Learning about Monetary Policy Rules when Long-Horizon Expectations Matter

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This paper considers the implications of an important source of model misspecification for the design of monetary policy rules: the assumed manner of expectations formation. In the model considered here, private agents seek to maximize their objectives subject to standard constraints and the restriction of using an econometric model to make inferences about future uncertainty. Because agents solve a multiperiod decision problem, their actions depend on forecasts of macroeconomic conditions many periods into the future, unlike the analysis of Bullard and Mitra (2002) and Evans and Honkapohja (2002). A Taylor rule ensures convergence to the rational expectations equilibrium associated with this policy if the so-called Taylor principle is satisfied. This suggests the Taylor rule to be desirable from the point of view of eliminating instability due to self-fulfilling expectations.

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Over the past decade, monetary policy theory and central banking practice have underscored various desiderata for judicious policy. It is often argued that social welfare can be improved by arranging for the central bank to conduct monetary policy according to a suitably chosen instrument rule, dictating how interest rates should be adjusted in response to particular disturbances to the economy. The discussions of Clarida, Galí, and Gertler (1999) and Woodford (2003) present a coherent theory of monetary policy and make the case for such rules. In practice, however, monetary policy contends with many difficulties. Among these, the absence of a correctly specified model of the economy with which to formulate policy is paramount.

This paper considers a potentially important source of model misspecification in the design of instrument rules: the assumed manner in which expectations are formed. The motivation is two-fold. First, even if rational expectations provide a reasonably accurate description of economic agents' behavior, a prudent policy should be robust to small deviations from rationality. Given two policies that both implement a particular desired equilibrium, the policy that results in this equilibrium under more general assumptions on expectations formation is presumably preferred.

Second, some have argued that policies that appear to be desirable, because they are consistent with a desirable equilibrium, will almost surely have disastrous consequences in practice, by allowing self-fulfilling expectations to propagate. For example, Friedman (1968) argued that a monetary policy aimed at pegging the nominal interest rate would inevitably lead to economic instability via a Wicksellian cumulative process. Moreover, he argued that due to small implementation errors, this would occur even if the nominal interest rate target was optimally chosen. The argument proceeds as follows: Suppose the monetary authority pegs the nominal interest rate below the natural rate of interest. This policy would give rise to expectations of future inflation, with the resulting lower real rate of interest tending to stimulate output and prices. Such price rises would engender expectations of further price inflation, in turn further lowering the expected real rate and so on—generating self-fulfilling expectations of ever higher inflation. This paper seeks to build on the research of Howitt (1992) by providing a formal analysis of such self-fulfilling expectations in the context of a model with optimizing behavior. Unlike the analysis of Howitt, this paper postulates a framework where agents optimally make forecasts of macroeconomic conditions many periods into the future when making current decisions.

The rational expectations paradigm comprises two stipulations: (1) agents optimize given their beliefs about the joint probability distribution for various state variables that are independent of their actions and that matter for their payoffs and (2) the probabilities that they assign coincide with the predictions of the model. Following a considerable literature on learning (see Sargent [1993] and Evans and Honkapohja [2001] for reviews), this paper retains the first stipulation while replacing the second with the assumption that the joint probabilities are formed using an econometric model. The predictions of this econometric model need not coincide with the predictions of the theoretical model. The central question posed by the analysis is whether, given sufficient data, the predictions of the econometric model eventually converge to those of the economic model.

Having departed from the rational expectations paradigm, some care must be taken in specifying an individual agent's knowledge. Agents are assumed to know what they need to know to behave according to the first stipulation above: they know their own preferences and constraints, and, more generally, they correctly understand the mapping from their actions to their expected payoff, given a probability distribution for the variables that are outside their control. However, they are not assumed to know anything of the true economic model of how those variables outside of their control are determined. For instance, they do not know that other agents have preferences just like their own and that agents form expectations the way that they do, even if these things are true within the model. It therefore is not appropriate to assume agents use knowledge that other agents' consumption decisions satisfy a subjective Euler equation (for example) in deciding what to do themselves. This has the crucial implication that agents have to make long-horizon forecasts in the framework proposed by this paper.

Recent work by Bullard and Mitra (2002) and Evans and Honkapohja (2003) is similarly motivated. These authors, however, assume a log-linear model of the monetary transmission mechanism in which agents need only forecast inflation and aggregate income one period in advance. In contrast, this paper assumes that agents face a multiperiod decision problem, as in the microfoundations used in recent analysis of the implications of monetary policy rules under rational expectations (see Bernanke and Woodford 1997; Clarida, Galí, and Gertler 1999; and Woodford 1999). This paper demonstrates that the aggregation of rationally modeled decisions, when these decisions are based on subjective expectations, does not predict the aggregate dynamics that depend only on forecasts a single period in the future, even though the aggregate dynamics under rational expectations can be described in that way. In fact, in making current decisions about spending and pricing of their output, agents must make forecasts of macroeconomic conditions many periods into the future. This prediction is a direct result of agents not being able to base their decisions on knowledge of the actions of other agents in the economy. The central methodological contribution of this paper is demonstrating that long-horizon forecasts matter in the determination of current economic conditions in a simple model of output gap and inflation determination with subjective expectations. As such, it builds on the work of Marcet and Sargent (1989), which shows that the optimal decision rule in a partial equilibrium model of investment determination necessarily depends on long-horizon forecasts.

In the model proposed here, learning occurs in the following manner. Agents conjecture the form of the equilibrium dynamics of state variables and estimate an econometric model of this form. This econometric model describes the agents' perceived law of motion. The estimated model is then used to evaluate forecasts of the future paths of state variables that are exogenous to private agents' decision problems. These forecasts, in conjunction with agents' optimal decision rules, can be solved to provide a solution for the actual path of aggregate variables as a function of the current state. This is the actual law of motion. Each period, this process is repeated as additional data become available. A principal focus of the analysis is the manner in which agents update their decision rules, and whether additional data lead them to adopt perceived laws of motion that are closer to the actual laws of motion of the economy. In particular, do agents learn the rational expectations dynamics over time?

The criterion by which this paper judges convergence of learning dynamics to rational expectations dynamics is the notion of expectational stability, or E-stability, proposed by Evans and Honkapohja (2001). Given the requirements of E-stability and the aggregate economic dynamics implied by the model's microfoundations, the analysis considers the implications of learning for several standard prescriptions for monetary policy—specifically, whether certain policy rules are able to ensure least-squares convergence to the associated rational expectations dynamics.

In this paper, monetary policy is specified as a commitment to one of two classes of state-contingent instrument rules: (1) nominal interest-rate rules that depend only on the history of exogenous disturbances and (2) Taylor rules that specify a path for the nominal interest rate that depends on the model's endogenous variables. The former class of rule is of considerable interest, as it has been argued to be a natural way to implement optimal monetary policy, by specifying the optimal action in each possible state of the world. However, such rules, which include nominal interest-rate pegs as a special case, are subject to the critique of Friedman (1968) and also Sargent and Wallace (1975), who showed that commitment to exogenously determined interest-rate paths can lead to multiple rational expectations equilibria. The latter feedback rules, introduced by Taylor (1993), have been used in monetary policy both as a prescriptive and descriptive tool. As initially demonstrated by McCallum (1983), interest-rate rules that possess sufficient feedback from endogenous variables can often deliver a determinate equilibrium. In the present model under rational expectations, Woodford (2003, chap. 4) shows that a Taylor rule leads to a determinate equilibrium if the so-called Taylor principle is satisfied.

Two main results emerge from the analysis of learning dynamics. First, interest-rate rules that are specified as depending only on the history of exogenous disturbances are not expectationally stable under learning dynamics. Such rules are therefore subject to self-fulfilling expectations, consistent with the concerns of Friedman (1968). This, combined with the indeterminacy of rational expectations equilibrium of this class of policy rule, suggests such rules to be ineffective in eliminating economic instability due to self-fulfilling expectations, and therefore undesirable as a means to implement optimal monetary policy. Second, for the Taylor rule, expectational stability hinges critically on satisfaction of the so-called Taylor principle (which stipulates that feedback from endogenous variables to nominal interest rates be sufficiently strong to ensure that increases in inflation be associated with increases in the real interest rate). These findings are invariant to the nature of learning dynamics considered and suggest the Taylor principle to be a remarkably robust feature of the policy environment in the context of this model.

The analysis of this paper is most closely related to the work of Bullard and Mitra (2002) and Evans and Honkapohja (2003), who analyze a log-linear model of the monetary transmission mechanism, where agents forecast inflation and aggregate spending one period in advance.[1] The latter show that instrument rules that require the nominal interest rate to respond only to the history of exogenous disturbances are not stable under learning dynamics. Moreover, they show how to implement the optimal rational expectations equilibrium when the monetary authority is constrained to be a discretionary optimizer and that this equilibrium is also E-stable. Bullard and Mitra (2002) show in the same model that for a monetary authority that is assumed to be able to commit to a number of Taylor-type interest-rate rules, the associated rational expectations equilibrium is E-stable under learning dynamics so long as the Taylor principle is satisfied. That these findings concur with the results of this paper is not necessarily to be expected. The presence of long-horizon forecasts in the present paper gives rise to dynamics that are distinct from those predicted by these analyses.

The paper proceeds as follows. Section 1 sketches the microfoundations of a simple dynamic stochastic general equilibrium model under a general assumption on expectations, provides commentary on the irreducibility of long-horizon forecasts, and highlights some attractive features of the framework proposed to model learning dynamics. Section 2 develops the expectations formation mechanism adopted in this paper. Section 3 discusses the notion of expectational stability and provides a simple abstract example of learning analysis. Section 4 considers the robustness of some common prescriptions for monetary policy to the presence of learning dynamics. The final section concludes.

1. The Framework

To develop a framework suitable for the analysis of monetary policy under alternative assumptions on expectations formation, we

[1] See also Bullard and Mitra (2000), Evans and Honkapohja (2002), Honkapohja and Mitra (2004), and Honkapohja and Mitra (2005) for further analyses of issues in monetary policy under learning dynamics in the same framework.

make use of a simple dynamic stochastic general equilibrium model with microfoundations found in Clarida, Galí, and Gertler (1999) and Woodford (2003). To simplify the exposition, the analysis abstracts from monetary frictions that would allow money to be held despite being dominated in rate of return, as in the "cashless" baseline model developed in Woodford (2003, chap. 2). The model is developed in several steps. The household's intertemporal allocation problem is considered, followed by the firm's optimal pricing problem. The implications of the assumed expectations formation mechanism for monetary policy are then explored.

1.1 Household's Intertemporal Problem

The economy is populated by a continuum of households that seek to maximize future expected discounted utility

$$\hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[U(C_T^i; \xi_T) - \int_0^1 v(h_T^i(j); \xi_T) dj \right], \tag{1}$$

where utility depends on a consumption index, C_t^i, of the economy's available goods (to be specified); a vector of aggregate preference shocks, ξ_t; and the amount of labor supplied for the production of each good j, $h^i(j)$. The second term in the brackets therefore captures the total disutility of labor supply. The consumption index, C_t^i, is the Dixit-Stiglitz constant-elasticity-of-substitution aggregator of the economy's available goods and has an associated price index written, respectively, as

$$C_t^i \equiv \left[\int_0^1 c_t^i(j)^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \quad \text{and} \quad P_t \equiv \left[\int_0^1 p_t(j)^{1-\theta} dj \right]^{\frac{1}{1-\theta}},$$

where $\theta > 1$ is the elasticity of substitution between any two goods, and $c_t^i(j)$ and $p_t(j)$ denote household i's consumption and the price of good j. The absence of real money balances from the period utility function (1) reflects the assumption that there are no transaction frictions that can be mitigated by holding money balances. However, agents may nonetheless choose to hold money if it provides comparable returns to other available financial assets.

$\hat{E}_t^i$ denotes the subjective beliefs of household i about the probability distribution of the model's state variables—that is, variables that are beyond agents' control, though relevant to their decision problems. The presence of a hat, "ˆ", denotes nonrational expectations, and the special case of rational expectations will be denoted by the usual notation, E_t. Beliefs are assumed to be homogenous across households for the purposes of this paper (though this is not understood to be the case by agents) and to satisfy standard probability laws so that $\hat{E}_t^i \hat{E}_{t+1}^i = \hat{E}_t^i$. In forming beliefs about future events, agents do not take into account that they will update their own beliefs in subsequent periods, and this is the source of nonrational behavior in this model. However, when households solve their decision problem at time t, beliefs held at that time satisfy standard probability laws, so that standard solution methods apply. The specific details of beliefs and the manner in which agents update beliefs are developed in section 2. The discount factor is assumed to satisfy $0 < \beta < 1$. The function $U(C_t; \xi_t)$ is concave in C_t for a given value of ξ_t, and $v(h_t(i); \xi_t)$ is convex in $h_t(i)$ for a given value of ξ_t.

Asset markets are assumed to be *incomplete*: there is a single one-period riskless nonmonetary asset available to transfer wealth intertemporally. Under this assumption, the household's flow budget constraint can be written as

$$M_t^i + B_t^i \leq \left(1 + i_{t-1}^m\right) M_{t-1}^i + \left(1 + i_{t-1}\right) B_{t-1}^i + P_t Y_t^i - T_t - P_t C_t^i, \quad (2)$$

where M_t^i denotes the household's end-of-period holdings of money, B_t^i denotes the household's end-of-period nominal holdings of riskless bonds, i_t^m and i_t are the nominal interest rates paid on money balances and bonds held at the end of period t, Y_t^i is the period income (real) of households, and T_t denotes lump sum taxes and transfers. The household receives income in the form of wages paid, $w(j)$, for labor supplied in the production of each good, j. Furthermore, all households i are assumed to own an equal part of each firm and therefore receive a common share of profits $\Pi_t(j)$ from the sale of each firm's good j (though agents do not know this to be true). Period nominal income is therefore determined as

$$P_t Y_t^i = \int_0^1 [w_t(j) h_t^i(j) + \Pi_t(j)] dj$$

for each household i. The flow budget constraint indicates that financial assets at the end of period t can be no more than the value of assets brought into this period, plus nonfinancial income after taxes and consumption spending. This constraint must hold in all future dates and states of uncertainty. Fiscal policy is assumed to be Ricardian so that goods prices, asset prices, and output are determined independently of fiscal variables.[2] It will be assumed that the fiscal authority pursues a zero-debt policy, so that bonds are in zero net supply.

To summarize, the household's problem in each period t is to choose $\left\{c_t^i(j), h_t^i(j), M_t^i, B_t^i\right\}$ for all $j \in [0,1]$ so as to maximize (1) subject to the constraint (2), taking as parametric the variables $\left\{p_T(j),\ w_T(j),\ \Pi_T,\ i_{T-1},\ i_{T-1}^m,\ \xi_T\right\}$ for $T \geq t$. The first-order conditions characterizing the solution to this optimization problem are detailed in appendix 1.

1.1.1 A Consumption Rule Derived

In order to derive a linear decision rule describing the household's optimal intertemporal allocation of consumption, a log-linear approximation to the household's first-order conditions is employed. Appendix 1 shows that a log-linear approximation to the household's Euler equation and the intertemporal budget constraint imply the relations

$$\hat{C}_t^i = \hat{E}_t^i \hat{C}_{t+1}^i - \sigma(\hat{\imath}_t - \hat{E}_t^i \hat{\pi}_{t+1}) + g_t - \hat{E}_t^i g_{t+1} \tag{3}$$

and

$$\hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \hat{C}_T^i = \varpi_t^i + \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \hat{Y}_T^i, \tag{4}$$

where $\sigma \equiv -U_c/(U_{cc}\bar{C})$ is the intertemporal elasticity of substitution, $g_t \equiv \sigma U_{c\xi}\xi_t/U_c$, and where for any variable z_t, $\hat{z}_t \equiv \ln(z_t/\bar{z})$ denotes the log deviation of the variable from its steady-state value, $\bar{z}$, defined in appendix 1. $\varpi_t^i \equiv W_t^i/(P_t\bar{Y})$ is the share of the household's real wealth as a fraction of steady-state income, where $W_t^i \equiv (1 + i_{t-1}) B_{t-1}^i$. Solving (3) backwards recursively from date T to date t and taking expectations at that time gives

[2] Preston (2002) considers the fiscal theory of the price level under learning dynamics.

$$\hat{E}_t^i \hat{C}_T^i = \hat{C}_t^i - g_t + \hat{E}_t^i \left[g_T + \sigma \sum_{T=t}^{T-1} (\hat{\imath}_t - \hat{\pi}_{t+1}) \right],$$

which on substitution into the intertemporal budget constraint yields

$$\hat{C}_t^i = (1-\beta)\,\varpi_t^i + \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \times \left[(1-\beta)\hat{Y}_T^i - \beta\sigma(\hat{\imath}_T - \hat{\pi}_{T+1}) + \beta(g_T - g_{T+1}) \right] \tag{5}$$

as the desired decision rule: it describes optimal behavior given arbitrary beliefs (so long as such beliefs satisfy standard probability laws). It follows that households necessarily make long-horizon forecasts of macroeconomic conditions to determine their optimal current consumption choice. Consumption varies across households according to differences in wealth and income. Section 5 discusses why optimizing agents necessarily make decisions according to (5), rather than just making use of the Euler equation (3) as has been assumed in the recent literature.

It is useful to contrast this derived decision rule to the predicted consumption allocation under the permanent income hypothesis. Indeed, the first two terms capture precisely the basic insight of the permanent income hypothesis that agents should consume a constant fraction of the expected future discounted wealth, given a constant real interest rate equal to $\beta^{-1} - 1$. The third term arises from the assumption of a time-varying real interest rate, and represents deviations from this constant real rate due to either variation in the nominal interest rate or inflation. The final term results from allowing stochastic disturbances to the economy.

To determine aggregate behavior, integrate over i to give

$$\hat{C}_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)\hat{Y}_T - \beta\sigma(\hat{\imath}_T - \hat{\pi}_{T+1}) + \beta(g_T - g_{T+1}) \right],$$

using the fact that $\int_i \varpi_t^i di = 0$ from market clearing (bonds are in zero net supply) and introducing the notation $\int_i z_t^i di = z_t$ for any variable z and, specifically, $\int_i \hat{E}_t^i di = \hat{E}_t$ to define the average expectations operator. (In aggregating, we have made use of the equilibrium property that all agents will receive the same wage for each type of labor supplied. Since all agents hold the same diversified portfolio of firm profits, it is necessarily true that $\hat{Y}_t^i = \hat{Y}_t^j$ for all i, j and we call this common income stream $\hat{Y}_t$.) It is important to note that the expectations operator, $\hat{E}_t$, possesses no behavioral content, and simply defines the average expectations of a distribution of agents in the economy. That this is true follows immediately from the assumed knowledge of agents: they do not know the tastes and beliefs of other agents in the economy and therefore do not have a complete economic model with which to infer the true aggregate probability laws and how state variables beyond their control are determined.

Since equilibrium requires $\hat{C}_t = \hat{Y}_t$, the aggregation of household decision rules can be written in terms of the output gap, $x_t \equiv \hat{Y}_t - \hat{Y}_t^n$, to give

$$x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} - \sigma(\hat{\imath}_T - \hat{\pi}_{T+1}) + r_T^n \right], \qquad (6)$$

where $\hat{Y}_t^n$ is the natural rate of output (to be defined) and $r_t^n \equiv (\hat{Y}_{t+1}^n - g_{t+1}) - (\hat{Y}_t^n - g_t)$ is a composite of exogenous disturbances. The current output gap is therefore determined by the current nominal interest rate and exogenous disturbance and the average of households' long-horizon forecasts of both these variables and also output and inflation into the indefinite future.

1.2 Optimal Price Setting

Now consider the firm's problem, again relegating details to the appendix. Calvo price setting is assumed so that a fraction $0 < \alpha < 1$ of goods prices are held fixed in any given period, while a fraction $1 - \alpha$ of goods prices are adjusted. Given homogeneity of beliefs, all firms having the opportunity to change their price in period t face the same decision problem and therefore set a common price p_t^*. The Dixit-Stiglitz aggregate price index must therefore evolve according to the relation

$$P_t = \left[\alpha P_{t-1}^{1-\theta} + (1-\alpha)\, p_t^{*1-\theta} \right]^{\frac{1}{1-\theta}}. \qquad (7)$$

Firms setting prices in period t face a demand curve $y_t(i) = Y_t(p_t(i)/P_t)^{-\theta}$ for their good and take aggregate output Y_t and aggregate prices P_t as parametric. Good i is produced using a single labor input $h(i)$ according to the relation $y_t(i) = A_t f(h_t(i))$, where A_t is an exogenous technology shock and the function $f(\cdot)$ satisfies the standard Inada conditions.

When setting prices in period t, firms are assumed to value future streams of income at the marginal value of aggregate income in terms of the marginal value of an additional unit of aggregate income today. That is, a unit of income in each state and date T is valued by the stochastic discount factor

$$Q_{t,T} = \beta^{T-t} \cdot \frac{P_t}{P_T} \cdot \frac{U_c(Y_T, \xi_T)}{U_c(Y_t; \xi_t)}.$$

This simplifying assumption is appealing in the context of the symmetric equilibrium that is examined in this model. Since all agents are assumed to have common beliefs and tastes, and because all households are assumed to own an equal share of firm profits, it follows that in equilibrium each receives a common income stream that is necessarily equal to aggregate income. Having firms value future profits at the marginal value of aggregate income therefore corresponds to each shareholder's valuation.[3]

The firm's price-setting problem in period t is therefore to maximize the expected present discounted value of profits

$$\hat{E}_t^i \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left[\Pi_T^i \left(p_t(i) \right) \right], \tag{8}$$

where

$$\Pi_T^i (p) = Y_t P_t^{\theta} p^{1-\theta} - w_t(i) f^{-1}(Y_t P_t^{\theta} p^{-\theta}/A_t), \tag{9}$$

with the notation $f^{-1}(\cdot)$ denoting the inverse function of $f(\cdot)$. The factor α^{T-t} in the firm's objective function is the probability that

[3]This assumption is not particularly important. In the employed log-linear approximation, firms only use knowledge of the long-run average value of $Q_{t,T}$, which equals the discount factor β. So long as firms know β, any number of assumptions on the price-setting behavior of firms would be consistent with the presented analysis. Firms could hold different beliefs about fluctuations in $Q_{t,T}$ so long as they all know the long-run average to be equal to β.

the firm will not be able to adjust its price for the next $(T-t)$ periods.

To summarize, the firm's problem is to choose $\{p_t(i)\}$ to maximize (8), taking as given the variables $\{Y_T,\ P_T,\ w_T(j),\ A_T,\ Q_{t,T}\}$ for $T \geq t$ and $j \in [0,1]$. The first-order conditions characterizing optimality are contained in appendix 2.

1.2.1 Price Decision Rule Derived

As for the household problem, we seek a log-linear approximation to firms' price-setting behavior. Appendix 2 demonstrates that the first-order condition of the firm's optimal pricing problem satisfies the approximate log-linear relation

$$\widehat{p}_t^*(i) = \hat{E}_t^i \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\frac{1-\alpha\beta}{1+\omega\theta} \cdot (\omega + \sigma^{-1}) x_T + \alpha\beta\hat{\pi}_{T+1}\right], \tag{10}$$

where $\omega > 0$ is the elasticity of firm i's real marginal cost function (defined in the appendix) with respect to its own output, $y_t(i)$. Thus firm i's optimal price is determined as a linear function of the future expected paths of the output gap and inflation. Analogously to the household's problem, firms optimally make long-horizon forecasts of general macroeconomic conditions in deciding their current price, $p_t^*(i)$.

To infer the aggregate implications of the maintained theory of pricing, integrate (10) over i to give

$$\widehat{p}_t^* = \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\frac{1-\alpha\beta}{1+\omega\theta} \cdot (\omega + \sigma^{-1}) x_T + \alpha\beta\hat{\pi}_{T+1}\right].$$

Noting that a log-linear approximation to the price index (7) gives $\hat{\pi}_t = \widehat{p}_t^* \cdot (1-\alpha)/\alpha$, the above expression can be written as

$$\hat{\pi}_t = \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\kappa\alpha\beta \cdot x_{T+1} + (1-\alpha)\beta\hat{\pi}_{T+1}\right], \tag{11}$$

where

$$\kappa \equiv \frac{(1-\alpha)}{\alpha}\frac{1-\alpha\beta}{1+\omega\theta}(\omega + \sigma^{-1}) > 0.$$

Equation (11) indicates that current inflation is determined by today's output gap and the average of firms' expectations of the future time path of both the output gap and the inflation rate. As for the households' problem, since private agents do not know the tastes and beliefs of others and therefore are unable to infer the true aggregate probability laws, this relation cannot be quasi-differenced to deliver a relationship between current inflation and expectations of next period's inflation rate. To simplify notation, for the remainder of the paper the "ˆ" is omitted, with the understanding that all variables are defined as log deviations from steady-state values.

It is worth noting that the foregoing methodology is not specific to the model at hand. Different theories of price setting or consumer behavior could be adopted. For instance, perfect competition could be assumed to give fully flexible prices. Alternatively, it could be assumed that some fraction γ of firms have flexible prices, while a fraction $1 - \gamma$ set prices a period in advance. This would give the Lucas supply curve, as shown by Woodford (2003, chap. 3). Long-horizon forecasts do not matter under these theories of pricing because firms do not face a multiperiod decision problem—they are static and two-period problems, respectively. Assuming Calvo pricing is a tractable way to develop a minimally realistic model for the analysis of monetary policy and facilitates comparison to the recent literature on monetary policy and learning. It is an open question whether other, possibly more realistic theories of pricing have important implications for monetary policy under learning dynamics.

1.3 The Irreducibility of Long-Horizon Forecasts

A number of previous papers have proposed analyses of learning dynamics in the context of models where agents solve multiperiod (indeed, infinite horizon) decision problems, but without requiring that agents make forecasts regarding outcomes more than one period in the future. In these papers, agents' decisions depend only on forecasts of future variables that appear in the Euler equations that can be used to characterize rational expectations equilibrium. For example, Bullard and Mitra (2002) propose an analysis of learning dynamics in a model that is intended to have the same underlying microfoundations as the model presented above—that is, intended to consider the consequences of least-squares learning in the context of the standard New Keynesian model of inflation and output gap determination. However, section 1 demonstrates that under learning dynamics, private-sector optimization implies the aggregate structural relations (6) and (11) so that long-horizon expectations of general macroeconomic conditions matter for the evolution of aggregate output and inflation.

Since these relations hold for arbitrary beliefs satisfying standard probability laws, they must also hold under rational expectations. Under this assumption, (6) and (11) can be simplified by application of the law of iterated expectations, as agents—having complete knowledge of the tastes and beliefs of other agents—are able to compute the equilibrium probabilities and associated laws, ensuring that individual beliefs coincide with the aggregate probability laws implied by the economic model. Leading the aggregate demand relation (6) one period and taking rational expectations at date t gives

$$\begin{aligned} E_t x_{t+1} &= E_t E_{t+1} \sum_{T=t+1}^{\infty} \beta^{T-t-1} \left[(1-\beta) x_{T+1} - \sigma(i_T - \pi_{T+1}) + r_T^n\right] \\ &= E_t \sum_{T=t+1}^{\infty} \beta^{T-t-1} \left[(1-\beta) x_{T+1} - \sigma(i_T - \pi_{T+1}) + r_T^n\right], \end{aligned} \tag{12}$$

where the second equality follows from the law of iterated expectations. It follows that

$$\begin{aligned} x_t &= E_t[(1-\beta)x_{t+1} - \sigma(i_t - \pi_{t+1}) + r_t^n] + \\ &\qquad E_t \sum_{T=t+1}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} - \sigma(\beta \cdot i_T - \pi_{T+1}) + r_T^n\right] \\ &= E_t[(1-\beta)x_{t+1} - \sigma(i_t - \pi_{t+1}) + r_t^n] + \beta E_t x_{t+1} \\ &= E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1}) + r_t^n, \end{aligned}$$

where the second equality makes use of (12). Similar manipulations for the Phillips curve relation give the rational expectations model of the monetary transmission mechanism

$$x_t = E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1}) + r_t^n$$
$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1}.$$

This simple model of the economy has been used in recent studies of monetary policy rules by Bernanke and Woodford (1997), Clarida, Galí, and Gertler (1999), and Woodford (1999). A rational expectations equilibrium analysis therefore predicts that only one-period-ahead forecasts of inflation and the output gap matter for the evolution of the economy. The approach of Bullard and Mitra (2002) is to take these relations and replace the rational expectations assumption with the learning assumption outlined in section 2. This gives the system

$$x_t = \hat{E}_t x_{t+1} - \sigma(i_t - \hat{E}_t \pi_{t+1}) + r_t^n \tag{13}$$
$$\pi_t = \kappa x_t + \beta \hat{E}_t \pi_{t+1}, \tag{14}$$

obtained by substituting the rational expectations operator, E_t, with the learning dynamics operator, $\hat{E}_t$. But the system (13)–(14) is not equivalent to the model consisting of equations (6) and (11) under most possible specifications of subjective expectations.

The proposed learning procedure has the advantage that if the econometric model used by agents to produce forecasts is correctly specified, then the resulting behavior is *asymptotically* optimal. That is, behavior under the learning algorithm differs from what would be optimal behavior under the true probability laws by an amount that is eventually arbitrarily small. For the examined monetary policies, a correctly specified econometric model posits inflation, output, and the nominal interest rate to be linear functions of the lagged natural rate disturbance, with a residual term orthogonal to the natural rate. The consistency of the ordinary least squares estimator implies that the coefficients that agents use in forming their beliefs are eventually close to the true coefficients. Since the optimal decision rule is a continuous function of the coefficients of the agents' forecasting rule, beliefs that are arbitrarily close to the correct ones imply behavior that is arbitrarily close to being optimal.

In general, this is not a property of the Euler equation approach. To make this clear, recall that the optimal decision rule is given by

$$\tilde{C}_t^i = (1-\beta)\,\varpi_t^i + \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)x_T - \beta\sigma(i_T - \pi_{T+1}) + \beta r_T^n\right],$$

where we have defined $\tilde{C}_t^i \equiv C_t^i - Y_t^n$. Agents having a positive initial wealth endowment, $\varpi_t^i > 0$, will have higher than average consumption (given that the income process is the same for all agents in the equilibrium described in section 1), while those having a negative initial wealth endowment, $\varpi_t^i < 0$, will have lower than average consumption.

It is immediate, then, that using the Euler equation alone cannot lead individual households to make the optimal consumption allocation each period, since it does not lead them to take account of their wealth in any way whatsoever. Suppose we interpret the Euler equation (13) as saying that household i forecasts the aggregate output gap, x_{t+1}, then bases its consumption decision $\tilde{C}_t^i$ on this, so that

$$\tilde{C}_t^i = \hat{E}_t^i x_{t+1} - \sigma(i_t - \hat{E}_t^i \pi_{t+1}) + r_t^n \tag{15}$$

describes household behavior. Such a procedure will lead to systematic underconsumption by households with $\varpi_t^i > 0$ and overconsumption of those households with $\varpi_t^i < 0$. If the Euler equation approach is instead interpreted as saying that the household forecasts its own future consumption $\tilde{C}_{t+1}^i$ (based on the past time series of own consumption spending) and then bases current consumption on this, we have

$$\tilde{C}_t^i = \hat{E}_t^i \tilde{C}_{t+1}^i - \sigma(i_t - \hat{E}_t^i \pi_{t+1}) + r_t^n, \tag{16}$$

and similar conclusions present themselves.[4]

Such suboptimal behavior is a manifestation of the following more general point: forecasting $\hat{E}_t^i \tilde{C}_{t+1}^i$ as $\hat{E}_t^i x_{t+1}$ is internally inconsistent with household optimization. It represents a forecast of the agent's own future decision that differs from what it expects to be optimal given its current forecasts of future income, inflation, and interest rates and given the agent's understanding of its own decision rule. Moreover, if agents have internally consistent beliefs, such forecasts would differ from what the agent should now be forecasting

[4]Note that under this proposed learning mechanism, the interpretation of $\hat{E}_t^i$ is distinct from the behavior postulated in this paper: agents, rather than forecasting future state variables that are beyond their control though pertinent to their decision problem, adopt a pure statistical model of their own future consumption choice—expectations are not taken with respect to the probability distribution induced by the optimal decision rule and beliefs about exogenous state variables.

about their own period $t+1$ forecasts. Forecasts of this kind therefore represent a less sophisticated approach to forecasting, because they fail to make use of information that the agent necessarily possesses. The model of learning proposed in this paper induces a more sophisticated approach to forecasting that ensures consistency among the various things that the agent is assumed to simultaneously believe.

As an example, consider the model in the case of a zero initial wealth endowment. The optimal decision rule is

$$\tilde{C}_t^i = \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_T - \beta\sigma(i_T - \pi_{T+1}) + \beta r_T^n \right].$$

Since in the optimal program this rule governs consumption decisions in all future periods, it follows that households expect next period's optimal consumption choice to be

$$\hat{E}_t^i \tilde{C}_{t+1}^i = \hat{E}_t^i \sum_{T=t+1}^{\infty} \beta^{T-t} \left[(1-\beta) x_T - \beta\sigma(i_T - \pi_{T+1}) + \beta r_T^n \right], \quad (17)$$

obtained by forwarding the optimal decision rule one period and taking expectations at time t. It follows immediately that for the Euler equation to provide the optimal consumption allocation, under the interpretations in the preceding paragraph given to (15) or (16) above, $\hat{E}_t^i x_{t+1}$ and $\hat{E}_t^i \tilde{C}_{t+1}$, respectively, must coincide with this optimal forecast given by (17). But in general, there is no reason for forecasts of x_{t+1} and $\tilde{C}_{t+1}^i$ constructed from regression of past observations of these variables on observed aggregate disturbances to coincide with (17). The optimal forecast is a particular linear combination of forecasts of the state variables relevant to the household's decision problem. Importantly, such forecasts will only coincide in a rational expectations equilibrium; that is, when agents know the true probability laws—the very laws agents are attempting to learn. Thus consumption decisions made according to either Euler equation (15) or (16) lead to suboptimal behavior.

Honkapohja, Mitra, and Evans (2002) argue that such Euler equations can be derived from the framework of this paper (in the case of zero initial wealth endowments) with the additional assumption that agents understand that market clearing requires $\tilde{C}_t^i = x_t$

(or $C_t^i = Y_t$) in all periods. While this assumption might appear appealing in the context of a model with homogenous agents that are constrained in equilibrium to consume identical incomes, more generally it lacks appeal on two grounds. First, market-clearing conditions are part of the set of rational expectations equilibrium restrictions that agents are attempting to learn—why are they any more likely to be endowed with knowledge of one restriction over another? This will be particularly important in more general models when agents receive differing income streams and have incentives to trade assets in equilibrium. Second, even if it is assumed that agents are aware of this market-clearing condition, so that the Euler equation of the form (15) can be derived, such a decision rule does not describe optimal behavior: households would never choose to adopt such a learning rule given the maintained assumption that agents optimize conditional on their beliefs.

2. Expectations Formation

The previous section derives the aggregate implications of household and firm behavior. Equation (6) specifies the evolution of aggregate demand, while equation (11) is analogous to a forward-looking Phillips curve determining current inflation as a function of expected future inflation and the output gap. To close this stylized model of the macroeconomy, assumptions on the expectations formation mechanism and the nature of monetary policy—which determines the evolution of the nominal interest rate $\{i_t\}$—are required. Given expectations, so long as monetary policy is specified as being determined by the model's exogenous variable and/or permitted to depend only on the endogenous variables, inflation, and the output gap (including future expected and past values), then this equation together with (6) and (11) is sufficient to determine $\{\pi_t,\ x_t,\ i_t\}$. The monetary policies considered in this paper satisfy this requirement. It remains to specify the expectations formation mechanism.

2.1 Recursive Learning

To be precise about the learning dynamics of this model, adjoin an equation for the interest rate to equations (6) and (11) to give the system

$$x_t = -\sigma i_t + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)x_{T+1} - \sigma(\beta \cdot i_{T+1} - \pi_{T+1}) + r_T^n\right] \tag{18}$$

$$\pi_t = \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\kappa\alpha\beta \cdot x_{T+1} + (1-\alpha)\beta \cdot \pi_{T+1}\right] \tag{19}$$

$$i_t = i(x, \pi, r^n). \tag{20}$$

The final equation defines a general specification for monetary policy that satisfies the requirements discussed above. Conditional on expectations, there are three equations that determine the three unknown endogenous variables $\{\pi_t,\ x_t,\ i_t\}$. It is clear from equations (18) and (19) that agents require forecasts of the entire future path of each endogenous variable. Agents therefore estimate a linear model in inflation, the output gap, and the nominal interest rate, using as regressors variables that appear in the minimum-state-variable solution to the model under rational expectations. This conjectured model represents agents' beliefs of the equilibrium dynamics of the model's state variables.[5]

Under the classes of monetary policies considered in section 4, the minimum-state-variable solution is always linear in the disturbance term, r_t^n. Suppose the natural rate of interest, r_t^n, is determined by the stochastic process

$$r_t^n = f' s_t,$$

where

$$s_t = C s_{t-1} + \varepsilon_{s,t} \tag{21}$$

[5]One might query the assumption that agents construct forecasts using just variables that appear in the minimum-state-variable solution. After all, it is clear from the optimizing model developed that agents also observe (simultaneously) aggregate output and prices when making their own decisions about consumption and price setting. It follows that these aggregate variables might be thought useful in constructing forecasts about the future evolution of the economy. This informational assumption leads to the same substantive conclusions on E-stability as the case in which agents do not use this additional information, and results are available from the author. To keep the ideas at the fore and the algebra at bay, the main analysis works under the assumption that forecasts are constructed using only variables that appear in the minimum-state-variable solution.

and s_t is an $(n \times 1)$ vector, f is an $(n \times 1)$ coefficient vector, $\varepsilon_{s,t}$ is an i.i.d disturbance vector, and C is a matrix with all eigenvalues being real and inside the unit circle. Thus the natural rate shock is specified as a fairly arbitrary linear combination of exogenous disturbances. Defining $z_t \equiv (\pi_t, x_t, i_t)'$, the estimated linear model is assumed to be

$$z_t = a_t + b_t \cdot s_t + \epsilon_t,$$

where ϵ_t is the usual error term and (a_t, b_t) are coefficient vectors of dimension (3×1) and $(3 \times n)$, respectively. The estimation procedure makes use of the entire history of available data in period t, $\{z_t, 1, s_t\}_0^{t-1}$. As additional data become available in subsequent periods, agents update their estimates of the coefficients (a_t, b_t). This is neatly represented as the recursive least-squares formulation

$$\phi_t = \phi_{t-1} + t^{-1} R_t^{-1} w_{t-1}(z_{t-1} - \phi_{t-1}' w_{t-1}) \tag{22}$$

$$R_t = R_{t-1} + t^{-1}(w_{t-1} w_{t-1}' - R_{t-1}), \tag{23}$$

where the first equation describes how the forecast coefficients, $\phi_t = (a_t', vec(b_t)')'$, are updated with each new data point and the second equation describes the evolution of the matrix of second moments of the appropriately stacked regressors $w_t \equiv \{1, s_t\}_0^t$. The forecasts can then be constructed as

$$\hat{E}_t z_T = a_{t-1} + b_{t-1} \cdot C^{T-t} \cdot s_t \tag{24}$$

for $T \geq t$. The matrix C is assumed to be known to agents for algebraic convenience. This is not important to the conclusions of this paper—all results hold when agents have to learn the nature of the autoregressive process describing s_t, and the results are available from the author. That agents form beliefs using (24) makes clear their irrationality—at time t, agents make use of an econometric model to assign probabilities to the evolution of state variables that does not account for their own subsequent updating of beliefs at $t+1$ by use of (22) and (23). This completes the description of the model.

To summarize, the model of the macroeconomy comprises an aggregate demand equation, (6); a Phillips curve, (11); a monetary policy rule; and the forecasting system (22), (23), and (24).

3. Analyzing Learning Dynamics

Subsequent analysis answers two related questions for a given assumption on monetary policy: Under what conditions does a unique rational expectations equilibrium obtain? And, given the existence of such an equilibrium, what conditions guarantee convergence to this equilibrium when agents' expectations are formed using a recursive least-squares algorithm rather than using rational expectations? While analysis of determinacy is now commonplace in the monetary policy literature, the conditions for convergence under least-squares learning dynamics are less familiar.[6] The criterion adopted in this paper to judge convergence under recursive learning is the notion of expectational stability of rational expectations equilibrium, called E-stability by Evans and Honkapohja (2001). Evans and Honkapohja show that local real-time convergence of a broad class of dynamic models under recursive learning is governed by E-stability. The following section draws on Evans and Honkapohja (2001) to develop the ideas of E-stability.

3.1 Expectational Stability

Agents use their econometric model to construct forecasts of the future path of endogenous variables. For expositional purposes, this subsection assumes the evolution of r_t^n is a standard AR(1) process, with coefficient $|\rho| < 1$. If monetary policy is conducted so that the minimum-state-variable solution is linear in r_t^n, then forecasts can be constructed using

$$\hat{E}_t z_T = a_{t-1} + b_{t-1} \cdot \rho^{T-t} \cdot r_t^n$$

for $T \geq t$. To obtain the actual evolution of the economy, substitute (24) into the system of equations (18), (19), and (20). Collecting like terms gives a general expression of the form

$$z_t = (Q + Aa_{t-1}) + (Bb_{t-1} + D)\, r_t^n,$$

where the matrices A and B collect coefficients on the estimated parameter vectors (a_t', b_t'), Q collects constant terms, and D collects

[6] See Blanchard and Kahn (1980) for a detailed discussion of the conditions for uniqueness of rational expectations equilibrium.

remaining coefficients on the state variable, r_t^n. Leading this expression one period and taking expectations (rational) provides

$$E_t z_{t+1} = (Q + Aa_{t-1}) + \rho\,(Bb_{t-1} + D)\,r_t^n,$$

which describes the optimal rational forecast conditional on private-sector behavior. Comparison with (24) makes clear that agents are estimating a misspecified model of the economy—agents assume a stationary model when in fact the true model has time-varying coefficients. Taken together with (24) at $T = t + 1$, it defines a mapping that determines the optimal forecast coefficients given the current private-sector forecast parameters (a_{t-1}', b_{t-1}'), written as

$$T(a_{t-1}, b_{t-1}) = (Q + Aa_{t-1}, Bb_{t-1} + D). \tag{25}$$

A rational expectations equilibrium (REE) is a fixed point of this mapping. For such REE, we are then interested in asking, under what conditions does an economy with learning dynamics converge to this equilibrium? Using stochastic approximation methods, Evans and Honkapohja (2001) show that the conditions for convergence of the learning algorithm (22) and (23) are neatly characterized by the local stability properties of the associated ordinary differential equation

$$\frac{d}{d\tau}(a, b) = T(a, b) - (a, b), \tag{26}$$

where τ denotes "notional" time. The REE is said to be expectationally stable, or E-stable, if this differential equation is locally stable in the neighborhood of the REE. From standard results for ordinary differential equations, a fixed point is locally asymptotically stable if all eigenvalues of the Jacobian matrix $D\,[T(a, b) - (a, b)]$ have negative real parts (where D denotes the differentiation operator and the Jacobian understood to be evaluated at the rational expectations equilibrium of interest). See Evans and Honkapohja (2001) for further details on expectational stability.

In the context of the above model, the Jacobian matrices are $(A - I)$ and $(B - I)$ and have dimension (3×3) (corresponding to the number of state variables that agents are forecasting). For such matrices to have roots all having negative real parts, the coefficients of the associated characteristic equation must satisfy three restrictions. It follows that E-stability imposes six restrictions on model parameters. Details of these conditions are provided in appendix 3. The remainder of the paper concerns itself with the relationship between the conditions for expectational stability and the requirements for determinacy when monetary policy is specified as a commitment to a variety of interest-rate rules.

4. Monetary Policy and Learning

The first part of this paper develops a framework in which agents face multiperiod decision problems and have subjective expectations. It shows that the aggregation of rationally formed decisions of individual agents with such subjective expectations implies that current output and inflation are determined by long-horizon forecasts of general macroeconomic conditions. The remainder of the paper is devoted to the question of whether certain policy rules in such an economy lead the learning dynamics to converge to the dynamics predicted by rational expectations equilibrium analysis—that is, in the language of the previous section, whether given sufficient data agents adopt perceived laws of motion that converge to the actual laws of motion of the economy.

Since Taylor (1993) there has been a revived interest in monetary policy rules, both as a prescriptive and descriptive tool. Taylor proposed a simple rule of the form

$$i_t = \bar{\imath}_t + \psi_\pi \pi_t + \psi_x x_t \tag{27}$$

prescribing the nominal interest rate to be adjusted in response to variations in inflation and in the output gap and $\bar{\imath}_t$ is a stochastic constant.[7] This work and Taylor (1999) provides evidence that a rule of this form gives a remarkably good characterization of U.S. monetary policy from the mid-1980s onward. More generally, some have argued that interest-rate rules should be an integral part of a framework for monetary policy, as such rules provide a possible solution to the pitfalls of discretionary behavior by the central bank: by providing a systematic response to economic shocks, the central bank might be better able to stabilize inflation and output and

[7]The actual rule proposed by Taylor (1993) was $i_t = \pi_t + 0.5(\pi_t - 2) + 0.5x_t$, interpreting π_t as the four-quarter-ended inflation rate.

therefore improve social welfare. Furthermore, by specifying the optimal choice of the nominal interest rate in each state of the world, interest-rate rules can, in principle, be designed to implement optimal monetary policy. Clarida, Galí, and Gertler (1999) and Woodford (2003) develop these ideas in considerable detail and present a coherent theory of monetary policy that makes the case for such rules.

However, much of the literature on monetary policy rules regarding the desirability of one rule versus another rests on the assumption of rational expectations. And while rational expectations has obvious appeal as a modeling device, there is good reason to be cautious about policy recommendations derived under its assumption. The model of this paper provides a natural framework to evaluate the desirability of monetary policy rules given alternative assumptions on the expectations formation mechanism. Indeed, the adaptive learning framework has considerable appeal, as it includes the rational expectations paradigm as a special limiting case. It follows that such an expectations formation mechanism presents a minimal deviation from rational expectations and, therefore, that any rules that are found to induce economic instability under its assumption are likely to be undesirable as a recommended policy. Indeed, Howitt (1992), Evans and Honkapohja (2003), and Bullard and Mitra (2002) argue convergence of least-squares learning to the predictions of rational expectations equilibrium analysis to be a minimal requirement of any proposed policy.

The use of interest-rate rules as a means to conduct monetary policy has also been criticized on the grounds that even though a policy is consistent with a desirable equilibrium, it will almost surely have disastrous consequences in practice by allowing for self-fulfilling expectations to propagate. For example, Friedman (1968) argued that any attempt by the monetary authority to peg the nominal interest rate, even at an optimally chosen value, would inevitably lead to economic instability via a cumulative Wicksellian process. Following Howitt (1992), the basic logic of this criticism can be neatly formulated in a model where agents form expectations of the future path of the economy by extrapolating from historical relationships in observed data. The remaining analysis examines the possibility of self-fulfilling expectations when agents must form long-horizon forecasts in order to make current decisions. Thus, given a candidate monetary policy, the central question of interest is whether, given sufficient data, agents with subjective expectations will be able to learn the predictions of rational expectations equilibrium analysis.

4.1 Monetary Policy Rules

Consider two classes of instrument rules: (1) nominal interest-rate rules that depend only on the history of exogenous disturbances and (2) Taylor-type feedback rules that specify a path for the nominal interest rate that depends on the path of endogenous variables. The former class of rule is of considerable interest since specifying the optimal action of the monetary authority in each state of the world is a natural way to implement optimal monetary policy. However, such rules are an example of the type of rule critiqued by Friedman (1968) and have also been criticized by Sargent and Wallace (1975), who showed that commitment to exogenously determined interest-rate paths can lead to indeterminacy of rational expectations equilibria. The possibility of indeterminacy of rational expectations equilibria raises an important challenge for the design of optimal monetary policy as underscored by the work of Svensson and Woodford (2002), Woodford (1999), Giannoni and Woodford (2002a), and Giannoni and Woodford (2002b): even though an optimal interest-rate rule, expressed as a function of the history of exogenous disturbances, can be designed to be consistent with the optimal equilibrium, such rules are also equally consistent with many other undesirable equilibria. In the context of the monetary policy literature under learning, Evans and Honkapohja (2003) have demonstrated an analogous result: such policy rules are in fact subject to self-fulfilling expectations as argued by Friedman.

Importantly, indeterminacy of rational expectations equilibrium is not a general property of interest-rate rules. McCallum (1983) showed that rules that allow appropriate feedback from endogenous variables can deliver a unique equilibrium. As mentioned, a prominent recent example due to Taylor (1993) is given by (27). Woodford (2003, chap. 4) shows that this rule leads to determinacy of rational expectations equilibrium if the so-called Taylor principle is satisfied. This rule will be the central focus of our study of learning dynamics in this economy.

There are clearly many other possible rules for the conduct of monetary policy. Clarida, Galí, and Gertler (1998) and Clarida, Galí, and Gertler (2000) have found that estimated central bank reaction functions often find an important role for expectations of future inflation in the setting of the current interest rate. This suggests rules of the form

$$i_t = \bar{\imath}_t + \psi_x E_t x_{t+1} + \psi_\pi E_t \pi_{t+1} \tag{28}$$

to be of practical interest.[8]

Alternatively, as argued by McCallum (1999), the informational assumptions implicit in the Taylor rule are tenuous in practice. Monetary authorities typically do not have available current-dated observations on the output gap and inflation rate when setting the current interest rate. Many researchers have responded to this criticism by modifying the information set available to the monetary authority when determining its instrument setting. Hence, the nominal interest rate could be argued to be better modeled as being determined by lagged expectations of current-dated output and inflation to give an instrument rule of the form

$$i_t = \bar{\imath}_t + \psi_x E_{t-1} x_t + \psi_\pi E_{t-1} \pi_t. \tag{29}$$

Finally, for a monetary authority concerned with stabilizing variation in output and inflation, the optimal commitment equilibrium in the present model under rational expectations can be shown to be implemented by a rule of the form

$$\begin{aligned} i_t = \frac{1}{\sigma} \Bigg[& E_t x_{t+1} - \frac{\lambda}{\lambda + \kappa^2} x_{t-1} + \left(\frac{\beta\kappa}{\lambda + \kappa^2} + \sigma \right) E_t \pi_{t+1} \\ & + \frac{\kappa}{\lambda + \kappa^2} u_t + r_t^n \Bigg], \end{aligned}$$

where λ gives the weight placed on stabilizing output variation and u_t a cost-push shock.[9] All three classes of monetary policy rules certainly warrant careful analysis when private agents have subjective

[8] This is not the form of rule that these authors find best characterizes the central bank's policy reaction function. It is presented as being illustrative of a type of rule that might be of practical interest.

[9] As done in many recent analyses of this model under rational expections, a cost-push shock can be introduced into the aggregate supply curve to ensure a nontrivial optimal monetary policy problem.

expectations and these rules are interpreted as responding to observed private forecasts. Indeed, Evans and Honkapohja (2002) and Bullard and Mitra (2002) have examined rules of these types under learning dynamics in the context of a model where expectations of inflation and output one period in advance matter. In the context of the model of this paper, analysis of these rules leads to quite different conclusions about their desirability as a guideline for the conduct of monetary policy. For this reason, discussion of these classes of rules is contained in companion papers, Preston (forthcoming) and Preston (2004), that seek to understand the desirability of central bank decision procedures that attempt to implement monetary policy using forecast-based instrument rules.

4.2 Learning Dynamics

4.2.1 Exogenous Interest-Rate Processes

To begin analysis of the model under learning, consider a monetary policy specified as a commitment to an instrument rule of the form

$$i_t = \bar{\imath}_t + \psi_c + \psi_r r_t^n \tag{30}$$

that posits the nominal interest rate to be set in response to the disturbance in the natural rate of interest. In the following propositions, we assume the exogenous variables $(\bar{\imath}_t, r_t^n)$ are determined as

$$\begin{bmatrix} \bar{\imath}_t \\ r_t^n \end{bmatrix} = f' s_t,$$

redefining f as an $(n \times 2)$ matrix and with s_t determined as in (21). This postulates both the natural rate disturbance and the stochastic constant of the Taylor rule to be a particular linear combination of the elements of the disturbance vector s_t.

Proposition 1. Under the interest-rate rule (30), the associated REE of the economy given by (18) and (19) is linear in the state variables s_t and is not E-stable under least-squares learning dynamics.

Proof. It is easy to verify the existence of an REE that is linear in the state variable, s_t. Therefore, assume that agents have forecast functions of the form (24). Substituting the assumed instrument rule and forecast functions (24) into the system (18), (19) gives

$$z_t = Aa_{t-1} + \{\text{terms independent of } a_{t-1}\}, \tag{31}$$

where

$$A = \begin{bmatrix} \frac{(1-\alpha)\beta}{1-\alpha\beta} + \frac{\kappa\sigma}{1-\beta} & \frac{\kappa\alpha\beta}{1-\alpha\beta} + \kappa & -\frac{\kappa\sigma\beta}{1-\beta} \\ \frac{\sigma}{1-\beta} & 1 & -\frac{\sigma\beta}{1-\beta} \\ 0 & 0 & 0 \end{bmatrix}.$$

Leading (31) and taking expectations delivers the optimal forecast of the evolution of the endogenous variables given the current forecast parameters of private agents. The required mapping between the perceived and optimal laws of motion follows immediately. E-stability requires $\det(A - I) < 0$, but

$$\det(A - I) = \frac{\kappa\sigma}{1-\beta} + \frac{\alpha\beta\kappa\sigma}{(1-\beta)(1-\alpha\beta)} > 0.$$

The desired result follows.

Not only do exogenous interest-rate rules suffer from an indeterminacy of equilibrium, but also any such equilibrium fails to be expectationally stable, giving credence to Friedman's critique of nominal interest-rate pegs (a form of interest-rate rule).[10] This result is related to that of Evans and Honkapohja (2003), who find an analogous result in the context of a model with a more restrictive class of learning dynamics. The present paper assumes agents know less about the economy and, as one might expect, this does not make agents better able to learn the rational expectations equilibrium. It is also worth noting that one of the motivations of the bounded rationality literature in macroeconomics was the possibility that learning mechanisms would provide an equilibrium selection criterion in the case of multiple rational expectations equilibria.[11] In the context of this model, learning is not able to overcome indeterminacy of equilibrium induced by an exogenous interest-rate rule.

[10] Honkapohja and Mitra (2004) also show in their model that all nonfundamentals-based equilibria are unstable under learning dynamics. However, given that no rational expectations equilibria are then learnable in that model, this class of rule has little to recommend itself.

[11] Sargent (1993) provides several examples where learning dynamics provide a criterion for equilibrium selection.

Proposition 1 also implies that to design optimal monetary policy rules, it is generally not enough to specify a rule in terms of exogenous disturbances to implement optimal monetary policy. While such rules are consistent with the desired equilibrium, they are equally consistent with the propagation of self-fulfilling expectations. The challenge to design rules that are immune to such instability is taken up in Preston (2004) and Preston (forthcoming).

4.2.2 The Taylor Rule

In contrast to interest-rate rules that depend only on the history of exogenous disturbances, Taylor rules can deliver determinacy of rational expectations equilibrium so long as the Taylor principle is satisfied. Under learning dynamics, the Taylor principle is necessary and sufficient for E-stability.

Proposition 2. Suppose agents construct forecasts using the perceived law of motion given by (24). Under the Taylor rule (27), the model given by (18) and (19) has minimum-state-variable rational expectations equilibria that are linear in the state variables, s_t, for which the Taylor principle

$$\kappa(\psi_\pi - 1) + (1 - \beta)\psi_x > 0$$

is necessary and sufficient for E-stability under least-squares learning dynamics.

Proof. A sketch of the proof now follows. Appendix 4 shows that the E-stability mapping implies the associated ordinary differential equation

$$\frac{\partial \phi}{\partial \tau} = \begin{bmatrix} A_3 - I_3 & \mathbf{0} \\ \mathbf{0} & H_{3n} - I_{3n} \end{bmatrix} \phi,$$

where $\phi = \left(a', vec\,(b)'\right)'$, all matrices are square and of indicated dimension, and I is an identity matrix. E-stability requires all $3n + 3$ eigenvalues of this system to have negative real parts. It is immediate that the eigenvalues are determined by the properties of the matrices $A_3 - I_3$ and $H_{3n} - I_{3n}$. The proof establishes that these two matrices have negative real roots so long as the Taylor principle holds.

The following treats the matrix $A_3 - I_3$, which characterizes the stability properties of the constant dynamics, leaving $H_{3n} - I_{3n}$ to appendix 4.

Since the constant dynamics are independent of the dynamics describing the forecast parameters $vec\,(b)$, we can analyze the subsystem

$$\frac{\partial a}{\partial \tau} = [A_3 - I_3]\, a.$$

Noting that an REE implies the coefficient restriction $a_i = \psi_x a_x + \psi_\pi a_\pi$, make a change of variables according to the relation $a_j = \psi_x a_x + \psi_\pi a_\pi - a_i$, where $a_j = b_j = 0$ in an REE. This yields the system

$$\frac{\partial \tilde{a}}{\partial \tau} = \begin{bmatrix} \tilde{A} - I_2 & \tilde{A}_2 \\ \mathbf{0} & -1 \end{bmatrix} \tilde{a},$$

where $\tilde{a} = (a_\pi, a_x, a_j)'$ and all matrices are of dimension (2×2). $\tilde{A}_2$ has elements that are composites of model primitives. The matrix $\tilde{A}$ can be shown to have elements

$$\begin{aligned}
\tilde{a}_{11} &= (\kappa\sigma\,(1-\alpha\beta)\,(1-\beta\psi_\pi) + \beta\,(1-\alpha)\,(1-\beta)\,(1+\beta\sigma\psi_x))\,/\Gamma_1 \\
\tilde{a}_{12} &= \kappa\,(1-\beta(1+(1-\alpha)\,\sigma\psi_x)\,/\Gamma_1 \\
\tilde{a}_{21} &= \sigma\left(1-\alpha\beta\,(1-\psi_\pi) - 2\beta\psi_\pi + \beta^2\psi_\pi\right)/\Gamma_1 \\
\tilde{a}_{22} &= ((1-\beta)\,(1-\alpha\beta) - \sigma\beta\psi_x\,(1-\alpha\beta) - \alpha\beta\kappa\sigma\psi_\pi\,(1-\beta))\,/\Gamma_1,
\end{aligned}$$

where

$$\Gamma_1 = (1-\beta)\,(1-\alpha\beta)\,(1+\sigma\psi_x + \sigma\kappa\psi_\pi)$$

and a_{ij} denotes the (i,j) element of the matrix $\tilde{A}$.

For E-stability, the Jacobian $D\frac{\partial \tilde{a}}{\partial \tau}$ must have roots with negative real parts. It is immediate that one root is equal to negative unity. $\tilde{A} - I_2$ must have positive determinant and negative trace for the remaining two eigenvalues to have the desired property. These restrictions imply the inequalities

$$\psi_\pi + \frac{(1-\beta)}{\kappa}\psi_x > 1$$

and

$$\psi_\pi + \frac{\psi_x}{\kappa} \cdot \frac{(1-\alpha\beta) + (1-\beta)^2}{(1-\alpha\beta) + (1-\beta)} > \frac{\kappa\sigma(1-\alpha\beta) - (1-\beta)^2}{\kappa\sigma\,[(1-\alpha\beta) + (1-\beta)]},$$

respectively. The first inequality clearly establishes the Taylor principle to be necessary for E-stability. To show that it is sufficient, note that the right-hand side of the second restriction is necessarily less than one. Furthermore, the slope coefficient on the parameter ψ_x is necessarily greater than $(1-\beta)/\kappa$; for it to be less than this value requires $\beta < 0$, contradicting the maintained model assumptions. It follows that any policy parameter pairs (ψ_x, ψ_π) satisfying the Taylor principle must also satisfy this second inequality. Appendix 4 applies similar arguments to the restrictions implied by $H_{3n} - I_{3n}$ to establish the desired result.

To give some intuition, particularly for the presence of the eigenvalues equal to negative one, which relate to learning the interest-rate dynamics, consider the following. Suppose that agents, serendipitously, happen to forecast a nominal interest-rate path coinciding with what would be determined by the Taylor rule given the agents' forecasts for the output gap and inflation. It follows that the economy would produce data for the output gap and inflation that are in turn consistent with estimating parameters that would generate forecasts in subsequent periods for the path of the nominal interest rate that would again be obtained under the Taylor rule. It follows that the Taylor rule itself cannot be a source of instability, and agents—by observing the realized values for output, inflation, and the nominal interest rate—can easily discern the restriction between these variables that is required by the Taylor rule in a rational expectations equilibrium. It follows that only the inflation and output gap dynamics are relevant for E-stability. This basic insight is important more generally: Preston (forthcoming) shows that a common property of desirable optimal monetary policies is that they ensure that the instrument rule itself is not a source of instability—that is, the associated eigenvalues are independent of private agents' beliefs as for the Taylor rule examined here.

Bullard and Mitra (2002) show a similar result in a learning analysis based on equations (13) and (14) and assuming the natural rate, r_t^n, to be specified as an $AR(1)$ stochastic process. It should be emphasized that this is not obviously to be expected. The framework given by (18) and (19) allows for both significantly more general out-of-equilibrium behavior, with output, inflation, and the nominal interest rate depending on average expectations of these same variables into the indefinite future and a more general stochastic process for the disturbances. The presence of additional expectational variables relative to the analysis of Bullard and Mitra (2002) is a potential source of instability under learning dynamics. That the Taylor principle continues to be the relevant condition for E-stability in the more general framework developed in this paper suggests it to be a robust result for this class of instrument rule.

As discussed by Honkapohja and Mitra (2004), results of this kind also provide an alternative interpretation of the performance of monetary policy in the United States in the 1970s relative to later decades. Clarida, Galí, and Gertler (1998) argue that the inflationary episode of the 1970s was the result of a monetary policy that was not consistent with a determinate price level. As a result, the economy was prone to "sunspot" equilibria and self-fulfilling expectations. In contrast, the results of this paper suggest that an equally consistent interpretation of this episode is that monetary policy, rather than inducing indeterminacy, was conducive to agents expecting ever higher inflation on the basis of their experience with past inflation—and hence to propagating self-fulfilling expectations.

5. Conclusion

This paper develops a framework to analyze the robustness of monetary policy rules to an important source of model misspecification—the assumed form of expectations formation. The principal contribution is methodological in nature: the solution to a simple microfounded model under a nonrational expectations assumption. Analysis of the multiperiod decision problems of households and firms under subjective beliefs shows that the predicted aggregate model dynamics are qualitatively different from those obtained under rational expectations. Indeed, the determination of inflation and output depends on the average of agents' long-horizon forecasts of the model's endogenous variables into the indefinite future.

The principal substantive contribution is the analysis of whether instrument rules that have been of particular interest to the monetary policy literature over the past decade are robust to deviations from rational expectations. When policy is specified as a commitment to an exogenous interest-rate rule, agents are unable to learn the associated rational expectations dynamics. Such rules are therefore undesirable both due to inducing multiple equilibria under rational expectations and to being subject to the Friedman (1968) critique that nominal interest-rate rules of this type are subject to self-fulfilling expectations. In contrast, for Taylor-type feedback rules, agents are able to learn the associated rational expectations dynamics so long as the Taylor principle is satisfied. Interestingly, this finding is invariant to a number of different information assumptions on the agent's forecasting model, making it a robust feature of the policy environment in this model. This suggests the Taylor rule to be desirable from the point of view of eliminating instability due to self-fulfilling expectations.

Companion papers, Preston (2004) and Preston (forthcoming), demonstrate for a number of more complicated rules that conclusions differ markedly in the framework developed here as compared to the Euler equation approach to modeling learning. The latter paper shows that forecast-based instrument rules, including the classes of rules proposed by Bullard and Mitra (2002) and Evans and Honkapohja (2002), are frequently prone to self-fulfilling expectations in the present model if the central bank responds to observed private-sector forecasts. However, if the central bank responds to the determinants of these expectations, this instability can be mitigated. The former paper shows that optimal monetary policy can always be implemented using specific targeting rules if the central bank correctly understands agents' behavior. However, without such knowledge, decision procedures that seek to control directly the path of the price level, rather than the inflation rate, tend to perform better under learning dynamics, even though these policies are equivalent in terms of the rational expectations equilibrium they imply.

Appendix 1. Household Optimality

Defining $W_{t+1}^i = (1 + i_t^m)\, M_t^i + (1 + i_t)\, B_t^i$ as the total beginning-of-period wealth at time $t+1$ allows the flow budget constraint (2) to be written as

$$P_t C_t^i + \Delta_t M_t^i + \frac{1}{1 + i_t} \cdot W_{t+1}^i \leq W_t^i + \left[P_t Y_t^i - T_t\right], \qquad (32)$$

where

$$\Delta_t \equiv \frac{i_t - i_m}{1 + i_t}$$

is the opportunity cost of holding wealth in a monetary form. Since (32) must hold in all states, s, and dates, t, the flow budget constraint can be solved forward recursively, given the appropriate no-Ponzi constraint $\lim_{j\to\infty} R_{t,t+j} W_{t+j+1} = 0$, to give

$$W_t^i \geq \sum_{j=0}^{\infty} R_{t,t+j} \left[P_{t+j} C_{t+j}^i + \Delta_{t+j} M_{t+j}^i - (P_{t+j} Y_{t+j}^i - T_{t+j}) \right],$$

where

$$R_{t,t+j} = \prod_{s=1}^{j} \left(\frac{1}{1 + i_{t+s-1}} \right).$$

Standard analysis shows that household intertemporal optimality is characterized by the first-order conditions for consumption and labor supply:

$$\frac{1}{1+i_t} = \beta E_t [\frac{P_t}{P_{t+1}} \cdot \frac{U_c(C_{t+1}^i, \xi_{t+1})}{U_c(C_t^i; \xi_t)}] \tag{33}$$

and

$$\frac{v_h(h_t(j); \xi_t)}{u_c(C_t^i; \xi_t)} = \frac{w_t(j)}{P_t} \tag{34}$$

for dates, t, and goods $j \in [0, 1]$. Since we are assuming a cashless economy, where the transaction frictions that money is usually held to mitigate are essentially zero, optimization also requires

$$M_t^i = 0$$

or

$$i_t = i_t^m.$$

In each period t, households also face the intratemporal problem of allocating expenditures across goods j. Optimality for all $j \in [0, 1]$ implies

$$c_t(j) = C_t^i \left(\frac{p_t(j)}{P_t} \right)^{-\theta} \tag{35}$$

so that total expenditure is given by $P_t C_t^i$.[12] To obtain the total consumption demand for good j, integrate over i to obtain

$$c_t(j) = C_t \left(\frac{p_t(j)}{P_t}\right)^{-\theta},$$

introducing the notation $\int_i z_t^i di = z_t$ for any variable z. In equilibrium, markets must clear for each good and aggregate output. This requires $y_t(j) = c_t(j)$ for all j and $C_t = Y_t$. Substitution of the market-clearing conditions into the above relation gives the demand curve for output produced by firm j. Asset market clearing implies $M_t = M_t^s$ and $B_t = B_t^s$, where $\int B_t^i di = B_t$ and similarly for M_t. Since $M_t^s > 0$, this implies $i_t = i_t^m$. Finally, the intertemporal budget constraint and the transversality condition must hold with equality. Since we have a zero-debt fiscal policy with bonds in zero net supply, it follows that

$$T_t = (1 + i_{t-1}) M_{t-1} - M_t$$

so that the intertemporal budget constraint can be written as

$$W_t^i = \sum_{j=0}^{\infty} R_{t,t+j} \left[P_{t+j} C_{t+j}^i - P_{t+j} Y_{t+j}^i\right],$$

redefining $W_t^i \equiv (1 + i_{t-1}) B_{t-1}$.

To obtain a log-linear approximation to the household's decision problem, define the linearization point to be the steady state characterized by $\xi_t = 0$ and $Y_t = \bar{Y}$ (defined in appendix 2) for all t.[13] Inspection of the household's first-order conditions implies a solution of the form $\pi_t = P_t/P_{t-1} = 1$, and $\bar{\imath}_t = \beta^{-1} - 1$ for all t, where a bar denotes steady-state value. For any variable z, define log deviation as $\hat{z}_t \equiv \log(z_t/\bar{z})$, except for the nominal interest rate for which $\hat{\imath} = \log[(1 + i)/(1 + \bar{\imath})]$ is used.[14] The analysis seeks a log-linear

[12] Total expenditure is obtained by multiplying (35) by $p_t(j)$ and integrating over j. Applying the definition of the price index delivers the result.

[13] Given that this paper explores a form of bounded rationality that is a minimal deviation from rational expectations, and, moreover, that the analysis will be later concerned with whether an economy under learning dynamics can converge to the associated rational expectations equilibrium, the linearization point is chosen to coincide with that same rational expectations equilibrium.

[14] Thus all hatted variables are interpreted as percentage deviations. The nominal interest rate is treated differently so that it corresponds to percentage point deviations of the continuously compounded nominal interest rate.

solution in which all variables fluctuate forever near these steady-state values.

Appendix 2. Firm Problem

This appendix characterizes the firm's optimal pricing problem, defines the notion of the natural rate of output, and provides some details of the log-linear relations used in section 1.2. For a thorough analysis, see Woodford (2003, chap. 3). Differentiating (8) with respect to $p_t(i)$ gives firm i's first-order condition,

$$\hat{E}_t^i \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} Y_T P_T^{\theta} \left[\hat{p}_t^*(i) - \mu P_T s_{t,T}(i)\right] = 0, \tag{36}$$

where $\mu = \theta/(\theta - 1)$, $s_{t,T}(i)$ the firm i's real marginal cost function (defined below) in period $T \geq t$ given the optimal price $p_t^*(i)$ determined in period t. To derive a log-linear approximation to the firm's optimal pricing condition, recall that the steady state is defined as $\xi_t = 0$ and $Y_t = \bar{Y}$ for all t. Inspection of (36) indicates there exists a solution with $p_t^*/P_t = P_t/P_{t-1} = 1$ in each period t. Therefore, we look for a log-linear approximation in which P_t/P_{t-1} and p_t^*/P_t remain forever close to one. Before deriving this log-linear approximation, several other useful relations are derived.

Combining the household's optimal labor supply condition (34) with the firm's production function and differentiating with respect to $p_t(i)$ gives the firm's real marginal cost function

$$s(y, Y; \tilde{\xi}) = \frac{v_h(f^{-1}(y/A; \xi))}{u_c(Y; \xi)A} \cdot \frac{1}{f'(f^{-1}(y))}, \tag{37}$$

where $\tilde{\xi} \equiv (A_t, \xi_t)'$ is a composite vector of all preference and technology shocks. Real marginal costs therefore depend on both firm-specific and aggregate conditions.

Now suppose that firms have full information about the current state of the economy and are able to set prices each period—the case of fully flexible price setting. Then, a standard result from a model of monopolistic competition is that prices are optimally set according to the mark-up relation $\frac{p_t(i)}{P_t} = \frac{\theta}{\theta-1} \cdot s(y_t(i), Y_t; \tilde{\xi}_t) = \mu s(y_t(i), Y_t; \tilde{\xi}_t)$. Under this assumption on price-setting behavior, firms—regardless of beliefs—face a symmetric problem. It follows that, in equilibrium, $p_t(i) = P_t$ and $y_t(i) = Y_t$ for all i and t, and combining with the optimality condition (36) implies

$$s(Y_t^n, Y_t^n; \tilde{\xi}_t) = \mu^{-1}, \tag{38}$$

where the level of output Y_t^n that satisfies this condition is called the natural rate of output. It is the rate of output that occurs under fully flexible prices and varies in accordance with fundamental shocks $\tilde{\xi}_t$. The quantity of output, $\bar{Y}_t$, used in the definition of the steady state satisfies $s(\bar{Y}, \bar{Y}; 0) = \mu^{-1}$.

To obtain a log-linear approximation to (36), log-linearize equation (37) to give

$$\hat{s}_{t,T}(i) = \omega \hat{y}_T(i) + \sigma^{-1}\hat{Y}_T - (\omega + \sigma^{-1})\hat{Y}_T^n.$$

It follows that the real marginal cost of producing average or aggregate output, $y_t(i) = Y_t$, is

$$\hat{s}_T(i) = (\omega + \sigma^{-1})(\hat{Y}_T - \hat{Y}_T^n) = (\omega + \sigma^{-1})x_T,$$

where the latter equality implicitly defines the output gap $x_t = \hat{Y}_t - \hat{Y}_t^n$. This provides a relationship between the marginal cost of producing output $y_t(i)$ and the average marginal cost of producing total output $\hat{Y}_t$ of the following form:

$$\hat{s}_{t,T}(i) = \hat{s}_T - \omega\theta \left[p_t(i) - \sum_{\tau=t+1}^{T} \pi_\tau \right], \tag{39}$$

making clear that a firm's marginal cost in producing its good differs from average marginal cost to the extent that its price differs from the aggregate price.

Substituting into the firm's first-order condition, (36), for the discount factor using (33), linearizing and substituting for real marginal costs using (39) gives the prediction that optimal price of firm i satisfies the approximate log-linear relation

$$\widehat{p_t^*}(i) = \hat{E}_t^i \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\frac{1-\alpha\beta}{1+\omega\theta} \cdot (\omega + \sigma^{-1})x_T + \alpha\beta\hat{\pi}_{T+1} \right].$$

Thus firm i's optimal price is determined as a linear function of the future expected paths of the output gap and inflation. Variation in the optimal prices set by firms in period t can be due only to differences in beliefs.

Appendix 3. Conditions for Eigenvalues to Have Negative Real Parts

Consider the matrix A with dimension (3×3). From $|A - \lambda I| = 0$, the characteristic equation is

$$\lambda^3 - c_1 \lambda^2 + c_2 \lambda - c_3 = 0,$$

where $c_1 = \text{Trace}(A)$, c_2 is the sum of all second-order principal minors of A, and $c_3 = |A|$. The following restrictions on the coefficients c_i must be satisfied for all eigenvalues to have negative real parts:

$$\begin{aligned} c_1 &< 0 \\ c_3 - c_1 c_2 &> 0 \\ c_3 &< 0. \end{aligned}$$

For a matrix A with dimension (2×2), $|A - \lambda I| = 0$ implies the characteristic equation is

$$\lambda^2 - c_1 \lambda + c_2 = 0,$$

where $c_1 = \text{Trace}(A)$ and $c_2 = |A|$. For both eigenvalues to have negative real parts, $c_1 < 0$ and $c_2 > 0$ must be satisfied.

Appendix 4. Proof of Proposition 2

Agents are assumed to construct forecasts according to the relation

$$E_t z_{T+1} = a_z + b_z C^{T-t} s_t, \tag{40}$$

where $z_T = (\pi_T,\ x_T,\ i_T)'$and a_z and b_z are estimated coefficient matrices of appropriate dimension. It follows that

$$\hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} z_{T+1} = a_z \left(1 - \beta\right)^{-1} + b_z \left(I_n - \beta C\right)^{-1} s_t$$

and

$$\hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} z_{T+1} = a_z \left(1 - \alpha\beta\right)^{-1} + b_z \left(I_n - \alpha\beta C\right)^{-1} s_t.$$

Denoting these infinite sums by f_β and f_α, respectively, and substituting into (18) and (19) implies:

$$\begin{aligned} x_t &= -\frac{\sigma\psi_\pi}{w}\left[(1-\alpha)\beta,\ \kappa\alpha\beta,\ \beta\right]\cdot f_\alpha \\ &\quad + \frac{1}{w}\left[\sigma, (1-\beta), -\sigma\right]\cdot f_\beta - \frac{\sigma}{w}\cdot\left[1\ \ -1\right] f_\gamma \\ \pi_t &= \left(1-\frac{\kappa\sigma\psi_\pi}{w}\right)\left[(1-\alpha)\beta,\ \kappa\alpha\beta,\ \beta\right]\cdot f_\alpha \\ &\quad + \frac{\kappa}{w}\left[\sigma, (1-\beta), -\sigma\right]\cdot f_\beta - \frac{\kappa\sigma}{w}\cdot\left[1\ \ -1\right] f_\gamma, \end{aligned}$$

where $w = 1 + \sigma\psi_x + \sigma\kappa\psi_\pi$ and

$$f_\gamma = f'\left(I_n - \beta C\right)^{-1} s_t.$$

Recalling that the nominal interest rate is given by the Taylor rule

$$i_t = \bar{\imath}_t + \psi_\pi \pi_t + \psi_x x_t,$$

the system can be written compactly as

$$z_t = A_1 f_\alpha + A_2 f_\beta + A_3 f_\gamma,$$

where A_1, A_2, and A_3 collect obvious coefficients and have dimension (3×3). Substituting for $f_\alpha, f_\beta, f_\gamma$ gives

$$\begin{aligned} z_t &= \left[(1-\alpha\beta)^{-1} A_1 + (1-\beta)^{-1} A_2\right] a_z \\ &\quad + \left[A_1 b_z \left(1-\alpha\beta C\right)^{-1} + A_2 b_z (1-\beta)^{-1} + A_3 f'\left(I_n - \beta C\right)^{-1}\right] s_t. \end{aligned}$$

This expression combined with (40) for $T = t$ defines the E-stability mapping from current private forecast parameters to the optimal forecast coefficients as

$$T\begin{pmatrix} a_z \\ b_z \end{pmatrix} = \begin{pmatrix} \left[(1-\alpha\beta)^{-1} A_1 + (1-\beta)^{-1} A_2\right] a_z \\ \left[A_1 b_z \left(1-\alpha\beta C\right)^{-1} + A_2 b_z (1-\beta)^{-1}\right] C \end{pmatrix}.$$

The associated ordinary differential equation can then be written as

$$\frac{\partial\phi}{\partial\tau} = \begin{bmatrix} A - I_3 & \mathbf{0} \\ \mathbf{0} & H - I_{3n} \end{bmatrix}\phi,$$

where $\phi = \left(a_z', vec\,(b_z)'\right)'$ and

$$\begin{aligned} A &= (1-\alpha\beta)^{-1} A_1 + (1-\beta)^{-1} A_2 \\ H &= \left[(I_n - \alpha\beta C_n)^{-1} C_n\right]' \otimes A_1 + \left[(I_n - \beta C_n)^{-1} C_n\right]' \otimes A_2 \end{aligned}$$

are (3×3) and $(3n \times 3n)$ matrices, respectively. The Jacobian is then given as

$$D\frac{\partial\phi}{\partial\tau} = \begin{bmatrix} A - I_3 & \mathbf{0} \\ \mathbf{0} & H - I_{3n} \end{bmatrix}.$$

To complete the proof of this proposition, it remains to consider the properties of the eigenvalues of the matrix $H - I_{3n}$, since $A - I_3$ was considered in the main text. Note that C can be diagonalized to give

$$C = S\Lambda S^{-1}, \tag{41}$$

where Λ is a diagonal matrix with elements given by the eigenvalues, ρ_k, of C, and S is a matrix composed of the corresponding eigenvectors, v_k. Also note that we can write

$$\begin{aligned} (I_n - \alpha\beta C)^{-1} C &= S\Lambda\,(I_n - \alpha\beta\Lambda)^{-1} S^{-1} \\ (I_n - \beta C)^{-1} C &= S\Lambda\,(I_n - \beta\Lambda)^{-1} S^{-1}. \end{aligned}$$

The matrix H can therefore be written as

$$\begin{aligned} H &= \left[S\Lambda\,(I_n - \alpha\beta\Lambda)^{-1} S^{-1}\right]' \otimes A_1 \\ &\quad + \left[S\Lambda\,(I_n - \beta\Lambda)^{-1} S^{-1}\right]' \otimes A_2 \\ &= \left(S^{-1\prime} \otimes I_3\right)\left[\Lambda\,(I_n - \alpha\beta\Lambda)^{-1} \otimes A_1\right. \\ &\quad \left. +\ \Lambda\,(I_n - \beta\Lambda)^{-1} \otimes A_2\right]\left(S' \otimes I_3\right). \end{aligned}$$

Note that

$$G \equiv \Lambda\,(I_n - \alpha\beta\Lambda)^{-1} \otimes A_1 + \Lambda\,(I_n - \beta\Lambda)^{-1} \otimes A_2$$

is block diagonal with elements

$$G_k\,(\rho_k) = \rho_k\,(1 - \alpha\beta\rho_k)^{-1} \otimes A_1 + \rho_k\,(1 - \beta\rho_k)^{-1} \otimes A_2,$$

where each $G_k(\rho_k)$ is (3×3). Let v_k be an eigenvector of C associated with the eigenvalue ρ_k and let $\lambda_i(\rho_k)$ be an eigenvector of the associated diagonal block $G_k(\rho_k)$ (note that there are three such eigenvectors).

Conjecture that the matrix H has eigenvectors of the form $v_k \otimes \lambda_i(\rho_k)$. Then, in the particular case of $v_1 \otimes \lambda_i(\rho_1)$ (where, without loss of generality, assume ρ_1 to be the first diagonal element of Λ and v_1 the first column vector of S), we have

$$\begin{aligned}
(v_1 \otimes \lambda_i(\rho_1))' H &= (v_1 \otimes \lambda_i(\rho_1))' \left(S^{-1\prime} \otimes I_2\right) \Big[\Lambda (I_n - \alpha\beta\Lambda)^{-1} \\
&\quad \otimes A_1 + \Lambda (I_n - \beta\Lambda)^{-1} \otimes A_2\Big] \left(S' \otimes I_2\right) \\
&= \left(\left(S^{-1} v_1\right)' \otimes \lambda_i(\rho_1)'\right) \Big[\Lambda (I_n - \alpha\beta\Lambda)^{-1} \otimes A_1 \\
&\quad + \Lambda (I_n - \beta\Lambda)^{-1} \otimes A_2\Big] \left(S' \otimes I_2\right) \\
&= [\lambda_i(\rho_1)' \vdots 0_{1\times(3n-3)}] \Big[\Lambda (I_n - \alpha\beta\Lambda)^{-1} \otimes A_1 \\
&\quad + \Lambda (I_n - \beta\Lambda)^{-1} \otimes A_2\Big] \left(S' \otimes I_2\right) \\
&= [\lambda_i(\rho_1)' G_1(\rho_1) \vdots 0_{1\times(3n-3)}] \left(S' \otimes I_2\right) \\
&= [\lambda_i(\rho_1)' \gamma_i(\rho_1) \vdots 0_{1\times(3n-3)}] \left(S' \otimes I_2\right) \\
&= \gamma_i(\rho_1) (v_1 \otimes \lambda_i(\rho_1))'.
\end{aligned}$$

Thus $v_1 \otimes \lambda_i(\rho_1)$ is in fact an eigenvector of H with associated eigenvalue $\gamma_i(\rho_1)$. Since for each ρ_k there are three eigenvalues $\gamma_i(\rho_k)$ and corresponding eigenvectors v_k and $\lambda_i(\rho_k)$, there are therefore $3n$ eigenvectors of the form $v_k \otimes \lambda_i(\rho_k)$ that span the space of H.

To complete the proof requires demonstrating that all $3n$ eigenvalues $\gamma_i(\rho_k)$ are less than unity. Consider the properties of the matrix

$$G_k(\rho_k) - I_3$$

formed from the kth diagonal block of G. Using Mathematica, it is easily shown that one eigenvalue (corresponding to the coefficient relevant to the interest-rate dynamics) is equal to negative unity, while the remaining two have properties determined by the quadratic equation

$$a\lambda^2 + b\lambda + c.$$

A sufficient condition for there to be two roots with negative real parts is $a, b, c > 0$. It is easily shown that

$$a = (1 - \beta\rho_k)(1 - \alpha\beta\rho_k)(1 + \sigma\psi_x + \kappa\sigma\psi_\pi).$$

For the remaining two conditions to hold, the restrictions

$$\psi_\pi + \frac{(1 - \beta\rho_k)}{\kappa}\psi_x > \rho_k - \frac{(1 - \rho_k)(1 - \beta\rho_k)}{\kappa\sigma}$$

and

$$\psi_\pi + \frac{\psi_x}{\kappa} \cdot \frac{(1 - \alpha\beta\rho_k) + (1 - \beta\rho_k)^2}{(1 - \alpha\beta\rho_k) + (1 - \beta\rho_k)}$$
$$> \frac{\kappa\sigma(1 - \alpha\beta\rho_k) - \left[(1 - \alpha\beta\rho_k)(1 - \rho_k) + (1 - \beta\rho_k)^2\right]}{\kappa\sigma\left[(1 - \alpha\beta\rho_k) + (1 - \beta\rho_k)\right]}$$

must be satisfied. To show that satisfaction of the Taylor principle is necessary and sufficient for this first restriction to hold, note that the constant term of first inequality is less than unity by inspection and that the slope coefficient on ψ_x is necessarily greater than $(1 - \beta)/\kappa$ for $\rho_k \in (-1, 1)$. Thus for all positive (ψ_π, ψ_x), the Taylor principle ensures satisfaction of this inequality. Finally, consider the second inequality above. The constant is again less than unity by inspection. For the Taylor principle to be sufficient, consider the slope coefficient. If the restriction

$$\frac{(1 - \alpha\beta\rho_k) + (1 - \beta\rho_k)^2}{(1 - \alpha\beta\rho_k) + (1 - \beta\rho_k)} > 1 - \beta$$

holds, then the Taylor principle is indeed sufficient. Rearranging yields the restriction

$$f(\rho_k) \equiv \beta(1 - \alpha\beta\rho_k) + (1 - \beta\rho_k)^2 - (1 - \beta)(1 - \beta\rho_k) > 0,$$

which satisfies $f(-1),\ f(0),\ f(1) > 0$. Since $f'(\rho_k) < 0$ for all $\rho_k \in (-1, 1)$, $f(\rho_k) > 0$ for all $\rho_k \in (-1, 1)$. The Taylor principle is therefore necessary and sufficient for the restrictions $b, c > 0$ to hold. Thus, all eigenvalues of $G_k(\rho_k) - I_3$ have negative real parts and $G_k(\rho_k)$ all have eigenvalues less than unity if and only if the Taylor principle holds. It follows that all eigenvalues of H must be less than unity and, therefore, that the eigenvalues of $H - I_{3n}$ all have negative real parts. The conditions for E-stability are therefore satisfied if and only if the Taylor principle holds.

References

Bernanke, Ben S., and Michael Woodford. 1997. "Inflation Forecasts and Monetary Policy." *Journal of Money, Credit, and Banking* 29 (4): 653–86.

Blanchard, Olivier J., and Charles Kahn. 1980. "The Solution of Linear Difference Models Under Rational Expectations." *Econometrica* 48:1305–11.

Bullard, James, and Kaushik Mitra. 2000. "Determinacy, Learnability and Monetary Policy Inertia." Working Paper No. 2000-030B, Federal Reserve Bank of St. Louis.

———. 2002. "Learning About Monetary Policy Rules." *Journal of Monetary Economics* 49 (6): 1105–29.

Clarida, Richard, Jordi Galí, and Mark Gertler. 1998. "Monetary Policy Rules in Practice: Some International Evidence." *European Economic Review* 42:1033–67.

———. 1999. "The Science of Monetary Policy: A New Keynesian Perspective." *Journal of Economic Literature* 37:1661–1707.

———. 2000. "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory." *Quarterly Journal of Economics* 115:147–80.

Evans, George W., and Seppo Honkapohja. 2001. *Learning and Expectations in Macroeconomics.* Princeton, NJ: Princeton University Press.

———. 2002. "Monetary Policy, Expectations and Commitment." Working Paper.

———. 2003. "Expectations and the Stability Problem for Optimal Monetary Policies." *Review of Economic Studies* 70 (4).

Friedman, Milton. 1968. "The Role of Monetary Policy." *American Economic Review* 58 (1): 1–17.

Giannoni, Marc P., and Michael Woodford. 2002a. "Optimal Interest-Rate Rules: I. General Theory." Unpublished, Columbia University.

———. 2002b. "Optimal Interest-Rate Rules: II. Applications." Unpublished, Columbia University.

Honkapohja, Seppo, and Kaushik Mitra. 2004. "Are Non-Fundamental Equilibria Learnable in Models of Monetary Policy?" *Journal of Monetary Economics* 58 (8): 1743–70.

———. 2005. "Performance of Monetary Policy with Internal Central Bank Forecasting." *Journal of Economic Dynamics and Control* 29 (4): 627–58.

Honkapohja, Seppo, Kaushik Mitra, and George W. Evans. 2002. "Notes on Agents' Behavioral Rules Under Adaptive Learning and Recent Studies of Monetary Policy." Unpublished Working Paper.

Howitt, Peter. 1992. "Interest Rate Control and Nonconvergence to Rational Expectations." *Journal of Political Economy* 100 (4): 776–800.

Marcet, Albert, and Thomas J. Sargent. 1989. "Convergence of Least-Squares Learning in Environments with Hidden State Variables and Private Information." *Journal of Political Economy,* 1306–22.

McCallum, Bennett T. 1983. "On Non-uniqueness in Rational Expectations Models: An Attempt at Perspective." *Journal of Monetary Economics* 11:139–68.

———. 1999. "Issues in the Design of Monetary Policy Rules." In *Handbook of Macroeconomics*, Vol. 1C, ed. John B. Taylor and Michael Woodford. Amsterdam: North-Holland.

Preston, Bruce. 2002. "Interaction of Monetary and Fiscal Policy Rules under Adaptive Learning." Unpublished, Princeton University.

———. 2004. "Adaptive Learning and the Use of Forecasts in Monetary Policy." Unpublished, Columbia University.

———. Forthcoming. "Adaptive Learning, Forecast-Based Instrument Rules and Monetary Policy." *Journal of Monetary Economics.*

Sargent, Thomas J. 1993. *Bounded Rationality in Macroeconomics.* Oxford University Press.

Sargent, Thomas J., and Neil Wallace. 1975. "Rational Expectations, the Optimal Monetary Instrument, and the Optimal Monetary Supply Rule." *Journal of Political Economy* 83:241–54.

Svensson, Lars E.O., and Michael Woodford. 2002. "Implementing Optimal Monetary Policy Through Inflation-Forecast Targeting." Unpublished Manuscript, Princeton University.

Taylor, John. 1993. "Discretion Versus Policy Rules in Practice." *Carnegie-Rochester Conference Series on Public Policy* 39:195–214.

———. 1999. "A Historical Analysis of Monetary Policy Rules." In *Monetary Policy Rules*, ed. John Taylor. Chicago: University of Chicago Press.

Woodford, Michael. 1999. "Optimal Monetary Policy Inertia." NBER Working Paper No. 7261.

———. 2003. *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton, NJ: Princeton University Press.