the following hold:*

1. *The multiplier effect is equal to*

$$\mathcal{M} = \frac{1}{1 - \sum\limits_{n=1}^{N} w_n \nabla_{\underline{\mathbf{1}}}\beta_n(\mathbf{x}^*)},$$

where $\mathbf{x}^* = (x_1^*, ..., x_N^*)$ is the vector of equilibrium signal thresholds and

$$w_n = \frac{\partial\theta^*/\partial x_n^*}{\Sigma_{l=1}^{N}\partial\theta^*/\partial x_l^*}$$

measures the relative sensitivity of θ^* to changes in x_n^*, $n = 1, ..., N$.

2. *If $\tau_n < \infty$ for all $n \in \{1, ..., N\}$, then*

$$\nabla_{\underline{\mathbf{1}}}\beta_n(\mathbf{x}^*) < 1 \text{ and } \mathcal{M} < \infty$$

Moreover, if $\tau_n \to \infty$ for all $n \in \{1, ..., N\}$, then

$$\nabla_{\underline{\mathbf{1}}}\beta_n(\mathbf{x}^*) \to 1 \text{and } \mathcal{M} \to \infty$$

The result establishes the link between the equilibrium degree of strategic complementarities in the model and the multiplier effect. It tells us that the multiplier effect is strong precisely when the "equilibrium strategic complementarities" are strong (Part 1 of the Proposition). This identifies $\nabla_{\underline{\mathbf{1}}}\beta_n(\mathbf{x}^*)$ as the relevant measure of strategic complementarities in the model. Second, Proposition 4 indicates that the equilibrium strategic complementarities are maximized in the limit as $\tau \to \infty$ which explains why the multiplier effect tends to ∞ in this case.

One may wonder how, in the limit as information becomes arbitrarily precise, the strength of strategic complementarities in the global game compares with the strength of strategic complementarities in the complete information game. In the Appendix (Section D.3) I show that they are equally strong. This observation underscores the important difference between global games and complete information models, namely the presence of "strategic uncertainty" in global games, which is missing in complete information frameworks. Thus, while it is true that the strength of strategic complementarities increases with the precision of private signals, so does the strategic uncertainty, which is maximized precisely in the limit as the noise in the signals disappears (see Morris and Shin (2003)). In other words, even though the incentives to coordinate in global game model are the highest when $\tau \to \infty$, agents are unable to coordinate their actions effectively.

5 When Is the Multiplier Effect Strong?

As argued in Section 3, the multiplier effect acts as an amplification mechanism in the model, always magnifying the initial effect of changes to parameters. In the applied global game literature, such changes in parameters are often interpreted as policy adjustments or unexpected shocks to an economic environment. One of the key aspects of such analysis is understanding when the amplification mechanism is strong, so that impacts of such shocks greatly exceed their direct effect. The goal of this section is to answer this question.

I first consider the setup of Section 2.1 and provide intuition as to when we should expect the multiplier effect to be large. However, at this level of generality, it is difficult to establish sharp predictions regarding the size of the multiplier effect. Therefore, in what follows I consider a simplified model with a single type of agent, where the regime change condition is linear in θ and m, and where agents payoffs are piecewise-constant. Within this simple setup, I provide a full characterization of the conditions under which multiplier effect is large. It is worth stressing that despite its simplicity, this "simple model" is popular in applications (for the recent applications see Morris and Shin, 2016, or Vives, 2014). Finally, in Section 5.3 I show how the results established below can help us understand when a small shock to banks' capital can result in a credit freeze, as was the case during the Great Recession (see Bebchuk and Goldstein (2011), Duchin, Ozbas, and Sensoy (2010), or Ivashina and Scharfstein (2010)).

5.1 The General Model

As shown in Section 4.2, the multiplier effect is strong when the equilibrium strategic complementarities are strong. Thus, understanding under which conditions the multiplier effect is strong boils down to understanding when agents have strong motives to coordinate their actions. This happens when a small change in x_n^* results in a relatively large adjustment in θ^* (i.e., $\partial\theta^*/\partial x_n^*$ is large) and, in turn, the change in θ^* has a relatively large impact on x_n^* (i.e., $\partial x_n^*/\partial\theta^*$ is large).

By inspection of the equilibrium regime change condition, we see that

$$\frac{\partial\theta^*}{\partial x_n^*} \propto \frac{\partial}{\partial m} R\left(\theta^*, m^*\left(\theta\right); \boldsymbol{\psi}\right) f\left(\frac{x_n^* - \theta^*}{\tau_n^{-1/2}}\right),$$

implying that $\partial\theta^*/\partial x_n^*$ is large when a change in x_n^* results in a large change in the proportion of agents attacking the regime (i.e., $f\left(\tau_n^{1/2}\left(x_n^* - \theta^*\right)\right)$ is high) and the regime is sensitive to changes in the proportion of agents attacking $\left(\text{i.e., } \partial R\left(\theta^*, \Sigma_{n=1}^N F^n\left(\tau_n^{1/2}\left(x^* - \theta^*\right)\right); \boldsymbol{\psi}\right)/\partial m \text{ is high}\right)$.

Similarly, by inspection of the agents' indifference condition, we see that

$$\frac{\partial x_n^*}{\partial\theta^*} \propto \left[H^n\left(\theta^*; \boldsymbol{\psi}\right) - L^n\left(\theta^*; \boldsymbol{\psi}\right)\right] f\left(\frac{x_n^* - \theta^*}{\tau_n^{-1/2}}\right),$$

implying that x_n^* is sensitive to changes in θ^* when the payoff difference between successful attack and unsuccessful attack is large at θ^* (large $H^n(\theta^*;\boldsymbol{\psi}) - L^n(\theta^*;\boldsymbol{\psi})$), and when, conditional on observing the threshold signal x_n^*, agents assign a high probability to θ lying in a close neighborhood of θ^*. This is because, in this case, a small change in θ^* results in a large increase in the expected utility difference between attacking and not attacking the regime at the critical signal x_n^*, prompting agents to increase their threshold signals sharply.

Beyond this broad intuition, little more can be said without imposing more structure on the model. Thus, in what follows I consider a simple setup which is more amenable to analysis.

5.2 The Simple Model

In this section, I consider a setup with only one type of agent, where the agents' payoff functions are constant in θ, that is, $H(\theta) = H > 0$ and $L(\theta) = L < 0$, and where the regime change function is linear in θ and in the proportion of agents that attack the regime, m, that is

$$R(\theta, m) = \theta - zm,$$

where $z > 0$ is a parameter that captures the sensitivity of the regime to actions of agents. Each agent receives a private signal $x_i = \theta + \tau_x^{-1/2}\varepsilon_i$, $\varepsilon_i \sim N(0,1)$, with the ε_i independent across agents and independent of θ, and they all share a common prior $\theta \sim N\left(\mu_\theta, \tau_\theta^{-1/2}\right)$. As mentioned earlier, this setup is common in applications.[24] Finally, it is convenient to define $\gamma \equiv -L/(H-L)$.

Notation 1 *Let* $\gamma \equiv \frac{-L}{H-L}$.

The parameter γ measures the relative benefit of a successful attack to the cost of unsuccessful attack. Note that $\gamma \in (0,1)$, tends to 0 as $H \to \infty$ or $L \to 0$ and tends to 1 as $H \to 0$ or $L \to -\infty$. In this setup, the multiplier effect is given by

$$\mathcal{M} = \frac{1}{1 - \frac{\tau_x + \tau_\theta}{\tau_x} \frac{\tau_x^{1/2}\phi\left(\tau_x^{1/2}(x^* - \theta^*)\right)}{1 + \tau_x^{1/2}\phi\left(\tau_x^{1/2}(x^* - \theta^*)\right)}}.^{25} \tag{3}$$

[24] The analysis of this section can be extended to the case of an arbitrary distribution of signals, but only when the prior is uninformative (i.e., uniform improper prior). Below, I limit myself to the case of a Gaussian information structure (with a proper prior), as such an information structure is by far the most popular in applications.

[25] See Section A of the Appendix

Finally, define $\overline{\tau}_{\theta}(\boldsymbol{\psi})$ as the highest value of τ_{θ} for which the model has a unique equilibrium for a given $\boldsymbol{\psi}$; that is, $\overline{\tau}_{\theta}(\boldsymbol{\psi}) = \tau_x^{1/2}\sqrt{2\pi}\left(-\overline{R}_2/\underline{R}_1\right)$ where $\underline{R}_1$ is the lower bound on $\partial R/\partial\theta$ and $\overline{R}_2$ is the upper bound on $\partial R/\partial m$.[26]

5.2.1 The multiplier effect as a function of γ, z, and μ_{θ}

Let $\boldsymbol{\psi_{-m}}$ denote the vector containing all the model's parameters except ψ_m. The next proposition characterizes how, for a given information structure (i.e., holding τ_x and τ_{θ} fixed), the multiplier effect varies with the parameters.

Proposition 5 *For a fixed information structure, define a function $g : \mathbb{R}^3 \rightarrow \mathbb{R}_+$ by*

$$g\left(\mu_{\theta,} z, \gamma\right) = \left|\mu_{\theta} - \frac{1}{2}z + \frac{\sqrt{\tau_x + \tau_{\theta}}}{\tau_{\theta}}\Phi^{-1}(\gamma)\right|$$

1. *The multiplier effect is strong when g is low and achieves its maximum strength when $g\left(\mu_{\theta,} z, \gamma\right) = 0$.*
2. *For each $\psi_m \in \left\{\mu_{\theta,} z, \gamma\right\}$, with $\boldsymbol{\psi_{-m}}$ **held** constant, there exists $\widehat{\psi}_m\left(\boldsymbol{\psi_{-m}}\right) \in \mathbb{R}$ such that the multiplier effect is increasing in ψ_m for all $\psi_m < \widehat{\psi}_m\left(\boldsymbol{\psi_{-m}}\right)$, achieves a maximum at $\widehat{\psi}_m\left(\boldsymbol{\psi_{-m}}\right)$, and is decreasing in ψ_m for all $\psi_m > \widehat{\psi}_m\left(\boldsymbol{\psi_{-m}}\right)$.*

The above proposition follows from the observation that $\mathcal{M}$ is a decreasing function of $|x^* - \theta^*|$. In the simple model considered in this section, there is a one-to-one mapping between the value of g and the distance between x^* and θ^*. In other words, as g increases so does $|x^* - \theta^*|$, which translates into a low $\mathcal{M}$. The second part of the proposition follows directly from the first part. It states that, holding other parameters constant, the multiplier effect is weak when $\psi_m \in \left\{\mu_{\theta,} z, \gamma\right\}$ takes extreme values and increases as ψ_m moves closer to $\widehat{\psi}_m\left(\boldsymbol{\psi_{-m}}\right)$, where $\widehat{\psi}_m\left(\boldsymbol{\psi_{-m}}\right)$ is the value of ψ_m for which $g\left(\mu_{\theta,} z, \gamma\right) = 0$.

To understand why the multiplier effect is weak when γ, z, or μ_{θ} take extreme values, in light of Proposition 4, it suffices to understand why in these cases the strategic complementarities are weak. Note that agent i has weak incentives to coordinate his action with others when, from the ex-ante perspective, one of the actions is more attractive than the other. For example, when γ is high, the benefit from a successful attack compared to the loss from an unsuccessful attack is large, and thus an agent is willing to take the risk and attack even if he believes that few agents will. When z is high, the regime is likely to collapse even if only few agents will attack, and hence again the agent is willing to attack even if he believs that few other agents will. Finally, when μ_{θ} is low, agent i believs that the regime change will occur regardless of the actions of other agents, and again he is inclined to attack

[26] Derivations of $\overline{\tau}_{\theta}(\boldsymbol{\psi})$ are standard and hence omitted from the paper.

irrespective of the proportion of agents that attacks the regime. It follows that in all these cases, agents have weak incentives to coordinate their actions, and hence both the strategic complementarities and the multiplier effect are weak. When γ, z, or μ_θ take intermediate values, then the expected payoff from each action depends greatly on the actions of others, and hence each agent has strong incentives to coordinate with others.

5.2.2 The multiplier effect as a function of τ_x and τ_θ

Next, I investigate how the multiplier effect varies with τ_x and τ_θ. This is a more subtle question, since a change in τ_θ or τ_x affects the multiplier effect through two channels. First, it affects the sensitivity of x^* to changes in the regime change threshold θ^* ($\left(\tau_x + \tau_\theta\right)/\tau_x$ in the denominator in Equation (3)). Second, it affects the sensitivity of the regime's strength to changes in the proportion of agents attacking (as captured by the other factor in that term in the denominator in Equation (3)). These effects often work in the opposite direction, making it challenging to establish how a change in τ_θ or τ_x will affect the multiplier effect.

First, consider changes in τ_x.

Proposition 6 *There exists $\overline{\tau}_x$ such that for all $\tau_x > \overline{\tau}_x$ we have $\frac{\partial \mathcal{M}}{\partial \tau_x} > 0$.*

This result states that for sufficiently high τ_x the multiplier effect is strictly increasing in τ_x; a result that should not be surprising in light of Theorem 1. If $\tau_\theta \to 0$, one can strengthen this result and show that $\mathcal{M}$ is always increasing in τ_x. However, once we allow for informative public information, it is possible that the multiplier effect will be a non-monotone function of τ_x for intermediate levels of private precision.

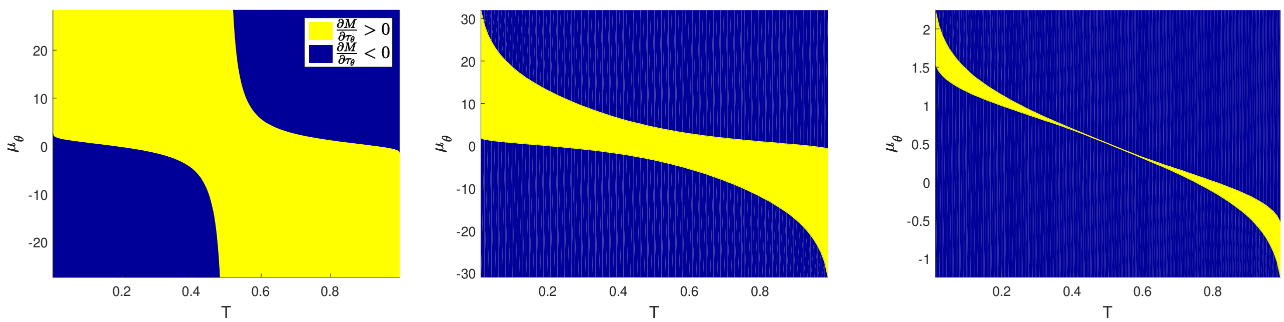
Next, consider changes in τ_θ.

Proposition 7 *For each τ_θ, there exists $\mu_L\left(\tau_\theta\right), \mu_H\left(\tau_\theta\right) \in \mathbb{R}$ with $\mu_L\left(\tau_\theta\right) < \mu_H\left(\tau_\theta\right)$ such that $\frac{\partial \mathcal{M}}{\partial \tau_\theta} \geq 0$ if and only if $\mu_\theta \in \left[\mu_L\left(\tau_\theta\right), \mu_H\left(\tau_\theta\right)\right]$, with a strict inequality holding if $\mu_\theta \in \left(\mu_L\left(\tau_\theta\right), \mu_H\left(\tau_\theta\right)\right)$.*[27]

This result establishes that an increase in the precision of public information increases the multiplier effect if μ_θ takes intermediate values, and decreases it otherwise. This is because a change in τ_θ affects $\mathcal{M}$ through two channels, which tend to work in opposite directions. On the one hand, a higher τ_θ increases $\left(\tau_x + \tau_\theta\right)/\tau_x$, which tends to increase $\mathcal{M}$, the effect which is independent of μ_θ. On the other hand, for extreme values of μ_θ, an increase in τ_θ tends to increase the distance between x^* and θ^*, which decreases the sensitivity of θ^* to changes in x^* decreasing $\mathcal{M}$.

[27] The bounds $\mu_L\left(\tau_\theta\right)$ and $\mu_H\left(\tau_\theta\right)$ depend on all other parameters of the model, besides τ_θ. Each panel of Figure 2 shows for example how they vary with γ for a given value of τ_θ. However, for notational convenience I suppress this dependence.

To understand why the distance between x^* and θ^* increases in response to an increase in τ_θ, consider the case when μ_θ is high (the case of low μ_θ is analogous). When μ_θ is high agents expect the regime to be strong, and hence they are willing to attack it only if their signals are low, implying that $x^* < \theta^*$. In this situation, an increase in τ_θ reinforces the belief that the regime is strong, further decreasing x^* and further increasing the distance $|x^* - \theta^*|$. This effect becomes stronger as μ_θ increases, since the same increase in τ_θ leads to a larger increase in the agents' posterior beliefs in the strength of the regime, and hence to a larger fall in x^*. For sufficiently large μ_θ, this second effect dominates and, as a consequence $\mathcal{M}$ is decreasing in τ_θ.



Panel A: Small τ_θ ($\tau_\theta \to 0$) Panel B: Medium τ_θ ($\tau_\theta = \frac{1}{32}\overline{\tau}_\theta$) Panel C: Large τ_θ ($\tau_\theta = \overline{\tau}_\theta$)

Figure 2: The effect of an increase in τ_θ on the multiplier effect

Figure 2 depicts how the region where $d\mathcal{M}/d\tau_\theta > 0$ changes as τ_θ increases. We see that this region shrinks rapidly as τ_θ increases, and thus at high values of τ_θ a further increase in the precision of public information tends to decrease the multiplier effect. To understand why this is the case, note that the posterior belief of an agent who receives the threshold signal, which is given by

$$\frac{\tau_x x^* + \tau_\theta \mu_\theta}{\tau_x + \tau_\theta},$$

is a convex function of τ_θ. Thus, when the precision of public information is already high, a further increase in τ_θ has a larger effect on the posterior belief of such a player than when τ_θ is low. As a consequence, the negative effect of an increase in τ_θ on $\mathcal{M}$ described above is stronger for all values of μ_θ when τ_θ is already high. On the other hand, the positive effect of an increase in τ_θ on $\mathcal{M}$ is independent of τ_θ. As a consequence the region where $d\mathcal{M}/d\tau_\theta > 0$ shrinks as τ_θ increases. Note that Figure 2 implies that, for most parameter values, $\mathcal{M}$ is a non-monotone function of τ_θ as τ_θ is varied from 0 to $\overline{\tau}_\theta$.

5.3 Application: Credit Freezes and Amplification of Small Shocks

Current accounts of the Great Recession tend to emphasize how a relatively small shock to the economy resulted in the deepest recession since the Great Depression (see e.g., Brunnermeier (2009)) and the freeze of interbank and credit markets (see, for example, Duchin, Ozbas, and Sensoy (2010) or Ivashina and Scharfstein (2010)). Using a global game model, Bebchuk and Goldstein (2011) emphasize how a small shock to banks' capital (when amplified by strategic complementarities) can lead to a freeze in lending to private sector, and analyze policies that can help to prevent such an outcome.[28] In this section I provide conditions under which such an amplification mechanism is likely to be strong.

In particular, note that the framework of Bebchuk and Goldstein (2011) fits into the simple framework considered in Section 5.2. Therefore, we obtain the following result, which is an immediate corollary of Propositions 5 and 6.

Corollary 2 *For a fixed information structure define*

$$g^{BG}\left(\mu_\theta, \alpha, r, R\right) = \left|\mu_\theta - \frac{1}{2}z + \frac{\sqrt{\tau_x+\tau_\theta}}{\tau_\theta}\Phi^{-1}\left(\frac{1+r}{1+R}\right)\right|$$

As g^{BG} decreases, the strength of the amplification mechanism increases, achieving its maximum when $g\left(\mu_\theta, z, r, R\right) = 0$. Moreover, if the precision of private information is high ($\tau_x > \overline{\tau}_x$), then the strength of the amplification mechanism is increasing in τ_x.

Corollary 2 provides potentially important insights for design of macroeconomic prudential policies and financial regulations. First, it stresses that strategic complementarities in lending can be large even if the complementarities at the macroeconomic level are weak (small z). Moreover, it suggests that even if the credit market look robust (high R or high μ_θ), a small shock can still have a large effect on the provision of credit if g^{BG} takes a low value. Thus, when analyzing the vulnerability of a market, regulators should consider all the above variables jointly, and explicitly take into account the links between them. Finally, Corollary 2 states that resolving informational asymmetries, as captured by an increase in τ_x, may increase the strength of the amplification mechanism present in the credit market, making the market more vulnerable.

6 Implications for Comparative Statics Analysis

In this section, I show how Theorem 1 can shed light on the following important questions about comparative statics results: (1) When can comparative statics results in a global game model be deduced directly from the model primitives? (2) When are comparative statics

[28] See Section 2.2 for a description of this environment.

predictions of global games independent of the particular information structure chosen for the analysis (i.e., when are they "robust" to changes in the information structure)? (3) When do comparative statics differ from predictions under complete information based on analysis of the extremal equilibria?

6.1 Robust Predictions and Predictions Based on the Model's Primitives

As discussed in Section 3, Theorem 1 implies that in order to determine whether a change in a parameter of the model increases or decreases θ^*, it suffices to focus on the direct effect. In this section, I go a step further and analyze the direct effect to determine when the comparative statics results can be deduced from the primitives and when they do not depend on the assumed information structure.

Note that there is a close relationship between the conditions under which the sign of comparative statics can be deduced from the model's primitives and those under which their sign does not depend on the assumed information structure. This is because, in order to deduce comparative statics results from the primitives, it must be the case that the effect of a change in a parameter on the regime change function R and the payoff functions π does not depend on θ and m. Otherwise, we would need to know θ^* and $m^*(\theta^*)$ in order to determine the effect of a change in ψ_m, and these are objects that we can compute only by solving for the equilibrium. However, changes in the information structure affect precisely θ^* and $m^*(\theta^*)$ and not the model's primitives. It follows that if we can deduce comparative statics from the model's primitives then these results are "robust" to alternative information structures and vice versa.

The next corollary provide a general, easy-to-check, condition under which $sgn\left(d\theta^*/d\psi_m\right)$ can be determined from the model's primitives and does not depend on the assumed information structure.

Corollary 3 *Fix $\boldsymbol{\psi}$. Suppose that for all $\theta \in \mathbb{R}$ and $m \in [0,1]$, we have*

$$\frac{\partial R\left(\theta, m; \boldsymbol{\psi}\right)}{\partial \psi_m} \leq (\geq)\, 0 \quad \textit{and} \quad \frac{\partial \pi^n\left(\theta; \boldsymbol{\psi}\right)}{\partial \psi_m} \geq (\leq)\, 0, \quad n = 1, ..., N \tag{4}$$

Then

1. *$\frac{d\theta^*}{d\psi_m} \geq (\leq)\, 0$.*
2. *$sgn\left(\frac{d\theta^*}{d\psi_m}\right)$ is independent of the assumed information structure (i.e., unchanged for any choice of $\{F_n\}_{n=1}^N$).*

This result follows from the observation that

$$sgn\left(D\left(\psi_m\right)\right) = sgn\left(\frac{\partial R}{\partial \psi_m} + \sum_{n=1}^{N} \frac{\partial R}{\partial x_n^*} \frac{\partial x_n^*}{\partial \psi_m}\right),$$

where $\partial R / \partial x_n^* < 0$ for all $n \in \{1, ..., N\}$, as a higher threshold implies a higher proportion of agents attacking the regime. The above condition is intuitive: It states that $d\theta^* / d\psi_m > 0$ if an increase in ψ_m increases the relative payoffs from attacking the regime and/or it decreases the resistance of the regime to attack. It should be stressed that while the hypothesis of Corollary 3 is simple, it is satisfied in many applications. For example, I show below that all the parameters (with exception of σ) in Morris and Shin (1998) satisfy this condition. This is also true of the simple model considered in Section 5.2 (again with exception of precision parameters τ_x and τ_θ).

What if the hypothesis of Corollary 3 is not satisfied? In that case, without imposing either more structure on the model or further conditions on the model's primitives, we might be unable to deduce comparative statics from the model's primitives and cannot guarantee that they do not depend on the imposed information structure. In the Appendix I provide two examples of non-robust predictions. In the first example, R is increasing in ψ_m for some values of θ and m but decreasing for others. In that case, a change in the information structure may shift the equilibrium threshold θ^* and the equilibrium proportion of agents that attack the regime $m^*(\theta^*)$, from the region where $\partial R / \partial \psi_m > 0$ to the region where $\partial R / \partial \psi_m < 0$. In the second example, a change in ψ_m decreases the payoff functions ($\partial \pi_n / \partial \psi_m < 0$) but also decreases the resilience of the regime ($\partial R / \partial \psi_m$). In this situation, the information structure may determine which effect dominates. However, it should be stressed that such situations tends to arise only in relatively complex models and most parameters, and that in most of the canonical models the sufficient condition for "robustness" identified in Corollary 3 is satisfied.

A stronger result regarding the robustness of predictions to the changes in information structure can be achieved if we assume that payoff differential functions are piecewise constant, that is $H^n(\theta; \boldsymbol{\psi}) = H^n > 0$ and $L^n(\theta; \boldsymbol{\psi}) = L^n < 0$.

Lemma 1 *Suppose that $H^n(\theta; \boldsymbol{\psi}) = H^n > 0$ and $L(\theta; \boldsymbol{\psi}) = L^n < 0$ for all $n \in \{1, ..., N\}$. Then the model's predictions do not depend on the information structure.*

When the payoffs do not depend on θ, the proportion of agents that attacks the regime at the equilibrium threshold θ^* is determined by the payoffs $\{H^n, L^n\}_{n=1}^N$ only (see part (3) of Proposition 1). As such, the equilibrium threshold θ^*, and hence its comparative statics, also do not depends on the information structure. Note that this "robustness" result holds without the need for additional restrictions on the regime change function R that were needed in Corollary 3. However, it does requires the strong assumption of constant payoffs. Furthermore, it will not hold if we assume that players have a proper prior belief. Further analysis of "robustness" of comparative statics results in global games, while important, is beyond the scope of this paper.

6.2 Comparison with Predictions of the Model with Complete Information

Finally, I close this section by comparing the predictions of the global game model with models with complete information. In the complete information version of the setup described in Section 2, all agents observe θ once it has been realized. It is well-known that in this case any threshold $\widehat{\theta} \in \left[\underline{\theta}(\boldsymbol{\psi}), \overline{\theta}(\boldsymbol{\psi})\right]$ can be supported as an equilibrium. The boundaries $\underline{\theta}(\boldsymbol{\psi})$ and $\overline{\theta}(\boldsymbol{\psi})$ of this "multiplicity region" constitute the smallest and the largest equilibrium, respectively, of the compete information model.[29] The predictions of the complete information model are then often based on the behavior of $\underline{\theta}(\boldsymbol{\psi})$ and $\overline{\theta}(\boldsymbol{\psi})$ in response to changes in $\psi_m \in \boldsymbol{\psi}$.

Let $\boldsymbol{\psi}^R = \{\psi_m \in \boldsymbol{\psi} | \partial R/\partial \psi_m \neq 0\}$ and $\boldsymbol{\psi}^P = \{\psi_m \in \boldsymbol{\psi} | \partial \pi^n/\partial \psi_m \neq 0 \text{ for some } n \in (1, ..., N)\}$ so that $\boldsymbol{\psi}^R$ is the vector of all the parameters that affect the regime change condition while $\boldsymbol{\psi}^P$ is the vector of all parameters that affect the payoff functions. Note that a change in $\psi_m \in \boldsymbol{\psi}^R$ affects the extremal equilibria if and only if ψ_m affects the regime change condition ($\partial R/\partial \psi_m \neq 0$). This is because $\underline{\theta}(\boldsymbol{\psi})$ and $\overline{\theta}(\boldsymbol{\psi})$ are defined as solutions to $0 = R\left(\underline{\theta}(\boldsymbol{\psi}), 0; \boldsymbol{\psi}\right)$ and $0 = R\left(\overline{\theta}(\boldsymbol{\psi}), 1; \boldsymbol{\psi}\right)$, respectively. Thus, in contrast to the global game model, a change in $\psi_m \in \boldsymbol{\psi}^P \backslash \boldsymbol{\psi}^R$ has no effect on the extremal equilibria. This last observation is worth emphasizing, as it constitutes one of the advantages of global game selection over other selection mechanisms based on the complete information game.

Corollary 4 *Consider the effect of change in ψ_m on θ^*, $\underline{\theta}$ and $\overline{\theta}$. Suppose $\psi_m \in \boldsymbol{\psi}^R$ and that $\partial R/\partial \psi_m > (<)\, 0$ and $\partial \pi_n/\partial \psi_m \leq (\geq)\, 0$ for all θ and m. Then,*

$$sgn\left(\frac{d\theta^*}{d\psi_m}\right) = sgn\left(\frac{d\underline{\theta}(\boldsymbol{\psi})}{d\psi_m}\right) = sgn\left(\frac{d\overline{\theta}(\boldsymbol{\psi})}{d\psi_m}\right)$$

On the other hand, if $\psi_m \in \boldsymbol{\psi}^P \backslash \boldsymbol{\psi}^R$ then the predictions of the two models will differ.

6.3 Example: Comparative Statics Result in Morris and Shin (1998)

In this section, I show how Corollaries 3 and 4 can be used to simplify and extend analysis of the global game model. As an example, I consider the model of Morris and Shin (1998) described in Section 2.2.1. In that setup we have $\boldsymbol{\psi} = \{v, e^*, t, \sigma\}$, where v captures the benefit of maintaining the currency peg, e^* is the prevailing fixed exchange rate, t is the transaction cost of attacking the peg, and σ is the precision of agents information; the resilience of the

[29] To be precise, the strategy profile $\left\{\underline{\beta}_i\right\}_{i\in[0,1]}$ where $\underline{\beta}_i(\theta) = 1$ if $\theta \leq \underline{\theta}(\boldsymbol{\psi})$ and $\underline{\beta}_i(\theta) = 0$ otherwise, $i \in [0,1]$, constitutes the smallest equilibrium of the complete information setup, with $\underline{\theta}(\boldsymbol{\psi})$ being the implied regime change threshold. Similarly, the strategy profile $\left\{\overline{\beta}_i(\theta)\right\}_{i\in[0,1]}$, where $\overline{\beta}_i(\theta) = 1$ if $\theta \leq \overline{\theta}(\boldsymbol{\psi})$ and $\beta_i(\theta) = 0$ otherwise, $i \in [0,1]$, constitutes the largest equilibrium of the complete information setup, with $\overline{\theta}(\boldsymbol{\psi})$ being the implied regime change threshold.

regime and the payoff functions are given by $R(\theta, m; \boldsymbol{\psi}) = v - c(\theta, m)$, $H(\theta; \boldsymbol{\psi}) = e^* - f(\theta) - t$, and $L(\theta; \boldsymbol{\psi}) = -t$. Finally, let θ^* denote the unique global game regime change threshold.

By inspection of functions R, H, and L, we see that an increase in v affects only R and always increases the resilience of the peg. On the other hand, changes in e^* and t affect only the payoff functions and do so monotone fashion. Thus, Corollary 3 implies that

$$\frac{d\theta^*}{dv} < 0, \ \frac{d\theta^*}{de^*} > 0, \text{ and } \frac{d\theta^*}{dt} < 0$$

Moreover, these predictions do not depend on the information structure assumed (i.e., they do not depend on the value of σ or the choice of the distribution of noise, F). How do these predictions compare with those based on the extremal equilibria of the complete information model? From Corollary 4, both models predict that increase in v makes a collapse of the peg more likely. On the other hand, changes in t or e^* have no effect on the extremal equilibria of the complete information model.

7 Extensions of Theorem 1

In this section, I provide two extensions of Theorem 1. First, I show that a similar decomposition applies to the case when several parameters are changed at the same time. Second, I discuss how Theorem 1 extends to environments with multiple equilibria.

7.1 A Simultaneous Change in Multiple Parameters

So far I have considered only the effect of a change in a single parameter on the regime change threshold θ^*. It turns out that Theorem 1 can be extended to describe the effect of a simultaneous change in several parameters of the model on the regime change threshold θ^*. Such a result might be particularly useful in finance applications, such as in the analysis of banking regulations where one is interested in the effect of changing several aspects of the model at once (for example liquidity requirements and maturity structure requirements).

Fix $K > 1$, and let $\{\psi_{m_1}, ..., \psi_{m_K}\} \subset \boldsymbol{\psi}$ be a subset of the parameters of the model. Suppose that we are interested in computing the effect that a simultaneous small change in $\psi_{m_1}, ..., \psi_{m_K}$ has on θ^*. To this end, let us write θ^* explicitly as a function of $\psi_{m_1}, ..., \psi_{m_K}$, that is $\theta^*(\psi_{m_1}, ..., \psi_{m_K})$ (since other parameters are kept constant, we ignore them in what follows). Now, denote by $\nabla_u \theta^*(\psi_{m_1}, ..., \psi_{m_K})$ the directional derivative of θ^* in the direction u, where $u \in \mathbb{R}^K$. Then a simultaneous small change in $\psi_{m_1}, ..., \psi_{m_K}$ is captured by directional derivative of $\theta^*(\psi_{m_1}, ..., \psi_{m_K})$ in the direction of $\underline{\mathbf{1}} = (1, ..., 1) \in \mathbb{R}^K$.[30] We can now state the following result, which extends Theorem 1 to the case of a simultaneous change in multiple parameters.

[30] The direction $\underline{\mathbf{1}} = (1, ..., 1)$ implies that all K parameters change at the same rate.

Proposition 8 *Let* $\{\psi_{m_1}, ..., \psi_{m_K}\}$ *be a subset of the parameters of the model and let* $\theta^*(\psi_{m_1}, ..., \psi_{m_K})$ *denote* θ^* *as a function of these parameters when other parameters are held constant. Then*

$$\nabla_{\underline{\mathbf{1}}}\theta^*(\psi_{m_1}, ..., \psi_{m_K}) = \mathcal{M}\sum_{k=1}^{K}\mathcal{D}(\psi_{m_k}),$$

where $\mathcal{M}$ *and* $\mathcal{D}(\psi_{m_i})$ *are defined as in Theorem* 1.

Thus, we see that the results established above extend to the case of simultaneous small changes in multiple parameters.

7.2 The Multiplier Effect in the Model with Multiple Equilibria

The analysis above was conducted under the assumption that the model has a unique equilibrium. One may wonder if Theorem 1 generalizes in some form to settings that feature multiple monotone equilibria, each characterized by a different regime change threshold $\widehat{\theta}^*$.[31]

The challenge of obtaining predictions based on a model with multiple equilibria stems from the fact that following any change in the model, the agents can simply coordinate on different equilibrium, which makes predictions in models with multiple equilibria problematic. However, one may still be interested in computing comparative statics results of a regime change threshold $\widehat{\theta}^*$ associated with a particular equilibrium. In this case $d\widehat{\theta}^*/d\psi_m$ for some $\psi_m \in \boldsymbol{\psi}$ should be interpreted as a change in a threshold that characterizes a particular equilibrium.[32]

Suppose that the model stated in Section 2 has multiple monotone equilibria. This can happen, for example, if all agents share a common prior belief G that is informative enough compared to private signals (see, for example, Hellwig (2002) or Morris and Shin (2004)). Below, I discuss how Theorem 1 extends to settings with multiple equilibria.

To state the main result of this section, I will need the following definition of stability of equilibria (see, for example, Vives (2004)).

Definition 1 (Stability) *Let* $\widehat{\theta}_0$ *denote agents' initial belief about the regime change threshold. A monotone equilibrium characterized by the regime change threshold* $\widehat{\theta}^*$ *is stable if*

[31] In what follows I denote by $\widehat{\theta}^*$ the regime change threshold associated with a particular monotone equilibrium, and by θ^* the actual threshold below which the regime changes in equilibrium (where θ^* corresponds to one of the $\widehat{\theta}^*$s). See also the discussion below.

[32] More precisely, let $\widehat{\theta}_1^*$ and $\widehat{\theta}_2^*$ be two distinct thresholds associated with two different monotone equilibria. Suppose that initially $\theta^* = \widehat{\theta}_1^*$ and that ψ_m increases. We can compute how a change in ψ_m affects $\widehat{\theta}_1^*$ and $\widehat{\theta}_2^*$ (that is $d\widehat{\theta}_1^*/d\psi_m$ and $d\widehat{\theta}_2^*/d\psi_m$). But since agents may switch from playing the equilibrium with threshold $\widehat{\theta}_1^*$ to the equilibrium with threshold $\widehat{\theta}_2^*$, we cannot associate $d\widehat{\theta}_1^*/d\psi_m$ with the change in θ^* induced by the change in ψ_m. In other words, with multiple equilibria, comparative statics of equilibrium thresholds are not necessarily equivalent to the actual model predictions.

$\exists \varepsilon > 0$ such that for all $\widehat{\theta}_0 \in (\theta^* - \varepsilon, \theta^* + \varepsilon)$ the best-response dynamics initiated at $\widehat{\theta}_0$ converge to $\widehat{\theta}^*$. A monotone equilibrium that is not stable is called unstable.

It turns out that Theorem 1 remains valid as long as we focus on stable equilibria.[33] In particular, one can show that $d\widehat{\theta}^*/d\psi_m = \mathcal{M} \times \mathcal{D}(\psi_m)$, where $\mathcal{M}$ and $\mathcal{D}(\psi_m)$ are defined as in Theorem 1; $\mathcal{M} \in (1, \infty)$ so that $sgn\left(d\widehat{\theta}^*/d\psi_m\right) = sgn\left(\mathcal{D}(\psi_m)\right)$; and the multiplier effect has the natural interpretation of an amplification mechanism that can be derived using best-response dynamics.[34] There are two reasons why Theorem 1 extends to stable equilibria. First, the actual derivation of the decomposition of $d\widehat{\theta}^*/d\psi_m$ in the proof of Theorem 1 did not utilize the fact that the equilibrium is unique. Indeed, they remain valid for any equilibrium that does not disappear following a change in ψ_m. Second, the necessary and sufficient condition for $\mathcal{M} \in (1, \infty)$ and for the best-response dynamics to converge is that $\sum_{n=1}^{N} \frac{\partial \theta^*}{\partial x_n^*} \frac{\partial x_n^*}{\partial \theta^*} < 1$ when evaluated at the initial equilibrium threshold. But, as discussed in Section G of the Appendix, this is exactly the necessary and sufficient condition for the equilibrium to be stable.

The situation is different when we consider an unstable equilibrium. While the decomposition of comparative statics results into the direct and multiplier effects remains valid, the multiplier effect loses its natural interpretation as capturing the change induced by adjustments in beliefs. This is because when an equilibrium is unstable the best-response dynamics diverge, as a small change in ψ_m will make agent switch from playing unstable equilibrium to playing the closest stable equilibrium. This implies a large discrete adjustment in the regime change threshold θ^* regardless of how small the initial change in ψ_m is. As a consequence, the multiplier effect computed using best-response dynamics is infinite. On the other hand, if we compute the multiplier effect directly using the implicit function theorem as in the proof of Theorem 1, the resulting multiplier effect is negative. The difference between the two approaches steps from the fact that the best-response dynamics describe the change in the equilibrium play following a change in a parameter, while the implicit theorem characterizes the change in the given equilibrium threshold. While in the case of stable equilibria these two approaches coincide, this does not occur in the case of an unstable equilibrium.

I summarize the above discussion in Proposition 9.

[33] From Van Zandt and Vives (2007) we know that global games belong to the class of so-called monotone supermodular games, which always have the smallest and largest equilibria that are in monotone strategies and stable. In global games, monotone equilibria can be ordered by the value of the regime change threshold θ^* associated with them.

[34] Part 3 of Theorem 1 does not generalize, since as we consider a sequence of precision levels that tends to infinity, at some point along such a sequence some equilibria may disappear, which means that for these equilibria $\lim_{\tau\to\infty} \mathcal{M}$ and $\lim_{\tau\to\infty} \mathcal{D}(\psi_m)$ are not defined.

Proposition 9 *Suppose that the model features multiple monotone equilibria. Consider a particular equilibrium and the associated regime change threshold* $\widehat{\theta}^*$. *Let* $\mathcal{M}$ *and* $\mathcal{D}(\psi_m)$ *be defined as in Theorem* 1.

1. *If a monotone equilibrium is stable, then for any* $\psi_m \in \boldsymbol{\psi}$ *we have* $d\widehat{\theta}^*/d\psi_m = \mathcal{M} \times \mathcal{D}(\psi_m)$, $sgn\left(d\widehat{\theta}^*/d\psi_m\right) = sgn(\mathcal{D}(\psi_m))$, $\mathcal{M} \in (1, \infty)$, *and the best-response dynamics following a small change in* ψ_m *converge.*

2. *If a monotone equilibrium is unstable, then for any* $\psi_m \in \boldsymbol{\psi}$ *we have* $d\widehat{\theta}^*/d\psi_m = \mathcal{M} \times \mathcal{D}(\psi_m)$, *but* $sgn\left(d\widehat{\theta}^*/d\psi_m\right) = -sgn(\mathcal{D}(\psi_m))$, $\mathcal{M} < 0$, *and the best-response dynamics following a small change in* ψ_m *diverge.*

8 Conclusions

In this paper, I provided a general analysis of comparative statics in a general global game model. The central result of the paper is the decomposition of the comparative statics results into the direct effect and the multiplier effect. Despite its simplicity, this decomposition proves to be a surprisingly powerful tool for deriving and interpreting comparative statics results in global games models.

In the remainder of the paper, I analyzed the direct effect and the multiplier effect. In particular, I related the multiplier effect to the "publicity multiplier" of Morris and Shin (2003, 2004) and to the strength of strategic complementarities present in the model. Furthermore, I used these insights to characterize conditions under which the multiplier effect is strong, so that a small shock to the model, when amplified by the endogenous adjustments in beliefs, results in large equilibrium adjustments. I then analyzed the direct effect. This analysis allowed me to identify conditions under which (1) comparative statics results can be deduced from the model primitives, (2) comparative statics results are independent of the assumed information structure, and (3) predictions based on the global game coincide with those of the underlying complete information model..

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Appendix

This appendix contains the proofs of the results stated in the paper and is divided into seven sections, with each section of the appendix corresponding to a particular section of the paper (with exception of Section A which contains the preliminary results that I use in my analysis). In Section B I provide the proof of Proposition 1, and in Section C I present the proof of Theorem 1. Section D includes the proofs of Propositions 3 and 4 as well as proofs of related results discussed in the paper. In Section E I provide proofs of Propositions 5−7. Finally, Section F contains the proofs of Corollaries 3 and 4, and Section G includes the proofs of the extensions of Theorem 1 (i.e., Propositions 8 and 9). Several of the more involved proofs are only sketched here; complete proofs can be found in the Online Appendix available at the author's website.

A Preliminary Results

In this section, I list the intermediate claims used to establish results of Section 2 and 5 of the paper. See the Online Appendix for their proofs.

A.1 Preliminary Claims for the Results of Section 2

Lemma A.1 is key to establishing the equilibrium thresholds $\{\theta^*, x_1^*, ..., x_N^*\}$ converge to a finite limit precision of private information tends to infinity.

Lemma A.1 *Consider a sequence* $\{x_n\}_{n=1}^{\infty} \subset [a, b]$*. If all the convergent subsequences of this sequence have the same finite limit L, then the full sequence also has*

Note that this result allows us to focus on convergent subsequences in our analysis of limiting behavior of thresholds. Then, if we confirm the hypothesis of this lemma, we know that this limiting behavior is passed onto the sequence itself.

A.2 Preliminary Claims for the Results of Section 5

In Section 5.2 I consider a simplified model with a single type of agent, where the payoff differential function is piecewise constant, but where agents have a proper prior. I further assume that the private signals and the prior belief have normal distribution. In that setup, τ_x denotes the precision of the private signal (common to all agents), τ_θ denotes the precision of the common prior, and μ_θ denotes the mean of the prior belief. The results listed below utilize all of these assumptions.

Lemma A.2 *Consider the setup of Section* 5.2*, and suppose that*

$$\frac{\tau_x^{1/2}}{\tau_\theta} > \frac{1}{z}\frac{1}{\sqrt{2\pi}}$$

Then the equilibrium signal threshold is given by

$$x^* = \frac{\tau_x + \tau_\theta}{\tau_x}\theta^* - \frac{\tau_\theta}{\tau_x}\mu_\theta - \frac{\sqrt{\tau_x + \tau_\theta}}{\tau_x}\Phi^{-1}(\gamma) \tag{5}$$

and the equilibrium regime change threshold θ^* *is the unique solution to*

$$\theta^* - z\Phi\left(\frac{\tau_\theta}{\tau_x^{1/2}}(\theta^* - \mu_\theta) - \frac{\sqrt{\tau_x + \tau_\theta}}{\tau_x}\Phi^{-1}(\gamma)\right) = 0 \tag{6}$$

Notation 2 *I refer to the condition*

$$\tau_x^{1/2}/\tau_\theta > (1/z)\left(1/\sqrt{2\pi}\right)$$

as the "uniqueness condition."

This condition is maintained for all the results of Section 5.2 (i.e., all the results pertaining to the simple model discussed there).

Lemma A.3 *In the simple framework of Section* 5.2*, the multiplier effect is given by*

$$\frac{1}{1-\frac{\tau_x+\tau_\theta}{\tau_x}\frac{z\tau_x^{1/2}\phi\left(\tau_x^{1/2}(x^*-\theta^*)\right)}{1+z\tau_x^{1/2}\phi\left(\tau_x^{1/2}(x^*-\theta^*)\right)}}$$

Lemma A.4 *Define*

$$\overline{\mu}_\theta \equiv \frac{1}{2}z-\frac{\sqrt{\tau_x+\tau_\theta}}{\tau_\theta}\Phi^{-1}(\gamma)$$

1. $x^*-\theta^*>0$ *if* $\mu_\theta<\overline{\mu}_\theta$.
2. $x^*-\theta^*=0$ *if* $\mu_\theta=\overline{\mu}_\theta$.
3. $x^*-\theta^*<0$ *if* $\mu_\theta>\overline{\mu}_\theta$.

Moreover, $\lim_{\mu_\theta\to\infty}(x^*-\theta^*)=-\infty$ *and* $\lim_{\mu_\theta\to-\infty}(x^*-\theta^*)=\infty$.

Corollary A.1 *Define*

$$\begin{aligned}\widehat{\gamma}(z,\mu_\theta) &\equiv \Phi\left(\frac{\tau_\theta}{\sqrt{\tau_x+\tau_\theta}}\left(\frac{1}{2}z-\mu_\theta\right)\right)\\ \widehat{z}(\gamma,\mu_\theta) &\equiv 2\left[\mu_\theta+\frac{\sqrt{\tau_x+\tau_\theta}}{\tau_\theta}\Phi^{-1}(\gamma)\right]\end{aligned}$$

1. *If* $\gamma<\widehat{\gamma}(z,\mu_\theta)$ *or* $z>\widehat{z}(\gamma,\mu_\theta)$*, then* $x^*-\theta^*>0$.
2. *If* $\gamma=\widehat{\gamma}(z,\mu_\theta)$ *or* $z=\widehat{z}(\gamma,\mu_\theta)$*, then* $x^*-\theta^*=0$.
3. *If* $\gamma>\widehat{\gamma}(z,\mu_\theta)$ *or* $z<\widehat{z}(\gamma,\mu_\theta)$*, then* $x^*-\theta^*<0$.

Lemma A.5 *Define*

$$\widetilde{\mu}_\theta(\gamma)\equiv-\frac{1}{2}\frac{1}{\sqrt{\tau_x+\tau_\theta}}\Phi^{-1}(\gamma)+z\Phi\left(-\frac{\tau_x+\frac{1}{2}\tau_\theta}{\tau_x+\tau_\theta}\sqrt{\frac{\tau_x+\tau_\theta}{\tau_x}}\Phi^{-1}(\gamma)\right)$$

We have:

1. $\theta^*-\mu_\theta-\frac{1}{2}\frac{1}{\sqrt{\tau_x+\tau_\theta}}\Phi^{-1}(\gamma)>0$ *if* $\mu_\theta<\widetilde{\mu}_\theta(\gamma)$.
2. $\theta^*-\mu_\theta-\frac{1}{2}\frac{1}{\sqrt{\tau_x+\tau_\theta}}\Phi^{-1}(\gamma)=0$ *if* $\mu_\theta=\widetilde{\mu}_\theta(\gamma)$.
3. $\theta^*-\mu_\theta-\frac{1}{2}\frac{1}{\sqrt{\tau_x+\tau_\theta}}\Phi^{-1}(\gamma)<0$ *if* $\mu_\theta>\widetilde{\mu}_\theta(\gamma)$.

Lemma A.6 *The derivative of the regime change threshold with respect to* μ_θ *is given by*

$$\frac{d\theta^*}{d\mu_\theta}=-\frac{z\frac{\tau_\theta}{\tau_x^{1/2}}\phi\left(\tau_x^{1/2}(x^*-\theta^*)\right)}{1-z\frac{\tau_\theta}{\tau_x^{1/2}}\phi\left(\tau_x^{1/2}(x^*-\theta^*)\right)}<0$$

Lemma A.7 *Consider* $\overline{\mu}_\theta(\gamma)$ *and* $\widetilde{\mu}_\theta(\gamma)$*. We have:*

1. $\overline{\mu}_\theta(\gamma) > \widetilde{\mu}_\theta(\gamma)$ *if* $\gamma < \frac{1}{2}$, $\overline{\mu}_\theta(\gamma) = \widetilde{\mu}_\theta(\gamma)$ *if* $\gamma = \frac{1}{2}$, *and* $\overline{\mu}_\theta(\gamma) < \widetilde{\mu}_\theta(\gamma)$ *if* $\gamma > \frac{1}{2}$.
2. $\frac{\partial}{\partial\gamma}\left[\overline{\mu}_\theta(\gamma) - \widetilde{\mu}_\theta(\gamma)\right] < 0$ *for all* $\gamma \in (0,1)$.
3. $\lim_{\gamma\to 0}\left(\overline{\mu}_\theta(\gamma) - \widetilde{\mu}_\theta(\gamma)\right) = \infty$ *and* $\lim_{\gamma\to 1}\left(\overline{\mu}_\theta(\gamma) - \widetilde{\mu}_\theta(\gamma)\right) = -\infty$.

Lemma A.8 *Consider* $d\theta^*/d\tau_x$. *Then*

$$\frac{d\theta^*}{d\tau_x} = -\frac{\tau_\theta}{\tau_x^{3/2}} \frac{(\theta^* - \mu_\theta) - \frac{1}{\sqrt{\tau_x+\tau_\theta}}\Phi^{-1}(\gamma)}{\frac{1}{\phi\left(\Phi^{-1}\left(\frac{1}{z}\theta^*\right)\right)}\frac{1}{z} - \frac{\tau_\theta}{\tau_x^{1/2}}}$$

Moreover,

1. *If* $\mu_\theta < \widehat{\mu}_\theta$ *then* $\frac{\partial\theta^*}{\partial\tau_x} > 0$.
2. *If* $\mu_\theta = \widehat{\mu}_\theta$ *then* $\frac{\partial\theta^*}{\partial\tau_x} = 0$.
3. *If* $\mu_\theta > \widehat{\mu}_\theta$ *then* $\frac{\partial\theta^*}{\partial\tau_x} < 0$,

where

$$\widehat{\mu}_\theta \equiv z\Phi\left(-\sqrt{\frac{\tau_x}{\tau_x+\tau_\theta}}\Phi^{-1}(\gamma)\right) - \frac{1}{\sqrt{\tau_x+\tau_\theta}}\Phi^{-1}(\gamma)$$

Lemma A.9 *Consider* $d\theta^*/d\tau_\theta$. *Then*

$$\frac{d\theta^*}{d\tau_\theta} = -\frac{\frac{1}{\tau_x^{1/2}}(\theta^* - \mu_\theta) - \frac{1}{2}\frac{1}{\tau_x^{1/2}}\frac{1}{\sqrt{\tau_x+\tau_\theta}}\Phi^{-1}(\gamma)}{\frac{1}{\phi\left(\Phi^{-1}\left(\frac{1}{z}\theta^*\right)\right)}\frac{1}{z} - \frac{\tau_\theta}{\tau_x^{1/2}}}$$

Moreover,

1. *If* $\mu_\theta < \widetilde{\mu}_\theta$ *then* $\frac{\partial\theta^*}{\partial\tau_\theta} > 0$.
2. *If* $\mu_\theta = \widetilde{\mu}_\theta$ *then* $\frac{\partial\theta^*}{\partial\tau_\theta} = 0$.
3. *If* $\mu_\theta > \widetilde{\mu}_\theta$ *then* $\frac{\partial\theta^*}{\partial\tau_\theta} < 0$,

where

$$\widetilde{\mu}_\theta \equiv \Phi\left(-\sqrt{\frac{\tau_\theta+\tau}{\tau}}\frac{\frac{1}{2}\tau_\theta+\tau}{\tau_\theta+\tau}\Phi^{-1}(T)\right) - \frac{1}{2}\frac{1}{\sqrt{\tau_\theta+\tau}}\Phi^{-1}(T)$$

B Proofs for Section 2

Proof of Proposition 1. The proof of uniqueness in the case $\tau_n < \infty$, for all $n \in \{1, ..., N\}$ is standard and consists of two steps. In the first step I show that there exists a unique symmetric equilibrium in monotone strategies (or threshold strategies). Next, it can be shown (using iterative deletion of strictly dominated strategies) that there exist no other equilibria. The proof of the uniqueness result when $\tau_n \to \infty$ for all $n \in \{1, ..., N\}$ is more technically involved than usual, on account of the presence of multiple types of agents. This is because with N types of agents and the indifference conditions, we have to make sure that the whole system converges as $\tau_n \to \infty$.

Note first that the posterior belief of agent i, who observed signal x_i, is given by

$$\Pr\left(\theta < \widehat{\theta}\,\middle|\, x_i = x\right) = \Pr\left(x_i - \tau_{s(i)}\varepsilon_i < \widehat{\theta}\,\middle|\, x_i = x\right) = 1 - F_{s(i)}\left(\frac{x-\widehat{\theta}}{\tau_{s(i)}^{-1/2}}\right)$$

Therefore, agent i's posterior density of $\theta|x_i$ is $f_{s(i)}\left(\tau_{s(i)}^{1/2}(x_i-\theta)\right)$. Note that a higher x shifts agent i's posterior belief upwards according to the first-order stochastic dominance ordering.

Let $\widehat{\theta}$ be a candidate equilibrium regime change threshold such that the regime changes if and only if $\theta \leq \widehat{\theta}$. I first show that for each $\widehat{\theta}$ there exists a unique signal thresholds x_n^* for each type s_n, $n=1,...,N$ such that agent of type s_n will attack the regime if and only if he observes a signal smaller than or equal to x_n^*. To this end, consider agent i. If agent i expects the regime to change if and only if $\theta \leq \widehat{\theta}$, then the expected payoff differential between attacking he regime and not attacking it is given by

$$\int_{-\infty}^{\widehat{\theta}} H^{s(i)}(\theta,\psi)\,\tau_{s(i)}^{1/2} f_{s(i)}\left(\frac{x_i-\theta}{\tau_{s(i)}^{-1/2}}\right)d\theta + \int_{\widehat{\theta}}^{\infty} L^{s(i)}(\theta,\psi)\,\tau_{s(i)}^{1/2} f_{s(i)}\left(\frac{x_i-\theta}{\tau_{s(i)}^{-1/2}}\right)d\theta$$

I now show that the expected payoff differential is strictly increasing in x_i. For this purpose, it is useful to perform a change of variables by defining $z=\tau_{s(i)}^{1/2}(x_i-\theta)$ in the expression for the expected payoff differential so that the expected payoff differential can be written as

$$\int_{\tau_{s(i)}^{1/2}\left(x_i-\widehat{\theta}\right)}^{\infty} H^{s(i)}\left(x_i-\tau_{s(i)}^{-1/2}z;\boldsymbol{\psi}\right) f_{s(i)}(z)\,dz + \int_{-\infty}^{\tau_{s(i)}^{1/2}\left(x_{s(i)}^*-\widehat{\theta}\right)} L^{s(i)}\left(x_{s(i)}^*-\tau_{s(i)}^{-1/2}z;\boldsymbol{\psi}\right) f_{s(i)}(z)\,dz$$

Differentiating with respect to x_i, we get

$$-H^{s(i)}(\theta^*;\boldsymbol{\psi})\,\tau_{s(i)}^{1/2} f_{s(i)}\left(\frac{x_i-\theta}{\tau_{s(i)}^{-1/2}}\right) + L^{s(i)}(\theta^*;\boldsymbol{\psi})\,\tau_{s(i)}^{1/2} f_{s(i)}\left(\frac{x_i-\theta}{\tau_{s(i)}^{-1/2}}\right)$$
$$+\underbrace{\int_{\tau_{s(i)}^{1/2}\left(x_i-\widehat{\theta}\right)}^{\infty} \frac{\partial H^{s(i)}\left(x_i-\tau_{s(i)}^{-1/2}z;\boldsymbol{\psi}\right)}{\partial x_i} f_{s(i)}(z)\,dz}_{<0} + \underbrace{\int_{-\infty}^{\tau_{s(i)}^{1/2}\left(x_{s(i)}^*-\widehat{\theta}\right)} \frac{\partial L^{s(i)}\left(x_{s(i)}^*-\tau_{s(i)}^{-1/2}z;\boldsymbol{\psi}\right)}{\partial x_i} f_{s(i)}(z)\,dz}_{<0}$$

Since $H^{s(i)}>0$, $L^{s(i)}<0$, and both $H^{s(i)}$ and $L^{s(i)}$ are non-increasing in θ, we conclude that the expected payoff difference is a decreasing function of x_i. It is straightforward to see that for $x_i \to \infty$ the payoff differential between the regime and not attacking it is strictly positive and that for $x_i \to -\infty$ it is strictly negative. Thus, we conclude that there exists a unique signal value, call it x_i^*, such that agent i will attack the regime if and only if $x_i \leq x_i^*$. Since all agents of the same type are ex-ante symmetric, it follows that for every $n \in \{1,...,N\}$ all agents of type s_n, use the same threshold, which I denote by x_n^* and which is the unique solution to

$$\int_{\tau_n^{1/2}\left(x_n^*-\widehat{\theta}\right)}^{\infty} H^n\left(x_n^*-\tau_n^{-1/2}z;\boldsymbol{\psi}\right) f_n(z)\,dz + \int_{-\infty}^{\tau_n^{1/2}\left(x_n^*-\widehat{\theta}\right)} L^n\left(x_n^*-\tau_n^{-1/2}z;\boldsymbol{\psi}\right) f_n(z)\,dz = 0, \tag{7}$$

which is often referred to as the payoff indifference equation.

Note that the above equation implicitly defines x_n^* as a function of $\widehat{\theta}$. Since the derivative of the LHS of the above equation with respect to $\widehat{\theta}$ is given by

$$H^n\left(\widehat{\theta};\boldsymbol{\psi}\right)\tau_n^{1/2} f_n\left(\frac{x_n^*-\theta}{\tau_n^{-1/2}}\right) - L^n(\theta^*;\boldsymbol{\psi})\,\tau_n^{1/2} f_n\left(\frac{x_n^*-\theta}{\tau_n^{-1/2}}\right) > 0$$

it follows from the implicit function theorem that

$$\frac{\partial x_n^*}{\partial \widehat{\theta}} \in (0,1)$$

We saw that the payoff indifference equation for type s_n defines a function $x_n^*\left(\widehat{\theta}\right)$. It follows that at a candidate equilibrium regime change threshold $\widehat{\theta}$ the proportion of agents that attacks the regime is given by

$$\sum_{n=1}^{N}\lambda_n F_n\left(\frac{x_n^*\left(\widehat{\theta}\right)-\widehat{\theta}}{\tau_n^{-1/2}}\right)$$

It follows that $\widehat{\theta}$ is an equilibrium regime change threshold if and only if it is a solution to

$$R\left(\widehat{\theta},\sum_{n=1}^{N}\lambda_n F_n\left(\frac{x_n^*\left(\widehat{\theta}\right)-\widehat{\theta}}{\tau_n^{-1/2}}\right),\boldsymbol{\psi}\right)=0 \tag{8}$$

By assumption, we know that the LHS of the above equation is strictly negative for all $\widehat{\theta}<\underline{\theta}$ and strictly positive for $\widehat{\theta}>\overline{\theta}$. Moreover

$$\begin{aligned}&\frac{\partial}{\partial\widehat{\theta}}R\left(\widehat{\theta},\sum_{n=1}^{N}\lambda_n F_n\left(\frac{x_n^*\left(\widehat{\theta}\right)-\widehat{\theta}}{\tau_n^{-1/2}}\right),\psi\right)\\ = \quad & R_1-R_2\left[\sum_{n=1}^{N}\lambda_n\tau_n^{1/2}f_n\left(\frac{x_n^*\left(\widehat{\theta}\right)-\widehat{\theta}}{\tau_n^{-1/2}}\right)\left(\frac{\partial x_n^*\left(\widehat{\theta}\right)}{\partial\widehat{\theta}}-1\right)\right]>0\end{aligned}$$

since $R_1>0$, $R_2<0$,and $\partial x_n^*\left(\widehat{\theta}\right)/\partial\widehat{\theta}<1$. Therefore, we conclude that indeed there is a unique $\widehat{\theta}$ (call it θ^*) that solves Equation (8). Then θ^* and $\{x_n^*(\theta^*)\}_{n=1}^N$ as defined by Equation (7) constitute the unique equilibrium in monotone strategies.

To establish that there are no other equilibria, one can use the standard iterative deletion of dominated strategies argument utilized in the literature on global games (see Morris and Shin (2004), for a particularly careful exposition of this argument).

(Part 3) I consider now what happens as $\tau_n\to\infty$ for each $n=1,...,N$. By Equation (7), it is easy to see that $\lim_{\tau_n\to\infty}(x_n^*-\theta^*)=0$, $n\in\{1,...,N\}$.[35] Note that by Assumption 1 we have $\theta^*\in\left[\underline{\theta},\overline{\theta}\right]$. Since $(x_n^*-\theta^*)\to 0$, there exists $\kappa>0$ and $\overline{\tau}_n<\infty$ such that for all $\tau_n>\overline{\tau}_n$ we have $x_n^*\in\left[\underline{\theta}-\kappa,\overline{\theta}+\kappa\right]$. Let $\overline{\tau}=\max_{n=1,...,N}\overline{\tau}_k$. Then for all $\tau>\overline{\tau}$ the vector of equilibrium thresholds $(\theta^*,x_1^*,...,x_N^*)\in\left[\underline{\theta},\overline{\theta}\right]\times\left[\underline{\theta}-\kappa,\overline{\theta}+\kappa\right]\times....\times\left[\underline{\theta}-\kappa,\overline{\theta}+\kappa\right]\equiv K$, where K is a compact subset of $\mathbb{R}^{N+1}$.

Next, let $\boldsymbol{\tau}=(\tau_1,...,\tau_N)$, and consider a sequence $\{\boldsymbol{\tau}_k\}_{k=1}^{\infty}$ such that along this sequence $\tau_n\to\infty$ for each $n\in\{1,...,N\}$. Let $\{\theta^*(\boldsymbol{\tau}_k),x_1^*(\boldsymbol{\tau}_k),...,x_N^*(\boldsymbol{\tau}_k)\}_{k=1}^{\infty}$ be the resulting sequence of thresholds. As shown above, for all $\boldsymbol{\tau}_k>\overline{\boldsymbol{\tau}}\equiv(\overline{\tau},...,\overline{\tau})$ we have $\{\theta^*(\boldsymbol{\tau}_k),x_1^*(\boldsymbol{\tau}_k),...,x_N^*(\boldsymbol{\tau}_k)\}\in K$ where K is compact, and thus $\{\theta^*(\boldsymbol{\tau}_k),x_1^*(\boldsymbol{\tau}_k),...,x_N^*(\boldsymbol{\tau}_k)\}_{k=1}^{\infty}$ has a convergent subsequence where each element of this vector converges to a finite limit. Call this subsequence $\left\{\boldsymbol{\tau}_{k_j}\right\}_{j=1}^{\infty}$.

Since $\{\theta^*(\boldsymbol{\tau}_k),x_1^*(\boldsymbol{\tau}_k),...,x_N^*(\boldsymbol{\tau}_k)\}$ has to be the solution to the $N+1$ equilibrium conditions for each $\boldsymbol{\tau}_k$ we know that these thresholds satisfy payoff indifference conditions. The payoff indifference condition for type s_n can be written as

$$\int_{-\infty}^{\infty}\left[\mathbf{1}_{\left\{z\in\left[\tau_n^{1/2}(x_n^*-\theta^*),\infty\right)\right\}}H^n\left(x_n^*-\tau_n^{-1/2}z;\boldsymbol{\psi}\right)+\mathbf{1}_{\left\{z\in\left(-\infty,\tau_n^{1/2}(x_n^*-\theta^*)\right)\right\}}L^n\left(x_n^*-\tau_n^{-1/2}z;\boldsymbol{\psi}\right)\right]f_n(z)\,dz=0$$

where the random variable

$$\mathbf{1}_{\left\{z\in\left[\tau_n^{1/2}(x_n^*-\theta^*),\infty\right)\right\}}H^n\left(x_n^*-\tau_n^{-1/2}z;\boldsymbol{\psi}\right)+\mathbf{1}_{\left\{z\in\left(-\infty,\tau_n^{1/2}(x_n^*-\theta^*)\right)\right\}}L^n\left(x_n^*-\tau_n^{-1/2}z;\boldsymbol{\psi}\right)$$

[35] This can be established by contradiction, that is, by supposing that $|x_k^*-\theta^*|$ does not converge to 0 as $\tau_n\to\infty$ and showing that under this assumption Payoff Indifference condition is violated for sufficiently high values of τ_n.

is bounded (as both H^n and L^n are bounded). Thus, by the bounded convergence theorem we can pass the limit as $\boldsymbol{\tau}_{k_j} \to \infty$ through the integral. Then the indifference condition converges to

$$\int_{-\infty}^{\infty} \left[\mathbf{1}_{\{z\in[\varsigma_n,\infty)\}} H^n\left(x_n^*;\boldsymbol{\psi}\right) + \mathbf{1}_{\{z\in(-\infty,\varsigma_n)\}} L^n\left(x_n^*;\boldsymbol{\psi}\right)\right] f_n(z)\, dz = 0$$

where

$$\varsigma_n \equiv \lim_{\boldsymbol{\tau}_{k_j}\to\infty} \tau_n^{1/2}\left(x_n^* - \theta^*\right) \in \mathbb{R}$$

Therefore,

$$H^n\left(x_n^*;\boldsymbol{\psi}\right)\int_{-\infty}^{\infty} \mathbf{1}_{\{z\in[\varsigma_n,\infty)\}} f_n(z)\, dz + L^n\left(x_n^*;\boldsymbol{\psi}\right)\int_{-\infty}^{\infty} \mathbf{1}_{\{z\in(-\infty,\varsigma_n)\}} f_n(z)\, dz = 0$$

We can write the above condition as

$$H^n\left(x_n^*;\boldsymbol{\psi}\right)\left[1 - F_n(\varsigma_n)\right] + L^n\left(x_n^*;\boldsymbol{\psi}\right) F_n(\varsigma_n) = 0$$

Rearranging, we get

$$F_n(\varsigma_n) = \frac{H^n\left(x_n^*;\boldsymbol{\psi}\right)}{H^k\left(x_n^*;\boldsymbol{\psi}\right) - L^n\left(x_n^*;\boldsymbol{\psi}\right)}$$

Note that $F_n(\varsigma_n)$ is the proportion of the agents of type s_n that attacks the regime as $\boldsymbol{\tau}_{k_j} \to \infty$. Denote by θ^*_∞ the limit of θ^* as $\boldsymbol{\tau}_{k_j} \to \infty$, and recall that $\lim_{\boldsymbol{\tau}_{k_j}\to\infty}\left(x_n^* - \theta^*\right) = 0$. Then θ^*_∞ has to be the unique solution to the regime change condition[36]

$$R\left(\theta^*_\infty, \sum_{n=1}^{N} \lambda_n \frac{H^n\left(\theta^*_\infty;\boldsymbol{\psi}\right)}{H^n\left(\theta^*_\infty;\boldsymbol{\psi}\right) - L^n\left(\theta^*_\infty;\boldsymbol{\psi}\right)};\boldsymbol{\psi}\right) = 0$$

So far I have considered a particular convergent subsequence. However, note that the limit derived above is independent of that convergent subsequence. Therefore,by Lemma A.1 we conclude that $\{\theta^*(\boldsymbol{\tau}_n), x_1^*(\boldsymbol{\tau}_n), ..., x_N^*(\boldsymbol{\tau}_n)\}$ converges to the above limit. That is, in the limit as $\tau_n \to \infty$ for all $n = 1, ..., N$ we have the regime change threshold θ^* converging to the unique solution to

$$R\left(\theta^*_\infty, \sum_{n=1}^{N} \lambda_n \frac{H_n\left(\theta^*_\infty;\boldsymbol{\psi}\right)}{H_n\left(\theta^*_\infty;\boldsymbol{\psi}\right) - L_n\left(\theta^*_\infty;\boldsymbol{\psi}\right)};\boldsymbol{\psi}\right) = 0$$

This completes the proof. ■

C Proofs for Section 3

In this section, I first provide a proof of Theorem 1 where the decomposition of comparative statics into the multiplier effect and the direct effect is obtained by applying the implicit function theorem to the system of equilibrium conditions. I then show how this decomposition can be obtained using best-response dynamics. Finally, I provide the proof of Proposition 2.

C.1 Proof of Theorem 1

Proof of Theorem 1. Fix the vector of parameters $\boldsymbol{\psi} = \{\psi_1, ..., \psi_M\} \in \mathbb{R}^M$, and let $\{\theta^*(\boldsymbol{\psi}), x_1^*(\boldsymbol{\psi}), ..., x_N^*(\boldsymbol{\psi})\}$ be the associated monotone equilibrium. The equilibrium thresholds have to satisfy the N indifference equa-

[36] Otherwise, there would exist values of $\boldsymbol{\tau}_{k_j}$ such that $\theta^*\left(\boldsymbol{\tau}_{k_j}\right)$ would violate regime change condition.

tions, which can be written succinctly as

$$P^1\left(\theta^*\left(\boldsymbol{\psi}\right), x_1^*\left(\boldsymbol{\psi}\right);\boldsymbol{\psi}\right) = 0$$
$$\vdots$$
$$P^N\left(\theta^*\left(\boldsymbol{\psi}\right), x_N^*\left(\boldsymbol{\psi}\right);\boldsymbol{\psi}\right) = 0$$

and the regime change condition, which can be written as[37]

$$R\left(\theta^*\left(\boldsymbol{\psi}\right), x_1^*\left(\boldsymbol{\psi}\right), ..., x_N^*\left(\boldsymbol{\psi}\right);\boldsymbol{\psi}\right) = 0.$$

Next, using the above equilibrium equations, I derive the total effect of a change in ψ_m on the regime change threshold. In what follows I will use $\partial\theta^*/\partial\psi_m$ for the effect of a change in ψ when the signal thresholds $\{x_n^*\left(\boldsymbol{\psi}\right)\}_{n=1}^N$ are held constant, and $d\theta^*/d\psi_m$ for the total effect that a change in ψ_m has on θ^*, including its effect through adjustments in $\{x_n^*\left(\boldsymbol{\psi}\right)\}_{n=1}^N$. Similarly, I will use $\partial\theta^*/\partial x_n^*$ for the direct effect that a change in x_n^* has on θ^* in the regime change condition. In a similar spirit, I define $\partial x_n^*/\partial\psi_m$ as the effect of a change in ψ_m on x_n^* when agents' beliefs about θ^* are held constant, and $\partial x_n^*/\partial\theta^*$ as the effect of changing agents' beliefs about θ^* on x_n^*, $n \in \{1, ..., N\}$.

The key observation is that the equilibrium conditions written above are identities, as they define implicitly equilibrium thresholds as a function of $\boldsymbol{\psi}$ (see De la Fuente (2000)). Therefore, differentiating the equilibrium conditions with respect to ψ_m we obtain

$$\begin{aligned}
\frac{\partial P^1}{\partial\theta^*}\frac{d\theta^*}{d\psi_m} + \frac{\partial P^1}{\partial x_1^*}\frac{dx_1^*}{d\psi_m} + \frac{\partial P^1}{\partial\psi_m} &= 0 \\
&\vdots \\
\frac{\partial P^N}{\partial\theta^*}\frac{d\theta^*}{d\psi_m} + \frac{\partial P^N}{\partial x_N^*}\frac{dx_N^*}{d\psi_m} + \frac{\partial P^N}{\partial\psi_m} &= 0
\end{aligned}$$

and

$$\frac{\partial R}{\partial\theta^*}\frac{d\theta^*}{d\psi_m} + \sum_{n=1}^N \frac{\partial R}{\partial x_n^*}\frac{dx_n^*}{d\psi_m} + \frac{\partial R}{\partial\psi_m} = 0 \tag{9}$$

Note that using the n-th indifference condition (the n-th equation above) we can express $dx_n^*/d\psi_m$ as

$$\frac{dx_n^*}{d\psi_m} = -\frac{\partial P^n/\partial\theta^*}{\partial P^n/\partial x_n^*}\frac{d\theta^*}{d\psi_m} - \frac{\partial P^n/\partial\psi_m}{\partial P^n/\partial x_n^*}$$

Then

$$\frac{\partial x_n^*}{\partial\psi_m} = -\frac{\partial P^n/\partial\psi_m}{\partial P^n/\partial x_1^*} \text{ and } \frac{\partial x_n^*}{\partial\theta^*} = -\frac{\partial P^n/\partial\theta^*}{\partial P^n/\partial x_n^*}$$

as these derivatives capture the change in x_n^* as a result of changes in ψ_m and θ^* in the n-th indifference equation. It follows that

$$\frac{dx_n^*}{d\psi_m} = \frac{\partial x_n^*}{\partial\theta^*}\frac{d\theta^*}{d\psi_m} + \frac{\partial x_n^*}{\partial\psi_m}$$

[37] Note that the equilibrium regime change condition is given by

$$R\left(\theta^*\left(\boldsymbol{\psi}\right), \sum_{n=1}^N \lambda_n F_n\left(\frac{x^*\left(\boldsymbol{\psi}\right) - \theta^*\left(\boldsymbol{\psi}\right)}{\tau_n^{-1/2}}\right);\boldsymbol{\psi}\right) = 0$$

Thus, with a slight abuse notation we can write the above condition as

$$R\left(\theta^*\left(\boldsymbol{\psi}\right), x_1^*\left(\boldsymbol{\psi}\right), ..., x_N^*\left(\boldsymbol{\psi}\right);\boldsymbol{\psi}\right) = 0$$

Substituting this into Equation (9), I obtain

$$\frac{\partial R}{\partial \theta^*}\frac{d\theta^*}{d\psi_m}+\sum_{n=1}^{N}\frac{\partial R}{\partial x_n^*}\left[\frac{\partial x_n^*}{\partial \theta^*}\frac{d\theta^*}{d\psi_m}+\frac{\partial x_n^*}{\partial \psi_m}\right]+\frac{\partial R}{\partial \psi_m}=0$$

Dividing both sides by $\partial R/\partial\theta^*$ and rearranging, the above equation becomes

$$\left[1-\sum_{n=1}^{N}\frac{\partial R/\partial x_n^*}{\partial R/\partial \theta^*}\frac{\partial x_n^*}{\partial \theta^*}\right]\frac{d\theta^*}{d\psi_m}=-\sum_{n=1}^{N}\frac{\partial R/\partial x_n^*}{\partial R/\partial \theta^*}\frac{\partial x_n^*}{\partial \psi_m}-\frac{\partial R/\partial \psi_m}{\partial R/\partial \theta^*}$$

Note that

$$\frac{\partial \theta^*}{\partial \psi_m}=-\frac{\partial R/\partial \psi_m}{\partial R/\partial \theta^*}\text{ and }\frac{\partial \theta^*}{\partial x_n^*}=-\frac{\partial R/\partial x_n^*}{\partial R/\partial \theta^*},$$

as these ratios of derivatives simply capture the effects that changes in ψ_m and x_n^* in the regime change equilibrium condition have on θ^*. Thus, the above equation can be written as

$$\frac{d\theta^*}{d\psi_m}=\frac{1}{1-\sum_{n=1}^{N}\frac{\partial \theta^*}{\partial x_n^*}\frac{\partial x_n^*}{\partial \theta^*}}\times\left[\sum_{n=1}^{N}\frac{\partial \theta^*}{\partial x_n^*}\frac{\partial x_n^*}{\partial \psi_m}+\frac{\partial \theta^*}{\partial \psi_m}\right],$$

or

$$\frac{d\theta^*}{d\psi_m}=\mathcal{M}\times\mathcal{D}\left(\psi_m\right)$$

as claimed.[38]

Next, I show that the multiplier effect is always positive. Once this is established the fact that $sgn\left(d\theta^*/d\psi_m\right)=sgn\left(\mathcal{D}\left(\psi_m\right)\right)$ follows immediately. To see that $\mathcal{M}\left(\psi_m\right)>0$, recall first that, as shown in the proof of uniqueness, we have $\partial x_n^*/\partial\theta^*<1$. Moreover, since the equilibrium regime change condition is given by

$$R\left(\theta^*,\sum_{n=1}^{N}\lambda_n F_n\left(\frac{x_n^*-\theta^*}{\tau_n^{-1/2}}\right);\boldsymbol{\psi}\right)=0,$$

it follows that

$$\frac{\partial \theta^*}{\partial x_n^*}=\frac{-R_2\lambda_n\tau_n^{1/2}f_n\left(\frac{x_n^*-\theta^*}{\tau_n^{-1/2}}\right)}{R_1-R_2\sum_{k=1}^{N}\lambda_k\tau_k^{1/2}f_k\left(\frac{x_k^*-\theta^*}{\tau_k^{-1/2}}\right)}\in(0,1),$$

since $R_1>0$ and $R_2<0$. Therefore

$$0<\sum_{n=1}^{N}\frac{\partial \theta^*}{\partial x_n^*}\frac{\partial x_n^*}{\partial \theta^*}\leq\sum_{n=1}^{N}\frac{\partial \theta^*}{\partial x_n^*}=\frac{-R_2\sum_{n=1}^{N}\lambda_n\tau_n^{1/2}f_n\left(\frac{x_n^*-\theta^*}{\tau_n^{-1/2}}\right)}{R_1-R_2\sum_{k=1}^{N}\lambda_k\tau_k^{1/2}f_k\left(\frac{x_k^*-\theta^*}{\tau_k^{-1/2}}\right)}<1$$

This establishes that $\mathcal{M}\in(1,\infty)$, thus providing proofs of parts 1 and 2 of Theorem 1.

Finally, suppose that $\tau_n=\tau$, for all $n\in\{1,...,N\}$ and consider what happens to the multiplier effect as $\tau\to\infty$. From the expression for $\partial\theta^*/\partial x_n^*$, it is clear that

$$\lim_{\tau\to\infty}\frac{\partial \theta^*}{\partial x_n^*}=1,\ n=1,...,N$$

Recall from the proof of Proposition 1 (Section B of this Appendix) that

$$\frac{\partial x_n^*}{\partial \theta^*}=\frac{-H^n\left(\theta^*;\boldsymbol{\psi}\right)\tau^{1/2}f_n\left(\frac{x_n^*-\theta}{\tau^{-1/2}}\right)+L^n\left(\theta^*;\boldsymbol{\psi}\right)\tau^{1/2}f_n\left(\frac{x_n^*-\theta}{\tau^{-1/2}}\right)}{-H^n\left(\theta^*;\boldsymbol{\psi}\right)\tau^{1/2}f_n\left(\frac{x_n^*-\theta}{\tau^{-1/2}}\right)+L^n\left(\theta^*;\boldsymbol{\psi}\right)\tau^{1/2}f_n\left(\frac{x_n^*-\theta}{\tau^{-1/2}}\right)+\Lambda^n},$$

where

$$\Lambda^n\equiv\int_{\tau^{1/2}\left(x_n^*-\theta^*\right)}^{\infty}\frac{\partial H^n\left(x_n^*-\tau^{-1/2}z;\boldsymbol{\psi}\right)}{\partial x_n^*}f_n(z)\,dz+\int_{-\infty}^{\tau^{1/2}\left(x_n^*-\theta^*\right)}\frac{\partial L^n\left(x_n^*-\tau^{-1/2}z;\boldsymbol{\psi}\right)}{\partial x_n^*}f_n(z)\,dz<0$$

[38] Below, I show that the above decomposition can also be derived using myopic best-response dynamics.

In the proof of Proposition 1 I showed that $\tau^{1/2}(x_n^* - \theta^*)$ converges to a finite constant as $\tau \to \infty$. Thus, $\lim_{\tau\to\infty} \Lambda^n$, is finite while

$$\begin{aligned}
&\lim_{\tau\to\infty}\left[-H^n(\theta^*;\boldsymbol{\psi})\,\tau^{1/2} f_n\left(\frac{x_n^*-\theta}{\tau^{-1/2}}\right) + L^n(\theta^*;\boldsymbol{\psi})\,\tau^{1/2} f_n\left(\frac{x_n^*-\theta}{\tau^{-1/2}}\right)\right] \\
= &\lim_{\tau\to\infty}\tau^{1/2} f_n\left(\frac{x_n^*-\theta}{\tau^{-1/2}}\right)\underbrace{\left[-H^n(\theta^*;\boldsymbol{\psi}) + L^n(\theta^*;\boldsymbol{\psi})\right]}_{<0} \\
= &-\infty
\end{aligned}$$

It follows that

$$\lim_{\tau_n\to\infty}\frac{\partial x_n^*}{\partial\theta^*} = 1,$$

implying that

$$\frac{\partial\theta^*}{\partial x_n^*}\frac{\partial x_n^*}{\partial\theta^*} \to 1 \text{ as } \tau\to\infty$$

Thus, we conclude that

$$\lim_{\tau_n\to\infty}\mathcal{M} = \infty$$

This completes the proof of Theorem 1. ■

C.2 Derivations Using Best-Response dynamics

Here, I show how one can derive the multiplier and direct effects by considering best-response dynamics following a change in model parameters. This approach leads to a more intuitive derivation of the multiplier effect.

Suppose that initially the vector of parameters is $\boldsymbol{\psi}$. Given $\boldsymbol{\psi}$, agents believe that the regime will collapse if and only if $\theta \le \theta^*(\boldsymbol{\psi})$, and each agent of type s_n plans to attack the regime if and only if he receives a signal lower than x_n^*. Suppose that suddenly a parameter $\psi_m \in \boldsymbol{\psi}$ increases. How do agents respond to this change?

A small increase in ψ_m has two effects on the regime outcome. First, holding agents' beliefs about θ^* constant, it leads to a change in the signal threshold used by type s_n equal to $\partial x_n^*/\partial\psi_m = -P^n_{\psi_m}/P^n_{x_n^*}$, for each n. This in turn leads to a change in the regime change threshold equal to $(\partial\theta^*/\partial x_n^*)(\partial x_n^*/\partial\psi_m)$. Second, it directly affects the resilience of the status quo, and hence leads to a change in the regime threshold equal to $\partial\theta^*/\partial\psi_m = -R_{\psi_m}/R_{\theta^*}$. It follows that the change in ψ_m leads to an initial change in θ^* equal to

$$\frac{\partial\theta^*}{\partial\psi_m} + \sum_{n=1}^N \frac{\partial\theta^*}{\partial x_n^*}\frac{\partial x_n^*}{\partial\psi_m},$$

which is exactly the direct effect $\mathcal{D}(\psi_m)$ identified above.

The above change in θ^* initiates further adjustments to the new equilibrium. In response to a change in θ^* by $\mathcal{D}(\psi_m)$, each agent adjust their threshold. This adjustment is approximately equal to $(\partial x_n^*/\partial\theta^*)\,\mathcal{D}(\psi_m)$, that is the product of sensitivity of x_n^* to changes in θ^* and the actual change in θ^*. This leads to an additional change in θ^* equal to

$$S_1 \equiv \sum_{n=1}^N \frac{\partial\theta^*}{\partial x_n^*}\frac{\partial x_n^*}{\partial\theta^*}\mathcal{D}(\psi_m),$$

where I am using S_1 to denote the first step of best-response dynamics initiated by the direct effect. This change in θ^* equal to S_1 induces a further change in agents' beliefs about the regime outcome and hence to a further adjustment. This process continues ad infinitum, with the adjustment of θ^* in the k-th round of this process equal to

$$\mathcal{D}(\psi_m)\left(\sum_{n=1}^N \frac{\partial\theta^*}{\partial x_n^*}\frac{\partial x_n^*}{\partial\theta^*}\right)^k$$

The total change in θ^* is then obtained by adding up the adjustments in θ^* in all rounds (including the initial response), hence (using the convention that $S_0 \equiv \mathcal{D}(\psi_m)$) is given by:

$$\frac{d\theta^*}{d\psi_m} = \sum_{k=1}^{\infty} S_k = \frac{1}{1 - \sum_{n=1}^{N} \frac{\partial\theta^*}{\partial x_n^*}\frac{\partial x_n^*}{\partial\theta^*}} \left[\frac{\partial\theta^*}{\partial\psi_m} + \sum_{n=1}^{N} \frac{\partial\theta^*}{\partial x_n^*}\frac{\partial x_n^*}{\partial\psi_m}\right],$$

where the second equality is valid if $\Sigma_{n=1}^{N}(\partial\theta^*/\partial x_n^*)(\partial x_n^*/\partial\theta^*) < 1$ (which is always the case when there is a unique equilibrium). Thus, we obtain the result as stated in Theorem 1. However, the above derivations have the advantage that they make it clear that the multiplier effect captures the adjustment in θ^* driven by the adjustment in agents' beliefs about the regime change threshold.

C.3 Proof of Proposition 2

Proof of Proposition 2. The planner's problem is given by

$$\min_{\{v_k\}_{n=1}^{N}} \sum_{n=1}^{N} \lambda_n v_n$$
$$s.t.\ \theta^*(v_1, ..., v_N) \geq \widehat{\theta}$$
$$v_k \in [0, \overline{v}],\ n = 1, ..., N$$

The Lagrangian associated with the above problem is given by

$$L = -\sum_{n=1}^{N} \lambda_n v_n + \eta\left[-\widehat{\theta} + \theta^*(v_1, ..., v_N)\right] + \sum_{n=1}^{N} \xi_n v_n + \sum_{n=1}^{N} \varrho_n [\overline{v} - v_n],$$

where I wrote a Lagrangian for maximization of $-\Sigma_{n=1}^{N}\lambda_n v_n$ subject to the above constraints. The first-order conditions are

$$-\lambda_1 + \eta\frac{d\theta^*(v_1, ..., v_N)}{dv_1} + \xi_1 - \varrho_1 = 0$$
$$\vdots$$
$$-\lambda_N + \eta\frac{d\theta^*(v_1, ..., v_N)}{dv_N} + \xi_N - \varrho_N = 0,$$

where as before I denote by $d\theta^*/dv_n$ the total change of θ^* with respect to v_n. Now, from Theorem 1 we know that

$$\frac{d\theta^*(v_1, ..., v_N)}{dv_n} = \mathcal{M} \times \mathcal{D}(v_n)$$

Therefore, the above system of equation can be written as

$$-\lambda_1 + \eta[\mathcal{M}\ \mathcal{D}(v_1; \mathbf{v})] + \xi_1 - \varrho_1 = 0$$
$$\vdots$$
$$-\lambda_N + \eta[\mathcal{M}\ \mathcal{D}(v_N; \mathbf{v})] + \xi_N - \varrho_N = 0,$$

Let $\mathbf{v}^* = \{v_1^*, ..., v_N^*\}$ denote the vector of optimal subsidies and suppose that $v_n^* < \overline{v}$ and $v_{n+1}^* > 0$. In that case $\varrho_n = 0$ (as $v_n^* < \overline{v}$) and $\xi_{n+1} = 0$ (as $v_n^* > 0$). Since v_n^* and v_{n+1}^* are optimal it follows that

$$\begin{aligned} -1 + \eta\left[\mathcal{M}\ \frac{\mathcal{D}(v_n; \mathbf{v})}{\lambda_n}\right] + \frac{\xi_n}{\lambda_n} &= 0 \\ -1 + \eta\left[\mathcal{M}\ \frac{\mathcal{D}(v_{n+1}; \mathbf{v})}{\lambda_{n+1}}\right] - \frac{\varrho_{n+1}}{\lambda_{n+1}} &= 0 \end{aligned}$$

Since ξ_n, $\varrho_{n+1} > 0$, it follows that

$$-1 + \eta\left[\mathcal{M}\ \frac{\mathcal{D}(v_n; \mathbf{v})}{\lambda_n}\right] < -1 + \eta\left[\mathcal{M}\ \frac{\mathcal{D}(v_{n+1}; \mathbf{v})}{\lambda_{n+1}}\right]$$

Since $\eta > 0$, the above inequality implies that

$$\frac{\mathcal{D}\left(v_n;\mathbf{v}\right)}{\lambda_n} < \frac{\mathcal{D}\left(v_{n+1};\mathbf{v}\right)}{\lambda_{n+1}},$$

which is a contradiction. It follows that the planner always targets the types of agents who have the highest direct effect adjusted by the size of the group.

Despite the fact that the above condition suggests that the optimal subsidy scheme depends on the measures of the types, the optimal subsidy scheme is not affected by the values of the $\lambda_n's$. This is because

$$\mathcal{D}\left(v_n;\mathbf{v}\right) = \frac{\partial\theta^*}{\partial x_n^*}\frac{\partial x_n^*}{\partial v_k}$$

where $\partial x_n^*/\partial v_n$ only depends on $\lambda_n's$ indirectly via θ^* (as this effect is the same for all types of agents) while

$$\frac{\partial\theta^*}{\partial x_n^*} \propto \lambda_k$$

Therefore, λ_n's cancel out when we compute the direct effect. ■

D Proofs for Section 4

D.1 Publicity Multiplier

Proof of Proposition 3. This is simply a corollary of Theorem 1. In particular, from the proof of Theorem 1 we know that for any parameter ψ_m,

$$\frac{dx^*}{d\psi_m} = \frac{\partial x^*}{\partial\psi_m} + \frac{\partial x^*}{\partial\theta^*}\frac{d\theta^*}{d\psi_m}$$

Then using the expression for $d\theta^*/d\psi_m$ derived above and setting $\partial\theta^*/\partial\psi_m = 0$, we have

$$\frac{\frac{dx^*}{d\psi_m}}{\frac{\partial x^*}{\partial\psi_m}} = \frac{\frac{\partial x^*}{\partial\psi_m} + \frac{\partial x^*}{\partial\theta^*}\frac{\frac{\partial\theta^*}{\partial x^*}\frac{\partial x^*}{\partial\psi_m}}{1-\frac{\partial\theta^*}{\partial x^*}\frac{\partial x^*}{\partial\theta^*}}}{\frac{\partial x^*}{\partial\psi_m}} = \frac{1}{1-\frac{\partial\theta^*}{\partial x^*}\frac{\partial x^*}{\partial\theta^*}} = \mathcal{M}$$

■

In Section 4.1 I restricted my attention to parameters of the model that affect only the payoff indifference condition. I also considered the setup with a single type of agent, to make the model closely comparable to the setup considered by Morris and Shin (2003). The next result provides the decomposition of $dx^*/d\psi_m$ into the multiplier effect and the "partial effect" when both of these assumptions are relaxed, that is, when there are N distinct types of players and ψ_m may affect payoff differential functions $\{\pi^n\}_{n=1}^N$ as well as the regime change function.

Proposition D.1 *Fix $\boldsymbol{\psi}$, and consider the effect of that changing ψ_m has on x_n^*. Then*

$$\frac{dx_n^*}{d\psi_m} = \mathcal{M}\left[\frac{\partial x_n^*}{\partial\psi_m} + \frac{\partial x_n^*}{\partial\theta^*}\frac{\partial\theta^*}{\partial\psi_m} + \sum_{k\neq n}\frac{\partial x_n^*}{\partial\theta^*}\frac{\partial\theta^*}{\partial x_k^*}\left(\frac{\partial x_k^*}{\partial\psi_m} - \frac{\partial x_n^*}{\partial\psi_m}\right)\right]$$

Proof. From the proof of Theorem 1 we know that

$$\frac{dx_n^*}{d\psi_m} = \frac{\partial x_n^*}{\partial\psi_m} + \frac{\partial x_n^*}{\partial\theta^*}\frac{d\theta^*}{d\psi_m}$$

Using the fact that $d\theta^*/d\psi_m = \mathcal{M} \times \mathcal{D}(\psi_m)$ and the definition of $\mathcal{D}(\psi_m)$ we get

$$\begin{aligned}
\frac{dx_n^*}{d\psi_m} &= \frac{\partial x_n^*}{\partial \psi_m} + \frac{\partial x_n^*}{\partial \theta^*}\mathcal{M}\left[\frac{\partial \theta^*}{\partial \psi_m} + \sum_{k=1}^{N} \frac{\partial \theta^*}{\partial x_k^*}\frac{\partial x_k^*}{\partial \psi_m}\right] \\
&= \frac{\partial x_n^*}{\partial \psi_m}\left[1 - \mathcal{M}\frac{\partial x_n^*}{\partial \theta^*}\frac{\partial \theta^*}{\partial x_n^*}\right] + \frac{\partial x_n^*}{\partial \theta^*}\mathcal{M}\left[\frac{\partial \theta^*}{\partial \psi_m} + \sum_{k\neq n} \frac{\partial \theta^*}{\partial x_k^*}\frac{\partial x_k^*}{\partial \psi_m}\right] \\
&= \frac{\partial x_n^*}{\partial \psi_m}\left[\frac{1 - \sum_{k\neq n}\frac{\partial \theta^*}{\partial x_k^*}\frac{\partial x_k^*}{\partial \theta^*}}{1 - \sum_{k=1}^{N}\frac{\partial \theta^*}{\partial x_k^*}\frac{\partial x_k^*}{\partial \theta^*}}\right] + \frac{\partial x_n^*}{\partial \theta^*}\mathcal{M}\left[\frac{\partial \theta^*}{\partial \psi_m} + \sum_{k\neq n} \frac{\partial \theta^*}{\partial x_k^*}\frac{\partial x_k^*}{\partial \psi_m}\right] \\
&= \mathcal{M}\left[\frac{\partial x_n^*}{\partial \psi_m} + \frac{\partial x_n^*}{\partial \theta^*}\frac{\partial \theta^*}{\partial \psi_m} + \frac{\partial x_n^*}{\partial \theta^*}\sum_{k\neq n}\frac{\partial \theta^*}{\partial x_k^*}\left(\frac{\partial x_k^*}{\partial \psi_m} - \frac{\partial x_n^*}{\partial \psi_m}\right)\right]
\end{aligned}$$

■

The above decomposition is intuitive. The direct effect consists of the effect that a change in ψ_m has on x_n^* through its effect on the regime change condition (as captured by $(\partial x_n^*/\partial\theta^*)(\partial\theta^*/\partial\psi_m)$), and through its effect on the payoff indifference condition (as captured by $\partial x_n^*/\partial\psi_m$). But note that a change in ψ_m also affects the payoff indifference conditions of other types of agents. Since the multiplier effect captures how changes in $\{x_k^*\}_{k=1}^N$, through their effects on θ^*, lead to further changes in x_n^*, we need to take into account the fact that types of agents different than type s_n may adjust their thresholds by different amounts in response to a change in ψ_m. The last term in the square brackets is thus simply an adjustment for this heterogeneity in the initial response of agents to the change in ψ_m.

D.2 Relation to Strategic Complementarities

Proof of Proposition 4. **(Part 1)** Let $\mathbf{x}^* = \{x_1^*, ..., x_N^*\}$, where x_n^* is the threshold used by agents of type s_n, and let $\theta^*(\mathbf{x}^*)$ be the implied regime change threshold. To establish the first part of the Proposition it suffices to show that

$$\sum_{n=1}^{N} w_n \nabla_{\underline{\mathbf{1}}}\beta_n(\mathbf{x}^*) = \sum_{n=1}^{N}\frac{\partial\theta^*(\mathbf{x}^*)}{\partial x_n^*}\frac{\partial x_n^*}{\partial\theta^*}$$

Note that

$$\sum_{n=1}^{N} w_n \nabla_{\underline{\mathbf{1}}}\beta_n(\mathbf{x}^*) = \sum_{n=1}^{N} w_n\left(\sum_{k=1}^{N}\frac{\partial\beta_n(\mathbf{x}^*)}{\partial x_k^*}\right) = \sum_{n=1}^{N} w_n\left(\sum_{k=1}^{N}\frac{\partial\beta_n(\mathbf{x}^*)}{\partial\theta^*}\frac{\partial\theta^*(\mathbf{x}^*)}{\partial x_k^*}\right)$$

where we used the observation that x_k^* affects $\beta_n(\mathbf{x}^*)$ only indirectly, through its effect on θ^*. But then

$$\sum_{n=1}^{N} w_n\frac{\partial\beta_n(\mathbf{x}^*)}{\partial\theta^*} = \left(\sum_{k=1}^{N}\frac{\partial\theta^*(\mathbf{x}^*)}{\partial x_k^*}\right)\sum_{n=1}^{N}\frac{\frac{\partial\theta^*(\mathbf{x}^*)}{\partial x_n^*}}{\sum_{l=1}^{N}\frac{\partial\theta^*(\mathbf{x}^*)}{\partial x_l^*}}\frac{\partial\beta_n(\mathbf{x}^*)}{\partial\theta^*} = \sum_{n=1}^{N}\frac{\partial\theta^*(\mathbf{x}^*)}{\partial x_n^*}\frac{\partial\beta_n(\mathbf{x}^*)}{\partial\theta^*} = \sum_{n=1}^{N}\frac{\partial\theta^*(\mathbf{x}^*)}{\partial x_n^*}\frac{\partial x_n^*}{\partial\theta^*}$$

since in equilibrium $x_n^* = \beta_n(\mathbf{x}^*)$.

(Part 2) Without loss of generality, let us focus on an of type s_n whose indifference condition is

$$\int_{-\infty}^{\theta^*(\mathbf{x}^*)} H^n(\theta;\boldsymbol{\psi}) f_n(\theta|\beta_n(\mathbf{x}^*))\, d\theta + \int_{\theta^*(\mathbf{x}^*)}^{\infty} L^n(\theta;\boldsymbol{\psi}) f_n(\theta|\beta_n(\mathbf{x}^*))\, d\theta = 0$$

In the proof of Proposition 1, I showed that as long as $\tau_n < \infty$, we have

$$\frac{\partial\beta_n(\mathbf{x}^*)}{\partial\theta^*} < 1 \quad\text{and}\quad \sum_{k=1}^{N}\frac{\partial\theta^*}{\partial x_k^*} < 1$$

It follows that

$$\nabla_{\underline{\mathbf{1}}}\beta_n(\mathbf{x}^*) = \sum_{k=1}^{N}\frac{\partial\beta_n(\mathbf{x}^*)}{\partial\theta^*}\frac{\partial\theta^*(\mathbf{x}^*)}{\partial x_k^*} = \frac{\partial\beta_n(\mathbf{x}^*)}{\partial\theta^*}\sum_{k=1}^{N}\frac{\partial\theta^*(\mathbf{x}^*)}{\partial x_k^*} < \frac{\partial\beta_n(\mathbf{x}^*)}{\partial\theta^*} < 1$$

In the proof of the Proposition 1, I also showed that

$$\lim_{\tau_n \to \infty} \frac{\partial \beta_n(\mathbf{x}^*)}{\partial \theta^*} \to 1,$$

and that if $\tau_n = \tau$ for all $n \in \{1, ..., N\}$ then

$$\lim_{\tau \to \infty} \sum_{k=1}^{N} \frac{\partial \theta^*(\mathbf{x}^*)}{\partial x_k^*} = 1$$

Therefore,

$$\lim_{\tau \to \infty} \nabla_{\mathbf{1}} \beta_n(\mathbf{x}^*) = 1$$

This establishes Part 2 of Proposition 4. ■

D.3 Strategic Complementarities Under the Complete Information

In this section, I show that strategic complementarities in the complete information model and in the global game model in the limit as $\tau_x \to \infty$ are equally strong. For simplicity, I assume that $N = 1$, so that there is a single type of agent.

Consider first the complete information version of the model described in Section 2. In this case I let $x_i = \theta$, so that the private signals reveal the truth, that is the information structure is complete. I restrict attention to monotone strategies of the form "attack the status quo if $x_i \leq \widehat{x}_i$, and refrain from attacking the status quo if $x_i > \widehat{x}_i$." Now, suppose that all agents use the same threshold $\widehat{x}$, and let $\beta_i^{CI}(\widehat{x})$ denote the optimal threshold of agent i when faced with such a strategy profile. It is easy to see that the best-response function of agent i is to choose $\beta_i^{CI}(\widehat{x}) = \widehat{x}$ if $\widehat{x} \in [\underline{\theta}, \overline{\theta}]$, $\beta_i^{CI}(\widehat{x}) = \underline{\theta}$ if $\widehat{x} \leq \underline{\theta}$, as the status quo collapses even for all $\theta \leq \underline{\theta}$, and $\beta_i^{CI}(\widehat{x}) = \overline{\theta}$ if $\widehat{x} \geq \overline{\theta}$ as the status quo always survives for all $\theta \geq \overline{\theta}$. These observations are summarized in the following lemma.

Lemma D.1 *When the information structure is complete, the best-response function of each agent i is given by*

$$\beta_i^{CI}(\widehat{x}) = \begin{cases} \underline{\theta} \text{ if } \widehat{x} \leq \underline{\theta} \\ \widehat{x} \text{ if } \widehat{x} \in [\underline{\theta}, \overline{\theta}] \\ \overline{\theta} \text{ if } \widehat{x} > \overline{\theta} \end{cases} \quad \text{and} \quad \frac{\partial \beta_i^{CI}(\widehat{x})}{\partial \widehat{x}} = \begin{cases} 0 \text{ if } \widehat{x} \leq \underline{\theta} \\ 1 \text{ if } \widehat{x} \in [\underline{\theta}, \overline{\theta}] \\ 0 \text{ if } \widehat{x} > \overline{\theta} \end{cases}$$

Next, consider the global game model, that is, the model described in Section 2, but set $N = 1$ and denote by τ the common precision level of the private signals.[39] Let $\widehat{x}$ denote the threshold used by all agents, and denote by $\beta_i(\widehat{x})$ the threshold used by all the agent i in response. We have the following result.

Lemma D.2 *Suppose that $\tau \to \infty$. Then*

$$\lim_{\tau \to \infty} \beta_i(\widehat{x}) = \begin{cases} \underline{\theta} \text{ if } \widehat{x} \leq \underline{\theta} \\ \widehat{x} \text{ if } \widehat{x} \in [\underline{\theta}, \overline{\theta}] \\ \overline{\theta} \text{ if } \widehat{x} > \overline{\theta} \end{cases} \quad \text{and} \quad \lim_{\tau \to \infty} \frac{\partial \beta_i(\widehat{x})}{\partial \widehat{x}} = \begin{cases} 0 \text{ if } \widehat{x} \leq \underline{\theta} \\ 1 \text{ if } \widehat{x} \in [\underline{\theta}, \overline{\theta}] \\ 0 \text{ if } \widehat{x} > \overline{\theta} \end{cases}$$

Proof. Let $\widehat{x}$ be the thresholds used by all agents and $\widehat{\theta}(\widehat{x})$ be the implied regime change threshold. I want to compute $\beta_i(\widehat{x})$ and $\partial \beta_i(\widehat{x}) / \partial \widehat{x}$.

Suppose first that all agents use threshold $\widehat{x} < \underline{\theta}$. In this case it is easy to show that $\lim_{\tau \to \infty} \widehat{\theta}(\widehat{x}) = \underline{\theta}$. As the consequence, it has to to be the case that $\lim_{\tau \to \infty} \beta_i(\widehat{x}) = \underline{\theta}$. To see that, consider agent i's indifference condition and note that if $\lim_{\tau \to \infty} \beta_i(\widehat{x}) > \underline{\theta}$ then the LHS of the indifference condition converges

[39] As there is only one type of agent, I drop the subscript n everywhere.

to $H\left(\lim_{\tau\to\infty}\beta_i\left(\widehat{x}\right);\boldsymbol{\psi}\right) > H\left(\underline{\theta};\boldsymbol{\psi}\right) > 0$, which would mean that for sufficiently high τ the payoff indifference condition is violated. Similarly, if $\lim_{\tau\to\infty}\beta_i\left(\widehat{x}\right) < \underline{\theta}$ then the LHS of the indifference condition converges to $L\left(\underline{\theta};\boldsymbol{\psi}\right) < 0$, which would again mean that for sufficiently high τ the payoff indifference condition is violated. Thus, $\lim_{\tau\to\infty}\beta\left(\widehat{x}\right) = \underline{\theta}$.[40] An analogous argument establishes that $\lim_{\tau\to\infty}\beta_i\left(\widehat{x}\right) = \overline{\theta}$ when $\widehat{x} > \overline{\theta}$.

Next, consider $\widehat{x} \in \left[\underline{\theta}, \overline{\theta}\right]$. First, I establish that if $\widehat{x} \in \left[\underline{\theta}, \overline{\theta}\right]$, then $\widehat{\theta}\left(\widehat{x}\right) \to \widehat{x}$ as $\tau \to \infty$. To see that this is the case, recall that the regime change equilibrium condition is given by

$$R\left(\widehat{\theta}, F\left(\frac{\widehat{x}-\widehat{\theta}}{\tau^{-1/2}}\right);\boldsymbol{\psi}\right) = 0$$

It follows that as $\tau \to \infty$ the LHS of the above equation converges to $R\left(\widehat{\theta}, 1;\boldsymbol{\psi}\right) < 0$ for all $\widehat{\theta} \in [\underline{\theta}, \widehat{x})$, and to $R\left(\widehat{\theta}, 0;\boldsymbol{\psi}\right) > 1$ for all $\widehat{\theta} \in [\underline{\theta}, \widehat{x})$. Thus, the only candidate limit is $\widehat{x}$. To see that $\lim_{\tau\to\infty}\widehat{\theta}\left(\widehat{x}\right)$ exists, note that $\widehat{\theta}\left(\widehat{x};\tau_x\right) \in \left[\underline{\theta}, \overline{\theta}\right]$, which is a compact subset of $\mathbb{R}$, and apply Lemma A.1. It follows that $\lim_{\tau\to\infty}\widehat{\theta}\left(\widehat{x}\right) = \widehat{x}$ and that

$$F\left(\frac{\widehat{x}-\widehat{\theta}}{\tau^{-1/2}}\right) \to m\left(\widehat{x}\right),$$

where $m\left(\widehat{x}\right)$ is defined as the solution to

$$R\left(\widehat{x}, m\left(\widehat{x}\right);\boldsymbol{\psi}\right) = 0$$

Next, consider $\beta_i\left(\widehat{x}\right)$. From the definition of $\beta_i\left(\widehat{x}\right)$ as the solution to the indifference equation it is clear that $\beta_i\left(\widehat{x}\right)$ can be written as $\beta_i\left(\widehat{\theta}\left(\widehat{x}\right)\right)$ since agent i's indifference equation depends on $\widehat{x}$ only indirectly through $\widehat{\theta}$. Using the same argument as above, we have $\lim_{\tau\to\infty}\beta_i\left(\widehat{\theta}\left(\widehat{x}\right)\right) = \widehat{\theta}\left(\widehat{x}\right)$ for all $\widehat{x} \in \left[\underline{\theta}, \overline{\theta}\right]$. But then it follows that

$$\beta\left(\widehat{x}\right) = \widehat{x} \text{ for all } \widehat{x} \in \left[\underline{\theta}, \overline{\theta}\right]$$

It remains to show that $\partial\beta_i\left(\widehat{x}\right)/\partial\widehat{x} \to 1$ as $\tau \to \infty$ for all $\widehat{x} \in \left[\underline{\theta}, \overline{\theta}\right]$. This cannot be concluded directly from the fact that $\lim_{\tau\to\infty}\beta\left(\widehat{x}\right) = \widehat{x}$ for all $\widehat{x} \in \left[\underline{\theta}, \overline{\theta}\right]$ since in general

$$\lim_{\tau\to\infty}\frac{\partial\beta_i\left(\widehat{x}\right)}{\partial\widehat{x}} \neq \frac{\partial}{\partial\widehat{x}}\left[\lim_{\tau\to\infty}\beta_i\left(\widehat{x}\right)\right]$$

However, as argued above, $\beta_i\left(\widehat{x}\right) = \beta_i\left(\widehat{\theta}\left(\widehat{x}\right)\right)$ and as shown in the proof of Proposition 1

$$\lim_{\tau\to\infty}\frac{\partial\beta\left(\widehat{\theta}\left(\widehat{x}\right)\right)}{\partial\widehat{\theta}} = 1$$

Furthermore,

$$\lim_{\tau\to\infty}\frac{\partial\widehat{\theta}\left(\widehat{x}\right)}{\partial\widehat{x}} = \lim_{\tau\to\infty} -\frac{R_2\tau^{1/2}f\left(\frac{\widehat{x}-\widehat{\theta}}{\tau_x^{-1/2}}\right)}{R_1 - R_2\tau^{1/2}f\left(\frac{\widehat{x}-\widehat{\theta}}{\tau_x^{-1/2}}\right)} = 1$$

This establishes the claim. ■

The above lemmas establish that the best-response function in the global game model with one type of agent converges to the best-response function of the complete information model as $\tau \to \infty$. Nevertheless, the global game model has a unique equilibrium, while the complete information model has multiple equilibria. As explained in the paper, the reason behind this is that the agents in the global game model face strategic uncertainty, that is, uncertainty about the behavior of others, which is absent under complete information.

[40] To establish that $\lim_{\tau\to\infty}\beta_i\left(\widehat{x}\right)$ exists one can follow the same argument as used in the proof of Proposition 1 (Section B of this Appendix) to show that the equilibrium thresholds converge when the noise in the signals is vanishingly small. In the interest of space, I omit this step.

E Proofs for Section 5

Proof of Proposition 5. By Lemma A.3, we know that in the simple framework of Section 4.2 the multiplier effect is given by

$$\mathcal{M} = \frac{1}{1 - \frac{\tau_\theta + \tau_x}{\tau_x} \frac{z\tau_x^{1/2}\phi\left(\tau_x^{1/2}(x^* - \theta^*)\right)}{1 + z\tau_x^{1/2}\phi\left(\tau_x^{1/2}(x^* - \theta^*)\right)}}$$

Note that $\mathcal{M}$ is high when $\phi\left(\tau_x^{1/2}(x^* - \theta^*)\right)$ is high, and tends to 1 (its minimum value) as $\phi\left(\tau_x^{1/2}(x^* - \theta^*)\right) \to 0$. Since $\phi(\cdot)$ is a symmetric function that achieves it maximum at 0, it follows that $\mathcal{M}$ is a decreasing function of $|x^* - \theta^*|$. Therefore, to understand how a change in γ, μ_θ, or z affects $\mathcal{M}$ it is enough to understand how a change in these parameters affects $|x^* - \theta^*|$. Furthermore, by Lemma A.2

$$x^* - \theta^* = \frac{\tau_\theta}{\tau_x}(\theta^* - \mu_\theta) - \frac{\sqrt{\tau_x + \tau_\theta}}{\tau_x}\Phi^{-1}(\gamma)$$

To establish how changes in parameters affect the distance between x^* and θ^*, suppose that $x^* - \theta^* = a$, where $a \in \mathbb{R}$. From the expression for $x^* - \theta^*$, we know that $x^* - \theta^* = a$ if and only if

$$\theta^* = \mu_\theta + \frac{\tau_x}{\tau_\theta}a + \frac{\sqrt{\tau_x + \tau_\theta}}{\tau_\theta}\Phi^{-1}(\gamma)$$

Next, note that θ^* takes such a value if and only if $\mu_\theta + \frac{\tau_x}{\tau_\theta}a + \frac{\sqrt{\tau_x+\tau_\theta}}{\tau_\theta}\Phi^{-1}(\gamma)$ is the solution to the regime change condition. Since (as established in Lemma A.2) the regime change condition is given by

$$\theta^* = z\Phi\left(\frac{\tau_\theta}{\tau_x^{1/2}}(\theta^* - \mu_\theta) - \sqrt{\frac{\tau_x + \tau_\theta}{\tau_x}}\Phi^{-1}(\gamma)\right),$$

we know that this happens if and only if

$$\mu_\theta + \frac{\sqrt{\tau_x + \tau_\theta}}{\tau_\theta}\Phi^{-1}(\gamma) = z\Phi\left(\tau_x^{1/2}a\right) - \frac{\tau_x}{\tau_\theta}a$$

Subtracting from $(1/2)\,z$ both sides, we obtain

$$\mu_\theta - \frac{1}{2}z + \frac{\sqrt{\tau_x + \tau_\theta}}{\tau_\theta}\Phi^{-1}(\gamma) = z\Phi\left(\tau_x^{1/2}a\right) - \frac{1}{2}z - \frac{\tau_x}{\tau_\theta}a \tag{10}$$

Note that the *RHS*of Equation (10) is equal to 0 at $a = 0$, and that its derivative is given by

$$z\tau_x^{1/2}\phi\left(\tau_x^{1/2}a\right) - \frac{\tau_x}{\tau_\theta} \leq z\tau_x^{1/2}\frac{1}{\sqrt{2\pi}} - \frac{\tau_x}{\tau_\theta} = \tau_x^{1/2}\left[z\frac{1}{\sqrt{2\pi}} - \frac{\tau_x^{1/2}}{\tau_\theta}\right] < 0,$$

where the last inequality holds as long as the equilibrium is unique (which is the maintained assumption in Section 5.2). It follows that the RHS of Equation (10) is strictly decreasing in a.

From the above observations, it follows that whenever $\mu_\theta - \frac{1}{2}z + \frac{\sqrt{\tau_x+\tau_\theta}}{\tau_\theta}\Phi^{-1}(\gamma) = 0$, then $a = 0$, meaning that $x^* - \theta^* = 0$, so that the multiplier effect achieves its maximum value (for given value of τ_θ and τ_x). Moreover, by applying the implicit function theorem to Equation (10), it is easy to see that $da/d\mu_\theta < 0$, $da/d\gamma < 0$ and $da/dz > 0$. It follows that a decreases as $\mu_\theta - \frac{1}{2}z + \frac{\sqrt{\tau_x+\tau_\theta}}{\tau_\theta}\Phi^{-1}(\gamma)$ increases. Thus, we conclude that Equation (10) implies a one-to-one mapping between $\mu_\theta - \frac{1}{2}z + \frac{\sqrt{\tau_x+\tau_\theta}}{\tau_\theta}\Phi^{-1}\left(\frac{1+r}{1+R}\right)$ and a, with a higher $\mu_\theta - \frac{1}{2}z + \frac{\sqrt{\tau_x+\tau_\theta}}{\tau_\theta}\Phi^{-1}\left(\frac{1+r}{1+R}\right)$ implying a lower a, and $a = 0$ when $\mu_\theta - \frac{1}{2}z + \frac{\sqrt{\tau_x+\tau_\theta}}{\tau_\theta}\Phi^{-1}\left(\frac{1+r}{1+R}\right) = 0$.

It follows that if we define

$$g(\mu_\theta, \alpha, \gamma) = \left|\mu_\theta - \frac{1}{2}z + \frac{\sqrt{\tau_x + \tau_\theta}}{\tau_\theta}\Phi^{-1}(\gamma)\right|,$$

then a higher value of $g(\mu_\theta, \alpha, \gamma)$ indicates a larger distance between x^* and θ^*, that is, a higher value of $|x^* - \theta^*|$. This proves the result. ■

Proof of Proposition 6. To understand how the multiplier effect varies with τ_x, it suffices to determine how a change in τ_x affects

$$\frac{\tau_\theta+\tau_x}{\tau_x}\frac{z\tau_x^{1/2}\phi\left(\tau_x^{1/2}\left(x^*-\theta^*\right)\right)}{1+z\tau_x^{1/2}\phi\left(\tau_x^{1/2}\left(x^*-\theta^*\right)\right)}$$

The derivative of the above expression with respect to τ_x is proportional to

$$\begin{aligned}
&\frac{\tau_\theta}{\left(\tau_x+\tau_\theta\right)}+\frac{\frac{1}{2}-\tau_x\left(x^*-\theta^*\right)\left[\frac{dx^*}{d\tau_x}-\frac{d\theta^*}{d\tau_x}\right]}{\tau_x^{-1/2}+z\phi\left(\tau_x^{1/2}\left(x^*-\theta^*\right)\right)}\\
=\ &\frac{\tau_x^{1/2}\tau_\theta}{\left(\tau_x+\tau_\theta\right)}+\frac{\frac{1}{2}-\tau_x^2\left(x^*-\theta^*\right)\left[\frac{\tau_\theta}{\tau_x^2}\left(\left(\theta^*-\mu_\theta\right)-\frac{1}{2}\frac{1}{\sqrt{\tau_x+\tau_\theta}}\Phi^{-1}\left(\gamma\right)\right)\frac{\frac{1}{z}\frac{1}{\phi\left(\Phi^{-1}\left(\frac{1}{z}\theta^*\right)\right)}}{\frac{\tau_\theta}{\tau_x^{1/2}}-\frac{1}{z}\frac{1}{\phi\left(\Phi^{-1}\left(\frac{1}{z}\theta^*\right)\right)}}+\frac{\sqrt{\tau_x+\tau_\theta}}{2\tau_x^2}\Phi^{-1}\left(\gamma\right)\right]}{\tau_x^{-1/2}+z\phi\left(\tau_x^{1/2}\left(x^*-\theta^*\right)\right)}
\end{aligned}$$

First, note that if we set $\tau_\theta=0$, the above expression is positive, since in this case $\left(x^*-\theta^*\right)=-\frac{1}{\tau_x^{1/2}}\Phi^{-1}\left(\gamma\right)$. It follows that when the prior is uninformative, the multiplier effect is always increasing in τ_x.

Next, note that $\tau_x^{1/2}\tau_\theta/\left(\tau_x+\tau_\theta\right)$ converges to 0 as $\tau_x\to\infty$. Moreover,

$$\lim_{\tau_x\to\infty}\tau_x^{1/2}\left(x^*-\theta^*\right)\to-\Phi^{-1}\left(\gamma\right)$$

while

$$\lim_{\tau_x\to\infty}\tau_x^{3/2}\left[\frac{\tau_\theta}{\tau_x^2}\left(\left(\theta^*-\mu_\theta\right)-\frac{1}{2}\frac{1}{\sqrt{\tau_x+\tau_\theta}}\Phi^{-1}\left(\gamma\right)\right)\frac{\frac{1}{z}\frac{1}{\phi\left(\Phi^{-1}\left(\frac{1}{z}\theta^*\right)\right)}}{\frac{\tau_\theta}{\tau_x^{1/2}}-\frac{1}{z}\frac{1}{\phi\left(\Phi^{-1}\left(\frac{1}{z}\theta^*\right)\right)}}+\frac{\sqrt{\tau_x+\tau_\theta}}{2\tau_x^2}\Phi^{-1}\left(\gamma\right)\right]\to\Phi^{-1}\left(\gamma\right)$$

It follows that

$$\frac{1}{2}-\tau_x^2\left(x^*-\theta^*\right)\left[\frac{\tau_\theta}{\tau_x^2}\left(\left(\theta^*-\mu_\theta\right)-\frac{1}{2}\frac{1}{\sqrt{\tau_x+\tau_\theta}}\Phi^{-1}\left(\gamma\right)\right)\frac{\frac{1}{z}\frac{1}{\phi\left(\Phi^{-1}\left(\frac{1}{z}\theta^*\right)\right)}}{\frac{\tau_\theta}{\tau_x^{1/2}}-\frac{1}{z}\frac{1}{\phi\left(\Phi^{-1}\left(\frac{1}{z}\theta^*\right)\right)}}+\frac{\sqrt{\tau_x+\tau_\theta}}{2\tau_x^2}\Phi^{-1}\left(\gamma\right)\right]$$

converges to

$$\frac{1}{2}+\left[\Phi^{-1}\left(\gamma\right)\right]^2>0$$

Thus, for sufficiently high τ_x the multiplier effect is necessarily increasing in τ_x. ■

Proof of Proposition 7. The proof of this result can be found in the Online Appendix. Here, I briefly explain the approach used to prove this result.

To understand how $\mathcal{M}$ varies with τ_θ, we need to show how

$$\frac{\tau_\theta+\tau_x}{\tau_x}\frac{z\tau_x^{1/2}\phi\left(\tau_x^{1/2}\left(x^*-\theta^*\right)\right)}{1+z\tau_x^{1/2}\phi\left(\tau_x^{1/2}\left(x^*-\theta^*\right)\right)}$$

varies with τ_θ. This is a challenging task, since a change in τ_θ affects the above expression both directly and indirectly (via its effect on x^* and θ^*) and the resulting derivative is a complex object. The approach I take below is nevertheless straightforward if tedious. I first compute and simplify the derivative of the above object with respect to τ_θ. I then show that (for any parameters of the model) (1) the resulting expression is negative as $\mu_\theta\to\pm\infty$; (2) there exists a non-empty closed interval I of μ_θ on which this expression is positive; and (3) as μ_θ increases, the resulting expression crosses the 0 line from below when $\mu_\theta<\min\{I\}$ and from above when $\mu_\theta>\min\{I\}$. The Proposition follows from these observations. ■

Proof of Corollary 2. Note that in Bebchuk and Goldstein (2011), $\gamma=(1+r)/(1+R)$. The result then follows from Propositions 5 and 6. ■

F Proofs for Section 6

Proof of Corollary 3. Note that

$$\mathcal{D}\left(\psi_m\right) = -\frac{\frac{\partial R\left(\theta^*, m\left(\theta^*;\left\{x_n^*\right\}_{n=1}^N\right);\boldsymbol{\psi}\right)}{\partial \psi_m}}{\frac{\partial R\left(\theta^*, m\left(\theta^*;\left\{x_n^*\right\}_{n=1}^N\right),\psi\right)}{\partial \theta^*}} + \sum_{n=1}^{N} \frac{\frac{\partial R\left(\theta^*, m\left(\theta^*;\left\{x_n^*\right\}_{n=1}^N\right);\boldsymbol{\psi}\right)}{\partial x_n^*}}{\frac{\partial R\left(\theta^*, m\left(\theta^*;\left\{x_n^*\right\}_{n=1}^N\right),\psi\right)}{\partial \theta^*}} \frac{\frac{\partial P_n\left(\theta^*, x_n^*, \boldsymbol{\psi}\right)}{\partial \psi_m}}{\frac{\partial P_n\left(\theta^*, x_n^*, \boldsymbol{\psi}\right)}{\partial x_n^*}}$$

Note first that, given the assumed information structure, when player i receives a higher signal x_i, he expects a higher value of θ on average. In other words, a higher x_i shifts his posterior belief upward according to the first-order stochastic dominance ordering (see Section B of this Appendix). Therefore, it follows that a higher x_i always decreases agent i's incentives to attack the regime. Now, if we write agent i's indifference condition as $P_{s(i)}\left(\theta^*, x^*, \boldsymbol{\psi}\right) = 0$, it follows that $\partial P_{s(i)}\left(\theta^*, x_{s(i)}^*, \boldsymbol{\psi}\right)/\partial x_{s(i)}^* < 0$. Next, suppose that an increase in ψ_m always increases the payoff differential between attacking and not attacking, that is, $\partial \pi^{s(i)}\left(\theta;\boldsymbol{\psi}\right)/\partial \psi_m \geq 0$ for all θ. In this case, it is immediate that $\partial P_{s(i)}\left(\theta^*, x_{s(i)}^*, \boldsymbol{\psi}\right)/\partial \psi_m \geq 0$ so that

$$\frac{\partial x_{s(i)}^*}{\partial \psi_m} = -\frac{\partial P_{s(i)}\left(\theta^*, x_{s(i)}^*, \boldsymbol{\psi}\right)/\partial \psi_m}{\partial P_{s(i)}\left(\theta^*, x_{s(i)}^*, \boldsymbol{\psi}\right)/\partial x_{s(i)}^*} \geq 0,$$

and that $\partial x_{s(i)}^*/\partial \psi_m \geq 0$ does not depend on the assumed distribution of signal noise $F_{s(i)}$.

Next, consider the direct effect of a change in ψ_m that operates through the regime change function $R\left(\theta^*, m\left(\theta^*;\{x_n^*\}_{n=1}^N\right),\psi\right)$, where $m\left(\theta^*, \{x_n^*\}_{n=1}^N\right) = \sum_{n=1}^N \lambda_n F_n\left(\tau_n^{1/2}\left(x_n^* - \theta^*\right)\right)$. Note that a higher θ^* always decreases the proportion of agents attacking the regime when $\{x_n^*\}_{n=1}^N$ is held constant. Since $R_1 > 0$ by assumption, it follows that $\partial R\left(\theta^*, m\left(\theta^*;\{x_n^*\}_{n=1}^N\right),\psi\right)/\partial\theta^* \geq 0$. Therefore, it follows that if $\partial R\left(\theta, m;\boldsymbol{\psi}\right)/\partial \psi_m \leq 0$, then

$$\frac{\partial \theta^*}{\partial \psi_m} = -\frac{\partial R\left(\theta^*, m\left(\theta^*;\{x_n^*\}_{n=1}^N\right);\boldsymbol{\psi}\right)/\partial \psi_m}{\partial R\left(\theta^*, m\left(\theta^*;\{x_n^*\}_{n=1}^N\right),\psi\right)/\partial \theta^*} \geq 0$$

Again $sgn\left(\partial\theta^*/\partial\psi_m\right)$ does not depend on the assumed $\{F_n\}_{n=1}^N$.

Finally, we always have $\partial\theta^*/\partial x_n^* > 0$, $n = 1, ..., N$, as higher signal thresholds imply a higher proportion of agents attacking the regime for a given θ regardless of $\{F_n\}_{n=1}^N$. Thus, we conclude that $\mathcal{D}\left(\psi_m\right) \geq 0$, and its signs does not depend on the particular noise structure as characterized by $\{F_n\}_{n=1}^N$. ■

Proof of Lemma 1. Suppose that $H^n\left(\theta\right) = H^n > 0$ and $L^n\left(\theta\right) = L^n < 0$. It is easy to show that in monotone equilibrium, a player of type s_n uses the threshold x_n^* that solves

$$H^n F_n\left(\frac{x_n^* - \theta^*}{\tau_n^{-1/2}}\right) + L^n\left(1 - F_n\left(\frac{x_n^* - \theta^*}{\tau_n^{-1/2}}\right)\right) = 0$$

Therefore, the proportion of agents of type s_n that attacks the regime is given by

$$\lambda_n \Pr\left(x_i \leq x_n^* | \theta^*\right) = \lambda_n F_n\left(\frac{x_n^* - \theta^*}{\tau_n^{-1/2}}\right) = \lambda_n \frac{-L^n}{H^n - L^n} \in \left(0, \lambda_n\right)$$

It follows that the total mass of agents attacking the regime in equilibrium is given by

$$m^*\left(\theta^*\right) = \sum_{n=1}^N \lambda_n \frac{-L^n}{H^n - L^n} \in \left(0, 1\right)$$

The key observation here is that the proportion of agents that attack the regime at $\theta = \theta^*$ does not depends on the assumed information structure $\{F_n\}_{n=1}^N$. But then it follows that the unique equilibrium threshold θ^*, which is determined by

$$R\left(\theta^*, m^*\left(\theta^*\right);\boldsymbol{\psi}\right) = 0$$

also does not depend on $\{F_n\}_{n=1}^{N}$. As such, the comparative statics are unchanged when the information structure changes. Moreover, note that this "robustness" result holds without the need for additional restrictions on the regime change function R that were needed in Corollary 3. ■

Proof of Corollary 4. This results follows immediately from the definitions of $\underline{\theta}(\boldsymbol{\psi})$ and $\overline{\theta}(\boldsymbol{\psi})$ and the fact that (as discussed in the proof of Corollary 3) we have $d\theta^*/d\psi_m > 0$ when $\partial R/\partial\psi_m > (<) \, 0$ and $\partial\pi^n/\partial\psi_m \leq (\geq) \, 0$ for all θ and m. ■

F.1 Predictions that Depend on the Information Structure: Examples

In this section, I provide two examples to show how the comparative statics depend on information structure when the hypothesis of Corollary 3 is not satisfied. In the first example, which is based on Morris and Shin (1998), whether a change in ψ_m increases or decreases the resilience of the regime depends on the mass of agents that attack the regime (i.e., the sign of $\partial R/\partial\psi_m$ depends on m). In the second example, which is based on Szkup (2016), a change in ψ_m tends to decrease the resilience of the regime (i.e., $\partial R/\partial\psi_m < 0$), but at the same time it also decreases agents' incentives to attack the regime ($\partial\pi^n/\partial\psi_m > 0$).

In both cases, I compare comparative statics predictions derived under two distinct information structures. Under the first information structure, agents have the prior belief that $\theta \sim unif\,[\mu_\theta - \eta, \mu_\theta + \eta]$, and each of them receives a private signal $x_i = \theta + \varepsilon_i$, with $\varepsilon_i \sim unif\,[-\varepsilon, \varepsilon]$.[41] I refer to this information structure as "uniform-uniform." Under the second information structure, agents have the prior belief that $\theta \sim N\left(\mu_\theta, \tau_\theta^{-1}\right)$, and each of them receives a private signal $x_i = \theta + \varepsilon_i$, with $\varepsilon_i \sim N\left(0, \tau_x^{-1/2}\right)$. I refer to this information structure as "normal-normal."

F.2 Example 1: Non-monotone Regime Change Function

Consider the following example adapted from Morris and Shin (1998). Suppose for simplicity that the payoffs to speculators from attacking the regime are $1-t$ if an attack is successful and $-t$ if an attack is unsuccessful, while choosing not to attack yields 0. The central bank will keep the peg if

$$\theta - c\,(m, L) > 0$$

and abandon it otherwise. Here, θ stands for the benefit of keeping the peg, m is the fraction of speculators that attacks the peg, and L is the amount of foreign reserves that the central bank can raise quickly in order to prevent the attack. Finally, $c\,(m, L)$ is the cost of defending the regime when the size of attack is m and the foreign reserves are L, with $c_m > 0$.

Assume that $c_L\,(m, L) < 0$ for $m < \overline{m}$, and $c_L\,(m, L) = 0$ if $m = \overline{m}$, and $c_L\,(m, L) > 0$ if $m > \overline{m}$, so that raising additional liquidity to prevent the attack decreases the cost of defending the peg if and only if m is relatively low. These assumptions capture the idea that when the attack is expected to be small, the central bank can borrow foreign reserves from other foreign central banks cheaply (since the loans are almost risk-free). On the other hand, if the attack is large, then raising additional foreign reserves is costly, as other banks expect that the peg will collapse and results in an economic crisis, in which case they are unlikely to recover their loans. As such ,other central banks will charge a high interest rate on those loans.

Let θ^* denote the threshold level of θ such that the peg is abandoned if and only if $\theta < \theta^*$. We then have the following result.[42]

[41] Implicitly, I assume that η is large enough so that $\mu_\theta - \eta < \underline{\theta} - 2\varepsilon$ and $\mu_\theta + \eta > \overline{\theta} + 2\varepsilon$ (i.e., $\left[\underline{\theta} - 2\varepsilon, \overline{\theta} + 2\varepsilon\right] \subset (\mu_\theta - \eta, \mu_\theta + \eta)$), which is required for the equilibrium to be unique.

[42] The proof of this result can be found in the Online Appendix.

Lemma F.1 *Consider the effects that changes in μ_θ and L have on θ^* (i.e., $d\theta^*/d\mu_\theta$ and $d\theta^*/dL$).*

1. *Under the "uniform-uniform" information structure:*

   (a) $\frac{d\theta^*}{d\mu_\theta} = 0$, *that is, a change in μ_θ has no effect on θ^**

   (b) $\frac{d\theta^*}{dL} < 0$ *if and only if* $1 - t < \overline{m}$.

2. *Under the "normal-normal" information structure:*

   (a) $\frac{d\theta^*}{d\mu_\theta} < 0$, *that is a higher μ_θ always decreases θ^*.*

   (b) $\frac{d\theta^*}{dL} < 0$ *if and only if*

$$\mu_\theta + \frac{\sqrt{\tau_x + \tau_\theta}}{\tau_\theta}\Phi^{-1}(t) + \frac{\sqrt{\tau_x}}{\tau_\theta}\Phi^{-1}(\overline{m}) \geq c(\overline{m}, L)$$

Thus, we see that predictions of the model differ substantially under the two information structures considered. When the prior and the noise in the signals have uniform distributions, changes in the mean of the prior have no effect on θ^*, while an increase in L decrease the probability of a currency crisis if and only if the transaction cost t is sufficiently high. On the other hand, when the prior and the noise in the signals have normal distribution, an increase in the mean of the prior always leads to a decrease in the currency crisis threshold. Furthermore, whether an increase in L decreases or increases θ^* depends on all the parameters of the model

F.2.1 Example 2: Counteracting Effects of an Increase in ψ_m

In this example, I consider a case where an increase in ψ_m negatively affects the regime's resistance but also discourages the agents from attacking the regime. As before, I consider the uniform-uniform and normal-normal information structures. As

There is a single firm and a continuum of investors indexed by i, $i \in [0, 1]$. The firm owns a risky project with a return of θ and with total liquidation value 1. The project can be partially liquidated to meet early withdrawals if such a need arises. The firm financed its project by issuing an amount α of short-term debt with face value 1 and $(1 - \alpha)$ of long-term debt with face value D_L, with $\alpha \in [0, 1]$. Before the project matures, the short-term debt holders have to decide whether to roll over their debt or withdraw their funds early.

Short-term debt holders who withdraw their funds early get their funds back, that is their are paid back 1 unit of funds. Short-term debt holders who roll over their loans are promised $D_S > 1$ if the firm's return from the project exceeds its debt obligation. Otherwise, the firm defaults and all the debt-holders receive nothing.[43] It follows that short-term debt holders face the following payoffs:

	Repayment	Default
Roll over	D_S	0
Withdraw	1	1

Let m denote the fraction of short-term debt holders that withdraw their funds early. The firm repays its debt if and only if

$$\theta(1 - m\alpha) - \alpha(1 - m) D_S - (1 - \alpha) D_L > 0$$

Here $\theta(1 - m\alpha)$ is the return from the scaled-down investment, where a fraction $m\alpha$ of the investment was liquidated to meet early withdrawals. In this setting, "attacking the regime" is associated with withdrawing funds early.

[43] This is a substantially simplified version of the model in Szkup (2016), which abstracts from issues of early default, the optimal choice of maturity structure, and the optimal choice of face values of debt.

Fix the vector of the parameters of the model $\boldsymbol{\psi}$. Let $\theta^*(\boldsymbol{\psi})$ denote the default threshold such that the firm repays its debt if and only if $\theta \geq \theta^*(\boldsymbol{\psi})$, and let $m^*(\theta^*)$ denote the proportion of agents withdrawing early when $\theta = \theta^*(\boldsymbol{\psi})$. [44] Now, suppose that the firm would like to avoid early withdrawals and considers increasing D_S in order to discourage investors from withdrawing their loans early. The next result characterizes the effect of increasing D_S on θ^* under the two information structures. The proof of this result can be found in the Online Appendix.

Lemma F.2 *Consider the effect of changing D_S on the equilibrium default threshold.*

1. *Under the "uniform-uniform" information structure, we have $\frac{d\theta^*}{dD_S} \geq 0$ if and only if*

$$\alpha \geq \min\left\{\frac{D_L - D_S^2}{D_L + 1 - 2D_S}, 0\right\}$$

2. *Under the "normal-normal" information structure, we have $\frac{d\theta^*}{dD_S} < 0$ if and only if*

$$\mu_\theta < \widehat{\mu}_\theta(\boldsymbol{\psi})$$

where $\widehat{\mu}_\theta(\boldsymbol{\psi})$ is the value of μ_θ that is the unique solution to

$$-(1 - m^*(\theta^*(\boldsymbol{\psi}))) - (\theta^*(\boldsymbol{\psi}) - D_S)\frac{\partial m^*(\theta^*(\boldsymbol{\psi}))}{\partial D_S} = 0.$$ [45]

We see that under the uniform-uniform information structure, whether D_S decreases or increases depends on α, D_L, and D_S but not on the parameters that affect the information structure. On the other hand, under the normal-normal information structure, all parameters of the model (including the precision of the private signal, the precision of the prior, and the mean of the prior) matter through their impact on $\widehat{\mu}_\theta$.

G Proofs for Section 7

Proof of Proposition 8. This result follows from Theorem 1 and the definition of the directional derivative. More precisely, the effect of a simultaneous small change in each $\psi_{m_1}, ..., \psi_{m_K}$ on θ^* is captured by the directional derivative of $\theta^*\left(\psi_{m_1}, ..., \psi_{m_K}\right)$ in the direction $\underline{\mathbf{1}} = (1, ..., 1) \in \mathbb{R}^K$ and is given by

$$\nabla_{\underline{\mathbf{1}}}\theta^*\left(\psi_{m_1}, ..., \psi_{m_K}\right) = \sum_{k=1}^{K}\frac{d\theta^*}{d\psi_{m_k}} = \mathcal{M}\sum_{k=1}^{K}\mathcal{D}(\psi_k)$$

This establishes the result. ■

To prove Proposition 9, I need the following result. Its proof can be found in the Online Appendix.

Lemma G.1 *The monotone equilibrium characterized by the regime change threshold $\widehat{\theta}^*$ is stable if and only if $\exists \varepsilon > 0$ such that*

$$R\left(\widehat{\theta}, \sum_{n=1}^{N}\lambda_n F_n\left(\frac{x_n^*\left(\widehat{\theta}\right) - \widehat{\theta}}{\tau_n^{-1/2}}\right); \boldsymbol{\psi}\right)$$

is increasing for all $\widehat{\theta} \in \left(\widehat{\theta}^* - \varepsilon, \widehat{\theta}^* + \varepsilon\right)$.

[44] It can be shown that the default threshold s^* is unique as long as

$$\frac{\tau_x^{1/2}}{\tau_\theta} > \frac{\alpha(D_L - D_S)}{(1-\alpha)}\frac{1}{\sqrt{2\pi}}$$

I assume that this condition is satisfied.

[45] Since $\mu_\theta \in \boldsymbol{\psi}$, a change in μ_θ affects this condition via its impact on $\theta^*(\boldsymbol{\psi})$.

With this result in hand, I provide the proof of Proposition 9.

Proof of Proposition 9. Consider a monotone equilibrium with associated regime change $\widehat{\theta}^*$. Thenthe decomposition of a change in $\widehat{\theta}^*$ into the mutliplier and direct effects can be derived by applying the implicit function theorem in exactly the same way as in the proof of Theorem 1.

Next, I show that the multiplier effect associated with $\widehat{\theta}^*$ is positive if and only if the equilibrium is stable. Recall that the multiplier effect is defined as

$$\mathcal{M} = \frac{1}{1 - \sum_{n=1}^{N} \frac{\partial \widehat{\theta}^*}{\partial x_n^*} \frac{\partial x_n^*}{\partial \widehat{\theta}^*}},$$

where $\widehat{\theta}^*$ solves

$$R\left(\widehat{\theta}^*, \sum_{n=1}^{N} \lambda_n F_n \left(\frac{x_n^*\left(\widehat{\theta}^*\right) - \widehat{\theta}^*}{\tau_n^{-1/2}}\right); \boldsymbol{\psi}\right) = 0$$

and$\left\{x_n^*\left(\widehat{\theta}^*\right)\right\}_{n=1}^{N}$ are the unique solutions to the indifference conditions when agents believe that the regime change threshold will collapse if and only if $\theta \leq \widehat{\theta}^*$. The multiplier effect $\mathcal{M} \in (1, \infty)$ if and only if $\sum_{n=1}^{N} \frac{\partial \widehat{\theta}^*}{\partial x_n^*} \frac{\partial x_n^*}{\partial \widehat{\theta}^*} < 1$. But note that

$$\sum_{n=1}^{N} \frac{\partial \widehat{\theta}^*}{\partial x_n^*} \frac{\partial x_n^*}{\partial \widehat{\theta}^*} = -\frac{R_2 \sum_{n=1}^{N} \lambda_n \tau_n^{1/2} f_n \left(\frac{x_n^*\left(\widehat{\theta}^*\right) - \widehat{\theta}^*}{\tau_n^{-1/2}}\right) \frac{\partial x_n^*}{\partial \widehat{\theta}^*}}{R_1 - R_2 \sum_{n=1}^{N} \lambda_n \tau_n^{1/2} f_n \left(\frac{x_n^*\left(\widehat{\theta}^*\right) - \widehat{\theta}^*}{\tau_n^{-1/2}}\right)},$$

where the numerator and the denominator are always positive (since $R_2 < 0$ and $\partial x_n^* / \partial \widehat{\theta}^* > 0$). Thus, $\sum_{n=1}^{N} \frac{\partial \widehat{\theta}^*}{\partial x_n^*} \frac{\partial x_n^*}{\partial \widehat{\theta}^*} < 1$ if and only if

$$R_1 + R_2 \sum_{n=1}^{N} \lambda_n \tau_n^{1/2} f_n \left(\frac{x_n^* - \widehat{\theta}^*}{\tau_n^{-1/2}}\right) \left[\frac{\partial x_n^*}{\partial \widehat{\theta}^*} - 1\right] > 0$$

However, by Lemma G.1, we see that this condition is the same as the necessary and sufficient condition for a monotone equilibrium to be stable. Thus, $\sum_{n=1}^{N} \frac{\partial \widehat{\theta}^*}{\partial x_n^*} \frac{\partial x_n^*}{\partial \widehat{\theta}^*} < 1$ (so that $M \in (1, \infty)$) if and if $\widehat{\theta}^*$ is associated with a stable equilibrium. By the same argument it follows that when equilibrium is unstable we have $\sum_{n=1}^{N} \frac{\partial \widehat{\theta}^*}{\partial x_n^*} \frac{\partial x_n^*}{\partial \widehat{\theta}^*} > 1$, and hence $\mathcal{M} < 0$.

By the above argument, it follows that for a stable equilibrium we have $sgn\left(d\widehat{\theta}^* / d\psi_m\right) = sgn\left(\mathcal{D}\left(\psi_m\right)\right)$, while if $\widehat{\theta}$ is associated with an unstable equilibrium then $sgn\left(d\widehat{\theta}^* / d\psi_m\right) = -sgn\left(\mathcal{D}\left(\psi_m\right)\right)$.

Finally, consider the best-response dynamics following a change in ψ_m. Note that the derivations of $d\widehat{\theta}^* / d\psi_m$ using best-response dynamics presented in Section C.2 are also valid even if there are multiple equilibria regardless, whether $\widehat{\theta}^*$ is associated with a stable or an unstable equilibrium. In either case

$$\frac{d\widehat{\theta}^*}{d\psi_m} = \sum_{k=0}^{\infty} \mathcal{D}\left(\psi_m\right) \left[\sum_{n=1}^{N} \frac{\partial \widehat{\theta}^*}{\partial x_n^*} \frac{\partial x_n^*}{\partial \widehat{\theta}^*}\right]^k$$

In light of the above discussion, if the equilibrium is stable then $\frac{\partial \widehat{\theta}^*}{\partial x_n^*} \frac{\partial x_n^*}{\partial \widehat{\theta}^*} \in (0, 1)$, and hence the best-response dynamics converge. On the other hand, if the equilibrium is unstable, then, as was shown above, we have $\sum_{n=1}^{N} \frac{\partial \widehat{\theta}^*}{\partial x_n^*} \frac{\partial x_n^*}{\partial \widehat{\theta}^*} > 1$, and hence the best-response dynamics diverge. This completes the proof. ■