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Yokote, Koji

School of Political Science and Economics, Waseda University

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Koji Yokote*

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Abstract

The separation theorem in discrete convex analysis states that two disjoint discrete convex sets can be separated by a hyperplane with a 0-1 normal vector. We apply this theorem to an auction model and provide a unified approach to existing results. When p is not an equilibrium price vector, i.e., aggregate demand and aggregate supply are disjoint, the separation theorem indicates the existence of excess demand/supply. This observation yields a refined analysis of a characterization of competitive price vectors by Gul and Stacchetti (2000). Adjusting the prices of items in excess demand/supply corresponds to Ausubel's (2006) auction.

JEL classification: C78, D44

1 Introduction

The purpose of the present paper is to apply the separation theorem in discrete convex analysis (Murota 2003) to an auction model and provide a unified approach to existing results. The discrete separation theorem states that two disjoint discrete convex sets can be separated by a hyperplane with a 0-1 normal vector. As recognized in the literature, under the gross-substitutes condition, aggregate demand forms a discrete convex set called the *$M^\natural$-convex set*. Geometrically, Walrasian equilibrium can be described as a situation where aggregate demand and aggregate supply intersect. To put it differently, p is not an equilibrium price vector if and only if aggregate demand and aggregate supply are disjoint. Applying the discrete separation theorem to the two sets, the "slope" of the separating

*School of Political Science and Economics, Waseda University, 1-6-1, Nishi-Waseda, Shinjuku-ku, Tokyo 169-8050, Japan (koji.yokote@aoni.waseda.jp)

hyperplane turns out to describe excess demand or excess supply. This observation yileds a refined analysis of Hall's (1935) theorem and a characterization of competitive price vectors by Gul and Stacchetti (2000). We further show that Ausubel's (2006) auction proceeds by increasing/decreasing the prices of items in excess demand/supply, thereby providing an economic interpretation of the auction.

The rest of the paper is organized as follows. Section 2 presents preliminaries. Section 3 presents an application of the discrete separation theorem to an auction model. Section 4 presents concluding remarks. Section 5 presents proofs of the main results.

2 Preliminaries

2.1 Discrete convex analysis

Let $\mathbb{Z}$ be the set of integers and K be an arbitrary finite set. For $x \in \mathbb{Z}^K$, we define

$$\text{supp}^+ x = \{k \in K : x_k > 0\}, \ \text{supp}^- x = \{k \in K : x_k < 0\}.$$

For each $A \subseteq K$, let $\chi_A \in \{0,1\}^K$ denote the **characteristic vector of $\boldsymbol{A}$**, i.e.,

$$(\chi_A)_k = \begin{cases} 1 & \text{if } k \in A, \\ 0 & \text{otherwise.} \end{cases}$$

For a singleton set $\{k\} \subseteq K$, we write χ_k for $\chi_{\{k\}}$.

We say that a function $v : \{0,1\}^K \to \mathbb{R}$ is an **M$^\natural$-concave function** if, for any $x, y \in \{0,1\}^K$ and $k \in \text{supp}^+(x - y)$, we have

(i) $v(x) + v(y) \leq v(x - \chi_k) + v(y + \chi_k)$, or

(ii) there exists $\ell \in \text{supp}^-(x - y)$ such that $v(x) + v(y) \leq v(x - \chi_k + \chi_\ell) + v(y + \chi_k - \chi_\ell)$.

Section 3 of Kojima et al. (2017) provides an interpretation for M$^\natural$-concavity.

We say that $X \subseteq \mathbb{Z}^K$ with $X \neq \emptyset$ is an **M$^\natural$-convex set** if, for any $x, y \in X$ and $k \in \text{supp}^+(x - y)$, we have

(i) $x - \chi_k \in X$, $y + \chi_k \in X$, or

(ii) there exists $\ell \in \text{supp}^-(x - y)$ such that $x - \chi_k + \chi_\ell \in X$, $y + \chi_k - \chi_\ell \in X$.

The following is a discrete analogue of the separation theorem.

Theorem 1. Let $X_1, X_2 \subseteq \mathbb{Z}^K$ be M$^\natural$-convex sets. If $X_1 \cap X_2 = \emptyset$, there exists $\alpha \in \{0,1\}^K \cup \{0,-1\}^K$ such that

$$\sup_{x \in X_1} \alpha \cdot x < \inf_{x \in X_2} \alpha \cdot x.$$

Proof. See Section 5.1. □

As is the case for continuous settings, this theorem states that two "convex" sets can be separated by a hyperplane. The key difference is that the normal vector α can be taken as a characteristic vector (i.e., a vector with 0-1 coordinates).

The following figure shows an example of the discrete separation theorem for $K = \{k, \ell\}$. Note that each "edge" of X_1 and X_2 is parallel to $\chi_k - \chi_\ell$ or χ_k or χ_ℓ.[1] The normal vector of the separating hyperplane is taken as $\alpha = (1, 1)$.

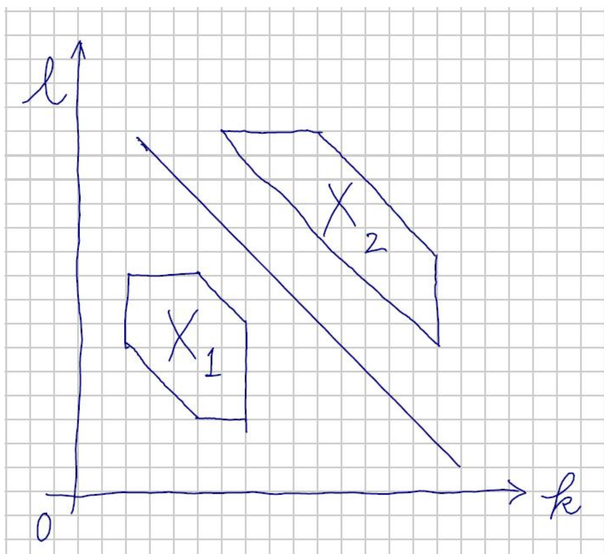

Figure 1: Discrete separation theorem for $K = \{k, \ell\}$

Remark 1. The separation theorem in discrete settings is already proved for two sets satisfying *M-convexity,* which is stronger than (but essentially equivalent to) M$^\natural$-convexity (see Murota 2003, Theorem 4.21). We provide another proof for its fundamental importance.

For two sets $X_1, X_2 \subseteq \mathbb{Z}^K$, we define the **Minkowski sum** $X_1 + X_2$ by

$$X_1 + X_2 = \{x_1 + x_2 : x_1 \in X_1, x_2 \in X_2\}.$$

The following theorem says that M$^\natural$-convexity is preserved in Minkowski sum.

[1] An M$^\natural$-convex set in a general n-dimensional space can be characterized in terms of the direction of edges; see Murota (2003), p.119.

Theorem 2 (Murota 2003, Theorem 4.23). *Let $X_1, X_2 \subseteq \mathbb{Z}^K$ be M$^\natural$-convex sets. Then, $X_1 + X_2$ is an M$^\natural$-convex set.*

3 Application to an auction model

We show that the discrete separation theorem clarifies the mathematical structure behind an auction model.

Let N be a finite set of **agents** and K be a finite set of **items**. Each agent i has a **valuation function** $v_i : \{0,1\}^K \to \mathbb{Z}$; we identify a subset of items $A \subseteq K$ with χ_A. For each $i \in N$, we define the **demand correspondence** $D_i : \mathbb{R}^K_+ \to \{0,1\}^K$ by

$$D_i(p) = \left\{x \in \{0,1\}^K : v_i[p](x) \geq v_i[p](y) \text{ for all } y \in \{0,1\}^K\right\} \text{ for all } p \in \mathbb{R}^K_+,$$

where $v_i[p](x) = v_i(x) - p \cdot x$.

We say that v_i is **monotonic** if for any $A, B \subseteq K$ with $A \subseteq B$, we have $v_i(\chi_A) \leq v_i(\chi_B)$. We say that v_i satisfies the **gross substitutes condition** (Kelso and Crawford 1982) if for any $p, q \in \mathbb{R}^K_+$ with $p \leq q$ and $x \in D_i(p)$, there exists $y \in D_i(q)$ such that $x_k \leq y_k$ if $p_k = q_k$.

Theorem 3 (Fujishige and Yang 2003). *Suppose v_i is monotonic. Then v_i satisfies the gross substitutes condition if and only if v_i is M$^\natural$-concave.*

Throughout this paper, we assume that v_i is monotonic and satisfies the gross substitutes condition for all $i \in N$. By Theorem 3, v_i is M$^\natural$-concave for all $i \in N$.

Theorem 4 (Fujishige and Yang 2003; Murota 2003, Theorem 6.30). *Let v_i be an M$^\natural$-concave function. Then, for any $p \in \mathbb{R}^K_+$, $D_i(p)$ is an M$^\natural$-convex set.*

An **allocation** is a set of bundles $(x_i)_{i \in N}$ satisfying

$$x_i \in \{0,1\}^K \text{ for all } i \in N, \ \sum_{i \in N} x_i = \chi_K.$$

A **Walrasian equilibrium** is a pair $\left(p^*, (x_i^*)_{i \in N}\right)$, where $p^* \in \mathbb{R}^K_+$, x^* is an allocation, and $x_i^* \in D_i(p^*)$ for all $i \in N$. We say that p^* is a (Walrasian) **equilibrium price vector** if there exists an allocation $(x_i^*)_{i \in N}$ such that $\left(p^*, (x_i^*)_{i \in N}\right)$ is a Walrasian equilibrium.

For each $p \in \mathbb{R}^K_+$, we define the **aggregate demand** $D(p)$ by

$$D(p) = \sum_{i \in N} D_i(p).$$

The following lemma immediately follows from the definition of an equilibrium price vector.

Lemma 1. A price vector $p \in \mathbb{R}^K_+$ is an equilibrium price vector if and only if $\chi_K \in D(p)$.

Equivalently, p is not an equilibrium price vector if and only if

$$\chi_K \notin D(p). \tag{1}$$

By Theorems 2-4, $D(p)$ is an M$^\natural$-convex set. Regarding χ_K as a singleton set, $\{\chi_K\}$ is an M$^\natural$-convex set. Hence (1) refers to two disjoint M$^\natural$-convex sets. By Theorem 1, there exists $\alpha \in \{0,1\}^K \cup \{0,-1\}^K$ such that

$$\alpha \cdot \chi_K < \min_{x \in D(p)} \alpha \cdot x. \tag{2}$$

Assume $\alpha \in \{0,1\}^K$ and let $A \subseteq K$ be such that $\alpha = \chi_A$. Then (2) is equivalent to

$$\begin{aligned} |A| &< \min_{x \in \sum_{i \in N} D_i(p)} \sum_{k \in A} x_k \\ &= \sum_{i \in N} \min_{x_i \in D_i(p)} \sum_{k \in A} (x_i)_k \\ &= \sum_{i \in N} \min_{x_i \in D_i(p)} \big|\{k \in A : (x_i)_k = 1\}\big|. \end{aligned} \tag{3}$$

For each $i \in N$, Gul and Stacchetti (2000) called the above minimum value the *requirement function* and interpreted it as follows: "the minimal number of objects in A that she would need to construct any of her optimal consumption bundles". In the above inequality, the sum of the minimal numbers among agents is greater than the number of items in A, which means *excess demand*.

When $\alpha \in \{0,-1\}^K$, letting $A \subseteq K$ be such that $\alpha = -\chi_A$, (2) is equivalent to

$$|A| > \sum_{i \in N} \max_{x_i \in D_i(p)} \big|\{k \in A : (x_i)_k = 1\}\big|,$$

which means *excess supply*.

For $p \in \mathbb{R}^K_+$ and $A \subseteq K$, we define

$$\begin{aligned} R_i^{min}(p, A) &= \min_{x_i \in D_i(p)} \big|\{k \in A : (x_i)_k = 1\}\big|, \; R^{min}(p, A) = \sum_{i \in N} R_i^{min}(A, p), \\ R_i^{max}(p, A) &= \max_{x_i \in D_i(p)} \big|\{k \in A : (x_i)_k = 1\}\big|, \; R^{max}(p, A) = \sum_{i \in N} R_i^{min}(A, p). \end{aligned}$$

We summarize the above discussion as a theorem.[2]

[2] The above discussion proves the contrapositive of the *if* part of Theorem 5. The *only if* part immediately follows from the definition of a competitive price vector.

Theorem 5. *$p \in \mathbb{R}^K_+$ is an equilibrium price vector if and only if*

$$R^{min}(p, A) \leq |A| \leq R^{max}(p, A) \text{ for all } A \subseteq K.$$

This theorem has an intuitive interpretation that p is an equilibrium price vector if and only if excess demand/supply do not exist.

The above argument enables a refined analysis of a characterization of competitive price vectors by Gul and Stacchetti (2000). Indeed, their characterization immediately follows from the discrete separation theorem. Let $[\mathbf{0}, \chi_K]$ denote the **integer interval** between $\mathbf{0}$ and χ_K, i.e., $[\mathbf{0}, \chi_K] = \{x \in \mathbb{Z}^K : \mathbf{0} \leq x \leq \chi_K\}$. We say that p is a **quasi competitive price vector** if $[\mathbf{0}, \chi_K] \cap D(p) \neq \emptyset$. Namely, p is a quasi competitive price vector if all the items are consumed by at most one agent.

Corollary 1 (Gul and Stacchetti 2000, Corollary)**.** *p is a quasi competitive price vector if and only if*

$$R^{min}(p, A) \leq |A| \text{ for all } A \subseteq K.$$

Proof. The *only if* part immediately follows from the definition of a quasi competitive price vector. We prove the contrapositive of the *if* part. If p is not a quasi competitive price vector, then $[\mathbf{0}, \chi_k] \cap D(p) = \emptyset$. Since any integer interval is an M$^\natural$-convex set, by Theorem 1, there exists $\alpha \in \{0, 1\} \cup \{0, -1\}$ such that

$$\max_{x \in [\mathbf{0}, \chi_K]} \alpha \cdot x < \min_{x \in D(p)} \alpha \cdot x. \tag{4}$$

If $\alpha \in \{0, -1\}$, then we derive $\max_{x \in [\mathbf{0}, \chi_K]} \alpha \cdot x = 0 \geq \min_{x \in D(p)} \alpha \cdot x$, a contradiction to (4). Hence, $\alpha \in \{0, 1\}$. Since $\max_{x \in [\mathbf{0}, \chi_K]} \alpha \cdot x = \alpha \cdot \chi_K$, by choosing $A \subseteq K$ with $\alpha = \chi_A$, the same transformation as (3) yields the desired condition. □

Remark 2. Hall's (1935) theorem is a fundamental theorem in graph theory. See Demange et al. (1986) as well as Section 8.3 of Roth and Sotomayor (1990) for its application to auctions. As discussed by Gul and Stacchetti (2000), Corollary 1 is a generalization of Hall's (1935) theorem. Hence, Hall's (1935) theorem can be obtained by the discrete separation theorem.

Based on the description of excess demand and supply in Theorem 5, we provide an economic interpretation of Ausubel's (2006) auction. For each $i \in N$, we define the **indirect**

utility function $V_i : \mathbb{R}_+^K \to \mathbb{Z}$ by

$$V_i(p) = \max_{x \in \{0,1\}^K} v_i[p](x) \text{ for all } p \in \mathbb{R}_+^K$$

We define the **Lyapunov function** (Ausubel 2006) $L : \mathbb{Z}_+^K \to \mathbb{Z}$ by

$$L(p) = \sum_{i \in N} V_i(p) + p \cdot \chi_K \text{ for all } p \in \mathbb{Z}_+^K.$$

Ausubel's (2006) auction proceeds by decreasing the value of the Lyapunov function. The next theorem shows that the auction proceeds by increasing (res. decreasing) the prices of items in excess demand (res. excess supply).

Theorem 6. *Let $p \in \mathbb{Z}_+^K$ and $A \subseteq K$. Then,*

(i) $L(p + \chi_A) < L(p)$ if and only if $|A| < R^{min}(A, p)$.

(ii) $L(p - \chi_A) < L(p)$ if and only if $|A| > R^{max}(A, p)$.

Proof. See Section 5.2. □

Remark 3. The Lyapunov function $L(\cdot)$ satisfies a notion of discrete concavity called $L^\natural$-*concavity* (see Section 8 of Murota (2016) for a detailed discussion), where L refers to *lattice*. Combining L$^\natural$-concavity with Theorems 5 and 6, we can prove that p is a competitive price vector if and only if p is a minimizer of $L(\cdot)$, which was previously proved by Ausubel (2006). L$^\natural$-concavity also implies that the set of competitive price vectors has a lattice structure.

4 Concluding remarks

As proven by Murota et al. (2016), many existing iterative auctions can be embedded into Ausubel's (2006) auction. Hence our result shows that the discrete separation theorem is a critical mathematical tool in iterative auctions.

There are many papers that provide a characterization of the Walrasian equilibria; see, for example, Mishra and Talman (2010). It remains as a topic for future work to apply the discrtete separation theorem to other characterizations and clarify the mathematical structure behind them.

5 Proofs

5.1 Proof of Theorem 1

For $x \in \mathbb{Z}^K$ and $X \subseteq \mathbb{Z}^K$, we define

$$d^+(x, X) = \{k \in K : x + \chi_k \in X\},\ d^-(x, X) = \{k \in K : x - \chi_k \in X\}.$$

For $x \in \mathbb{Z}^K$ and $A \subseteq K$, we define[3]

$$||x||_1 = \sum_{k \in K} |x_k|,\ x(A) = \sum_{k \in A} x_k.$$

Lemma 2 (Murota Shioura 1999, Lemma 4.3)**.** Let $X \subseteq \mathbb{Z}^K$ be an M$^\natural$-convex set. Then, for any $x, y \in X$ with $x(K) < y(K)$, there exists $k \in \text{supp}^+(y - x)$ such that

$$x + \chi_k \in X \text{ and } y - \chi_k \in X.$$

Claim 1. Let $X \subseteq \mathbb{Z}^K$ be an M$^\natural$-convex set and $x \notin X$. If $d^+(x, X) \neq \emptyset$, then

$$x(d^+(x, X)) < y(d^+(x, X)) \text{ for all } y \in X.$$

Proof. We proceed by induction on $||x - y||_1$ for $y \in X$.
Induction base: Case 1: Suppose $||x - y||_1 = 1$. We consider two subcases.
Subcase 1-1: Suppose $y = x + \chi_k$ for some $k \in K$. Since $y = x + \chi_k \in X$, we have $k \in d^+(x, X)$. Then,

$$x(d^+(x, X)) < (x + \chi_k)(d^+(x, X)) = y(d^+(x, X)).$$

Subcase 1-2: Suppose $y = x - \chi_\ell$ for some $\ell \in K$. Let $k \in d^+(x, X)$. We apply Lemma 2 to $x + \chi_k, y \in X$ with $(x + \chi_k)(K) > y(K)$. Since $\text{supp}^+((x + \chi_k) - y) = \{k, \ell\}$, we have $(x + \chi_k) - \chi_k = x \in X$ or $y + \chi_\ell = x \in X$, either of which is a contradiction to $x \notin X$.
Case 2: Suppose $||x - y||_1 = 2$. Let $k \in d^+(x, X)$. We consider three subcases.
Subcase 2-1: Suppose $y = x + 2 \cdot \chi_\ell$ for some $\ell \in K$. We apply Lemma 2 to $y, x + \chi_k \in X$ with $y(K) > (x + \chi_k)(K)$. Since $\text{supp}^+(y - (x + \chi_k)) = \{\ell\}$, we have $y - \chi_\ell = x + \chi_\ell \in X$. Hence $\ell \in d^+(x, X)$, which implies

$$x(d^+(x, X)) < (x + 2 \cdot \chi_\ell)(d^+(x, X)) = y(d^+(x, X)).$$

[3] $||x||_1$ is called the ℓ_1-norm of x.

Subcase 2-2: Suppose $y = x + \chi_\ell - \chi_m$ for some $\ell, m \in K$ with $\ell \neq m$. We apply Lemma 2 to $x + \chi_k, y \in X$ with $(x + \chi_k)(K) > y(K)$. Note that $\text{supp}^+((x + \chi_k) - y) = \{k, m\}$ if $k \neq \ell$ and $\text{supp}^+((x + \chi_k) - y) = \{m\}$ if $k = \ell$. Since $(x + \chi_k) - \chi_k = x \notin X$, in eihter case, we have $y + \chi_m = x + \chi_\ell \in X$. Namely, $\ell \in d^+(x, X)$.

Suppose $m \in d^+(x, X)$. We apply Lemma 2 to $x + \chi_m, y \in X$ with $(x + \chi_m)(K) > y(K)$. Since $\text{supp}^+((x + \chi_m) - y) = \{m\}$, we have $(x + \chi_m) - \chi_m = x \in X$, a contradiction to $x \notin X$. Hence, $m \notin d^+(x, X)$.

By $\ell \in d^+(x, X)$ and $m \notin d^+(x, X)$,

$$x(d^+(x, X)) < (x + \chi_\ell - \chi_m)(d^+(x, X)) = y(d^+(x, X)).$$

Subcase 2-3: Suppose $y = x - 2 \cdot \chi_\ell$ for some $\ell \in K$. We apply Lemma 2 to $x + \chi_k, y \in X$ with $(x + \chi_k)(K) > y(K)$. Note that $\text{supp}^+((x + \chi_k) - y) = \{k, \ell\}$. Since $(x + \chi_k) - \chi_k = x \notin X$, we have $(y + \chi_\ell) = x - \chi_\ell \in X$. Following the same argument as Subcase 1-2, we obtain a contradiction to $x \notin X$.

Induction step: Let $t \geq 3$. Suppose the result holds for all $y \in X$ with $1 \leq ||x - y||_1 \leq t - 1$. We prove the result for $y \in X$ with $||x - y||_1 = t$.

Suppose by way of contradiction that there exists $y \in X$ such that $||x - y||_1 = t$ and

$$x(d^+(x, X)) \geq y(d^+(x, X)). \tag{5}$$

Let $k \in d^+(x, X)$ be such that $x_k - y_k \geq x_{k'} - y_{k'}$ for all $k' \in d^+(x, X)$. By (5), $x_k - y_k \geq 0$. By M$^\natural$-convexity applied to $x + \chi_k, y \in X$ and $k \in \text{supp}^+((x + \chi_k) - y)$, we have

(i) $x \in X$ and $y + \chi_k \in X$, or

(ii) there exists $\ell \in \text{supp}^-((x + \chi_k) - y)$ such that $x + \chi_\ell \in X$ and $y + \chi_k - \chi_\ell \in X$.

By $x \notin X$, (ii) holds. By $x + \chi_\ell \in X$, we have $\ell \in d^+(x, X)$. Together with (5),

$$x(d^+(x, X)) \geq y(d^+(x, X)) = (y + \chi_k - \chi_\ell)(d^+(x, X)). \tag{6}$$

By (ii), $y + \chi_k - \chi_\ell \in X$. Since $\ell \in d^+(x, X)$ and $y_\ell > x_\ell + (\chi_k)_\ell = x_\ell$, together with the choice of k and (5), we obtain $x_k - y_k > 0$. Hence, $||x - (y + \chi_k - \chi_\ell)||_1 = ||x - y||_1 - 2 = t - 2$. Then, (6) contradicts the induction hypothesis. □

Claim 2. *Let $X \subseteq \mathbb{Z}^K$ be an M$^\natural$-convex set and $x \notin X$. If $d^-(x, X) \neq \emptyset$, then*

$$x(d^-(x, X)) > y(d^-(x, X)) \textit{ for all } y \in X.$$

Proof. This claim can be proved in the same way as Claim 1 □

Claim 3. *Let $X \subseteq \mathbb{Z}^K$ be an M$^\natural$-convex set and $x \notin X$. Then, there exists $\alpha \in \{0,1\}^K \cup \{0,-1\}^K$ such that*

$$\alpha \cdot x < \inf_{y \in X} \alpha \cdot y. \tag{7}$$

Proof. Let $\bar{y} \in X$ be such that $||x - \bar{y}||_1 \leq ||x - y||_1$ for all $y \in X$. Set $m = ||x - \bar{y}||_1$ and let $k \in \text{supp}^+(\bar{y} - x) \cup \text{supp}^-(\bar{y} - x) \neq \emptyset$.
Case 1: Suppose $k \in \text{supp}^+(\bar{y} - x)$. By the choice of $\bar{y}$, we have $\bar{y} - \chi_k \notin X$. Set $\bar{x} = \bar{y} - \chi_k$. By $d^+(\bar{x}, X) \neq \emptyset$, Claim 1 implies

$$\bar{x}(d^+(\bar{x}, X)) < y(d^+(y, X)) \text{ for all } y \in X. \tag{8}$$

Suppose by way of contradiction that $x(d^+(\bar{x}, X)) > \bar{x}(d^+(\bar{x}, X))$. Then there exists $\ell \in d^+(\bar{x}, X)$ with $x_\ell > \bar{x}_\ell = \bar{y}_\ell - (\chi_k)_\ell$. Together with $\bar{y}_k > x_k$, we have $k \neq \ell$, from which follows $x_\ell > \bar{y}_\ell$. By $\bar{x} + \chi_\ell = \bar{y} - \chi_k + \chi_\ell \in X$ and $||(\bar{y} - \chi_k + \chi_\ell) - x||_1 = m - 2$, we obtain a contradiction to the choice of $\bar{y}$.

It follows that $x(d^+(\bar{x}, X)) \leq \bar{x}(d^+(\bar{x}, X))$. Together with (8),

$$x(d^+(\bar{x}, X)) < y(d^+(y, X)) \text{ for all } y \in X.$$

Set $A = d^+(\bar{x}, X)$. By the above inequality, together with the fact that $y(d^+(y, X))$ always takes an integer value, we obtain (7) for $\alpha = \chi_A$.
Case 2: Suppose $k \in \text{supp}^-(\bar{y} - x)$. This case can be proved in the same way as Case 1. Applying Claim 2 instead of Claim 1, we obtain $\alpha \in \{0, -1\}^K$ that satisfies (7). □

For $X \subseteq \mathbb{Z}^K$, we define $-X = \{x \in \mathbb{Z}^K : -x \in X\}$.

Claim 4. Let $X \subseteq \mathbb{Z}^K$ be an M$^\natural$-convex set. Then, $-X$ is also an M$^\natural$-convex set.

Proof. This claim immediately follows from the definition of M$^\natural$-convexity. □

Proof of Theorem 1 **.** By Theorem 2 and Claim 4, $-X_1 + X_2$ is an M$^\natural$-convex set. By $X_1 \cap X_2 = \emptyset$, $0 \notin -X_1 + X_2$. Applying Claim 3, there exists $\alpha \in \{0,1\}^K \cup \{0,-1\}^K$ such that

$$\begin{aligned} 0 &< \inf_{x \in -X_1 + X_2} \alpha \cdot x, \\ 0 &< \inf_{x_1 \in X_1} \alpha \cdot (-x_1) + \inf_{x_2 \in X_2} \alpha \cdot x_2, \\ 0 &< -\sup_{x_1 \in X_1} \alpha \cdot x_1 + \inf_{x_2 \in X_2} \alpha \cdot x_2, \\ \sup_{x_1 \in X_1} \alpha \cdot x_1 &< \inf_{x_2 \in X_2} \alpha \cdot x_2, \end{aligned}$$

which completes the proof. □

5.2 Proof of Theorem 6

Let $A \subseteq K$ and $p \in \mathbb{Z}_+^K$. We prove (i) and omit the proof of (ii) which can be obtained analogously.

Only-if: We prove the contrapositive. Suppose there exist $x_i \in D_i(p)$ for $i \in N$ such that

$$|A| \geq \chi_A \cdot \sum_{i \in N} x_i. \tag{9}$$

Then,

$$\begin{aligned}
L(p) &= \sum_{i \in N} V_i(p) + p \cdot \chi_K \\
&= \sum_{i \in N} \{v_i(x_i) - p \cdot x_i\} + p \cdot \chi_K \\
&\leq \sum_{i \in N} v_i(x_i) - p \cdot \sum_{i \in N} x_i + p \cdot \chi_K + |A| - \chi_A \cdot \sum_{i \in N} x_i \\
&= \sum_{i \in N} v_i(x_i) - (p + \chi_A) \cdot \sum_{i \in N} x_i + (p + \chi_A) \cdot \chi_K \\
&= \sum_{i \in N} \{v_i(x_i) - (p + \chi_A) \cdot x_i\} + (p + \chi_A) \cdot \chi_K \\
&\leq \sum_{i \in N} V_i(p + \chi_A) + (p + \chi_A) \cdot \chi_K \\
&= L(p + \chi_A).
\end{aligned}$$

where the first inequality follows from (9) and the second inequality follows from the definifion of the indirect utility function.

If:

For $i \in N$ and $a \in \mathbb{Z}_+$, we define

$$D_i^a(p) = \left\{x \in \{0,1\}^K : v_i[p](x) \geq V_i(p) - a\right\}.$$

Note that $D_i^0(p) = D_i(p)$.

Theorem 7 (Murota 2003, Theorem 6.15)**.** *Let $p \in \mathbb{R}^K$ and v_i be an $M^\natural$-concave function. Then, $v_i[p]$ is an $M^\natural$-concave function.*

Claim 5. Let $i \in N$. Then for any $a \in \mathbb{Z}_+$, we have

$$x \in D_i^a(p) \text{ implies } x \cdot \chi_A \geq R_i^{min}(A,p) - a.$$

Proof. For each $a \in \mathbb{Z}_+$, set

$$\mathcal{D}^a = \{x \in \{0,1\}^K : x \in D_i^a(p) \text{ and } x \cdot \chi_A < R_i^{min}(A,p) - a\}.$$

It suffices to prove that $\mathcal{D}^a = \emptyset$ for all $a \in \mathbb{Z}_+$. We proceed by induction on a. If $a = 0$, the result follows from $R_i^{min}(A,p) \geq 0$. Suppose that the result holds for $a = k$ and we prove the result for $a = k+1$, where $k \geq 1$.

Suppose by way of contradiction that there exists $x \in \mathcal{D}^{k+1}$. Let $\bar{x} \in D_i(p)$ be such that $|\text{supp}^+(\bar{x} - x) \cap A| \leq |\text{supp}^+(y - x) \cap A|$ for all $y \in D_i(p)$. By $\bar{x} \cdot \chi_A \geq R_i^{min}(A,p)$, it holds that $\bar{x} \cdot \chi_A > x \cdot \chi_A$. This means that there exists $k \in \text{supp}^+(\bar{x} - x) \cap A$. By Theorem 7, $v_i[p]$ is an M$^\natural$-concave function. With the notation $\chi_0 = \mathbf{0}$, by M$^\natural$-concavity, there exists $\ell \in \text{supp}^-(\bar{x} - x) \cup \{0\}$ such that

$$v_i[p](\bar{x}) + v_i[p](x) \leq v_i[p](\bar{x} - \chi_k + \chi_\ell) + v_i[p](x + \chi_k - \chi_\ell). \tag{10}$$

Since $(x + \chi_k - \chi_\ell) \cdot \chi_A \leq x \cdot \chi_A + 1 < R_i^{min}(A,p) - k$, by the induction hypothesis, $x + \chi_k - \chi_\ell \notin D_i^k(p)$. Together with $x \in D_i^{k+1}(p)$, we obtain $v_i[p](x) \geq v_i[p](x + \chi_k - \chi_\ell)$. This inequality and (10) imply $v_i[p](\bar{x}) \leq v_i[p](\bar{x} - \chi_k + \chi_\ell)$. Namely,

$$\bar{x} - \chi_k + \chi_\ell \in D_i(p) \text{ and } |\text{supp}^+((\bar{x} - \chi_k + \chi_\ell) - x) \cap A| = |\text{supp}^+(\bar{x} - x) \cap A| - 1,$$

a contradiction to the choice of $\bar{x}$. □

For any $i \in N$ and $x_i \in \{0,1\}^K$, by letting $a(x_i) := V_i(p) - v_i[p](x_i)$,

$$\begin{aligned} v_i[p + \chi_A](x_i) &= v_i[p](x_i) - \chi_A \cdot x_i \\ &= V_i(p) - a(x_i) - \chi_A \cdot x_i \\ &\leq V_i(p) - a(x_i) - R_i^{min}(A,p) + a(x_i) \\ &= V_i(p) - R_i^{min}(A,p), \end{aligned} \tag{11}$$

where the inequality follows from Claim 5. For each $i \in N$, let $\bar{x}_i \in D_i(p + \chi_A)$. Then,

$$\begin{aligned} L(p + \chi_A) &= \sum_{i \in N} V_i(p + \chi_A) + p \cdot \chi_K + |A| \\ &= \sum_{i \in N} v_i[p + \chi_A](\bar{x}_i) + p \cdot \chi_K + |A| \\ &\leq \sum_{i \in N} V_i(p) - \sum_{i \in N} R_i^{min}(A, p) + p \cdot \chi_K + |A| \\ &< \sum_{i \in N} V_i(p) + p \cdot \chi_K \\ &= L(p), \end{aligned}$$

where the first inequality follows from (11) and the second inequality follows from the assumption.

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