Weighted Shapley levels values

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Abstract

This paper presents a collection of four different classes of weighted Shapley levels values. All classes contain generalisations of the weighted Shapley values to cooperative games with a level structure. The first class is an upgrade of the weighted Shapley levels value in Gómez-Rúa and Vidal-Puga (2011), who use the size of components as weights. The following classes contain payoff vectors from the Harsanyi set. Hence they satisfy the dummy axiom, in contrary to the values in the first class in general. The second class contains extensions of the McLean weighted coalition structure values (Dragan, 1992; Levy and McLean, 1989; McLean, 1991). The first two classes satisfy the level game property (the payoff to all players of a component sum up to the payoff to the component in a game where components are the players) and the last two classes meet a null player out property. As a special case, the first three classes include the Shapley levels value and the last class contains a new extension of the Shapley value.

Keywords Cooperative game · Level structure · (Weighted) Shapley (levels) value · Weighted proportionality · Harsanyi set · Dividends

1 Introduction

Many organizations, companies, corporations, governments and so on are organized in hierarchical structures. Typically we have only one entity at the apex and in the following levels each entity is splitted up in two or more subordinates which normally have a lower rank as the superior one. A similar organizational structure, in some respects, show supply chains. Effectiveness can be increased by sharing or pooling of physical objects, resources and information. Queueing problems or electricity and other networks have a related background. A central characteristic of all such organizational forms are cooperation benefits. The question is how realized benefits should be shared and arising costs should be allocated.

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To distribute profits of cooperating coalitions the use of a cooperative game seems to be a natural approach. Winter (1989) formulated a model for cooperative games with hierarchical structure, called level structure, which consists of a sequence of coalition structures (the levels). In each level the player set is partitioned into components. Winter's value (Winter, 1989) for such a model, we call it Shapley levels value, extends the Owen value (Owen, 1977), itselves an extension of the Shapley value (Shapley, 1953b). So this value satisfies extensions of the symmetry axioms which are satisfied by the Owen value.

To treat symmetric players differently if there exist exogenous given weights for the players, Shapley (1953a) introduced the weighted Shapley values. Levy and McLean (1989) and McLean (1991) extended these values to coalition structures. Therefore they assigned weights to the components of the coalition structure. Dragan (1992) called one class of these values McLean weighted coalition structure values and presented for them a formula related to that of the Owen value. A different approach took Vidal-Puga (2012). He introduced a value for coalition structures with weights given by the size of the coalitions. Gómez-Rúa and Vidal-Puga (2011) extended it to level structures with a step by step top-down proceeding. Contrary to the McLean weighted coalition structure values their values don't satisfy the dummy axiom.

This paper introduces four classes[1] of values for level structures. All of them are extensions of the weighted Shapley values. Our first class, called weighted Shapley hierarchy levels values, generalizes the value from Gómez-Rúa and Vidal-Puga (2011) to values with arbitrary positive weights and therefore these values don't satisfy the dummy axiom in general. The computation needs to be done strictly hierarchical, step by step, top down.

Each of the following three classes can be represented by a formula with dividends (Harsanyi, 1959). The coefficients in the formulas form a dividend share system, meaning that all coefficients are non-negative and sum up to 1 for each coalition. Thus the values from these classes are all payoff vectors from the Harsanyi set (Hammer, 1977; Vasil'ev, 1978) and inherit all properties of these payoff vectors.

An unanimity game related to a coalition T gives a short impression how the values from these three classes work: In the first class, called weighted Shapley support levels values, each player of coalition T is supported by the weights of the components containing her; in the second class, called weighted Shapley alliance levels values, all players of coalition T who are elements of the same component form an alliance; in the last class, called weighted Shapley collaboration levels values, all players of a component which is a subset of coalition T collaborate.

The outline of the paper is structured as follows. Section 2 contains some preliminaries, section 3 presents the axioms and section 4 gives a quick look on the Shapley levels value. In each of the following sections, which form the main part of this paper, we introduce a new class of values, in section 5 the weighted Shapley hierarchy levels values, in section 6 the weighted Shapley support levels values, in section 7 the weighted Shapley alliance levels values and in section 8 the weighted Shapley collaboration levels values. Section 9 presents the Shapley collaboration levels value. Section 10 gives the conclusion. An

[1]All of these classes are special cases of values for level structures proposed in Besner (2016). Similar as Casajus (2017) presented a class of solutions φ^{ω}, which generalize the weighted Shapley values and the proportional Shapley value (Besner, 2016; Béal et al., 2017; Gangolly, 1981), Besner introduced classes of solutions which extend the weighted Shapley values and the proportional Shapley value to values for level structures. In this paper we concentrate only on extensions of weighted Shapley values and will present correlated extensions of the proportional Shapley value in a separate article.

appendix (section 11) provides all the proofs and some related lemmas.

2 Preliminaries

We denote by $\mathbb{R}$ the real numbers and by $\mathbb{R}_{++}$ the set of all positive real numbers. Let $\mathfrak{U}$ be a countably infinite set, the universe of all players, and denote by $\mathcal{N}$ the set of all non-empty and finite subsets of $\mathfrak{U}$. A cooperative game with transferable utility (**TU-game**) is a pair (N, v) consisting of a set of players $N \in \mathcal{N}$ and a **coalition function** $v\colon 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, where 2^N is the power set of N. We refer to a TU-game also by v. The subsets $S \subseteq N$ are called **coalitions**, $v(S)$ is the **worth** of coalition S and the set of all nonempty subsets of S is denoted by Ω^S. The set of all TU-games with player set N is denoted by $\mathcal{G}^N$. The **restriction** of (N, v) to the player set $S \in \Omega^N$ is denoted by (S, v).

Let $N \in \mathcal{N}$, $v \in \mathcal{G}^N$ and $S \subseteq N$. The **dividends** $\Delta_v(S)$ (Harsanyi, 1959) are defined inductively by

$$\Delta_v(S) := \begin{cases} v(S) - \sum_{R \subsetneq S} \Delta_v(R), \text{ if } S \in \Omega^N, \text{ and} \\ 0, \text{ if } S = \emptyset. \end{cases} \tag{1}$$

Another well-known formula of the dividends is given for all $S \in \Omega^N$ by

$$\Delta_v(S) = \sum_{R \subseteq S} (-1)^{|S|-|R|} v(R). \tag{2}$$

A game (N, u_T), $T \in \Omega^N$, with $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise is called an **unanimity game**. It is well-known that any $v \in \mathcal{G}^N$ has an unique presentation

$$v = \sum_{T \in \Omega^N} \Delta_v(T) u_T. \tag{3}$$

The **marginal contribution** $MC_i^v(S)$ of player $i \in N$ to $S \subseteq N \backslash \{i\}$ is given by $MC_i^v(S) := v(S \cup \{i\}) - v(S)$. We call a coalition $S \subseteq N$ **active** in v if $\Delta_v(S) \neq 0$. Player $i \in N$ is called a **dummy player** if $v(S \cup \{i\}) = v(S) + v(\{i\})$, $S \subseteq N \backslash \{i\}$; if in addition $v(\{i\}) = 0$, then i is called a **null player**; players $i, j \in N$, $i \neq j$, are called **symmetric** in v if $v(S \cup \{i\}) = v(S \cup \{j\})$, and (mutually) **dependent** (Nowak and Radzik, 1995) in v if $v(S \cup \{i\}) = v(S) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.

A **coalition structure** $\mathcal{B}$ on N is a partition of the player set N, i.e. a collection of nonempty, pairwise disjoint, and mutually exhaustive subsets of N. Each $B \in \mathcal{B}$ is called a **component** and $\mathcal{B}(i)$ denotes the component that contains a player $i \in N$. A **level structure** (Winter, 1989) on N is a finite sequence $\underline{\mathcal{B}} := \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$ of coalition structures $\mathcal{B}^r$, $0 \leq r \leq h+1$, on N such that:

- $\mathcal{B}^0 = \big\{\{i\} \colon i \in N\big\}$.
- $\mathcal{B}^{h+1} = \{N\}$.
- For each r, $0 \leq r \leq h$, $\mathcal{B}^r$ is a refinement of $\mathcal{B}^{r+1}$, i.e. $\mathcal{B}^r(i) \subseteq \mathcal{B}^{r+1}(i)$ for all $i \in N$.

$\mathcal{B}^r$ is called the r-th **level** of $\underline{\mathcal{B}}$; $\overline{\mathcal{B}}$ is the set of all components $B \in \mathcal{B}^r$ of all levels $\mathcal{B}^r \in \underline{\mathcal{B}}$, $0 \leq r \leq h$; $\mathcal{B}^r(B)$ is the component of the r-th level which contains the component $B \in \mathcal{B}^\ell$, $0 \leq \ell \leq r \leq h+1$.

The collection of all level structures with player set N is denoted by $\mathcal{L}^N$. A TU-game $(N,v)\in\mathcal{G}^N$ together with a level structure $\underline{\mathcal{B}}\in\mathcal{L}^N$ is an **LS-game** $(N,v,\underline{\mathcal{B}})$. If N and $\underline{\mathcal{B}}$ are clear, we refer to an LS-game also only by v. The set of all LS-games on N is defined by $\mathcal{GL}^N$. Note that each TU-game (N,v) corresponds to an LS-game $(N,v,\underline{\mathcal{B}_0})$ with a **trivial level structure** $\underline{\mathcal{B}_0}:=\{\mathcal{B}^0,\mathcal{B}^1\}$ and we would like to say that each LS-game $(N,v,\underline{\mathcal{B}_1})$, $\underline{\mathcal{B}_1}:=\{\mathcal{B}^0,\mathcal{B}^1,\mathcal{B}^2\}$, corresponds to a game with coalition structure (Aumann and Drèze, 1974), also known as "games with a priori unions" (Owen, 1977).

Let $N\in\mathcal{N}$, $\underline{\mathcal{B}}=\{\mathcal{B}^0,...,\mathcal{B}^{h+1}\}\in\mathcal{L}^N$, $v\in\mathcal{GL}^N$ and $T\in\Omega^N$:

- From a level structure on N follows a level structure on T by eliminating the players in $N\backslash T$. With coalition structures $\mathcal{B}^r|_T:=\{B\cap T: B\in\mathcal{B}^r,\ B\cap T\neq\emptyset\}$, $0\leq r\leq h+1$, the new level structure on T is given by $\underline{\mathcal{B}}|_T:=\{\mathcal{B}^0|_T,...,\mathcal{B}^{h+1}|_T\}\in\mathcal{L}^T$ and $(T,v,\underline{\mathcal{B}}|_T)\in\mathcal{GL}^T$ is called the **restriction** of $(N,v,\underline{\mathcal{B}})$ to player set T.
- We denote by $\mathcal{B}^r_{\mathcal{I}}|_T$, $0\leq r\leq h+1$, the coalition structure on T, given by
$$\mathcal{B}^r_{\mathcal{I}}|_T:=\begin{cases}\{T\}, & \text{if } r=h+1,\\ \big\{B\in\overline{\mathcal{B}}:\ B\subseteq(B^r\cap T),\ B^r\in\mathcal{B}^r,\ B\not\subseteq B'\in\overline{\mathcal{B}}, B'\subseteq(B^r\cap T)\big\}, & \text{else.}\end{cases}$$
With the level structure $\underline{\mathcal{B}_{\mathcal{I}}}|_T=\{\mathcal{B}^0_{\mathcal{I}}|_T,...,\mathcal{B}^{h+1}_{\mathcal{I}}|_T\}\in\mathcal{L}^T$ the LS-game $(T,v,\underline{\mathcal{B}_{\mathcal{I}}}|_T)\in\mathcal{GL}^T$ is called the **internally induced restriction** of $(N,v,\underline{\mathcal{B}})$ to player set T.
- We define $\underline{\mathcal{B}^r}:=\big\{\mathcal{B}^{r^0},...,\mathcal{B}^{r^{h+1-r}}\big\}\in\mathcal{L}^{\mathcal{B}^r}$, $0\leq r\leq h$, as the induced ***r*-th level structure** from $\underline{\mathcal{B}}$ by considering the components $B\in\mathcal{B}^r$ as players, where $\mathcal{B}^{r^k}:=\{\{B\in\mathcal{B}^r: B\subseteq B'\}: B'\in\mathcal{B}^{r+k}\}$, $0\leq k\leq h+1-r$. If $T=\bigcup_{B\subseteq T}B$, $B\in\mathcal{B}^r$, and we want to stress this property, T is denoted by T^r. Each such T^r is related to a coalition of all players $B\in\mathcal{B}^r$, $B\subseteq T^r$, in the induced r-th level structure, denoted by $\mathcal{T}^r:=\{B\in\mathcal{B}^r: B\subseteq T^r\}$ and vice versa. The induced ***r*-th level game** $\big(\mathcal{B}^r,v^r,\underline{\mathcal{B}^r}\big)\in\mathcal{GL}^{\mathcal{B}^r}$ is given by
$$v^r(\mathcal{T}^r):=v(T^r)\text{ for all }\mathcal{T}^r\in\Omega^{\mathcal{B}^r}.\tag{4}$$

Let $N\in\mathcal{N}$. A **TU-value** ϕ is an operator that assigns to any $v\in\mathcal{G}^N$ a payoff vector $\phi(N,v)\in\mathbb{R}^N$, an **LS-value** φ is an operator that assigns payoff vectors $\varphi(N,v,\ \underline{\mathcal{B}})\in\mathbb{R}^N$ to all LS-games $(N,v,\ \underline{\mathcal{B}})\in\mathcal{GL}^N$.

We define $\mathrm{W}:=\{f:\mathfrak{U}\rightarrow\mathbb{R}_{++}\}$ with $\mathrm{w}_i:=\mathrm{w}(i)$ for all $\mathrm{w}\in\mathrm{W}$ and $i\in\mathfrak{U}$ and $\mathcal{W}:=\{f:\mathcal{N}\rightarrow\mathbb{R}_{++}\}$ with $w_S:=w(S)$ for all $w\in\mathcal{W}$ and $S\in\mathcal{N}$, such that
$$w_{\mathcal{T}}=w_T,\ \mathcal{T},T\in\mathcal{N},\ \text{if }\mathcal{T}\text{ is a partition of }T.\tag{5}$$

Let $N\in\mathcal{N}$, $v\in\mathcal{G}^N$ and $\mathrm{w}\in\mathrm{W}$. The (simply) **weighted Shapley Value**[2] Sh^{w} (Shapley, 1953a) is defined by
$$Sh^{\mathrm{w}}_i(N,v):=\sum_{S\subseteq N,\,S\ni i}\frac{\mathrm{w}_i}{\sum_{j\in S}\mathrm{w}_j}\Delta_v(S)\ \text{ for all }\ i\in N.$$

A special case of a weighted Shapley value, all weights are equal, is the **Shapley value** Sh (Shapley, 1953b), defined by
$$Sh_i(N,v):=\sum_{S\subseteq N,\,S\ni i}\frac{\Delta_v(S)}{|S|}\ \text{ for all }\ i\in N.$$

[2]We desist from possibly null weights as in Shapley (1953a) or Kalai and Samet (1987)

The best-known LS-value is the Shapley levels value[3] (Winter, 1989). We introduce this value here with a formula presented in Calvo, Lasaga and Winter (1996, eq. (1)):

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$ and for all $T \subseteq N$, $T \ni i$,

$$K_T(i) := \prod_{r=0}^{h} K_T^r(i), \quad \text{where} \tag{6}$$
$$K_T^r(i) := \frac{1}{|\{B \in \mathcal{B}^r : B \subseteq \mathcal{B}^{r+1}(i),\, B \cap T \neq \emptyset\}|}.$$

The **Shapley Levels Value** Sh^L is given by

$$Sh_i^L(N, v, \underline{\mathcal{B}}) := \sum_{T \subseteq N, T \ni i} K_T(i) \Delta_v(T) \text{ for all } i \in N.$$

It is easy to see that Sh^L coincides with Sh if $\underline{\mathcal{B}} = \underline{\mathcal{B}}_0$.

All values above are payoff vectors from the **Harsanyi set** (Hammer, 1977; Vasil'ev, 1978), also called **selectope** (Derks, Haller and Peters, 2000), where the payoffs are obtained by distributing the dividends. The payoffs ϕ_i^p in this set, titled **Harsanyi payoffs**, are defined by

$$\phi_i^p(N, v) := \sum_{S \subseteq N, S \ni i} p_i^S \Delta_v(S), i \in N,$$

where the p_i^S are non-negative weights in a weight system $p = (p_i^S)_{S \in \Omega^N, i \in S}$ and sum up to 1 for each coalition S. The collection P^N on N of all such **dividend share systems** p is given by

$$P^N := \left\{ p = (p_i^S)_{S \in \Omega^N, i \in S} \middle| \sum_{i \in S} p_i^S = 1 \text{ and } p_i^S \geq 0 \text{ for each } S \in \Omega^N \text{ and all } i \in S \right\}.$$

3 Axioms

In this section we present axioms used in the main parts and point out:

Convention 3.1. In the case of using a subdomain, we require an axiom to hold when all games belong to this subdomain. If there are used weights for some coalitions, these weights are still valid for the same coalitions in the subdomain.

We refer to the following axioms for LS-values which are adaptions of standard-axioms for TU-values (with the exception of the last axiom):

Efficiency, E. For all $N \in \mathcal{N}$, $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}^N$, we have $\sum_{i \in N} \varphi_i(N, v, \underline{\mathcal{B}}) = v(N)$.

Dummy, D. For all $N \in \mathcal{N}$, $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}^N$ and $i \in N$ a dummy player in v, we have $\varphi_i(N, v, \underline{\mathcal{B}}) = v(\{i\})$.

Null player, N. For all $N \in \mathcal{N}$, $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}^N$ and $i \in N$ a null player in v, we have $\varphi_i(N, v, \underline{\mathcal{B}}) = 0$.

[3]The value is also known as level(s) structure value or Winter's (Shapley type) value. Our designation is used e. g. in Álvarez-Mozos et al. (2017).

Null player out, NO[4]**.** For all $N \in \mathcal{N}$, $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}^N$ and $j \in N$ a null player in v, we have $\varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_i(N \backslash \{j\}, v, \underline{\mathcal{B}}|_{N\backslash\{j\}})$ for all $i \in N \backslash \{j\}$.

Internal (induced restriction) **null player out, INO**[4]**.** For all $N \in \mathcal{N}$, $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}^N$, $j \in N$ a null player in v, we have $\varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_i(N \backslash \{j\}, v, \underline{\mathcal{B}_{\mathcal{I}}}|_{N\backslash\{j\}})$ for all $i \in N \backslash \{j\}$.

Additivity, A. For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} \in \mathcal{L}^N$, $v, v' \in \mathcal{GL}^N$, we have
$\varphi(N, v, \underline{\mathcal{B}}) + \varphi(N, v', \underline{\mathcal{B}}) = \varphi(N, v + v', \underline{\mathcal{B}})$.

Marginality, M. For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} \in \mathcal{L}^N$, $v, v' \in \mathcal{GL}^N$ and $i \in N$ such that $MC_i^v(S) = MC_i^{v'}(S)$ for all $S \subseteq N \backslash \{i\}$, we have $\varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_i(N, v', \underline{\mathcal{B}})$.

Coalitional strategic equivalence, CSE. For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} \in \mathcal{L}^N$, $v, v' \in \mathcal{GL}^N$ such that for any $T \in \Omega^N$, $c \in \mathbb{R}$ and all $S \subseteq N$,

$$v(S) = \begin{cases} v'(S) + c, & \text{if } S \supseteq T, \\ v'(S), & \text{else}, \end{cases} \tag{7}$$

we have $\varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_i(N, v', \underline{\mathcal{B}})$ for all $i \in N \backslash T$.

Balanced group contributions, BGC (Calvo, Lasaga and Winter, 1996)**.** For all $N \in \mathcal{N}, \underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $B_k, B_\ell \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$, we have

$$\sum_{i \in B_k} \varphi_i(N, v, \underline{\mathcal{B}}) - \sum_{i \in B_k} \varphi_i(N \backslash B_\ell, v, \underline{\mathcal{B}}|_{N\backslash B_\ell}) = \sum_{i \in B_\ell} \varphi_i(N, v, \underline{\mathcal{B}}) - \sum_{i \in B_\ell} \varphi_i(N \backslash B_k, v, \underline{\mathcal{B}}|_{N\backslash B_k}).$$

Weighted balanced group contributions, WBGC[5]**.** For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $w \in \mathcal{W}$, $B_k, B_\ell \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$, we have

$$\frac{\sum\limits_{i \in B_k} \varphi_i(N, v, \underline{\mathcal{B}}) - \sum\limits_{i \in B_k} \varphi_i(N \backslash B_\ell, v, \underline{\mathcal{B}}|_{N\backslash B_\ell})}{w_{B_k}} = \frac{\sum\limits_{i \in B_\ell} \varphi_i(N, v, \underline{\mathcal{B}}) - \sum\limits_{i \in B_\ell} \varphi_i(N \backslash B_k, v, \underline{\mathcal{B}}|_{N\backslash B_k})}{w_{B_\ell}}.$$

Symmetry between components, SymBC[6] (Winter, 1989)**.** For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $B_k, B_\ell \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$ and B_k, B_ℓ are symmetric in $(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}) \in \mathcal{GL}^{\mathcal{B}^r}$, we have

$$\sum_{i \in B_k} \varphi_i(N, v, \underline{\mathcal{B}}) = \sum_{i \in B_\ell} \varphi_i(N, v, \underline{\mathcal{B}}).$$

Symmetry within components, SymWC[7]**.** For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $B_k, B_\ell \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$ and all $i \in B_k \cup B_\ell$ are symmetric in v, we have

$$\sum_{i \in B_k} \varphi_i(N, v, \underline{\mathcal{B}}) = \sum_{i \in B_\ell} \varphi_i(N, v, \underline{\mathcal{B}}).$$

[4]These axioms are extensions from null player out in Derks and Haller (1999).
[5]This axiom extends weighted balanced contributions (Myerson, 1980).
[6]This axiom is called coalitional symmetry in Winter (1989).
[7]This axiom extends the symmetry within coalitions axiom (see e.g. Winter (2002)) to players from components of arbitrary levels.

Weighted proportionality between components, WPBC[8]**.** For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $w \in \mathcal{W}$, $B_k, B_\ell \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$, and B_k, B_ℓ are dependent in $(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}) \in \mathcal{GL}^{\mathcal{B}^r}$, we have

$$\sum_{i \in B_k} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_k}} = \sum_{i \in B_\ell} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_\ell}}.$$

Weighted proportionality within components, WPWC[8]**.** For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $w \in \mathcal{W}$, $B_k, B_\ell \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$ and all $i \in B_k \cup B_\ell$ are dependent in v, we have

$$\sum_{i \in B_k} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_k}} = \sum_{i \in B_\ell} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_\ell}}.$$

Level game property, LG (Winter, 1989). For all $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $B \in \mathcal{B}^r$, $0 \leq r \leq h+1$, we have

$$\sum_{i \in B} \varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_B(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}). \tag{8}$$

4 The Shapley levels value

Winter (1989) used the Owen value (Owen, 1977) as a starting point for his LS-value. Therefore Winter upgraded the efficiency, null player, symmetry and additivity axioms to axioms for level structures where symmetry is splitted up in coalitional symmetry and individual symmetry. If a level structure is defined as above, meaning that the singletons are the elements of the lowest level, in Winter (1989, remark 1.6) is pointed out that the individual symmetry can be omitted. In this sense we present Winter's first axiomatization of the Shapley levels value[9].

Theorem 4.1. *(Winter, 1989) Sh^L is the unique LS-Value that satisfies **E**, **N**, **SymBC** and **A**.*

There exist some further axiomatizations of the Shapley levels value (see Calvo, Lasaga and Winter (1996), Khmelnitskaya and Yanovskaya (2007) and Casajus (2010)). Interestingly, if some axiomatizations of the Shapley levels value will be adapted to extensions of the weighted Shapley values to weighted Shapley levels values, we get different LS-values.

5 Weighted Shapley hierarchy levels values

Gómez-Rúa and Vidal-Puga (2011) presented an extension of a weighted Shapley value to level structures where the weights are given by the size of the coalitions. They adapted the procedure from the value ζ in Vidal-Puga (2012) and replaced the two-step approach by a multi-step mechanism. We generalize their value to a value with arbitrary weights.

[8]In Nowak and Radzik (1995) the basic version of these axioms for TU-values is called ω-mutual dependence. We call it **weighted proportionality.**

[9]Winter (1989) introduced his value axiomatically and used this axiomatization as a definition.

Definition 5.1. *Let $N \in \mathcal{N}$, $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}^N$ and $w \in \mathcal{W}$*[10]*. The* ***weighted Shapley hierarchy levels value*** *Sh^{wHL} is defined by algorithm 5.1 below.*

The substantial idea behind the algorithm is as follows: We have a weight system $w \in \mathcal{W}$ for the coalitions. If coalitions S are regarded as players, we use a weight system $\mathrm{w} \in \mathrm{W}$ with $\mathrm{w}_S = w_S$, $w_S \in w$, $\mathrm{w}_S \in \mathrm{w}$. In a first step we distribute the worth of the grand coalition $v(N)$ among the components B^h of the $h-$th level as players using a weighted Shapley value with weight system w. In the second step each payoff to a component B^h from the first step is splitted up by the weighted Shapley value to all components B^{h-1} (the new players) which are subsets of the component B^h and so on for all levels. In the last step we distribute the payoff to components B^1 from the first level among the original players $i \in N$.

Algorithm 5.1. *Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $i \in N$, $w \in \mathcal{W}$, $\mathrm{w} \in \mathrm{W}$ such that $\mathrm{w}_S = w_S$, $w_S \in w$, $\mathrm{w}_S \in \mathrm{w}$, $S \in \Omega^{B^h}$, $B^h \in \mathcal{B}^h$, Sh^{w} a weighted Shapley value and $(\mathcal{R}, \tilde{v}_i^r)$ TU-games, where $\mathcal{R}$ are nonempty, pairwise disjoint sets of some subsets $S \in \Omega^{\mathcal{B}^r(i)}$, $1 \leq r \leq h+1$, with $\tilde{v}_i^r(\mathcal{Q}) := v_i^r\big(\bigcup_{S \in \mathcal{Q}} S\big)$ for all $\mathcal{Q} \in \Omega^{\mathcal{R}}$, $v_i^r \in \mathcal{G}^{\mathcal{B}^r(i)}$. Take $v_i^{h+1} := v$.*

- ***Step k, $1 \leq k \leq h$:*** *Let $r := h - k + 1$. We define the TU-game $\big(\mathcal{B}^r(i), v_i^r\big)$ by*

$$v_i^r(T) := Sh_T^{\mathrm{w}}\Big(\big\{B \in \mathcal{B}^r : B \subseteq \mathcal{B}^{r+1}(i),\ B \neq \mathcal{B}^r(i)\big\} \cup \{T\}, \tilde{v}_i^{r+1}\Big) \text{ for all } T \in \Omega^{\mathcal{B}^r(i)}.$$

  In particular, $v_i^r\big(\mathcal{B}^r(i)\big)$ is the payoff assigned to component $\mathcal{B}^r(i)$.

- ***Step h+1:*** *The payoff Sh^{wHL} assigned to player i is given by*

$$Sh_i^{wHL}(N, v, \mathcal{B}) := Sh_{\{i\}}^{\mathrm{w}}\big(\{B \in \mathcal{B}^0 : B \subseteq \mathcal{B}^1(i)\}, \tilde{v}_i^1\big).$$

If $h = 0$, we only execute step $h+1$.

Remark 5.2. *Sh^{wHL} coincides with Sh^{w} if $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$. It is obvious, by construction, that Sh^{wHL} satisfies **E** and **LG**. Sh^{w} is additive and thus Sh^{wHL} also meets **A**, but, as shown in example 1 in Gómez-Rúa and Vidal-Puga (2011), Sh^{wHL} doesn't satisfy **N**.*

Similar to Calvo, Lasaga and Winter (1996), which characterized the Shapley levels value by efficiency and balanced group contributions, Gómez-Rúa and Vidal-Puga (2011) characterized their value by efficiency and "balanced per capita contributions", a special case of weighted balanced group contributions where the weights are the size of the relevant components. We generalize their axiomatization:

Theorem 5.3. *Sh^{wHL} is the unique TU-Value that satisfies **E** and **WBGC**.*

The proof follows immediately by adapting the proofs of proposition 7 and theorem 8 in Gómez-Rúa and Vidal-Puga (2011), using weighted balanced contributions (Myerson, 1980; Hart and Mas-Colell, 1989) for the weighted Shapley values in proposition 7 and replacing the given weights (size of the components) by arbitrary weights in theorem 8.

Remark 5.4. *If all weights are equal, we use in fact in each step in algorithm 5.1 the Shapley value and **WBGC** equals **BGC**. So, by Calvo, Lasaga and Winter (1996), algorithm 5.1 defines in this case the Shapley levels value Sh^L.*

[10]In fact, we need only weights for coalitions $S \in \Omega^{B^h}$, $B^h \in \mathcal{B}^h$.

6 Weighted Shapley support levels values

In difference to a strong hierarchy where all actions are organized top down we define the following values more bottom up. The players join forces and form components of the first level, the players of the components of the first level federate to components of the second level and so on. The characterizing part in the following value is that each player is "supported" for her share of the related dividends by all components including her.

Definition 6.1. *Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $w \in \mathcal{W}$[11] and for all $T \subseteq N$, $T \ni i$,*

$$K_{w,T}(i) := \prod_{r=0}^{h} K_{w,T}^r(i), \text{ where} \tag{9}$$
$$K_{w,T}^r(i) := \frac{w_{\mathcal{B}^r(i)}}{\sum_{\substack{B \in \mathcal{B}^r:\, B \subseteq \mathcal{B}^{r+1}(i),\\ B \cap T \neq \emptyset}} w_B}.$$

The ***weighted Shapley support levels value*** *Sh^{wSL} is given by*

$$Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) := \sum_{T \subseteq N, T \ni i} K_{w,T}(i) \Delta_v(T) \text{ for all } i \in N. \tag{10}$$

Remark 6.2. *We see that the Shapley levels value is a weighted Shapley support levels value where all components have the same weight. Sh^{wSL} coincides with Sh^{w} if $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$ and, if $\underline{\mathcal{B}} = \underline{\mathcal{B}_1}$, the $K_{w,T}(i)$ coincide with the λ_i^S given in Dragan (1992, sec. 2(e)). Therefore, in this case, the Sh^{wSL} coincide with the McLean weighted coalition structure values (Dragan, 1992; Levy and McLean, 1989; McLean, 1991).*

Theorem 6.3. *The weighted Shapley support levels values Sh^{wSL} satisfy* ***E****,* ***D****,* ***N****,* ***A****,* ***M/CSE****,* ***LG****,* ***WPBC*** *and* ***WPWC****.*

For the proof, see appendix 11.1.

6.1 A characterization similar to Winter

The first axiomatization of the weighted Shapley values in Nowak and Radzik (1995) is based on efficiency, null player, weighted proportionality and additivity[12] for TU-games. So weighted proportionality replaces symmetry in the classical axiomatization of the Shapley value (Shapley, 1953b). Our axiomatization is based on the same axioms extended to LS-games, replacing symmetry between components in theorem 4.1 by weighted proportionality between components. So we have a "weighted" analogue to theorem 4.1.

Theorem 6.4. *Sh^{wSL} is the unique TU-Value that satisfies* ***E****,* ***N****,* ***WPBC*** *and* ***A****.*

For the proof[13], see appendix 11.2.

[11] In fact, we need only weights for the components $B \in \overline{\mathcal{B}}$.

[12] Nowak and Radzik (1995) used linearity instead additivity, but theorem 6.4 shows that additivity can substitute linearity.

[13] Replacing dependent by symmetric and **WPBC** by **SymBC** and using that players $i, j \in N$ are symmetric in v if $\Delta_v(S \cup \{i\}) = \Delta_v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$, we get a new proof of theorem 4.1.

6.2 A characterization similar to Khmelnitskaya and Yanovskaya

Khmelnitskaya and Yanovskaya (2007) characterized the Shapley levels value in the sense of Young (1985) by efficiency, symmetry between components[14] and marginality[15]. In Casajus and Huettner (2008) is shown that coalitional strategic equivalence and marginality are equivalent in TU-games. Their proof obviously holds for LS-games too. We obtain a characterisation which is also an extension of theorem 2.3 in Nowak and Radzik (1995).

Theorem 6.5. *Sh^{wSL} is the unique TU-Value that satisfies* ***E****,* ***WPBC*** *and* ***M/CSE****.*

For the proof, see appendix 11.3.

7 Weighted Shapley alliance levels values

The classes of weighted Shapley levels values above satisfy the level game property. But this is not the case for our two last classes. These classes of values allow the players, within the hierarchy of the level structure, to act more independent. So in the following class they can form subgroups with an own weight within the components containing them.

Look, for example, to a game where the whole world is the grand coalition. The world splits up in political unions like the European Union (EU) and countries which remain fully autonomous. Within the EU many countries are organized as a federal state or a comparable system and so on. But within the EU are also powerful subgroups possible like the euro area. Using a weighted Shapley support levels value the euro area has to get, outside of the EU, the same weight as the whole EU! Instead, the following class of values assigns the euro area exactly the worth it has itself.

Definition 7.1. *Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $w \in \mathcal{W}$[16] and for all $T \subseteq N$, $T \ni i$,*

$$A_{w,T}(i) := \prod_{r=0}^{h} A^r_{w,T}(i), \text{ where} \tag{11}$$

$$A^r_{w,T}(i) := \frac{w_{\mathcal{B}^r(i)\cap T}}{\sum_{\substack{B\in\mathcal{B}^r:\, B\subseteq\mathcal{B}^{r+1}(i),\\ B\cap T\neq\emptyset}} w_{B\cap T}}.$$

The ***weighted Shapley alliance levels value*** *Sh^{wAL} is given by*

$$Sh^{wAL}_i(N, v, \underline{\mathcal{B}}) := \sum_{T\subseteq N, T\ni i} A_{w,T}(i)\Delta_v(T) \text{ for all } i \in N. \tag{12}$$

Theorem 7.2. *The weighted Shapley alliance levels values Sh^{wAL} satisfy* ***E****,* ***D****,* ***N****,* ***NO****,* ***A****,* ***M/CSE*** *and* ***WPWC****.*

[14] Also here individual symmetry can be dropped if the singletons are the elements of the lowest level.

[15] Youngs original axiom is called strong monotonicity. In Chun (1989) the essential part of this axiom for the proof of the uniqueness is named marginality.

[16] In fact, we need only weights for coalitions $S \in \Omega^{B^h}$, $B^h \in \mathcal{B}^h$.

For the proof, see appendix 11.4. We get an axiomatization of the weighted Shapley alliance levels values which corresponds in case of a trivial level structure to an axiomatization of the weighted Shapley values too.

Theorem 7.3. *Sh^{wAL} is the unique LS-Value that satisfies **E**, **NO**, **WPWC** and **A**.*

For the proof, see appendix 11.5. We have an interesting special case if the weights are the size of the components.

Proposition 7.4. *Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$ and $\overline{w} \in \mathcal{W}$ such that $\overline{w}_S = |S|$ for all $S \in \Omega^{B^h}$, $B^h \in \mathcal{B}^h$, $\overline{w}_S \in \overline{w}$. Then we have*

$$Sh_i^{\overline{w}AL}(N, v, \underline{\mathcal{B}}) = Sh_i(N, v) \text{ for all } i \in N.$$

For the proof, see appendix 11.6.

7.1 A new characterization of the Shapley levels value

If all weights are equal, the coefficients $A_{w,T}(i)$ from def. 7.1 equal the $K_T(i)$ from expression (6) in the definition of the Shapley levels value. Thus we obtain, if we replace in the proof of theorem 7.3 dependent by symmetric and **WPWC** by **SymWC** the following corollary.

Corollary 7.5. *Sh^L is the unique TU-Value that satisfies **E**, **NO**, **SymWC** and **A**.*

8 Weighted Shapley collaboration levels values

In the following class of weighted Shapley levels values only subgroups of a component which are a component of a lower level can act more independently. For instance, we look at the regions of Europe, the regions merge to independent countries and most of the countries are members of the EU or autonomous nations like Norway. The Nordic Council is a geo-political forum where Denmark, Finland, Iceland, Norway and Sweden are full members. Denmark, Finland and Sweden are also members of the EU, but within the Nordic Council these countries don't act together with a weight support, given by the EU, as by the weighted Shapley support levels values, nor they form an alliance with an own weight as by the weighted Shapley alliance levels values. Here each of these countries act autonomous as a single component and the following class of values gives a good model for such situations.

Definition 8.1. *Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $w \in \mathcal{W}$[17] and for all $T \subseteq N$, $T \ni i$,*

$$C_{w,T}(i) := \prod_{r=0}^{h} C_{w,T}^r(i), \text{ with} \tag{13}$$
$$C_{w,T}^r(i) := \frac{w_{\mathcal{B}_T^r(i)}}{\sum_{B \in \widehat{\mathcal{B}_T^{r+1}(i)}} w_B},$$

[17]In fact, we need only weights for the components $B \in \overline{\mathcal{B}}$.

where $\mathcal{B}_T^r(i)$ *is the largest component* $\mathcal{B}^\ell(i)$, $0 \leq \ell \leq r$, *with* $\mathcal{B}^\ell(i) \subseteq T$, $\mathcal{B}_T^{h+1}(i) := T$ *and*

$$\widehat{\mathcal{B}_T^{r+1}(i)} := \begin{cases} \{\mathcal{B}_T^r(i)\}, & \text{if } \mathcal{B}_T^r(i) = \mathcal{B}_T^{r+1}(i), \\ \{B \in \overline{\mathcal{B}} : B \subsetneq \mathcal{B}_T^{r+1}(i),\, B \nsubseteq B' \in \overline{\mathcal{B}},\, B' \subsetneq \mathcal{B}_T^{r+1}(i)\}, & \text{else,} \end{cases}$$

is the set of all largest components which are subsets of $\mathcal{B}_T^{r+1}(i)$. *The* ***weighted Shapley collaboration levels value*** Sh^{wCL} *is given by*

$$Sh_i^{wCL}(N, v, \underline{\mathcal{B}}) := \sum_{T \subseteq N, T \ni i} C_{w,T}(i) \Delta_v(T) \text{ for all } i \in N. \tag{14}$$

Theorem 8.2. *The weighted Shapley collaboration levels values* Sh^{wCL} *satisfy* ***E***, ***D***, ***N***, ***INO***, ***A***, ***M/CSE*** *and* ***WPWC***.

For the proof, see appendix 11.7. Also the following axiomatization extends an axiomatization of the weighted Shapley values.

Theorem 8.3. Sh^{wCL} *is the unique TU-Value that satisfies* ***E***, ***INO***, ***WPWC*** *and* ***A***.

For the proof, see appendix 11.8. We obtain an interesting special case if the weights are the size of the components again.

Proposition 8.4. *Let* $N \in \mathcal{N}$, $(N, v, \underline{\mathcal{B}}) \in \mathcal{GL}^N$ *and* $\overline{w} \in \mathcal{W}$ *such that* $\overline{w}_B = |B|$ *for all* $B \in \overline{\mathcal{B}}$, $\overline{w}_B \in \overline{w}$. *Then we have*

$$Sh_i^{\overline{w}CL}(N, v, \underline{\mathcal{B}}) = Sh_i(N, v) \text{ for all } i \in N.$$

For the proof, see appendix 11.9

9 The Shapley collaboration levels value

As a special case of the weighted Shapley collaboration levels values we can present an extension of the Shapley value to level structures.

Definition 9.1. *Let* $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, *and for all* $T \subseteq N$, $T \ni i$,

$$C_T(i) := \prod_{r=0}^{h} C_T^r(i), \text{ with}$$
$$C_T^r(i) := \frac{1}{\left|\widehat{\mathcal{B}_T^{r+1}(i)}\right|},$$

where $\mathcal{B}_T^r(i)$ *is the largest component* $\mathcal{B}^\ell(i)$, $0 \leq \ell \leq r$, $\mathcal{B}^\ell(i) \subseteq T$, $\mathcal{B}_T^{h+1}(i) := T$ *and* $\widehat{\mathcal{B}_T^{r+1}(i)} := \{B \in \overline{\mathcal{B}} : B \subseteq \mathcal{B}_T^{r+1}(i),\, B \nsubseteq B' \in \overline{\mathcal{B}},\, B' \subsetneq \mathcal{B}_T^{r+1}(i)\}$ *is the set of all largest components which are subsets of* $\mathcal{B}_T^{r+1}(i)$. *The* ***Shapley collaboration levels value*** Sh^{CL} *is given by*

$$Sh_i^{CL}(N, v, \underline{\mathcal{B}}) := \sum_{T \subseteq N, T \ni i} C_T(i) \Delta_v(T) \text{ for all } i \in N.$$

Def. 9.1 coincides with def. 8.1 if all weights are equal. If we replace in the proof of theorem 8.3 dependent by symmetric and **WPWC** by **SymWC**, we obtain the following corollary.

Corollary 9.2. *Sh^{CL} is the unique TU-Value that satisfies **E**, **INO**, **SymWC** and **A**.*

10 Conclusion

The rapidly increasing volume of collected data and global networking make it possible and necessary to share benefits between cooperating participants, often structured hierarchical. To distribute generated surpluses the presented four new classes of LS-values contain alternatives to the Shapley levels value, founded on convincing axioms, if there exist exogenous given weights for some coalitions.

The weighted Shapley hierarchy levels values meet with the multi-stage mechanism a widespread idea how a hierarchical value has to operate. But, besides the Shapley levels value, these values don't satisfy the null player property in general.

In numerous situations it seems justified that a null player should receive a zero payoff. This is the case in our classes which are part of the Harsanyi set. The class of the weighted Shapley support levels values appears to be the most popular class of them. These values extend the McLean weighted coalition structure values, satisfy the level game property, can be axiomatized by adapted classical axiomatizations of the Shapley levels value (Winter, 1989; Khmelnitskaya and Yanovskaya, 2007) and contain the Shapley levels value as well, which satisfies a null player out property. In general, however, this property is not fulfilled by these values.

If we use the restriction of an LS-game (the same restriction as normally used for coalition structures), the weighted Shapley alliance levels values satisfy the null player out property and contain the Shapley levels value too.

So this paper also opens different perspectives on the Shapley levels value. But a level structure is more then just a sequence of coalition structures, the coalition structures are ordered. Thus we introduced an internally induced restriction, which should be used for example in the case that a component splits in the components next in size if one player quits the component. The weighted Shapley collaboration levels values satisfy the null player out property for internally induced restrictions. So we have found a situation where the Shapley levels value fails and a new extension of the Shapley value, called Shapley collaboration levels value, has been introduced.

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11 Appendix

Convention 11.1. To avoid cumbersome case distinctions in the proves using **WPBC** (**SymBC**) or **WPWC** (**SymWC**), if there is only one single player assessed in isolation, she is defined as dependent (symmetric) by herself. Then **WPBC** (**SymBC**) and

WPWC (**SymWC**) are trivially satisfied.

Lemma 11.2. *Let* $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $\mathcal{B}^r \in \underline{\mathcal{B}}$, $0 \leq r \leq h$. *Each* $S \in \Omega^N$ *is a subset of exactly one coalition* $T^r \in \Omega^N$, $T^r = \bigcup_{\substack{B^r \subseteq T^r,\, B^r \in \mathcal{B}^r,\\ B^r \cap S \neq \emptyset}} B^r$. *Thus each* $S \in \Omega^N$ *is also uniquely referred to as* S_{T^r}.

Proof. Each coalition $T^r \in \Omega^N$ is a union of components $B \in \mathcal{B}^r$. $\mathcal{B}^r$ is a partition and so each player $i \in S$, $S \in \Omega^N$, is contained in only one component $B \in \mathcal{B}^r$. Thus exists for each coalition $S \in \Omega^N$ exactly one coalition $T^r \in \Omega^N$ which is a union of all components $B \in \mathcal{B}^r$ containing at least one player $i \in S$. □

Lemma 11.3. *Let* $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $\mathcal{B}^r \in \underline{\mathcal{B}}$, $0 \leq r \leq h$, *and* S_{T^r} *the coalitions from lemma 11.2 with related coalitions* T^r. *Then we have in the induced game* $(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r)$ *for each* $\mathcal{T}^r \in \Omega^{\mathcal{B}^r}$, *related to* $T^r \in \Omega^N$,

$$\Delta_{v^r}(\mathcal{T}^r) = \sum_{S_{T^r} \subseteq T^r} \Delta_v(S_{T^r}). \tag{15}$$

Proof. Let $t = |\{B \in \mathcal{B}^r : B \subseteq T^r\}|$ the number of components $B \in \mathcal{B}^r$ which are subsets from a coalition $T^r \in \Omega^N$ with $\mathcal{T}^r \in \Omega^{\mathcal{B}^r}$. We use induction on the size t, $1 \leq t \leq |\mathcal{B}^r|$.

Initialisation: Let $t = 1$. T^r is a component $B \in \mathcal{B}^r$ and $\mathcal{T}^r$ is a player in v^r. We have

$$\Delta_{v^r}(\mathcal{T}^r) \underset{(1)}{=} v^r(\mathcal{T}^r) \underset{(4)}{=} v(T^r) \underset{(1)}{=} \sum_{S \subseteq T^r} \Delta_v(S) \underset{\substack{Lem.\\ 11.2}}{=} \sum_{S_{T^r} \subseteq T^r} \Delta_v(S_{T^r}).$$

Induction step: Assume that eq. (15) holds for an arbitrary $\hat{t} \geq 1$ (IH). Let now $\hat{\mathcal{T}}^r \in \Omega^{\mathcal{B}^r}$ with correlated $\hat{T}^r \in \Omega^N$, $\hat{t} = |\{B \in \mathcal{B}^r : B \subseteq \hat{T}^r\}|$ and $T^r = \hat{T}^r \cup \hat{B}$, $\hat{B} \in \mathcal{B}^r$, $\hat{B} \not\subseteq \hat{T}^r$. We have $t = \hat{t} + 1$ and it follows

$$\begin{aligned}
\Delta_{v^r}(\mathcal{T}^r) &\underset{(1)}{=} \sum_{\mathcal{Q}^r \subseteq \mathcal{T}^r} \Delta_{v^r}(\mathcal{Q}^r) \underset{\substack{(1)\\(4)}}{=} v(T^r) - \sum_{\mathcal{Q}^r \subsetneq \mathcal{T}^r} \Delta_{v^r}(\mathcal{Q}^r) \\
&\underset{\substack{(1)\\(IH)}}{=} \Delta_v(T^r) + \sum_{S \subsetneq T^r} \Delta_v(S) - \sum_{\substack{Q^r \subsetneq T^r,\\ \mathcal{Q}^r \subseteq \mathcal{B}^r}} \sum_{S_{Q^r} \subseteq Q^r} \Delta_v(S_{Q^r}) \\
&\underset{\substack{Lem.\\ 11.2}}{=} \Delta_v(T^r) + \sum_{S \subsetneq T^r} \Delta_v(S) - \sum_{\substack{S \subsetneq T^r,\\ S \neq S_{T^r}}} \Delta_v(S) \\
&= \Delta_v(T^r) + \sum_{S_{T^r} \subsetneq T^r} \Delta_v(S_{T^r}) = \sum_{S_{T^r} \subseteq T^r} \Delta_v(S_{T^r}).
\end{aligned}$$

□

Lemma 11.4. *Players* $i, j \in N$, $i \neq j$, *are dependent in* $v \in \mathcal{G}^N$, *iff* $\Delta_v(S \cup \{k\}) = 0$, $k \in \{i, j\}$, *for all* $S \subseteq N \backslash \{i, j\}$.

Proof. Let $i, j \in N$, $i \neq j$, and $v \in \mathcal{G}^N$. We show by induction on the size $s := |S|$ of all coalitions $S \subseteq N \backslash \{i, j\}$

$$\Delta_v(S \cup \{k\}) = 0 \quad \Leftrightarrow \quad v(S \cup \{k\}) = v(S) + v(\{k\}). \tag{16}$$

Initialisation: If $S = \emptyset$ and so $s = 0$, statement (16) is satisfied.

Induction step: Assume that equality in (16) and such equivalence hold for all coalitions $\tilde{S} \subseteq N \backslash \{i,j\}$, $|\tilde{S}| \leq s'$, $s' \geq 0$, (IH) and let $s = s'+1$ and $k \in \{i,j\}$. We get

$$\begin{aligned} & v(S \cup \{k\}) = \ v(S) + v(\{k\}) \\ \underset{(1)}{\Leftrightarrow} \quad & \Delta_v(S \cup \{k\}) + \sum_{R \subsetneq (S \cup \{k\})} \Delta_v(R) = \sum_{R \subseteq S} \Delta_v(R) + \Delta_v(\{k\}) \\ \underset{(IH)}{\Leftrightarrow} \quad & \Delta_v(S \cup \{k\}) + \Delta_v(\{k\}) + \sum_{R \subseteq S} \Delta_v(R) = \sum_{R \subseteq S} \Delta_v(R) + \Delta_v(\{k\}) \\ \Leftrightarrow \quad & \Delta_v(S \cup \{k\}) = 0. \end{aligned}$$

□

Remark 11.5. *It is well-known or easy to prove that statement (7) in* ***CSE*** *can be replaced equivalently by*

$$\Delta_v(S) = \begin{cases} \Delta_{v'}(T) + c, & \text{if } S = T, \\ \Delta_{v'}(S), & \text{otherwise.} \end{cases}$$

11.1 Proof of theorem 6.3

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v, v' \in \mathcal{GL}^N$, $w \in \mathcal{W}$ and $K_{w,T}$ and $K^r_{w,T}$ the expressions according to def. 6.1.

• **E, D, N, A, M/CSE**: Let $T \in \Omega^N, j \in T$. It is easy to show, by induction on r, that

$$\sum_{i \in \mathcal{B}^{r+1}(j),\, i \in T} \prod_{\ell=0}^{r} K^r_{w,T}(i) = 1.$$

So $\sum_{i \in T} K_{w,T}(i) = 1$ and, with $K_{w,T}(i) > 0$, $i \in T$, the $K_{w,T}(i)$ form a dividend share system $p \in P^N$ and Sh^{wSL} is a Harsanyi payoff. Therefore Sh^{wSL} satisfies all axioms which are satisfied by a Harsanyi payoff, in particular **E, D, N, A** and **M/CSE** are well-known matched axioms.

• **LG**: Let $B^r \in \mathcal{B}^r$, $0 \leq r \leq h+1$. If $r = 0$, eq. (8) trivially is satisfied because the 0-th level game corresponds to the original LS-game, if $r = h+1$, eq. (8) is satisfied by **E**.

Let now $1 \leq r \leq h$. We have for all $S \subseteq N$, $S \cap B^r \neq \emptyset$,

$$\sum_{i \in B^r,\, i \in S} \prod_{\ell=0}^{r-1} K^\ell_{w,S}(i) = 1. \tag{17}$$

In the game $(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r})$ we have for all $\mathcal{T}^r \in \Omega^{\mathcal{B}^r}$, $B^r \in \mathcal{T}^r$,

$$K_{w,\mathcal{T}^r}(B^r) = \prod_{\ell=r}^{h} K^{\ell-r}_{w,\mathcal{T}^r}(B^r). \tag{18}$$

Let $i \in B^r$, $r \leq \ell \leq h$ and S_{T^r} the coalitions from lemma 11.2 with related coalitions T^r. We have $\mathcal{B}^\ell(i) = \mathcal{B}^\ell(B^r)$. Notice that for each $\mathcal{T}^r \in \Omega^{\mathcal{B}^r}$, related to $T^r \in \Omega^N$, if $i \in S_{T^r}$, we have also $B^r \in \mathcal{T}^r$. It follows for all $S_{\mathcal{T}^r} \in \Omega^N$, $i \in S_{\mathcal{T}^r}$,

$$\begin{aligned} K^{\ell}_{w,S_{\mathcal{T}^r}}(i) &= \frac{w_{\mathcal{B}^{\ell}(i)}}{\sum_{\substack{B\in\mathcal{B}^{\ell}:B\subseteq\mathcal{B}^{\ell+1}(i),\\ B\cap S_{\mathcal{T}^r}\neq\emptyset}} w_B} \underset{\substack{Lem.\\ 11.2}}{=} \frac{w_{\mathcal{B}^{\ell}(B^r)}}{\sum_{\substack{B\in\mathcal{B}^{\ell}:B\subseteq\mathcal{B}^{\ell+1}(B^r),\\ B\cap T^r\neq\emptyset}} w_B} \\ &\underset{(5)}{=} \frac{w_{\mathcal{B}^{r\,\ell-r}(B^r)}}{\sum_{\substack{B\in\mathcal{B}^{r\,\ell-r}:B\subseteq\mathcal{B}^{r\,\ell+1-r}(B^r),\\ B\cap\mathcal{T}^r\neq\emptyset}} w_B} = K^{\ell-r}_{w,\mathcal{T}^r}(B^r). \end{aligned} \tag{19}$$

Thus we have for all $S_{\mathcal{T}^r} \in \Omega^N$, $B^r \in \mathcal{T}^r$, $\mathcal{T}^r \in \Omega^{\mathcal{B}^r}$,

$$\begin{aligned} \sum_{\substack{i\in B^r,\\ i\in S_{\mathcal{T}^r}}} K_{w,S_{\mathcal{T}^r}}(i) &\underset{(9)}{=} \sum_{\substack{i\in B^r,\\ i\in S_{\mathcal{T}^r}}} \prod_{\ell=0}^{h} K^{\ell}_{w,S_{\mathcal{T}^r}}(i) \underset{(19)}{=} \sum_{\substack{i\in B^r,\\ i\in S_{\mathcal{T}^r}}} \prod_{\ell=0}^{r-1} K^{\ell}_{w,S_{\mathcal{T}^r}}(i) \prod_{\ell=r}^{h} K^{\ell-r}_{w,\mathcal{T}^r}(B^r) \\ &\underset{(17)}{=} \prod_{\ell=r}^{h} K^{\ell-r}_{w,\mathcal{T}^r}(B^r) \underset{(18)}{=} K_{w,\mathcal{T}^r}(B^r). \end{aligned} \tag{20}$$

Finally we get

$$\begin{aligned} \sum_{i\in B^r} Sh^{wSL}_i(N,v,\underline{\mathcal{B}}) &\underset{(10)}{=} \sum_{i\in B^r} \sum_{\substack{S\subseteq N,\\ S\ni i}} K_{w,S}(i)\Delta_v(S) \underset{\substack{Lem.\\ 11.2}}{=} \sum_{i\in B^r} \sum_{\substack{S_{\mathcal{T}^r}\subseteq N,\\ S_{\mathcal{T}^r}\ni i}} K_{w,S_{\mathcal{T}^r}}(i)\Delta_v(S_{\mathcal{T}^r}) \\ &= \sum_{S_{\mathcal{T}^r}\subseteq N} \sum_{\substack{i\in B^r,\\ i\in S_{\mathcal{T}^r}}} K_{w,S_{\mathcal{T}^r}}(i)\Delta_v(S_{\mathcal{T}^r}) \underset{(20)}{=} \sum_{\substack{S_{\mathcal{T}^r}\subseteq N,\\ \mathcal{T}^r\ni B^r}} K_{w,\mathcal{T}^r}(B^r)\Delta_v(S_{\mathcal{T}^r}) \\ &\underset{\substack{Lem.\\ 11.2}}{=} \sum_{\mathcal{T}^r\subseteq\mathcal{B}^r,\,\mathcal{T}^r\ni B^r} K_{w,\mathcal{T}^r}(B^r) \sum_{S_{\mathcal{T}^r}\subseteq T^r} \Delta_v(S_{\mathcal{T}^r}) \\ &\underset{\substack{Lem.\\ 11.3}}{=} \sum_{\mathcal{T}^r\subseteq\mathcal{B}^r,\,\mathcal{T}^r\ni B^r} K_{w,\mathcal{T}^r}(B^r)\Delta_{v^r}(\mathcal{T}^r) = Sh^{wSL}_{B^r}(\mathcal{B}^r,v^r,\underline{\mathcal{B}}^r). \end{aligned}$$

• **WPBC**: Let $k,\ell \in N$, $0 \le r \le h$, $\mathcal{B}^r(\ell) \subseteq \mathcal{B}^{r+1}(k)$ and $\mathcal{B}^r(k), \mathcal{B}^r(\ell)$ be dependent in $(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r) \in \mathcal{GL}^{\mathcal{B}^r}$. If $r = 0$, we get

$$\begin{aligned} \frac{Sh^{wSL}_k(N,v,\underline{\mathcal{B}})}{w_{\{k\}}} &\underset{(10)}{=} \sum_{T\subseteq N,\,T\ni k} \frac{K_{w,T}(k)}{w_{\{k\}}}\Delta_v(T) \underset{\substack{Lem.\\ 11.4}}{=} \sum_{T\subseteq N,\,\{k,\ell\}\subseteq T} \frac{K_{w,T}(k)}{w_{\{k\}}}\Delta_v(T) \\ &\underset{\substack{Def.\\ 6.1}}{=} \sum_{T\subseteq N,\,\{k,\ell\}\subseteq T} \frac{K_{w,T}(\ell)}{w_{\{\ell\}}}\Delta_v(T) = \frac{Sh^{wSL}_\ell(N,v,\underline{\mathcal{B}})}{w_{\{\ell\}}}. \end{aligned}$$

Thus we have also in the r-th level game, $0 \le r \le h$,

$$\frac{Sh^{wSL}_{\mathcal{B}^r(k)}(\mathcal{B}^r,v^r,\underline{\mathcal{B}}^r)}{w_{\mathcal{B}^r(k)}} = \frac{Sh^{wSL}_{\mathcal{B}^r(\ell)}(\mathcal{B}^r,v^r,\underline{\mathcal{B}}^r)}{w_{\mathcal{B}^r(\ell)}}$$

and the claim follows by **LG**.

• **WPWC**: Let $k,\ell \in N$, $0 \le r \le h$, $\mathcal{B}^r(\ell) \subseteq \mathcal{B}^{r+1}(k)$ and all players $i \in \mathcal{B}^r(k) \cup \mathcal{B}^r(\ell)$ be dependent in v. We get

$$\sum_{i\in\mathcal{B}^r(k)} \frac{Sh^{wSL}_i(N,v,\underline{\mathcal{B}})}{w_{\mathcal{B}^r(k)}} \underset{\substack{Def.\\ 6.1}}{=} \frac{1}{w_{\mathcal{B}^r(k)}} \sum_{i\in\mathcal{B}^r(k)} \sum_{\substack{T\subseteq N,\\ T\ni i}} \left[\prod_{j=0}^{h} \frac{w_{\mathcal{B}^j(i)}}{\sum_{\substack{B\in\mathcal{B}^j:\,B\subseteq\mathcal{B}^{j+1}(i),\\ B\cap T\neq\emptyset}} w_B}\right]\Delta_v(T)$$

$$\underset{\substack{\text{Lem.}\\ 11.4}}{=} \frac{1}{w_{\mathcal{B}^r(k)}} \sum_{i\in\mathcal{B}^r(k)} \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \left[\prod_{j=0}^{h} \frac{w_{\mathcal{B}^j(i)}}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(i),\\ B\cap T\neq\emptyset}} w_B}\right] \Delta_v(T)$$

$$= \frac{1}{w_{\mathcal{B}^r(k)}} \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \sum_{i\in\mathcal{B}^r(k)} \left[\prod_{j=0}^{h} \frac{w_{\mathcal{B}^j(i)}}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(i),\\ B\cap T\neq\emptyset}} w_B}\right] \Delta_v(T)$$

$$= \frac{1}{w_{\mathcal{B}^r(k)}} \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \Delta_v(T) \left[\prod_{j=r}^{h} \frac{w_{\mathcal{B}^j(k)}}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(k),\\ B\cap T\neq\emptyset}} w_B} \cdot \sum_{i\in\mathcal{B}^r(k)} \prod_{j=0}^{r-1} \frac{w_{\mathcal{B}^j(i)}}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(i),\\ B\cap T\neq\emptyset}} w_B}{}^{18}\right]$$

$$= \frac{1}{w_{\mathcal{B}^r(k)}} \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \Delta_v(T) \prod_{j=r}^{h} \frac{w_{\mathcal{B}^j(k)}}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(k),\\ B\cap T\neq\emptyset}} w_B}$$

$$= \frac{1}{w_{\mathcal{B}^r(\ell)}} \sum_{\substack{T\subseteq N,\\ (\mathcal{B}^r(k)\cup\mathcal{B}^r(\ell))\subseteq T}} \Delta_v(T) \prod_{j=r}^{h} \frac{w_{\mathcal{B}^j(\ell)}}{\sum_{\substack{B\in\mathcal{B}^j:\, B\subseteq\mathcal{B}^{j+1}(\ell),\\ B\cap T\neq\emptyset}} w_B} = \sum_{i\in\mathcal{B}^r(\ell)} \frac{Sh_i^{wSL}(N,v,\underline{\mathcal{B}})}{w_{\mathcal{B}^r(\ell)}}.$$

□

11.2 Proof of theorem 6.4

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $w \in \mathcal{W}$, $S \in \Omega^N$ arbitrary and φ an LS-value which satisfies all axioms of theorem 6.4. Due to theorem 6.3, property (3) and **A**, it is sufficient to show that φ is uniquely defined on the game $v_S := \Delta_v(S) \cdot u_S$.

By lemma 11.2 exists for each level r, $0 \leq r \leq h$, exactly one coalition T_S^r, $\mathcal{T}_S^r \subseteq \mathcal{B}^r$, which is the smallest coalition of all R^r, $R^r \supseteq S$, with correlated $\mathcal{R}^r \subseteq \mathcal{B}^r$ and so in each game $\left(\mathcal{B}^r, v_S^r, \underline{\mathcal{B}^r}\right) \in \mathcal{GL}^{\mathcal{B}^r}$ we have $\Delta_{v_S^r}(\mathcal{T}_S^r) = \Delta_v(S)$ and $\Delta_{v_S^r}(\mathcal{R}^r) = 0$ for $\mathcal{R}^r \subseteq \mathcal{B}^r$, $\mathcal{R}^r \neq \mathcal{T}_S^r$. Therefore, by lemma 11.4, possibly using conv. 11.1, all components $B \in \mathcal{B}^r$, $B \cap S \neq \emptyset$, are dependent in v_S^r. If $B \in \mathcal{B}^r$, $B \cap S = \emptyset$, we have $\sum_{i\in B} \varphi_i(N, v_S, \underline{\mathcal{B}}) = 0$ by **N**.

We use induction on the size m, $0 \leq m \leq h$, for all levels r, $0 \leq r \leq h$, with $m := h - r$.

Initialisation: Let $m = 0$ and so $r = h$. We get for an arbitrary $i \in S$

$$\sum_{\substack{B\in\mathcal{B}^h,\\ B\cap S\neq\emptyset}} \sum_{j\in B} \varphi_j(N, v_S, \underline{\mathcal{B}}) \underset{(\mathbf{WPBC})}{=} \sum_{\substack{B\in\mathcal{B}^h,\\ B\cap S\neq\emptyset}} \frac{w_B}{w_{\mathcal{B}^h(i)}} \sum_{j\in\mathcal{B}^h(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) \underset{(\mathbf{E})}{=} \Delta_v(S)$$

$$\Leftrightarrow \sum_{j\in\mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) = \left[\prod_{k=h-m}^{h} \frac{w_{\mathcal{B}^k(i)}}{\sum_{\substack{B\in\mathcal{B}^k:\, B\subseteq\mathcal{B}^{k+1}(i),\\ B\cap S\neq\emptyset}} w_B}\right] \Delta_v(S). \tag{21}$$

Induction step: Assume that eq. (21) holds to φ with an arbitrary $m-1$, $0 \leq m-1 \leq h-1$

[18] If $r = 0$, we have an empty product, which is equal, by convention, to the multiplicative identity 1.

(IH). It follows for an arbitrary $i \in S$

$$\sum_{\substack{B\in\mathcal{B}^r,\, B\cap S\neq\emptyset,\\ B\subseteq\mathcal{B}^{r+1}(i)}} \sum_{j\in B} \varphi_j(N, v_S, \underline{\mathcal{B}}) \underset{(\mathbf{WPBC})}{=} \sum_{\substack{B\in\mathcal{B}^r,\, B\cap S\neq\emptyset,\\ B\subseteq\mathcal{B}^{r+1}(i)}} \frac{w_B}{w_{\mathcal{B}^r(i)}} \sum_{j\in\mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}})$$

$$\underset{(IH)}{=} \left[\prod_{k=h-m+1}^{h} \frac{w_{\mathcal{B}^k(i)}}{\sum_{\substack{B\in\mathcal{B}^k:\, B\subseteq\mathcal{B}^{k+1}(i),\\ B\cap S\neq\emptyset}} w_B}\right] \Delta_v(S)$$

$$\Leftrightarrow \quad \sum_{j\in\mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) = \left[\prod_{k=h-m}^{h} \frac{w_{\mathcal{B}^k(i)}}{\sum_{\substack{B\in\mathcal{B}^k:\, B\subseteq\mathcal{B}^{k+1}(i),\\ B\cap S\neq\emptyset}} w_B}\right] \Delta_v(S).$$

So φ is uniquely defined on v_S (take $m = h$ and so $r = 0$). □

11.3 Proof of theorem 6.5

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $w \in \mathcal{W}$ and φ an LS-value which satisfies all axioms of theorem 6.5. By theorem 6.3 we have only to show that φ satisfies eq. (10).

We use a first induction I_1 on $t := |\{T \subseteq N : T \text{ is active in } v\}|$.

Initialisation I_1: Let $t = 0$, then for all games $(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}) \in \mathcal{GL}^{\mathcal{B}^r}$, $0 \leq r \leq h$, v^r is identically zero on all coalitions. So all players, possibly using conv. 11.1, are dependent in each game v^r and for all $B^r_k, B^r_\ell \in \mathcal{B}^r$, $B^r_\ell \subseteq \mathcal{B}^{r+1}(B^r_k)$ we have

$$\sum_{i\in B^r_k} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B^r_k}} \underset{(\mathbf{WPBC})}{=} \sum_{i\in B^r_\ell} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B^r_\ell}}.$$

We use a second induction I_2 on the size $m := h - r$ to show

$$\sum_{i\in B^r} \varphi_i(N, v, \underline{\mathcal{B}}) = 0 \text{ for all } 0 \leq r \leq h \text{ and } B^r \in \mathcal{B}^r. \tag{22}$$

Initialisation I_2: Let $m = 0$ and so $r = h$. We get for an arbitrary $B^h_k \in \mathcal{B}^h$

$$\sum_{B^h\in\mathcal{B}^h} \sum_{i\in B^h} \varphi_i(N, v, \underline{\mathcal{B}}) \underset{(\mathbf{WPBC})}{=} \sum_{B^h\in\mathcal{B}^h} \frac{w_{B^h}}{w_{B^h_k}} \sum_{i\in B^h_k} \varphi_i(N, v, \underline{\mathcal{B}}) \underset{(\mathbf{E})}{=} 0.$$

Thus follows $\sum_{i\in B^h} \varphi_i(N, v, \underline{\mathcal{B}}) = 0$ for all $B^h \in \mathcal{B}^h$ because $w_{B^h} > 0$ and B^h_k was arbitrary.

Induction step I_2: Assume that eq. (22) holds to φ if $m \geq 0$ (IH_2). We get for an arbitrary $B^r_k \in \mathcal{B}^r$

$$\sum_{\substack{B^r\in\mathcal{B}^r,\\ B^r\subseteq\mathcal{B}^{r+1}(B^r_k)}} \sum_{i\in B^r} \varphi_i(N, v, \underline{\mathcal{B}}) \underset{(\mathbf{WPBC})}{=} \sum_{\substack{B^r\in\mathcal{B}^r,\\ B^r\subseteq\mathcal{B}^{r+1}(B^r_k)}} \frac{w_{B^r}}{w_{B^r_k}} \sum_{i\in B^r_k} \varphi_i(N, v, \underline{\mathcal{B}}) \underset{(IH_2)}{=} 0.$$

It follows $\sum_{i\in B^r} \varphi_i(N, v, \underline{\mathcal{B}}) = 0$ for all $0 \leq r \leq h$ and $B^r \in \mathcal{B}^r$. Therefore we have also $\varphi_i(N, v, \underline{\mathcal{B}}) = 0$ for all $i \in N$ and eq. (10) is satisfied for φ if $t = 0$.

Induction step I_1: Assume that eq. (10) holds to φ if $t \geq 0$, (IH_1). Let exactly $t + 1$ coalitions $Q_k \subseteq N$, $1 \leq k \leq t + 1$, active in v and denote

$$Q = \bigcap_{1\leq k\leq t+1} Q_k.$$

We distinguish two cases: (a) $i \in N \backslash Q$ and (b) $i \in Q$.

(a) Each player $i \in N \backslash Q$ is a member of at most t active coalitions Q_k and v gets at least one active coalition T_i, $i \notin T_i$. Hence exists a coalition function $v_i \in \mathcal{GL}^N$, where all coalitions have the same dividend in v_i as in v, except the coalition T_i, which gets the dividend $\Delta_{v_i}(T_i) = 0$, and there is existing a scalar $c \in \mathbb{R}$, $c \neq 0$, with

$$\Delta_v(S) = \begin{cases} \Delta_{v_i}(T_i) + c, & \text{if } S = T_i, \\ \Delta_{v_i}(S), & \text{else.} \end{cases}$$

By remark 11.5 and **CSE** we get $\varphi_i(v) = \varphi_i(v_i)$ with $i \in N \backslash T_i$ and, because there exists for all $i \in N \backslash Q$ a such T_i, it follows $\varphi_i(v) = \varphi_i(v_i)$ for all $i \in N \backslash Q$. All coalition functions v_i get at most t active coalitions and by (IH_1) we have

$$\varphi_i(v) = Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) \text{ for all } i \in N \backslash Q. \tag{23}$$

(b) Each player $j \in Q$ is a member of all $t+1$ active coalitions $Q_k \subseteq N$, $1 \leq k \leq t+1$, and therefore, by lemma 11.4 and conv. 11.1, all players $j \in Q$ are dependent in v. Now we define for each r, $0 \leq r \leq h$, a set

$$\mathcal{B}_Q^r = \{B^r \in \mathcal{B}^r : B^r \cap Q \neq \emptyset\}.$$

Note that all components $B_k^r, B_\ell^r \in \mathcal{B}_Q^r$, $B_\ell^r \subseteq \mathcal{B}^{r+1}(B_k^r)$, are dependent in v^r. We use a third induction I_3 on the size $s := h - r$ to show for all $B_k^r \in \mathcal{B}_Q^r$

$$\sum_{i \in B_k^r} \varphi_i(N, v, \underline{\mathcal{B}}) = \sum_{i \in B_k^r} Sh_i^{wSL}(N, v, \underline{\mathcal{B}}). \tag{24}$$

Initialisation I_3: Let $s = 0$ and so $r = h$. We get for an arbitrary $B_k^h \in \mathcal{B}_Q^h$

$$\begin{aligned} & \sum_{B^h \in \mathcal{B}_Q^h} \sum_{i \in B^h} \varphi_i(N, v, \underline{\mathcal{B}}) && \underset{(\mathbf{WPBC})}{=} \sum_{B^h \in \mathcal{B}_Q^h} \frac{w_{B^h}}{w_{B_k^h}} \sum_{i \in B_k^h} \varphi_i(N, v, \underline{\mathcal{B}}) \\ \underset{\substack{(\mathbf{E}) \\ (23)}}{=} & \sum_{B^h \in \mathcal{B}_Q^h} \sum_{i \in B^h} Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) && \underset{(\mathbf{WPBC})}{=} \sum_{B^h \in \mathcal{B}_Q^h} \frac{w_{B^h}}{w_{B_k^h}} \sum_{i \in B_k^h} Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) \\ \Leftrightarrow & \sum_{i \in B_k^h} \varphi_i(N, v, \underline{\mathcal{B}}) && = \sum_{i \in B_k^h} Sh_i^{wSL}(N, v, \underline{\mathcal{B}}). \end{aligned}$$

Induction step I_3: Assume that eq. (24) holds to φ if $s \geq 0$ (IH_3). We get for an arbitrary $B_k^r \in \mathcal{B}_Q^r$ and because $\mathcal{B}^{r+1}(B_k^r) \in \mathcal{B}_Q^{r+1}$

$$\begin{aligned} & \sum_{\substack{B^r \in \mathcal{B}_Q^r, \\ B^r \subseteq \mathcal{B}^{r+1}(B_k^r)}} \sum_{i \in B^r} \varphi_i(N, v, \underline{\mathcal{B}}) && \underset{(\mathbf{WPBC})}{=} \sum_{\substack{B^r \in \mathcal{B}_Q^r, \\ B^r \subseteq \mathcal{B}^{r+1}(B_k^r)}} \frac{w_{B^r}}{w_{B_k^r}} \sum_{i \in B_k^r} \varphi_i(N, v, \underline{\mathcal{B}}) \\ \underset{\substack{(IH_3) \\ (23)}}{=} & \sum_{\substack{B^r \in \mathcal{B}_Q^r, \\ B^r \subseteq \mathcal{B}^{r+1}(B_k^r)}} \sum_{i \in B^r} Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) && \underset{(\mathbf{WPBC})}{=} \sum_{\substack{B^r \in \mathcal{B}_Q^r, \\ B^r \subseteq \mathcal{B}^{r+1}(B_k^r)}} \frac{w_{B^r}}{w_{B_k^r}} \sum_{i \in B_k^r} Sh^{wSL}(N, v, \underline{\mathcal{B}}) \\ \Leftrightarrow & \sum_{i \in B_k^r} \varphi_i(N, v, \underline{\mathcal{B}}) && = \sum_{i \in B_k^r} Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) \end{aligned}$$

and so finally
$$\varphi_i(N, v, \underline{\mathcal{B}}) = Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) \text{ for all } i \in Q.$$
By Casajus and Huettner (2008) **M** and **CSE** are equivalent and the proof is complete. □

11.4 Proof of theorem 7.2

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v, v' \in \mathcal{GL}^N$ and $w \in \mathcal{W}$.

• **E, D, N, A, M/CSE**: Analogous to the proof of theorem 6.3, the $A_{w,T}(i)$ from def. 7.1 form a dividend share system $p \in P^N$ for all $T \subseteq N$, $T \ni i$, and so Sh^{wAL} is a Harsanyi payoff. Therefore Sh^{wAL} satisfies all axioms which are satisfied by a Harsanyi payoff, in particular **E, D, N, A** and **M/CSE**.

• **NO**: It is well-known that each coalition S containing a null player $j \in N$ in v is not active in v. In eq. (12) we have only to consider active coalitions. But for these coalitions is no change in the weights. We get $Sh_i^{wAL}(N, v, \underline{\mathcal{B}}) = Sh_i^{wAL}(N \backslash \{j\}, v, \underline{\mathcal{B}}|_{N \backslash \{j\}})$ for all $i \in N \backslash \{j\}$.

• **WPWC**: The proof is omitted because it is completely analogues to the proof of **WPWC** in theorem 6.3. We have only to replace the relevant weights w_B by $w_{B \cap T}$. □

11.5 Proof of theorem 7.3

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $w \in \mathcal{W}$, $S \in \Omega^N$ arbitrary and φ an LS-value which satisfies all axioms of theorem 7.3 and **N**, because **E** and **NO** imply obvious **N**. Due to theorem 7.2, property (3) and **A**, it is sufficient to show that φ is uniquely defined on the game $v_S := \Delta_v(S) \cdot u_S$.

All players $i \in N \backslash S$ are null players in v_S and we have $\varphi_i(N, v_S, \underline{\mathcal{B}}) = 0$ for all $i \in N \backslash S$ by **N**. All players $i \in S$, possibly using conv. 11.1, are dependent in v_S and, by **NO**, we get
$$\varphi_i(N, v_S, \underline{\mathcal{B}}) = \varphi_i(S, v_S, \underline{\mathcal{B}}|_S) \text{ for all } i \in S.$$
So we can use an analogue induction as in the proof of theorem 6.4, here on the restriction to player set S and where **WPBC** must be replaced by **WPWC**, and get that φ is uniquely defined on v_S. □

11.6 Proof of proposition 7.4

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^h\} \in \mathcal{L}^N$ and $\overline{w} \in \mathcal{W}$ such that $\overline{w}_S = |S|$ for all $S \in \Omega^{B^h}$, $B^h \in \mathcal{B}^h$, $\overline{w}_S \in \overline{w}$. We have only to show that
$$A_{\overline{w},T}(i) = \frac{1}{|T|} \text{ for all } T \subseteq N,\, T \ni i.$$

For all $T \subseteq N, T \ni i$, and $0 \leq r \leq h$ the set $\widetilde{\mathcal{B}_T^{r+1}(i)} := \{B \cap T : B \in \mathcal{B}^r, B \subseteq \mathcal{B}^{r+1}(i), B \cap T \neq \emptyset\}$ is a partition of $\mathcal{B}^{r+1}(i) \cap T$. So we have

$$\sum_{B \in \widetilde{\mathcal{B}_T^{r+1}(i)}} \overline{w}_{B \cap T} = \sum_{\substack{B \in \mathcal{B}^r\colon B \subseteq \mathcal{B}^{r+1}(i), \\ B \cap T \neq \emptyset}} \overline{w}_{B \cap T} = |\mathcal{B}^{r+1}(i) \cap T|.$$

By eq. (11) we get $A_{\overline{w},T}(i) = \dfrac{1}{|T|}$ as desired. □

11.7 Proof of theorem 8.2

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v, v' \in \mathcal{GL}^N$ and $w \in \mathcal{W}$.

• **E, D, N, A, M/CSE**: It is obvious, analogous to the proof of theorem 6.3, that the $C_{w,T}(i)$ from def. 8.1 form a dividend share system $p \in P^N$ for all $T \subseteq N$, $T \ni i$, and so Sh^{wCL} is a Harsanyi payoff. Therefore Sh^{wCL} satisfies all axioms which are satisfied by a Harsanyi payoff, in particular **E, D, N, A** and **M/CSE**.

• **INO**: Similar to the proof of **NO** in theorem 7.2 we have no change in the weights for eq. (14) and thus get $Sh_i^{wCL}(N, v, \underline{\mathcal{B}}) = Sh_i^{wCL}(N \backslash \{j\}, v, \underline{\mathcal{B}_{\mathcal{I}}}|_{N \backslash \{j\}})$ for all $i \in N \backslash \{j\}$, $j \in N$ a null player in v.

• **WPWC**: The proof is omitted because it is completely analogues to the proof of **WPWC** in theorem 6.3. □

11.8 Proof of theorem 8.3

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathcal{L}^N$, $v \in \mathcal{GL}^N$, $w \in \mathcal{W}$, $S \in \Omega^N$ arbitrary and φ an LS-value which satisfies all axioms of theorem 8.3 and **N**, because **E** and **INO** imply **N** what is easy to see. Due to theorem 8.2, property (3) and **A**, it is sufficient to show that φ is uniquely defined on the game $v_S := \Delta_v(S) \cdot u_S$.

All players $j \in N \backslash S$ are null players and we have $\varphi_j(N, v_S, \underline{\mathcal{B}}) = 0$ for all $j \in N \backslash S$ by **N**. All players $i \in S$, possibly using conv. 11.1, are dependent in v_S and, by **INO**, we get

$$\varphi_i(N, v_S, \underline{\mathcal{B}}) = \varphi_i(S, v_S, \underline{\mathcal{B}_{\mathcal{I}}}|_S) \text{ for all } i \in S.$$

So we can use an analogue induction as in the proof of theorem 6.4, here on the internally induced restriction to player set S and where **WPBC** must be replaced by **WPWC**, and obtain that φ is uniquely defined on v_S. □

11.9 Proof of proposition 8.4

Let $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, \mathcal{B}^1, ..., \mathcal{B}^h\} \in \mathcal{L}^N$ and $\overline{w} \in \mathcal{W}$ such that $\overline{w}_B = |B|$ for all $B \in \overline{\mathcal{B}}$, $\overline{w}_B \in \overline{w}$. We have only to show that

$$C_{\overline{w},T}(i) = \frac{1}{|T|} \text{ for all } T \subseteq N,\, T \ni i.$$

For all $T \subseteq N$, $T \ni i$, and $0 \leq r \leq h$ the set $\widehat{\mathcal{B}_T^{r+1}(i)}$ is a partition of $\mathcal{B}_T^{r+1}(i)$. So we have $\sum_{B \in \widehat{\mathcal{B}_T^{r+1}(i)}} \overline{w}_B = |\mathcal{B}_T^{r+1}(i)|$. By eq. (13) we get $C_{\overline{w},T}(i) = \dfrac{1}{|T|}$ as desired. □

11.10 Logical independence

All axiomatizations must also hold if $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$. In this case all axioms, used for axiomatization in this paper, coincide with usual axioms for TU-values. So the given axiomatizations coincide in this case with axiomatizations of the weighted Shapley values and the Shapley value, respectively. It is well-known or easy to proof that in this case the used axioms are logical independent. Therefore all axioms for LS-values must be also logical independent in the given axiomatizations.

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