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Abstract: Farinelli and Tibiletti (F-T) ratio, a general risk-reward performance measurement ratio, is popular due to its simplicity and yet generality that both Omega ratio and upside potential ratio are its special cases. The F-T ratios are ratios of average gains to average losses with respect to a target, each raised by a power index, p and q. In this paper, we establish the consistency of F-T ratios with any nonnegative values p and q with respect to first-order stochastic dominance. Second-order stochastic dominance does not lead to F-T ratios with any nonnegative values p and q, but can lead to F-T dominance with any $p < 1$ and $q \geq 1$. Furthermore, higher-order stochastic dominance ($n \geq 3$) leads to F-T dominance with any $p < 1$ and $q \geq n - 1$. We also find that when the variables being compared belong to the same location-scale family or the same linear combination of location-scale families, we can get the necessary relationship between the stochastic dominance with the F-T ratio after imposing some conditions on the means. Our findings enable academics and practitioners to draw better decision in their analysis.

Keywords: First-order Stochastic Dominance, High-order Stochastic Dominance, Upside Potential Ratio, Farinelli and Tibiletti ratio, Risk Measures.

JEL Classification: C0, D81, G10

1 Introduction

Due to its simplicity and easy interpretation, the Sharpe ratio has been widely used in practice (Sharpe 1966; Leung and Wong, 2008). However, the standard deviation, which is adopted in the Sharpe ratio, is not a good measure of risk because it penalizes upside deviation as well as downside deviation. In fact, most investors view upside deviation and downside deviation differently. They consider negative returns over the target return as risk and positive returns over the target return as profit. Downside risks, thus, become important components in the construction of performance measures. Risk measures based on below-target returns are first proposed by Fishburn (1977) in the context of portfolio optimization. Classic measures of downside risk include semi-deviation, (Markowitz, 1959, 1987), Value-at-Risk (Jorion, 2000) and the conditional Value-at-Risk (Rockafellar and Uryasev, 2000). Farinelli and Tibiletti (F-T, 2008) propose a general risk-reward ratio, which is suitable to compare skewed returns with respect to a benchmark. The F-T ratios are essentially ratios of average above-benchmark returns (gains) to average below-benchmark returns (losses), each raised by some power index, p and q, to proxy for the investor's degree of risk aversion. When the power index is equal to one for both numerator and denominator, the performance measure is the Omega ratio (Keating and Shadwick, 2002). On the other hand, when the power index is equal to one and two for numerator and denominator, respectively, the performance measure becomes the upside potential ratio (Sortino et al. 1999).

This paper focuses on the F-T ratio because of its intuitive simplicity and yet generality that both Omega ratio and upside potential ratio are its special cases. It is clear that the higher is an investment's F-T ratio, the more attractive it is to an investor who cares about downside risk. We call it F-T dominance. On the other hand, stochastic dominance (SD) theory can be used to compare different investments without assuming specific form of utility function. This raises an interesting following question: if we find an investment is preferred compared with another one by stochastic dominance theory, can its F-T ratios always higher than those of the other one? Or in another words, could stochastic dominance lead to F-T dominance and vice versa? In this paper, we show that the answer depends on the order of stochastic dominance.

Specifically, it is proven that first-order stochastic dominance is consistent with the F-T ratios with any nonnegative values p and q. Second-order stochastic dominance does not lead to F-T ratios with all nonnegative values p and q, but can lead to F-T dominance with any $p < 1$ and $q \geq 1$. We present a simple example in this paper to show that second-order stochastic dominance is not consistent with the F-T ratio at all time. Higher-order stochastic dominance ($n \geq 3$) leads to F-T dominance with any $p < 1$ and $q \geq n-1$. We find that when the variables being compared belong to the same location-scale family or the same linear combination of location-scale families, we can get the necessary relationship between the stochastic dominance with the F-T ratio after imposing some conditions on the means. Our findings enable academics and practitioners to draw better decision in their analysis.

The rest of this paper is organized as follows. Section 2 gives a brief introduction of SD theory. Section 3 contains our main result. Section 4 concludes the paper.

2 Definitions and Notations

Investors are called the j-order risk averters if their utility $u \in U_j = \{u : (-1)^i u^{(i)} \leq 0 \, , \, i = 1, \cdots, j\}$ and called the j-order risk seekers if $u \in U_j^R = \{u : u^{(i)} \geq 0 \, , \, i = 1, \cdots, j\}$ for any integer j in which $u^{(i)}$ is the i^{th} derivative of u. For any integer j, we define the j-order integral, $F_Z^{(j)}$, and the j-order reverse integral, $F_Z^{(j)R}$, of Z to be

$$\begin{aligned} F_Z^{(j)}(\eta) &= \int_{-\infty}^{\eta} F_Z^{(j-1)}(\xi) d\xi \; , \\ F_Z^{(j)R}(\eta) &= \int_{\eta}^{\infty} F_Z^{(j-1)R}(\xi) d\xi \; , \end{aligned} \tag{2.1}$$

respectively, with $F_Z^{(0)R} = F_Z^{(0)} = f_Z$ to be the probability density function (pdf) of Z for $Z = X$ and Y. When $j = 1$, $F_Z^{(1)} = F_Z$ is the cumulative distribution function (cdf) of Z.

Following the definition of stochastic dominance (SD), see, for example, Hanoch and Levy (1969), Levy (2015) and Guo and Wong (2016), prospect X first-order stochastically dominates prospect Y, denoted by

$$X \succeq_{FSD} Y \quad \text{if and only if} \quad F_X^{(1)}(\eta) \leq F_Y^{(1)}(\eta) \quad \text{for any} \quad \eta \in R, \tag{2.2}$$

and prospect X nth-order stochastically dominates prospect Y, denoted by

$$X \succeq_{nSD} Y \text{ if and only if } F_X^{(n)}(\eta) \le F_Y^{(n)}(\eta) \text{ for any } \eta \in R, \text{ and } F_X^{(k)}(\infty) \le F_Y^{(k)}(\infty) \quad (2.3)$$

with $2 \le k \le n$. Here, FSD and nSD stands for first- and nth-order stochastic dominance. For $n = 2$, 2SD can also be written as SSD (second-order SD). It is well known that

$$\text{if } X \succeq_{nSD} Y \quad \text{for any } n \ge 1 \ , \quad \text{then} \quad \mu_X \ge \mu_Y \ . \tag{2.4}$$

We need this property in the proofs of the theorems we developed in our paper.

Now, we follow Li and Wong (1999), Levy (2015), Guo and Wong (2016), and others to define risk-seeking stochastic dominance (RSD)[1] for risk seekers. Prospect X second-order risk-seeking stochastically dominates prospect Y, denoted by

$$X \succeq_{SRSD} Y \quad \text{if and only if} \quad F_X^{(2)R}(\eta) \ge F_Y^{(2)R}(\eta) \quad \text{for any} \quad \eta \in R. \tag{2.5}$$

Here, SRSD or 2RSD denotes second-order RSD.

We turn to define Farinelli and Tibiletti (F-T) ratio. Formally, for any prospect X, its F-T ratio $\phi_{FT,X}(\eta)$ is defined as:

$$\phi_{FT,X}(\eta) = \frac{(E[(X-\eta)_+^p])^{1/p}}{(E[(\eta-X)_+^q])^{1/q}}. \tag{2.6}$$

Here, $x_+ = \max\{0, x\}$ and η is called the return threshold. For any investor, return below her return threshold is considered loss and return above is gain. Furthermore, p and q are positive values to represent investor's degree of risk aversion. Thus, the F-T ratio is the ratio of average gain to average loss, each raised by some power index to proxy for the investor's degree of risk aversion.

As an illustration, we first consider Omega ratio, first discussed by Keating and Shadwick (2002). In fact, if we take $p = q = 1$, the above defined F-T ratio reduces to the Omega ratio:

$$O_X(\eta) = \frac{E[(X-\eta)_+]}{E[(\eta-X)_+]}.$$

[1]Levy (2015) denotes it as RSSD while we denote it as RSD.

Readers may refer to Guo, *et al.* (2017) to know more properties for the Omega ratio.

We turn to discuss the upside potential ratio (Sortino, *et al.,* 1999). In fact, if we take $p = 1$ and $q = 2$, the F-T ratio defined in (2.6) becomes the upside potential ratio, which is defined as:

$$U_X(\eta) = \frac{E[(X-\eta)_+]}{\sqrt{E[(\eta - X)_+^2]}}.$$

According to Proposition 1 in Ogryczak and Ruszczyński (2001), we have

$$F_X^{(n+1)}(\eta) = \int_{-\infty}^{\eta} F_X^{(n)}(x)dx = \frac{1}{n!}E[(\eta - X)_+^n].$$

Further, we note that

$$\begin{aligned} F_X^{(2)R}(\eta) &= \int_\eta^\infty (1 - F_X(\xi))d\xi = \int_\eta^\infty \int_\xi^\infty f_X(x)dxd\xi \\ &= \int_\eta^\infty \int_\eta^x d\xi f_X(x)dx = \int_\eta^\infty (x-\eta)f_X(x)dx = E[(X-\eta)_+]. \end{aligned}$$

In the above argument, the order of integration is changed by Fubini's theorem. Consequently, we can further rewrite the upside potential ratio as

$$U_X(\eta) = \frac{E[(X-\eta)_+]}{\sqrt{E[(\eta - X)_+^2]}} = \frac{F_X^{(2)R}(\eta)}{\sqrt{2!F_X^{(3)}(\eta)}}.$$

Thus, we consider the following general F-T ratio in our paper since the F-T ratio can be rewritten as:

$$\phi_{FT,X}(\eta) = \frac{(E[(X-\eta)_+^p])^{1/p}}{(E[(\eta - X)_+^q])^{1/q}} = \frac{(E[(X-\eta)_+^p])^{1/p}}{(q!F_X^{(q+1)}(\eta))^{1/q}} \ . \tag{2.7}$$

Another class of performance measurement is called Kappa ratio, which is first developed by Kaplan and Knowles (2004) as follows:

$$K_X^{(q)}(\eta) = \frac{\mu_X - \eta}{(E[(\eta - X)_+^q])^{1/q}}. \tag{2.8}$$

Thus the denominators of Kappa ratio and F-T ratio are the same, while the numerators of these two ratios are different. Compared with Kappa ratio, the numerator of F-T ratio measures the average above-benchmark returns. Readers may refer to Niu, *et al.* (2017) to know more properties for the Kappa ratio. More discussions about the differences and relationships between these two ratios can be found from Leon and Moreno (2017).

We state the following dominance rule by using the F-T ratio:

Definition 2.1 *For any two prospects X and Y with F-T ratios, $\phi_{FT,X}$ and $\phi_{FT,Y}$, respectively, X is said to dominate Y by the F-T ratio, denote by*

$$X \succeq_{FT} Y \quad \textit{if} \quad \phi_{FT,X}(\eta) \geq \phi_{FT,Y}(\eta), \ \textit{for any } \eta \in R. \tag{2.9}$$

3 The Theory

Is mean-risk rule consistent with stochastic dominance rule? Markowitz (1952) defines a mean-variance rule for risk averters and Wong (2007) defines a mean-variance rule for risk seekers. Wong (2007) further establishes consistency of mean-variance rules with second-order SD (SSD) rules under some conditions. Ogryczak and Ruszczyński (1999) show that under some conditions the standard semi-deviation and absolute semi-deviation make the mean-risk model consistent with the SSD. Ogryczak and Ruszczyński (2002) establish the equivalence between TVaR and the second-order stochastic dominance. In addition, Leitner (2005) further shows that AV@R as a profile of risk measures is equivalent to the SSD under certain conditions. Ma and Wong (2010) showed the equivalence between SSD and the C-VaR criteria.

3.1 Sufficient Conditions

Is F-T ratio consistent with SSD? This paper explores answer for this question. We first establish the following property to say the relationship between F-T ratio and SSD:

Property 3.1 *F-T ratio is not consistent with SSD for all positive p and q in the sense that for any two prospects X and Y with F-T ratios $\phi_{FT,X}$ and $\phi_{FT,Y}$, respectively, the following* *statement does not hold:*

$$X \succeq_{SSD} Y \Rightarrow X \succeq_{FT} Y \text{ for all positive } p,\ q. \tag{3.1}$$

We construct the following example to support the argument stated in Property 3.1.

Example 3.1 *Consider two prospects X and Y with the following distributions:*

$$X = 10 \quad \text{with prob. } 1 \quad \text{and} \quad Y = \begin{cases} 1 & \text{with prob. } 2/3 \\ 11 & \text{with prob. } 1/3 \end{cases} . \tag{3.2}$$

We have $\mu_X = 10$ and $\mu_Y = 13/3$ and obtain the following

$$F_X^{(2)}(\eta) = \begin{cases} 0 & \text{if } \eta < 10 \\ \eta - 10 & \text{if } \eta \geq 10 \end{cases} , \quad F_Y^{(2)}(\eta) = \begin{cases} 0 & \text{if } \eta < 1 \\ 2(\eta-1)/3 & \text{if } 1 \leq \eta < 11 \\ \eta - 13/3 & \text{if } \eta \geq 11 \end{cases} ,$$

$$F_X^{(2)R}(\eta) = \begin{cases} 10 - \eta & \text{if } \eta < 10 \\ 0 & \text{if } \eta \geq 10 \end{cases} , \quad F_Y^{(2)R}(\eta) = \begin{cases} 13/3 - \eta & \text{if } \eta < 1 \\ (11-\eta)/3 & \text{if } 1 \leq \eta < 11 \\ 0 & \text{if } \eta \geq 11 \end{cases} .$$

It is easy to observe that $F_X^{(2)}(\eta) \leq F_Y^{(2)}(\eta)$, for all $\eta \in R$; that is, $X \succeq_{SSD} Y$. However, for any $10 \leq \eta < 11$, we have $F_X^{(2)R}(\eta) \equiv 0 < F_Y^{(2)R}(\eta)$. Recall the definition of $U_X(\eta)$, we can conclude that $U_X(\eta) \equiv 0 < U_Y(\eta)$ for any $10 \leq x < 11$.

Thus, Example 3.1 shows that SSD is not sufficient to imply the F-T ratio dominance rule for all p and q. However, if we restrict the range of p and q, it can be shown that SSD can lead to F-T ratio dominance rule. While FSD is always consistent with the F-T ratio rule for any p and q. We state the results in the following Theorem:

Theorem 3.1 *For any two returns X and Y with F-T ratios $\phi_{FT,X}(\eta)$ and $\phi_{FT,Y}(\eta)$, respectively,*

1. *if $X \succeq_{FSD} Y$, then $X \succeq_{FT} Y$ for any nonnegative values p and q; .*

2. *if $X \succeq_{SSD} Y$, then $X \succeq_{FT} Y$ for any $p < 1$ and $q \geq 1$;*

3. if $X \succeq_{nSD} Y, n \geq 3$ then $X \succeq_{FT} Y$ for any $p < 1$ and $q \geq n - 1$.

Here, p and q are defined in either (2.6) or (2.7)

3.2 Necessary Conditions

We turn to study the necessary condition between SD and F-T ratio. We first show that the F-T ratios are strictly increasing functions of the Sharpe ratio for the location-scale family. To be precise, we obtain the following theorem:

Theorem 3.2 *Suppose that X belongs to a location-scale family, with its mean μ_X and standard deviation $\sigma_X > 0$. Then $\phi_{FT,X}(\eta)$ increases monotonically with $(\mu_X - \eta)/\sigma_X$.*

Theorem 3.2 proves the monotonicity of the Sharpe ratio and the F-T ratios when return follows a location-scale (LS) family, a family of univariate probability distributions parameterized by a location and a non-negative scale parameters, with several well-known distributions in finance including Cauchy, exponential, extreme value distribution of the maximum and the minimum, each of type I, Laplace, logistic and half-logistic, Maxwell-Boltzmann, normal and halfnormal, uniform distribution, etc.

Based on Theorem 3.2, we obtain the following result:

Theorem 3.3 *For any two returns X and Y that belong to the same location-scale family or same linear combination of location-scale families with means, μ_X and μ_Y, standard deviations, σ_X and σ_Y, and F-T ratios, $\phi_{FT,X}(\eta)$ and $\phi_{FT,Y}(\eta)$, respectively, we have*

1. *if $\mu_X > \mu_Y$ and*

   (a) *if there exists at least one η satisfying $\eta \geq \mu_X$ such that $\phi_{FT,X}(\eta) \leq \phi_{FT,Y}(\eta)$ for any $p, q > 0$, then $E\ [u(X)] \geq E\ [u(Y)]$ for any risk-averse investor with utility function $u \in U_k$ for any $k \geq 2$; and*

(b) *if there exists at least one* η *satisfying* $\mu_Y \geq \eta$ *such that* $\phi_{FT,X}(\eta) \leq \phi_{FT,Y}(\eta)$ *for any* $p, q > 0$, *then* $E[u(X)] \geq E[u(Y)]$ *for any risk-seeking investor with utility function* $u \in U_k^R$ *for any* $k \geq 2$*; and*

2. *if* $\mu_X = \mu_Y = \mu$ *and*

   (a) *if there exists at least one* η *satisfying* $\mu \geq \eta$ *such that* $\phi_{FT,X}(\eta) \geq \phi_{FT,Y}(\eta)$ *for any* $p, q > 0$, *then* $E[u(X)] \geq E[u(Y)]$ *for any risk-averse investor with utility function* $u \in U_k$ *for any* $k \geq 2$*; and*

   (b) *if there exists at least one* η *satisfying* $\eta \geq \mu$ *such that* $\phi_{FT,X}(\eta) \geq \phi_{FT,Y}(\eta)$ *for any* $p, q > 0$, *then* $E[u(X)] \geq E[u(Y)]$ *for any risk-seeking investor with utility function* $u \in U_k^R$ *for any* $k \geq 2$.

4 Conclusions

In practice, investors care about losses more than gains of similar magnitude. The gains and losses are relative to specified benchmarks. Returns below the benchmarks are considered as losses and returns above as gains. The F-T ratio encodes both of these features in a simple way.

We have shown that the simplicity of the F-T ratio belies its intimate connection with expected utility theory for all non-satiated investors (first-order stochastic dominance). However, the second-order stochastic dominance is, in general, not consistent with F-T ratio. A simple example is presented to illustrate this point. However, we find that second-order stochastic dominance can lead to F-T dominance with any $p < 1$ and $q \geq 1$. Further higher-order stochastic dominance ($n \geq 3$) leads to F-T dominance with any $p < 1$ and $q \geq n - 1$.

We find that when the variables being compared belong to the same location-scale family or the same linear combination of location-scale families, we can get the necessary relationship between the stochastic dominance with the F-T ratio after imposing some conditions on the means. Our findings enable academics and practitioners to draw better decision in their analysis.

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