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Singularities in Global Hyperbolic Space-time Manifold

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Abstract

If a space-time is timelike or null geodesically incomplete but cannot be embedded in a larger space-time, then we say that it has a singularity. There are two types of singularities in the space-time manifold. First one is called the Big Bang singularity. This type of singularity must be interpreted as the catastrophic event from which the entire universe emerged, where all the known laws of physics and mathematics breakdown in such a way that we cannot know what was happened during and before the big bang singularity. The second type is Schwarzschild singularity, which is considered as the end state of the gravitational collapse of a massive star which has exhausted its nuclear fuel providing the pressure gradient against the inwards pull of gravity. Global hyperbolicity is the most important condition on causal structure space-time, which is involved in problems as cosmic censorship, predictability, etc. Here both types of singularities in global hyperbolic space-time manifold are discussed in some details.

Keywords: Big Bang, global hyperbolicity, manifold, FRW model, Schwarzschild solution, space-time singularities.

1. Introduction

In the Schwarzschild metric and the Friedmann, Robertson-Walker (FRW) cosmological solution contained a space-time singularity where the curvature and density are infinite, and known all the physical laws would break down there. In the Schwarzschild solution such as a singularity is present at $r=0$ which is the final fate of a massive star (Mohajan 2013a), whereas in the FRW model it is found at the epoch $t=0$ (big bang), which is the beginning of the universe, where the scale factor $S(t)$ also vanishes and all objects are crushed to zero volume due to infinite gravitational tidal force (Mohajan 2013b).

Schwarzschild metric of Einstein equation is established assuming a star isolated from all the gravitating bodies. It is also important for the interpretation of black hole. Schwarzschild established his metric by considering asymptotically flat solutions to Einstein's equation (Mohajan 2013a).

Friedmann, Robertson-Walker (FRW) model is established on the basis of the assumption that the universe is homogeneous and isotropic in all epochs. Even though the universe is clearly inhomogeneous at the local scales of stars and cluster of stars, it is generally argued that an overall homogeneity will be achieved only at a large enough scale of about 14 billion light years. In the 1960s, Stephen W. Hawking and Roger Penrose discovered the singularities in the FRW model (Hawking and Ellis 1973, Mohajan 2013b). Hawking and Penrose (1970) explain that

singularities arise when a black hole is created due to gravitational collapse of massive bodies. A space-time singularity which cannot be observed by external observers is called a black hole. Poisson (2004) describes that when the black hole is formulated due to gravitational collapse, then space-time singularities must occur.

The existence of space-time singularities follows in the form of future or past incomplete non-spacelike geodesics in the space-time. Such a singularity would arise either in the cosmological scenarios, where it provides the origin of the universe or as the end state of the gravitational collapse of a massive star which has exhausted its nuclear fuel providing the pressure gradient against the inwards pull of gravity (Mohajan 2013c).

We consider a manifold M which is smooth, i.e., M is differentiable as permitted by M . We assume that M is Hausdorff and paracompact. Global hyperbolicity is the strongest and physically most important concept both in general and special relativity and also in relativistic cosmology. This notion was introduced by Jean Leray in 1953 (Leray 1953) and developed in the golden age of general relativity by A. Avez, B. Carter, Choquet-Bruhat, C. J. S. Clarke, Stephen W. Hawking, Robert P. Geroch, Roger Penrose, H. J. Seifert and others (Sánchez 2010).

Each generator of the boundary of the future has a past end point on the set one has to impose some global condition on the causal structure. This is relevant to Einstein's theory of general relativity, and potentially to other metric gravitational theories. In 2003, Antonio N. Bernal and Miguel Sánchez showed that any globally hyperbolic manifold M has a smooth embedded 3-dimensional Cauchy surface, and furthermore that any two Cauchy surfaces for M are diffeomorphic (Bernal and Sánchez 2003, 2005).

Despite many advances on global hyperbolicity however, some questions which affected basic approaches to this concept, remained unsolved yet. For example, the so-called folk problems of smoothability, affected the differentiable and metric structure of any globally hyperbolic space-time M (Sachs and Wu 1977). The Geroch, Kronheimer, and Penrose (GKP) causal boundary introduced a new ingredient for the causal structure of space-times, as well as a new viewpoint for global hyperbolicity (GKP 1972).

In this study, we describe how the matter fields with positive energy density affect the causality relations in a space-time and cause focusing in the families of non-spacelike trajectories. Here the main phenomenon is that matter focuses the non-spacelike geodesics of the space-time into pairs of focal points or conjugate points due to gravitational forces.

2. Some Related Definitions

In this section, we give some definitions which are related to our study (Mohajan 2013e). The definitions will provide necessary information to understand the paper perfectly.

Manifold: A manifold is essentially a space which is locally similar to Euclidean space in that it can be covered by coordinate patches but which need not be Euclidean globally. Map $\phi: O \rightarrow O'$ where $O \subset R^n$ and $O' \subset R^m$ is said to be a class C^r ($r \geq 0$) if the following conditions are satisfied. If we choose a point (event) p of coordinates (x^1, \dots, x^n) on O and its image $\phi(p)$ of

coordinates (x^1, \dots, x^n) on O' then by C^r map, we mean that the function ϕ is r -times differential and continuous. If a map is C^r for all $r \geq 0$ then we denote it by C^∞ ; also by C^0 map we mean that the map is continuous (Hawking and Ellis 1973, Mohajan 2016).

An n -dimensional, C^r , real differentiable manifold M is defined as follows:

M has a C^r atlas $\{U_\alpha, \phi_\alpha\}$ where U_α are subsets of M and ϕ_α are one-one maps of the corresponding U_α to open sets in R^n such that (figure 1);

- i. U_α cover M i.e., $M = \bigcup_\alpha U_\alpha$,
- ii. If $U_\alpha \cap U_\beta \neq \emptyset$, then the map $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta)$ is a C^r map of an open subset of R^n to an open subset of R^n .

Condition (ii) is very important for overlap of two local coordinate neighborhoods. Now suppose U_α and U_β overlap and there is a point p in $U_\alpha \cap U_\beta$. Now choose a point q in $\phi_\alpha(U_\alpha)$ and a point r in $\phi_\beta(U_\beta)$. Now $\phi_\beta^{-1}(r) = p$, $\phi_\alpha(p) = (\phi_\alpha \circ \phi_\beta^{-1})(r) = q$. Let coordinates of q be (x^1, \dots, x^n) and those of r be (y^1, \dots, y^n) . At this stage we obtain a coordinate transformation;

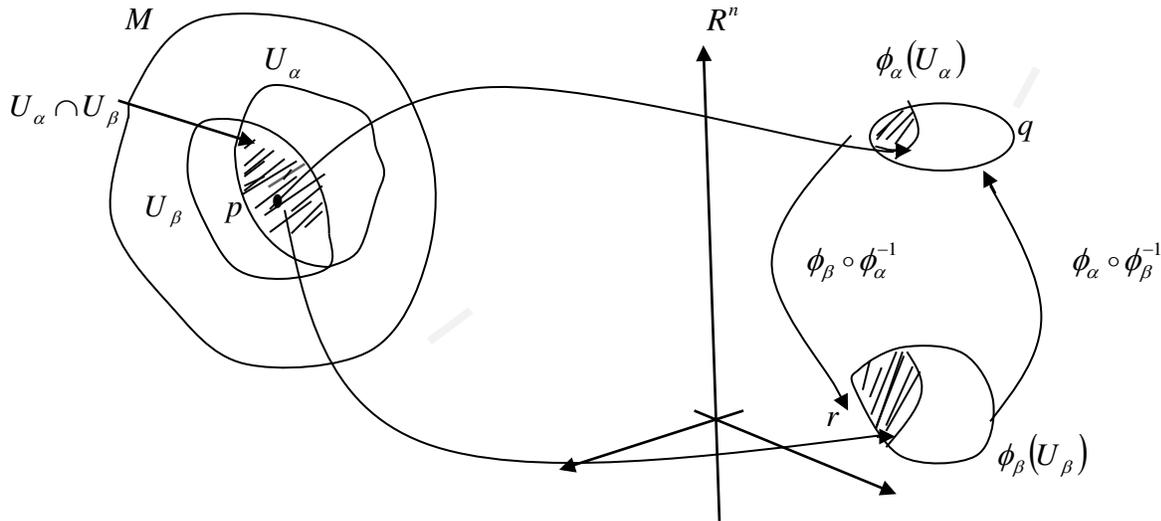


Figure 1: The smooth maps $\phi_\alpha \circ \phi_\beta^{-1}$ on the n -dimensional Euclidean space R^n giving the change of coordinates in the overlap region.

$$y^1 = y^1(x^1, \dots, x^n)$$

$$y^2 = y^2(x^1, \dots, x^n)$$

$$\dots \dots \dots$$

$$y^n = y^n(x^1, \dots, x^n).$$

The open sets U_α, U_β and maps $\phi_\alpha \circ \phi_\beta^{-1}$ and $\phi_\beta \circ \phi_\alpha^{-1}$ are all n -dimensional, so that C^r manifold M is r -times differentiable and continuous, i.e., M is a differentiable manifold (Hawking and Ellis 1973).

Hausdorff Space: A topological space M is a Hausdorff space if for a pair of distinct points $p, q \in M$ there are disjoint open sets U_α and U_β in M such that $p \in U_\alpha$ and $q \in U_\beta$ (Mohajan 2016).

Paracompact Space: An atlas $\{U_\alpha, \phi_\alpha\}$ is called locally finite if there is an open set containing every $p \in M$ which intersects only a finite number of the sets U_α . A manifold M is called a paracompact if for every atlas there is locally finite atlas $\{O_\beta, \psi_\beta\}$ with each O_β contained in some U_α . Let V^μ be a timelike vector, and then paracompactness of manifold M implies that there is a smooth positive definite Riemann metric $K_{\mu\nu}$ defined on M (Joshi 1996).

Compact Set: A subset A of a topological space M is compact if every open cover of A is reducible to a finite cover (Mohajan 2016).

Tangent Space: A C^k -curve in M is a map from an interval of R in to M (figure 2). A vector $\left(\frac{\partial}{\partial t}\right)_{\lambda(t_0)}$ which is tangent to a C^1 -curve $\lambda(t)$ at a point $\lambda(t_0)$ is an operator from the space of all smooth functions on M into R and is denoted by (Joshi 1996);

$$\left(\frac{\partial}{\partial t}\right)_{\lambda(t_0)}(f) = \left(\frac{\partial f}{\partial t}\right)_{\lambda(t_0)} = \lim_{s \rightarrow 0} \frac{f[\lambda(t+s)] - f[\lambda(t)]}{s}.$$

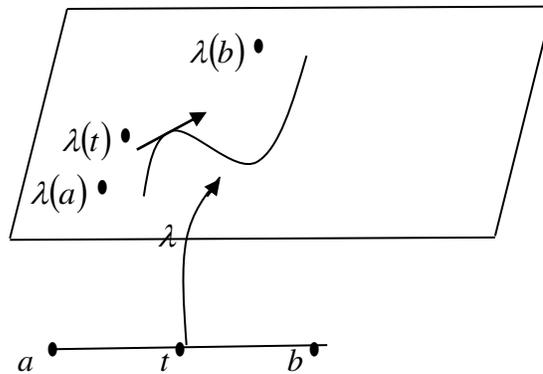


Figure 2: A curve in a differential manifold (Mohajan 2013d).

If $\{x^i\}$ are local coordinates in a neighborhood of $p = \lambda(t_0)$ then,

$$\left(\frac{\partial f}{\partial t}\right)_{\lambda(t_0)} = \frac{dx^i}{dt} \cdot \frac{\partial f}{\partial x^i} \Big|_{\lambda(t_0)}.$$

Thus, every tangent vector at $p \in M$ can be expressed as a linear combination of the coordinates derivatives, $\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^n}\right)_p$. Thus, the vectors $\left(\frac{\partial}{\partial x^i}\right)_p$ span the vector space T_p . Then the vector space structure is defined by $(\alpha X + \beta Y)f = \alpha(Xf) + \beta(Yf)$. The vector space T_p is also called the tangent space at the point p .

A metric is defined as;

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

where $g_{\mu\nu}$ is an indefinite metric in the sense that the magnitude of non-zero vector could be either positive, negative or zero (Mohajan 2013d). Then any vector $X \in T_p$ is called timelike, null, spacelike or non-spacelike respectively if;

$$g(X, X) < 0, \quad g(X, X) = 0, \quad g(X, X) > 0, \quad g(X, X) \leq 0. \tag{2}$$

Orientation: Let B be the set of all ordered basis $\{e_i\}$ for T_p , the tangent space at point p . If $\{e_i\}$ and $\{e_j\}$ are in B , then we have $e_j = a^i_j e_i$. If we denote the matrix (a_{ij}) then $\det(a) \neq 0$. An n -dimensional manifold M is called orientable if M admits an atlas $\{U_i, \varphi_i\}$ such that whenever $U_i \cap U_j \neq \emptyset$ then the Jacobian, $J = \det\left(\frac{\partial x^i}{\partial x^j}\right) > 0$, where $\{x^i\}$ and $\{x^j\}$ are local coordinates in U_i and U_j respectively. The Möbius strip is a non-orientable manifold. A vector defined at a point in Möbius strip with a positive orientation comes back with a reversed orientation in the negative direction when it traverses along the strip to come back to the same point (Mohajan 2015).

Space-time Manifold: General Relativity models the physical universe as a 4-dimensional C^∞ Hausdorff differentiable space-time manifold M with a Lorentzian metric g of signature $(-, +, +, +)$ which is topologically connected, paracompact and space-time orientable. These properties are suitable when we consider for local physics. As soon as we investigate global features then we face various pathological difficulties such as the violation of time orientation, possible non-Hausdorff or non-paracompactness, disconnected components of space-time, etc. Such pathologies are to be ruled out by means of reasonable topological assumptions only (Mohajan 2013d). However, we like to ensure that the space-time is causally well-behaved. We will consider the space-time Manifold (M, g) , which has no boundary. By the word ‘boundary’ we mean the ‘edge’ of the universe which is not detected by any astronomical observations. It is common to have manifolds without boundary; for example, for two-spheres S^2 in R^3 no point

in S^2 is a boundary point in the induced topology on the same implied by the natural topology on R^3 (Mohajan 2013d). All the neighborhoods of any $p \in S^2$ will be contained within S^2 in this induced topology. We shall assume M to be connected i.e., one cannot have $M = X \cup Y$, where X and Y are two open sets such that $X \cap Y \neq \emptyset$. This is because disconnected components of the universe cannot interact by means of any signal and the observations are confined to the connected component wherein the observer is situated (Mohajan 2014a). It is not known if M is simply connected or multiply connected. Manifold M is assumed to be Hausdorff, which ensures the uniqueness of limits of convergent sequences and incorporates our intuitive notion of distinct space-time events (Joshi 1996).

Hypersurface: In the Minkowski space-time $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, the surface $t = 0$ is a three-dimensional surface with the time direction always normal to it. Any other surface $t = \text{constant}$ is also a spacelike surface in this sense. Let S be an $(n-1)$ -dimensional manifold. If there exists a C^∞ map $\phi: S \rightarrow M$ which is locally one-one i.e., if there is a neighborhood N for every $p \in S$ such that ϕ restricted to N is one-one, and ϕ^{-1} is a C^∞ as defined on $\phi(N)$, then $\phi(S)$ is called an embedded sub-manifold of M . A hypersurface S of any n -dimensional manifold M is defined as an $(n-1)$ -dimensional embedded sub-manifold of M . Let V_p be the $(n-1)$ -dimensional subspace of T_p of the vectors tangent to S at any $p \in S$ from which follows that there exists a unique vector $n^a \in T_p$ and is orthogonal to all the vectors in V_p (Mohajan 2013d). Here n^a is called the normal to S at p . If the magnitude of n^a is either positive or negative at all points of S without changing the sign, then n^a could be normalized so that $g_{ab}n^an^b = \pm 1$. If $g_{ab}n^an^b = -1$ then the normal vector is timelike everywhere and S is called a spacelike hypersurface. If the normal is spacelike everywhere on S with a positive magnitude, S is called a timelike hypersurface. Finally, S is null hypersurface if the normal n^a is null at S (Mohajan 2015).

3. Basic Concept of General Relativity

The covariant differentiations of vectors are defined as;

$$A_{;v}^\mu = A_{,v}^\mu + \Gamma_{\nu\lambda}^\mu A^\lambda \quad (3a)$$

$$A_{\mu;v} = A_{\mu,v} - \Gamma_{\mu\nu}^\lambda A_\lambda \quad (3b)$$

where semi-colon denotes the covariant differentiation and coma denotes the partial differentiation (Mohajan 2014a).

By (3b) we can write;

$$A_{\mu;\nu;\sigma} - A_{\mu;\sigma;\nu} = R_{\mu\nu\sigma}^\alpha A_\alpha, \quad (4)$$

$$\text{where } R_{\mu\nu\sigma}^{\alpha} = \Gamma_{\mu\sigma,\nu}^{\alpha} - \Gamma_{\mu\nu,\sigma}^{\alpha} + \Gamma_{\beta\nu}^{\alpha} \Gamma_{\mu\sigma}^{\beta} - \Gamma_{\beta\sigma}^{\alpha} \Gamma_{\mu\nu}^{\beta} \quad (4a)$$

is a tensor of rank four and called Riemann curvature tensor. From (4) we observe that the curvature tensor components are expressed regarding the metric tensor and its second derivatives. From (4a) we get;

$$R_{[\mu\nu\sigma]}^{\alpha} = 0. \quad (5)$$

Taking inner product of both sides of (4a) with $g_{\rho\alpha}$ one gets covariant curvature tensor;

$$R_{\rho\mu\nu\sigma} = \frac{1}{2} \left(\frac{\partial^2 g_{\rho\sigma}}{\partial x^{\nu} \partial x^{\mu}} + \frac{\partial^2 g_{\mu\nu}}{\partial x^{\sigma} \partial x^{\rho}} - \frac{\partial^2 g_{\mu\sigma}}{\partial x^{\nu} \partial x^{\rho}} - \frac{\partial^2 g_{\rho\nu}}{\partial x^{\sigma} \partial x^{\mu}} \right) + g_{\alpha\lambda} (\Gamma_{\mu\nu}^{\alpha} \Gamma_{\rho\sigma}^{\lambda} - \Gamma_{\mu\sigma}^{\alpha} \Gamma_{\rho\nu}^{\lambda}). \quad (6)$$

Contraction of curvature tensor (6) gives Ricci tensor;

$$R_{\mu\nu} = g^{\lambda\sigma} R_{\lambda\mu\sigma\nu}. \quad (7)$$

Further contraction of (7) gives Ricci scalar;

$$\hat{R} = g^{\lambda\sigma} R_{\lambda\sigma}. \quad (8)$$

From which one gets Einstein tensor as;

$$G_{\nu}^{\mu} = R_{\nu}^{\mu} - \frac{1}{2} \delta_{\nu}^{\mu} R \quad (9)$$

where $div(G_{\nu}^{\mu}) = G_{\nu;\mu}^{\mu} = 0$.

The space-time (M, g) is said to have a flat connection if and only if;

$$R_{\nu\lambda\sigma}^{\mu} = 0. \quad (10)$$

This is the necessary and sufficient condition for a vector at a point p to remain unaltered after parallel transported along an arbitrary closed curve through p . This is because all such curves can be shrunk to zero, in which case the space-time is simply connected (Hawking and Ellis 1973).

The energy momentum tensor $T^{\mu\nu}$ is defined as;

$$T^{\mu\nu} = \rho_0 u^{\mu} u^{\nu} \quad (11)$$

where ρ_0 is the proper density of matter, and if there is no pressure. A perfect fluid is characterized by pressure $p = p(x^{\mu})$, then;

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}. \quad (12)$$

The principle of local conservation of energy and momentum states that;

$$T^{\mu\nu}_{;\nu} = 0. \quad (13)$$

The most appropriate tensor of the form required is the Einstein's tensor (9); then Einstein's field equation can be written as (Mohajan 2014b);

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T^{\mu\nu}. \quad (14)$$

where $G = 6.673 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ is the gravitational constant and $c = 10^8 \text{ m/s}$ is the velocity of light. Einstein introduced a cosmological constant $\Lambda (\approx 0)$ for static universe solutions as;

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}T^{\mu\nu}. \quad (15)$$

In relativistic unit $G = c = 1$, hence in relativistic units (15) becomes;

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi T_{\mu\nu}. \quad (16)$$

It is clear that divergence of both sides of (15) and (16) is zero. For empty space $T_{\mu\nu} = 0$ then $R_{\mu\nu} = \Lambda g_{\mu\nu}$, so that;

$$R_{\mu\nu} = 0 \text{ for } \Lambda = 0 \quad (17)$$

which is Einstein's law of gravitation for empty space.

4. Causal Structure of Space-time Manifold

In Lorentzian geometry causality plays an important role, as it displays a relativistic interpretation of space-time for both special and general relativity. Causality also appears as a fruitful interplay between relativistic motivations and geometric developments. Causal space-time is established at the end of the 1970s, after the works of Carter, Geroch, Hawking, Kronheimer, Penrose, Sachs, Seifert, Wu and others (Hawking and Sachs 1974).

No material particle can travel faster than the velocity of light. Hence, causality fixes the boundary of the space-time topology. We assume that the timelike curves to be smooth; with future-directed tangent vectors everywhere strictly timelike, including its end-points. A causal curve is a curve in space-time which is nowhere spacelike. A causal curve is continuous but not

necessarily everywhere smooth; its tangent vectors are either timelike or null. A causal curve will required end-points if it can be extended as a causal curve either into the past or the future. If a causal curve can be extended indefinitely and continuously into the past then it is called past-inextensible. The future-inextensible curve is defined similarly. If a causal curve is both past and future-inextensible then it is called simply inextensible (Hawking and Penrose 1970).

An event x chronologically precedes another event y , denoted by $x \ll y$, if there is a smooth future directed timelike curve from x to y . If such a curve is non-spacelike then x causally precedes y , i.e., $x < y$. The chronological future $I^+(x)$ is the set of all points of the space-time M that can be reached from x by future directed timelike curves. We can think of $I^+(x)$ as the set of all events that can be influenced by what happens at x . Now $I^+(x)$ and $I^-(x)$ of a point x are defined as (figure 3),

$$I^+(x) = \{ y \in M / x \ll y \}, \text{ and}$$

$$I^-(x) = \{ y \in M / y \ll x \}.$$

One can think of $I^+(x)$ as the set of all events that can be influenced by what happens at x . The causal future (past) of x can be defined as;

$$J^+(x) = \{ y \in M / x < y \},$$

$$J^-(x) = \{ y \in M / y < x \}.$$

Also $x \ll y$ and $y < z$ or $x < y$ and $y \ll z$ implies $x \ll z$. Hence, the closer and boundary of $I^+(x)$ and past $I^-(x)$ of a point x are defined respectively as (Penrose 1972);

$$\overline{I^+(x)} = \overline{J^+(x)} \text{ and } \dot{I}^+(x) = \dot{J}^+(x), \text{ where } \dot{I} \text{ is a topological boundary and } \bar{I} \text{ is the closure of } I.$$

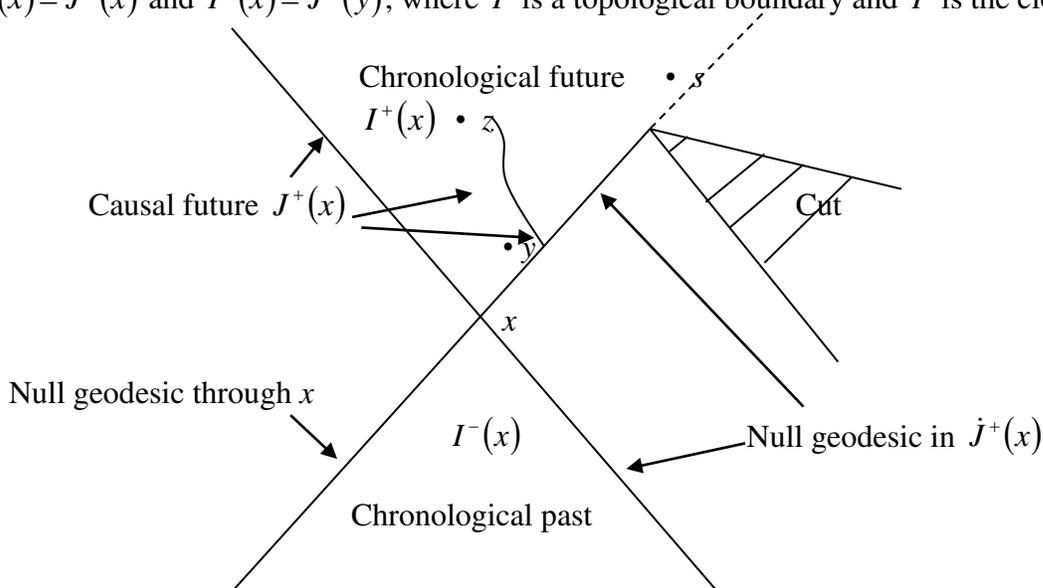


Figure 3: Removal of a closed set from the space-time gives a causal future $J^+(x)$ which is not closed. Events x and s are not causally connected.

Similarly, the chronological (causal) future of any set $S \subset M$ is defined as;

$$I^+(S) = \bigcup_{x \in S} I^+(x), \text{ and}$$

$$J^+(S) = \bigcup_{x \in S} J^+(x).$$

Similarly, we can define the past subsets of space-time.

The boundary of the future is null apart from at S itself. If x is in the boundary of the future but is not in the closure of S there is a past directed null geodesic segment through x lying in the boundary. Hence the boundary of the future of S is generated by null geodesics that have a future end point in the boundary and pass into the interior of the future if they intersect another generator and the null geodesic generators can have past end points only on S (Hawking 1994).

Causally Convex Set: Let S and T be open subsets of a space-time (M, g) , with $T \subset S$ then T is called causally convex in S if any causal curve contained in S with endpoints in T is entirely contained in T . In particular, when this holds for $S = M$, T is called *causally convex*. Again if T is causally convex in S and U is an open set such that $T \subset U \subset S$, then T is causally convex in U (Minguzzi and Sánchez 2008).

Future Set and Past Set: An open subset F is a future set if $I^+(F) = F$. The past set P is defined by $I^-(P) = P$. The boundary of a future set F is made of all events x such that $I^-(x) \subset F$ but $x \notin F$. If $x \in \dot{F}$ then of course $x \notin F$, since F is an open set.

Achronal Set: A set S in M is said to be achronal if no two points $x, y \in S$ may be joined by a piecewise timelike curve i.e., there do not exist $x, y \in S$ such that $y \in I^+(x)$. Let F be a future set, then the boundary of F is a closed, achronal C^0 -manifold that is a 3-dimensional embedded hypersurface.

Domain of Dependence of a Set: The future domain of dependence (the future Cauchy development) of a spacelike three-surface S , denoted by $D^+(S)$, is defined as the set of all points $x \in M$ such that every past-inextendible non-spacelike curve from x intersects S , i.e., $D^+(S) = \{x: \text{every past-inextendible timelike curve through } x \text{ meets } S\}$. It is clear that $S \subset D^+(S) \subset J^+(S)$ and S being achronal, $D^+(S) \cap I^-(S) = \emptyset$. The past domain of dependence $D^-(S)$ is defined similarly. The full domain of dependence for S is defined as; $D(S) = D^+(S) \cup D^-(S)$ (Joshi 1993).

Cauchy Surface: Let S be a closed achronal set. The edge of S is defined as a set of points $x \in S$ such that every neighborhood of x contains $y \in I^+(x)$ and $z \in I^-(x)$ with a timelike curve from z to y which does not meet S . A partial Cauchy surface S is defined as an acausal set without an edge. So that no non-spacelike curve intersects S more than once, and S is a spacelike hypersurface.

5. The Global Hyperbolic Space-Time

A partially Cauchy surface is called a Cauchy surface S or a global Cauchy surface if $D(S) = M$ i.e., if a set S is closed, achronal, and its domain of dependence is all of the space-time, $D(S) = M$. In another way, if $D(S) = M$ i.e., if every inextendible non-spacelike curve intersects S , then S is said to be a Cauchy surface (figure 4). For a Cauchy surface S , $edge(S) = \emptyset$. The Cauchy development is the region of spacetime that can be predicted from data on S . Here S must be an embedded topological hypersurface and must be also crossed by any inextendible causal curve γ (Hawking 1966a,b). The existence of a Cauchy hypersurface S implies that M is homeomorphic to $t \times S$, and all Cauchy hypersurfaces are homeomorphic.

Every non-spacelike curve in M meets S once and exactly once if S is a Cauchy surface. The relationship between the global hyperbolicity of M and the notion of Cauchy surface is shown in figure 4 (Hawking and Ellis 1973):

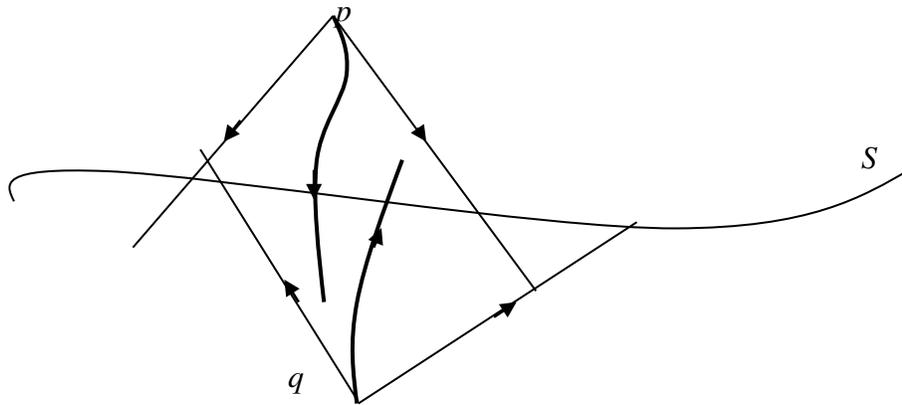


Figure 4: The spacelike hypersurface S is a Cauchy surface in the sense that for any p in future of S , all past non-spacelike curves from p intersect S . The same holds for all future-directed curves from any point q in past of S .

Time function is a continuous function $t : M \rightarrow \mathbb{R}$ which increases strictly on any future-directed causal curve. If the levels $t = \text{constant}$ are Cauchy hypersurfaces, then t is a Cauchy time function. The space-time manifold has a Cauchy surface S .

5.1 Globally Hyperbolicity

In mathematical physics, global hyperbolicity is a certain condition on the causal structure of a space-time manifold. If M is a smooth connected Lorentzian manifold with boundary, we say it

is globally hyperbolic if its interior is globally hyperbolic. Penrose has called globally hyperbolic space-times “the physically reasonable space-times” (Wald 1984). A space-time (M, g) which admits a Cauchy surface is called globally hyperbolic.

A space-time (M, g) which admits a Cauchy surface is called globally hyperbolic. An open set O is said to be globally hyperbolic if, i) for every pair of points x and y in O the intersection of the future of x and the past of y has compact closure, i.e., if a space-time (M, g) is said to be globally hyperbolic if the sets $J^+(x) \cap J^-(y)$ are compact for all $x, y \in M$ (i.e., no naked singularity can exist in space-time topology). In other words, it is a bounded diamond shaped region (diamond-compact) and ii) strong causality holds on O , i.e., there are no closed or almost closed time like curves contained in O (figure 4). Then it also satisfies that $J^+(x)$ and $J^-(y)$ are closed $\forall x, y \in M$. More precisely, consider two events x, y of the space-time (M, g) , and let $C(x, y)$ be the set of all the continuous curves which are future-directed and causal and connect x with y (Hawking and Ellis 1973).

Minkowski space-time, de Sitter space-time and the exterior Schwarzschild solution, Friedmann, Robertson-Walker (FRW) cosmological solutions and the steady state models are all globally hyperbolic. The Kerr solution is not globally hyperbolic, since it represents rotating model, i.e., not a static model. On the other hand anti de Sitter space-time and the Godel universe are not globally hyperbolic. The global hyperbolicity of M is closely related to the future or past development of initial data from a given spacelike hypersurface (Joshi 1996).

The physical significance of global hyperbolicity comes from the fact that it implies that there is a family of Cauchy surfaces $\Sigma(t)$ for globally hyperbolic open set O . A Cauchy surface for O is a spacelike or null surface that intersects every timelike curve in O once and only once. Let x and y be two points of O that can be joined by a timelike or null curve, then there is a timelike or null geodesic between x and y which maximizes the length of timelike or null curves from x to y (Hawking 1994).

5.2 Cauchy Horizons of a Set

Let S be a partial Cauchy surface. Then $N = D^+(S) \cup D^-(S) \neq M$ and N must be a proper subset of M . The boundary of N in M can be divided into two portions. Now suppose that the future Cauchy development was compact. This would imply that the Cauchy development would have a future boundary called the Cauchy horizon, $H^+(S)$. Since the Cauchy development is assumed to be compact, the Cauchy horizon will also be compact. The $H^+(S)$ and $H^-(S)$ which are respectively called the future and past Cauchy horizons of S . We can write (Hawking and Penrose 1970);

$$\begin{aligned} H^+(S) &= \{x / x \in D^+(S), I^+(x) \cap D^+(S) = \emptyset\} \\ &= D^+(S) - I^-[D^+(S)]. \end{aligned}$$

$H^-(S)$ is defined similarly. $H^+(S)$ is an achronal closed set. Also we can write, $I^+[H^+(S)] = I^+[S] - D^+(S)$.

The Cauchy horizon will be generated by null geodesic segments without past end points. Even though M may not be globally hyperbolic and S is not a Cauchy surface, the region $Int(D^+(S))$ or $Int(D^-(S))$ is globally hyperbolic in its own right and the surface S serves as a Cauchy surface for the manifold $Int(N)$. Thus $H^+(S)$ or $H^-(S)$ represents the failure of S to be a global Cauchy surface for M (figure 5).

If every geodesic can be extended to arbitrary values of its affine parameter, then it is geodesically complete. If a timelike or causal curve can be extended indefinitely and continuously into the past (future), then it is called past-inextendible (future-inextendible).

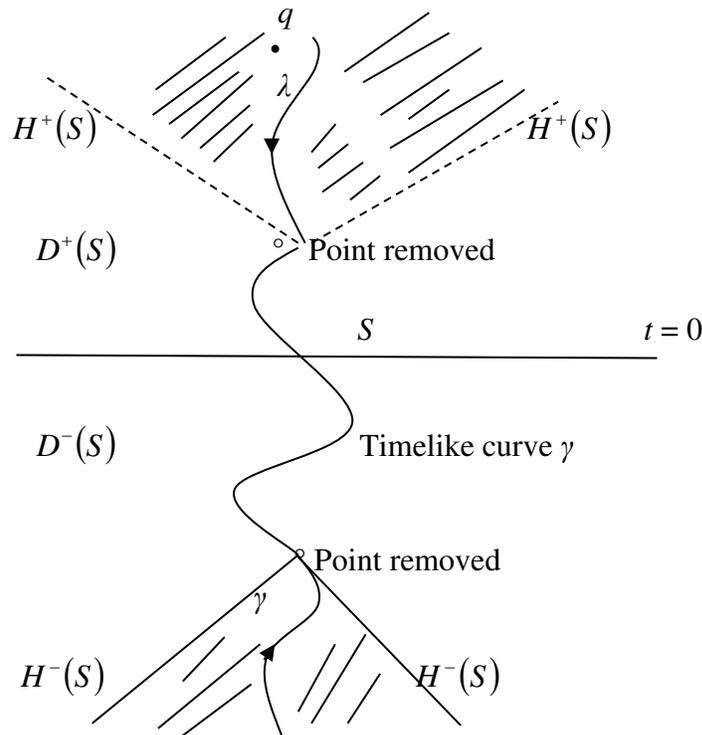


Figure 5: The space-time obtained by removing a point from the Minkowski space-time is not globally hyperbolic. The point q does not meet S in the past. The event $p \in D^+(S)$. The Cauchy horizon is the boundary of the shaded region which consists of points not in $D^+(S)$.

In globally hyperbolic space-times, there is a finite upper bound on the proper time lengths of non-spacelike curves two chronologically related events. Of course there is no lower limit of length for such curves except zero, because the chronologically related events can always be joined using broken null curves which could give an arbitrary small length curve between them. If S is Cauchy surface in globally hyperbolic space-time M , then for any point p in the future of

S , there is a past directed timelike geodesic from p orthogonal to S which maximizes the lengths of all non-spacelike curves from p to S (figure 6).

An important property of globally hyperbolic space-time that is relevant for the singularity theorems is the existence of maximum length non-spacelike geodesics between a pair of causally related events. In a complete Riemannian manifold with a positive definite metric any two points can be joined by a geodesic of minimum length and in fact such a geodesic need not be unique (Joshi 1996). (In a sphere paths of great circles are geodesics. Opposite poles can be joined by an infinite numbers of geodesics.)

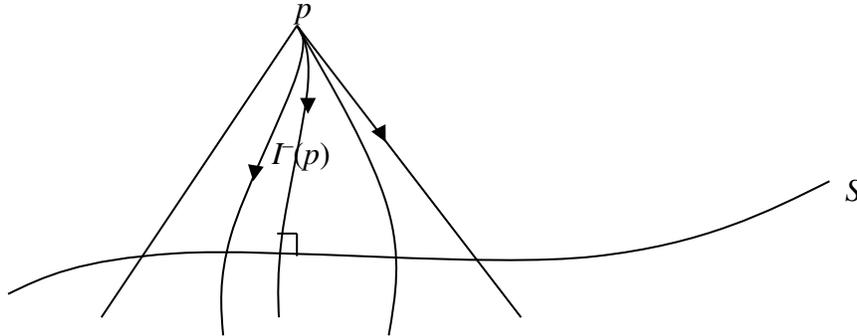


Figure 6: The spacelike hypersurface S is a Cauchy surface in the sense that for any p in future of S , all past directed non-spacelike curves from p intersect S .

6. Space-time Singularities

The existences of real singularities where the curvature scalars and densities diverge imply that all the physical laws break down. Let us consider the metric;

$$ds^2 = -\frac{1}{t^2} dt^2 + dx^2 + dy^2 + dz^2 \quad (18)$$

which is singular on the plane $t=0$. If any observer starting in the region $t > 0$ tries to reach the surface $t=0$ by traveling along timelike geodesics, he will not reach at $t=0$ in any finite time, since the surface is infinitely far into the future. If we put $t' = \ln(-t)$ in $t < 0$ then (18) becomes (Mohajan 2013e);

$$ds^2 = -dt'^2 + dx^2 + dy^2 + dz^2 \quad (19)$$

with $-\infty < t' < \infty$ which is Minkowski metric and there is no singularity at all (Clarke 1986).

A timelike geodesic which, when maximally extended, has no end point in the regular space-time and which has finite proper length, is called timelike geodesically incomplete. Now we shall discuss some definitions related with the singularity (Clarke 1986).

Definition: The generalized affine parameter (GAP) length of a curve $\gamma: [0, a) \rightarrow M$ with respect to a frame,

$$\mathbf{E} = \left(\mathbf{E}_a, a = 0,1,2,3 \right)$$

at $\gamma(0)$ is given by;

$$\ell_{\mathbf{E}}(\gamma) = \int_0^a \left(\sum_{i=0}^3 g(\dot{\gamma} E_i(s)) \right)^{1/2} ds$$

where $\dot{\gamma} = \frac{d\gamma}{ds}$ is tangent vector and $\mathbf{E}(s)$ is defined by parallel propagation along the curve, starting with an initial value $\mathbf{E}(0)$.

Definition: A curve $\gamma: [0, a] \rightarrow M$ is incomplete if it has finite GAP length with respect to some frame \mathbf{E} at $\gamma(0)$. If $\ell_{\mathbf{E}}(\gamma) < \infty$, then if we take any other frame \mathbf{E}' at $\gamma(0)$ we have $\ell_{\mathbf{E}'}(\gamma) < \infty$. This is because the corresponding parallel propagated frames satisfy (Mohajan 2013e);

$$E'_i = L_i^j E_j$$

for a constant Lorentz matrix \mathbf{L} and hence;

$$\ell_{\mathbf{E}'} \leq \|\mathbf{L}\| \ell_{\mathbf{E}},$$

where $\|\mathbf{L}\| = \text{Sup} \left((L_i^j X^i) \right)^{1/2}$.

Definition: A curve $\gamma: [0, a] \rightarrow M$ is termed inextensible if there is no curve $\gamma': [0, b] \rightarrow M$ with $b > a$ such that $\gamma' \upharpoonright [0, a] = \gamma$. This is equivalent to saying that there is no point p in M such that $\gamma(s) \rightarrow p$ as $s \rightarrow a$, i.e., γ has no end point in M .

Definition: A space-time is incomplete if it contains an incomplete inextensible curve. By the above definitions we can say that a space-time is called incomplete if it contains an incomplete timelike inextensible curve. The Friedman 'Big Bang' models are geodesically incomplete, since the curve defined by (Mohajan 2013e);

$$\gamma(s)^0 = S(t) - s$$

$$\gamma(s)^i = \text{Constant}, i = 1, 2, 3$$

is a geodesic which is incomplete, having no endpoint in M as $s \rightarrow S(t)$. Minkowski space is not incomplete. The region $r > 2m$ in the Schwarzschild metric is incomplete, while the region $0 < r < 2m$ is not a space-time, since the metric is not defined at $r = 2m$.

Definition: An extension of a space-time (M, g) is an isometric embedding $\theta: M \rightarrow M'$ where (M', g') is a space-time and θ is onto a proper subset of M' . By the above definition, Schwarzschild metric is not singular at $r=2m$ by Kruskal-Szekeres extension (Kruskal 1960, Szekeres 1960). A space-time is termed extensible if it has an extension.

Definition: A space-time is singular if it contains an incomplete curve $\gamma: [0, a) \rightarrow M$ such that there is no extension $\theta: M \rightarrow M'$ for which $\theta \circ \gamma$ is extensible.

7. Schwarzschild Singularity

The Schwarzschild metric which represents the outside metric for a star is given by (Mohajan 2013e);

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (20)$$

If r_0 is the boundary of a star then $r > r_0$ gives the outside metric as in (20). If there is no surface, (20) represents a highly collapsed object viz. a black hole of mass m (will be discussed later). The metric (20) has singularities at $r = 0$ and $r = 2m$, so it represents patches $0 < r < 2m$ or $2m < r < \infty$. If we consider the patches $0 < r < 2m$ then it is seen that as r tends to zero, the curvature scalar,

$$R^{\mu\nu\sigma\lambda} R_{\mu\nu\sigma\lambda} = \frac{48m^2}{r^6}$$

tends to ∞ and it follows that $r = 0$ is a genuine curvature singularity i.e., space-time curvature components tend to infinity (Mohajan 2013a).

8. Friedmann, Robertson–Walker (FRW) Model

The FRW model plays an important role in Cosmology. This model is established on the basis of the homogeneity and isotropy of the universe as described above. The current observations give a strong motivation for the adoption of the cosmological principle stating that at large scales the universe is homogeneous and isotropic and, hence, its large-scale structure is well described by the FRW metric. The FRW geometries are related to the high symmetry of these backgrounds. Due this symmetry numerous physical problems are exactly solvable, and a better understanding of physical effects in FRW models could serve as a handle to deal with more complicated geometries (Mohajan 2013b).

In (t, r, θ, ϕ) coordinates the Robertson-Walker line element is given by;

$$ds^2 = -dt^2 + S^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad (21)$$

where k is a constant which denotes the spatial curvature of the three-space and could be normalized to the values $+1, 0, -1$. When $k = 0$ the three-space is flat and (21) is called Einstein de-Sitter static model, when $k = +1$ and $k = -1$ the three-space are of positive and negative constant curvature; these incorporate the closed and open Friedmann models respectively (figure 7).

Let us assume the matter content of the universe as a perfect fluid then by (14) and (15), solving (21) we get;

$$\frac{3\ddot{S}}{S} + 4\pi(\rho + 3p) = 0, \text{ and} \tag{22}$$

$$\frac{3\dot{S}^2}{S^2} - \left(8\pi\rho - \frac{3k}{S^2}\right) = 0 \tag{23}$$

where we have considered $\Lambda = 0$. If $\rho > 0$ and $p \geq 0$ then $\ddot{S} < 0$. So $\dot{S} = \text{constant}$ and $\dot{S} > 0$ indicates the universe must be expanding, and $\dot{S} < 0$ indicates contracting universe. The observations by Hubble of the red-shifts of the galaxies were interpreted by him as implying that all of them are receding from us with a velocity proportional to their distances from us that is why the universe is expanding. For expanding universe $\dot{S} > 0$, so by (22) and (23) we get $\ddot{S} < 0$. Hence \dot{S} is a decreasing function and at earlier times the universe must be expanding at a faster rate as compared to the present rate of expansion. But if the expansion be constant rate as like the present expansion rate at all times then,

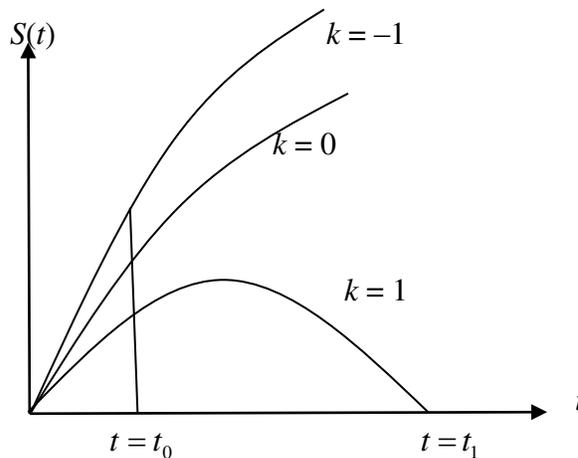


Figure 7: The behavior of the curve $S(t)$ for the three values $k = -1, 0, +1$; the time $t = t_0$ is the present time and $t = t_1$ is the time when $S(t)$ reaches zero again for $k = +1$.

$$\left(\frac{\dot{S}}{S}\right)_{t=t_0} \equiv H_0. \quad (24)$$

Now H_0^{-1} implies a global upper limit for the age of any type of Friedmann models. So the age of the universe will be less than H_0^{-1} . The quantity H_0 is called Hubble constant and at any given epoch it measures the rate of expansion of the universe. By observation H_0 has a value somewhere in the range of 50 to 120 $\text{kms}^{-1}\text{Mpc}^{-1}$.

At $S = 0$, the entire three-surface shrinks to zero volume and the densities and curvatures grow to infinity. Hence, by FRW models there is a singularity at a finite time in the past. This curvature singularity is called the big bang (Islam 2002, Hawking and Ellis 1973).

9. Conclusion

In this study we have discussed the global hyperbolic space-time manifold and the singularities therein. Here we have discussed two types of singularities: i) Big Bang singularity, which is found in Friedmann, Robertson-Walker's cosmological solution, and considers as the beginning of the universe; ii) Black hole type singularity is found in the Schwarzschild solution, which is the final fate of a massive star. In the beginning of the study we have provided some elementary definitions of differential geometry and topology. Then, we have discussed the basic concepts of general relativity. We have also discussed the causal structure of space-time manifold. Then we have discussed global hyperbolicity to make the paper interesting to the readers.

References

- Bernal, A.N. and Sánchez, M. (2003), On Smooth Cauchy Hypersurfaces and Geroch's Splitting Theorem, *Communications in Mathematical Physics*, 243(3): 461–470.
- Bernal, A.N. and Sánchez, M. (2005), Smoothness of Time Functions and the Metric Splitting of Globally Hyperbolic Spacetimes, arXiv:gr-qc/0401112v3 15.
- Clarke, C.J.S. (1986), Singularities: Global and Local Aspects in Topological Properties and Global Structure of Space-time (ed. P.G. Bergmann and V. de. Sabbata), Plenum Press, New York.
- Geroch, R.P. (1970), Domain of Dependence, *Journal of Mathematical Physics*, 11: 437–449.
- Geroch, R.P.; Kronheimer, E.H. and Penrose, R. (GKP) (1972), Ideal Points in Space-time, *Proceedings of the Royal Society, London, A* 237: 545–567.
- Hawking, S.W. (1966a), The Occurrence of Singularities in Cosmology I, *Proceedings of the Royal Society, London, A*294: 511–521.

Hawking, S.W. (1966b), The Occurrence of Singularities in Cosmology II, *Proceedings of the Royal Society, London*, A295: 490–493.

Hawking, S.W. (1994), Classical Theory, arXiv:hep-th/9409195v1 30 Sep 1994.

Hawking, S.W. and Ellis, G.F.R. (1973), *The Large Scale Structure of Space-time*, Cambridge University Press, Cambridge.

Hawking, S.W. and Penrose, R. (1970), The Singularities of Gravitational Collapse and Cosmology, *Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences*, 314: 529–548.

Hawking, S.W. and Sachs, R.K. (1974), Causally Continuous Spacetimes, *Communications in Mathematical Physics*, 35: 287–296.

Islam, J.N. (2002), *An Introduction to Mathematical Cosmology*, Cambridge University Press, Cambridge.

Joshi, P.S. (1996), *Global Aspects in Gravitation and Cosmology*, 2nd Edition, Clarendon Press, Oxford.

Kruskal, M.D. (1960), Maximal Extension of Schwarzschild Metric, *Physical Review*, 119(5): 1743–1745.

Leray, J. (1953), *Hyperbolic Differential Equations*, The Institute for Advanced Study, Princeton, N. J.

Minguzzi, E. and Sánchez, M. (2008), The Causal Hierarchy of Spacetimes, arXiv:gr-qc/0609119v3.

Mohajan, H.K. (2013a), Schwarzschild Geometry of Exact Solution of Einstein Equation in Cosmology, *Journal of Environmental Treatment Techniques*, 1(2): 69–75.

Mohajan, H.K. (2013b), Friedmann, Robertson-Walker (FRW) Models in Cosmology, *Journal of Environmental Treatment Techniques*, 1(3): 158–164.

Mohajan, H.K. (2013c), *Singularity Theorems in General Relativity*, M. Phil. Dissertation, Lambert Academic Publishing, Germany.

Mohajan, H.K. (2013d), Minkowski Geometry and Space-time Manifold in Relativity, *Journal of Environmental Treatment Techniques*, 1(2): 101–109.

Mohajan, H.K. (2013e), Space-time Singularities and Raychaudhuri Equations, *Journal of Natural Sciences*, 1(2): 1–13.

Mohajan, H.K. (2014a), General Upper Limit of the Age of the Universe, *ARPJ Journal of Science and Technology*, 4(1): 4–12.

Mohajan, H.K. (2014b), Upper Limit of the Age of the Universe with Cosmological Constant, *International Journal of Reciprocal Symmetry & Theoretical Physics*, 1(1): 43–68.

Mohajan, H.K. (2015), Basic Concepts of Differential Geometry and Fibre Bundles, *ABC Journal of Advanced Research*, 4(1): 57–73.

Mohajan, H.K. (2016), Global Hyperbolicity in Space-time Manifold, *International Journal of Professional Studies*, 1(1): 14–30.

Penrose, R. (1972), Techniques of Differential Topology in Relativity, A.M.S. Colloquium Publications, SIAM, Philadelphia.

Poisson E (2004). A relativist's toolkit: the mathematics of black hole mechanics, Cambridge Univ. Press, Cambridge.

Sachs, R.K. and Wu, H. (1977), General Relativity and Cosmology, *Bulletin of the American Mathematical Society*, 83(6): 1101–1164.

Sánchez, M. (2010), Recent Progress on the Notion of Global Hyperbolicity, arXiv:0712.1933v2 [gr-qc] 25 Feb 2010.

Szekeres, P. (1960), On the Singularities of a Riemannian Manifold, *Publicationes Mathematicae, Debrecen*, 7: 285–301.

Wald, R.M. (1984), *General Relativity*, The University of Chicago Press, Chicago.