Behavioral economics and auto-images of distributions of random variables

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Distributions of random variables defined on finite intervals were considered in connection with some problems of behavioral economics. To develop the results obtained for finite intervals, auto-image distributions of random variables defined on infinite or semi-infinite intervals are proposed in this article. The proposed auto-images are intended for constructing reference auto-image distributions for preliminary considerations and estimates near the boundaries of semi-infinite intervals and on finite intervals.

Contents

1. Introduction

1.1. Bounds for functions and moments of random variables

Bounds for functions and moments of random variables are considered in a number of works.

Bounds for the probabilities and expectations of convex functions of discrete random variables with finite support are studied in [8].

Inequalities for the expectations of functions are studied in [9]. These inequalities are based on information of the moments of discrete random variables.

A class of lower bounds on the expectation of a convex function using the first two moments of the random variable with a bounded support is considered in [1].

Bounds on the exponential moments of $\min(y, X)$ and $X\,I\{X < y\}$ using the first two moments of the random variable X are considered in [7].

1.2. Problems of applied sciences

There are some basic problems concerned with the mathematical description of the behavior of a man. They are the most actual in behavioral economics, decision sciences, social sciences and psychology. They are pointed out, e.g., in [6].

Examples of the problems are the underweighting of high and the overweighting of low probabilities, risk aversion, the Allais paradox, risk premium, the four-fold pattern paradox, etc.

The essence of the problems consists in biases of preferences and decisions of a man in comparison with predictions of the probability theory.

These biases are maximal near the boundaries of the probability scale, that is, at high and low probabilities.

1.3. Bounds for the expectation and solution of the problems

1.3.1. Bounds for the expectation

Bounds on the expectation of a random variable that takes on values in a finite interval are considered as well (see, e.g., a plenary report [4] and paper [5]). These bounds are based on information of the variance of the variable.

Suppose a random variable takes on values in a finite interval. An existence theorem was proven. The theorem states: if there is a non-zero lower bound on the variance of the variable, then non-zero bounds on its expectation exist near the boundaries of the interval.

The discovered non-zero bounds (or strict bounding inequalities) can be treated as non-zero forbidden zones for the expectation near the boundaries of the interval. These bounds (forbidden zones) cause biases of experimental data and, hence, biases of preferences and decisions of a man in comparison with predictions of the probability theory.

Note. It is intuitively evident that the minimal distance between the expectation of a random variable and the nearest boundary of the interval can be equal to zero only if the support of the distribution is a sole point. Nevertheless, for the sake of the mathematical rigor, this statement should be proven and will be proven in some future article of this series.

1.3.2. Partial solution of the problems

A non-zero noise can be associated with the above non-zero lower bound on the variance of the random variable.

So, a noise can lead to bounds (restrictions) on the expectations of experimental data. This influence of a noise should be taken into account when dealing with data obtained in real circumstances.

The works [2] and [3] were devoted to the well-known problems of utility and prospect theories. Such problems had been pointed out, e.g., in [6]. In [2] and [3] some examples of typical paradoxes were studied. Similar paradoxes may concern problems such as the underweighting of high and the overweighting of low probabilities, risk aversion, the Allais paradox, etc. A noise and data scattering are usual circumstances of the experiments. The proposed bounds explain, at least partially, the analyzed examples of paradoxes.

1.4 Need of further research. Aims of the present article

The discovered non-zero forbidden zones for the expectations near the boundaries of the intervals are proven only for random variables that are defined on the finite intervals. So there is a problem of expansion of the above theorem and considerations to cover more general cases.

The aim of the present article is to make first steps to generalizing of the above theorem and considerations.

2. Auto-images of distributions of random variables

2.1. Definitions

Suppose the distribution of a random variable is defined on an infinite or half-infinite interval. Further this distribution will be referred to as the mainframe one (**MF-distribution** or **MFD**). This interval will be referred to as the mainframe one (**MF-interval** or **MFI**).

Half-infinite or finite parts of the infinite MF-distributions and finite parts of the half-infinite MF-distributions will be referred to as **intervals of auto-image** or **auto-image intervals** (**AII**), if they comprise (at least on their boundaries) at least one of the main features of the MFI, such as the expectations, medians or modes.

An image of the mainframe distribution that is mapped into the auto-image interval will be referred to as an **auto-image distribution** (**AID**).

The AII boundary that is the nearest to the expectation of the variable will be referred to as $\boldsymbol{b \equiv b_{Boundary}}$. So the minimal distance between the expectation of the auto-image distribution and the nearest boundary $b_{Boundary}$ of the auto-image interval will be referred to as $min(|E[X_{AID}] - b_{Boundary}|)$ or $min(|E[X_{AID}] - b|)$.

An auto-image of the mainframe distribution that is mapped into the finite auto-image interval will be referred to as a **compact** auto-image distribution.

The auto-image distributions are composed of two parts. The parts of MFD, that are determined in the scopes of AII, are mapped into the AII without any modifications. The mainframe distributions parts that are determined on the MFI beyond the AII (the **out-auto-image-interval parts** or **out-AII parts**), are mapped into the AII with some modifications.

2.2. Modifications of out-AII parts

Let us consider some possible cases of the modifications of the out-auto-image parts of the mainframe distributions into the auto-image distributions. For the sake of the definiteness, let us consider the case of continuous distributions.

Uniformity

The out-AII parts are mapped into the AII in the forms of the uniform multiplication rising coefficient or of the uniform addition part.

Examples. Let us choose the mainframe interval as *(-∞, +∞)* and the auto-image interval as *[a, b]*. Let us denote the integrals on the out-AII parts as δ

$$\int_{-\infty}^{a} f(x)dx + \int_{b}^{+\infty} f(x)dx = \delta .$$

Multiplication. The mainframe interval part of the mainframe distribution in whole is multiplied by the uniform normalizing rising coefficient

$$f_{AII}(x \mid x \in [a,b]) = f_{MFI}(x) + \delta \times f_{MFI}(x) = f_{MFI}(x)(1+\delta) .$$

Addition. The out-AII parts are uniformly added to the mainframe interval part of the mainframe distribution

$$f_{AII}(x \mid x \in [a,b]) = f_{MFI}(x) + \frac{\delta}{b-a} .$$

Reflection

The out-auto-image part of the mainframe distribution is reflected with respect to the boundary of AII. In the case of a finite AII the reflection may be multiple. For example, in the case of the AII *[0, ∞)* the probability density function f_{AII} of the auto-image distribution can be expressed in terms of PDF f_{MFI} of the mainframe distribution as

$$f_{AII}(x) = f_{MFI}(x) + f_{MFI}(-x) .$$

The case of the reflection auto-image is, in a sense, similar to the reflection of a wave of light from a mirror.

Adhesion

The out-AII part of the mainframe distribution is adhered to the boundary of AII. In the case of the continuous mainframe PDF, it is transformed to the auto-image PDF of the mixed type, such that the discrete part of auto-image PDF is adhered to the boundary of AII.

The hypothetical adhesion situation is in a sense similar to the absorption of a wave of light by a black body.

2.3. About the expectations of adhered auto-images

One can prove (see Appendix 1) that, for the adhered auto-images, the distances from the expectations of auto-image distributions to the boundaries of AII are the minimal among all types of the auto-images.

2.4. Sufficient auto-images and intervals

An auto-image (and auto-image interval) can be referred to as the **sufficient** (concerning some parameter) auto-image (and auto-image interval), if the difference between the values of this parameter calculated for the MFD and AID is negligibly small in comparison with the value of this parameter calculated for the MFD.

For example:

An auto-image will be referred to as the **V-sufficient** auto-image (where "V" denotes Variation) if the part(s) of the variation of the mainframe distribution that is (are) calculated outside the auto-image interval is (are) much less than the part of the variation of the mainframe distribution that is calculated within the auto-image interval. An example of V-sufficient auto-image can be written for the auto-image interval *[a, b]* as

$$\int_{-\infty}^{a} \{x - E[X]\}^2 f(x)dx + \int_{b}^{+\infty} \{x - E[X]\}^2 f(x)dx << \\ << \int_{a}^{b} \{x - E[X]\}^2 f(x)dx \, .$$

An auto-image will be referred to as the **SD-sufficient** auto-image (where "SD" denotes Standard Deviation) if the part(s) of the standard deviation of the mainframe distribution, that is (are) calculated outside the auto-image interval, is (are) much less than the part of the standard deviation of the mainframe distribution that is calculated within the auto-image interval.

The sum of all probabilities must be equal to one. Therefore the area under the PDF curve must be equal to one.

$$\int_{-\infty}^{+\infty} f(x)dx = 1 .$$

An auto-image will be referred to, e.g., as the **Sum-sufficient** or **S-sufficient** (or **Norm-sufficient** or **N-sufficient**) auto-image if the integral of part(s) of the PDF of the mainframe distribution that is (are) calculated outside the auto-image interval is (are) much less than the integral of the part of the PDF of the mainframe distribution that is calculated within the auto-image interval.

An example of S-sufficient auto-image can be written for the auto-image interval *[a, b]* as

$$\int_{-\infty}^{a} f(x)dx + \int_{b}^{+\infty} f(x)dx << \int_{a}^{b} f(x)dx \le \int_{-\infty}^{+\infty} f(x)dx \, .$$

For the normal distribution, the auto-image interval that corresponds to the "three-sigma rule" can be used as S-sufficient auto-image interval.

3. About the expectations of reflected auto-images

3.1. Goal and idea

Let us search for conditions that ensure the minimal distance from the expectation of the reflected auto-image distribution to the nearest (finite) boundary of the auto-image interval (that is to the point of reflection). General considerations of all possible cases of this goal are too vast. Let us start from a particular case.

Let us consider continuous mainframe distributions defined on the infinite interval $(-\infty, +\infty)$ and the case of the reflection auto-images. The point of reflection can be denoted, e.g., as $\pi_{Rfl} \equiv \pi_{Reflection}$. The mainframe distribution can be reflected, e.g., to the right.

The PDF f_{AID} of the auto-image distribution is

$$f_{AID}(x \mid x \geq \pi_{Rfl}) = f(x) + f(2\pi_{Rfl} - x) \equiv f_{MFD}(x) + f_{MFD}(2\pi_{Rfl} - x) .$$

The expectation of the auto-image distribution (as a function of the point of reflection) is

$$\mu_{AID}(\pi_{Rfl}) = \int_{\pi_{Rfl}}^{+\infty} x[f(x) + f(2\pi_{Rfl} - x)]dx = .$$

First of all, let us consider a simple example that is similar to a reflected wave. Suppose a hypothetic "wave" that consists of two discrete values f_1 and f_2.

If f_1 and f_2 of the MFD move toward a mirror, then the expectation of the AID moves toward it as well. If f_1 and f_2 of the MFD have reflected and move away from the mirror, then the expectation of the AID moves away from it as well.

Suppose f_2 moves toward the mirror and f_1 has reflected and moves from it.

Suppose $f_2 \neq f_1$. In this case the direction of the motion of the expectation of the AID corresponds to that of the "heaviest" value (irrespective of the location of MFD expectation).

If $f_2 = f_1$, then there is no motion of the expectation of the AID while these values are located on the opposite sides of the mirror (irrespective of the location of MFD expectation).

A natural presupposition is that the minimal distance from the expectation of AID to the point of reflection is determined by the expectation of MFD. Nevertheless, an idea, that this distance can depend, at least also, on the median of the MFD, follows from the above example of the reflected two-values-wave.

3.2. Proof

The statement, that the minimal distance from the expectation of the reflected AID to the boundary of the AII is attained when the point of reflection coincides with the median of the mainframe distribution, is proven in Appendix 2.

A scheme of a moving mirror (moving point of reflection) will be used further due to its convenience.

4. Conclusions

The given article proposes and considers the auto-images of distributions of random variables.

The consideration has included the cases of the uniformity, reflection and accumulation. The sufficient auto-images are defined as well.

In the case of the reflection, the distance from the expectation of the auto-images of distributions has been shown to be minimal when the median of the mainframe distributions coincides with the points of reflection.

The proposed auto-images of distributions of random variables are intended for constructing reference auto-image distributions for preliminary considerations and estimates near the boundaries of semi-infinite intervals and on finite intervals of such AI distributions that were initially defined on infinite or semi-infinite intervals. They can assist to develop the results of works for finite intervals (see, for example, [2-5]), in particular in the field of the behavioral economics.

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5. Appendices

5.1. Appendix 1. Adhesion

5.1.1. Discrete distributions

Let us consider a discrete mainframe distribution which probability mass function (PMF) f is defined on the mainframe interval $[a, +\infty)$. Choose an auto-image interval $[a, b]$, where b will be referred to as $b_{Boundary}$.

Let us compare two cases:

1) The first is the case when the out-AII part is adhered to the boundary b of the AII. This adhered part can be determined as

$$f_b \equiv \sum_{x \in (b,\infty)} f(x_k) \equiv f_{Out} .$$

2) The second is the case when the out-AII part is mapped into the AII, at least not only on the boundary $b_{Boundary}$ of auto-image interval. This mapped part can be determined with the aid of the additions of the auto-image PMF Δf_{AII} as

$$\sum_{x \in [a,b]} \Delta f_{AII}(x_k) = \sum_{x \in (b,\infty)} f(x_k) \equiv f_{Out} .$$

The coordinates of every point x_k are not more than the coordinate of the boundary $b_{Boundary}$. Therefore the comparison of the expectations of these two auto-image distributions gives

$$\sum_{x \in [a,b]} x_k \Delta f_{AII}(x_k) \le \sum_{x \in [a,b]} b \Delta f_{AII}(x_k) = b \sum_{x \in [a,b]} \Delta f_{AII}(x_k) = b f_{Out} = b f_b .$$

Therefore, for the discrete case, the expectation of the adhered part of the AID is not less than that of the mapped parts. The same is true for the expectations of the whole auto-image distributions. Therefore the distance from the boundary $b_{Boundary}$ to the expectation of the whole AID is the minimal for the AID which out-AII part is adhered.

5.1.2. Continuous distributions

Let us consider a continuous mainframe distribution which probability density function f is defined on the mainframe interval $[a, +\infty)$. Choose an auto-image interval $[a, b]$, where b will be referred to as $b_{Boundary}$.

Let us compare two cases:

1) The first is the case when the out-AII part is adhered to the boundary b of the AII. This adhered part can be determined as

$$f_b \equiv f_{Out} \equiv \int_b^{+\infty} f_{MFD}(x)dx\,.$$

2) The second is the case when the out-AII part is mapped into the AII, at least not only on the boundary $b_{Boundary}$ of auto-image interval. This mapped part can be determined with the aid of the additions of the auto-image PDF Δf_{AII} as

$$\int_a^b \Delta f_{AII}(x)dx = \int_b^{+\infty} f_{MFD}(x)dx \equiv f_{Out}\,.$$

The coordinates of every point x are not more than the coordinate of the boundary $b_{Boundary}$. Therefore the comparison of the expectations of these two auto-image distributions gives

$$\int_a^b x\Delta f_{AII}(x)dx \le \int_a^b b\Delta f_{AII}(x)dx = b\int_a^b \Delta f_{AII}(x)dx = bf_{Out} = bf_b\,.$$

Therefore, for the continuous case, the expectation of the adhered part of the AID is not less than that of the mapped parts. The same is true for the expectations of the whole auto-image distributions. Therefore the distance from the boundary $b_{Boundary}$ to the expectation of the whole AID is the minimal for the AID which out-AII part is adhered.

For mixed cases the considerations can use superpositions of the above discrete and continuous cases.

5.2. Appendix 2. Reflection

5.2.1. Right median

Suppose the location of the point of reflection is zero (that is $\pi_{Rfl} = 0$), $m \geq 0$ (that is the median of the MFD is on the right of the point of reflection) and the mainframe distribution is "reflected" to the right. This situation can be referred to as the "initial" one.

The expectation of the AID (as a function of the point of reflection) is

$$\mu_{AID}(0) = \int_0^{+\infty} x[f(x) + f(-x)]dx .$$

Displacement

Suppose the point of reflection is shifted from the initial location to the right or to the left by some displacement $\varepsilon : 0 \leq \varepsilon \leq m$.

The PDF of the shifted auto-image distribution is

$$f_{AID}(x \mid x \geq \pi_{Rfl} = \varepsilon) = f(x) + f(2\varepsilon - x) \equiv f_{MFD}(x) + f_{MFD}(2\varepsilon - x) .$$

The expectation of the auto-image distribution may be rewritten as

$$\mu_{AID}(\varepsilon) = \int_\varepsilon^{+\infty} xf(x)dx + \int_\varepsilon^{+\infty} xf(2\varepsilon - x)dx .$$

Consider the last integral. Let us introduce temporarily a new argument y such that

$$y = x - 2\varepsilon \quad \text{and} \quad x = y + 2\varepsilon .$$

This leads to

$$\int_\varepsilon^{+\infty} xf(2\varepsilon - x)dx = \int_{\varepsilon-2\varepsilon}^{+\infty} (y + 2\varepsilon) f(-y)dy = \int_{-\varepsilon}^{+\infty} (y + 2\varepsilon) f(-y)dy = \\ = \int_{-\varepsilon}^{+\infty} yf(-y)dy + 2\varepsilon \int_{-\varepsilon}^{+\infty} f(-y)dy .$$

The last integral can be rewritten as

$$2\varepsilon \int_{-\varepsilon}^{+\infty} f(-x)dx = 2\varepsilon \int_{-m}^{+\infty} f(-x)dx - 2\varepsilon \int_{-m}^{-\varepsilon} f(-x)dx = 2\varepsilon \times \frac{1}{2} - 2\varepsilon \int_{-m}^{-\varepsilon} f(-x)dx = \\ = \varepsilon - 2\varepsilon \int_{-m}^{-\varepsilon} f(-x)dx .$$

So, the expectation of AID may be rewritten as

$$\mu_{AID}(\varepsilon) = \int_{\varepsilon}^{+\infty} xf(x)dx + \int_{-\varepsilon}^{+\infty} xf(-x)dx + \varepsilon - 2\varepsilon \int_{-m}^{-\varepsilon} f(-x)dx =$$

$$= \int_{0}^{+\infty} x[f(x) + f(-x)]dx -$$

$$- \int_{0}^{\varepsilon} xf(x)dx + \int_{-\varepsilon}^{0} xf(-x)dx + \varepsilon - 2\varepsilon \int_{-m}^{-\varepsilon} f(-x)dx .$$

The three last integrals can be rewritten as

$$\int_{-\varepsilon}^{0} xf(-x)dx = -\int_{-\varepsilon}^{0} |x| f(-x)dx =$$

$$= -\int_{0}^{\varepsilon} xf(x)dx .$$

and

$$-2\varepsilon \int_{-m}^{-\varepsilon} f(-x)dx = -2\varepsilon \int_{\varepsilon}^{m} f(x)dx .$$

and

$$- \int_{0}^{\varepsilon} xf(x)dx + \int_{-\varepsilon}^{0} xf(-x)dx + \varepsilon - 2\varepsilon \int_{-m}^{-\varepsilon} f(-x)dx =$$

$$= -2\int_{0}^{\varepsilon} xf(x)dx + \varepsilon - 2\varepsilon \int_{\varepsilon}^{m} f(x)dx .$$

So

$$\mu_{AID}(\varepsilon) = \int_{0}^{+\infty} x[f(x) + f(-x)]dx -$$

$$-2\int_{0}^{\varepsilon} xf(x)dx + \varepsilon - 2\varepsilon \int_{\varepsilon}^{m} f(x)dx .$$

The sought distance from the expectation of the reflected auto-image to the nearest finite-end boundary of the auto-image interval (that is to the point of reflection) is

$$\mu_{AID}(\varepsilon) - \varepsilon = \int_{0}^{+\infty} x[f(x) + f(-x)]dx -$$

$$-2\int_{0}^{\varepsilon} xf(x)dx - 2\varepsilon \int_{\varepsilon}^{m} f(x)dx .$$

Difference

The difference between the minimal distances from the points of reflection to the initial and shifted expectations of the auto-image distribution is

$$[\mu_{AID}(\varepsilon)-\varepsilon]-[\mu_{AID}(0)-0]=$$
$$=\int_0^{+\infty}x[f(x)+f(-x)]dx-2\int_0^{\varepsilon}xf(x)dx-2\varepsilon-\int_0^{+\infty}x[f(x)+f(-x)]dx=.$$
$$=-2\int_0^{\varepsilon}xf(x)dx-2\varepsilon\int_{\varepsilon}^{m}f(x)dx$$

If m is sufficiently large (that is the median of the MFD is located sufficiently far leftward the points of reflection) and

$$\int_{\varepsilon}^{m}f(x)dx\approx\frac{1}{2}.$$

takes place, then

$$-2\varepsilon\int_{\varepsilon}^{m}f(x)dx\approx\varepsilon\ .$$

and the difference is approximately equal to $-\varepsilon$. That is the expectation of the AID is practically motionless, and the point of reflection is shifted to the expectation of the AID by the same displacement ε. This is similar to the above case, when f_1 and f_2 of the MFD move toward a mirror.

As for the main goal of the section, the both above integrals are not negative until $\varepsilon\le m$ and the difference is not positive.

Therefore, for the right location of the median, the minimal distance from the expectation $\mu_{AID}\equiv\mu_{Im}\equiv\mu_{Image}$ of the auto-image distribution to the point of reflection is attained at $\varepsilon = m$.

5.2.2. Left median

Suppose $\pi_{Rfl} = 0$, the median is located at $-m \leq 0$ (that is the median is on the left of the point of reflection) and the MFD is "reflected" to the right as in the preceding case such that the PDF f_{AI} of the AID is

$$f_{AID}(x \mid x \geq \pi_{Rfl} = 0) = f(x) + f(-x) \equiv f_{MFD}(x) + f_{MFD}(-x).$$

The expectation of the auto-image distribution is

$$\mu_{AID}(m) = \int_0^{+\infty} x[f(x) + f(-x)]dx.$$

This situation can be referred to as the "initial" one.

Displacement

Suppose the point of reflection is shifted to the right or to the left by some displacement $\varepsilon : |\varepsilon| \geq 0$. The PDF of the shifted auto-image distribution is

$$f_{AID}(x \mid x \geq \pi_{Rfl} = \varepsilon) = f(x) + f(2\varepsilon - x) \equiv f_{MFD}(x) + f_{MFD}(2\varepsilon - x).$$

The expectation of the auto-image distribution may be rewritten as

$$\mu_{AID}(m) = \int_\varepsilon^{+\infty} xf(x)dx + \int_\varepsilon^{+\infty} xf(2\varepsilon - x)dx.$$

Let us introduce temporarily a new argument y such that

$$y = x - 2\varepsilon \quad \text{and} \quad x = y + 2\varepsilon.$$

For the last integral this leads to

$$\int_\varepsilon^{+\infty} xf(2\varepsilon - x)dx = \int_{\varepsilon-2\varepsilon}^{+\infty} (y + 2\varepsilon) f(-y)dy = \\ = \int_{-\varepsilon}^{+\infty} (x + 2\varepsilon) f(-x)dx.$$

The expectation may be rewritten as

$$\mu_{AID}(m) = \int_\varepsilon^{+\infty} xf(x)dx + \int_{-\varepsilon}^{+\infty} (x + 2\varepsilon) f(-x)dx.$$

The first integral can be rewritten as

$$\int_\varepsilon^{+\infty} xf(x)dx = \int_0^{+\infty} xf(x)dx - \int_0^{\varepsilon} xf(x)dx.$$

The expectation is

$$\mu_{AID}(m) = \int_\varepsilon^{+\infty} xf(x)dx + \int_{-\varepsilon}^{+\infty} (x + 2\varepsilon) f(-x)dx = \\ = \int_0^{+\infty} xf(x)dx - \int_0^{\varepsilon} xf(x)dx + \int_{-\varepsilon}^{+\infty} (x + 2\varepsilon) f(-x)dx.$$

The last integral can be rewritten as

$$\int_{-\varepsilon}^{+\infty}(x+2\varepsilon)f(-x)dx =$$
$$= \int_{0}^{+\infty} xf(-x)dx + \int_{0}^{+\infty} 2\varepsilon f(-x)dx + \int_{-\varepsilon}^{0}(x+2\varepsilon)f(-x)dx \; .$$

The last integral can be rewritten as

$$\int_{-\varepsilon}^{0}(x+2\varepsilon)f(-x)dx = \int_{-\varepsilon}^{0}(-|x|+2\varepsilon)f(-x)dx =$$
$$= \int_{0}^{\varepsilon}(2\varepsilon - x)f(-x)dx \; .$$

The expectation can be rewritten as

$$\mu_{AID}(m) = \int_{0}^{+\infty} xf(x)dx - \int_{0}^{\varepsilon} xf(x)dx + \int_{-\varepsilon}^{+\infty}(x+2\varepsilon)f(-x)dx =$$
$$= \int_{0}^{+\infty} xf(x)dx + \int_{0}^{+\infty} xf(-x)dx +$$
$$+ \int_{0}^{+\infty} 2\varepsilon f(-x)dx - \int_{0}^{\varepsilon} xf(x)dx + \int_{0}^{\varepsilon}(2\varepsilon - x)f(-x)dx = \; .$$
$$= \int_{0}^{+\infty} x[f(x)+f(-x)]dx + 2\int_{0}^{\varepsilon}(\varepsilon - x)f(-x)dx + 2\varepsilon\int_{0}^{+\infty} f(-x)dx$$

The last integral can be rewritten as

$$2\varepsilon\int_{0}^{+\infty} f(-x)dx = 2\varepsilon\int_{0}^{m} f(-x)dx + 2\varepsilon\int_{m}^{+\infty} f(-x)dx = 2\varepsilon\int_{0}^{m} f(-x)dx + 2\varepsilon\times\frac{1}{2} =$$
$$= 2\varepsilon\int_{0}^{m} f(-x)dx + \varepsilon \; .$$

So the expectation is

$$\mu_{AID}(m) = \int_{0}^{+\infty} x[f(x)+f(-x)]dx + 2\int_{0}^{\varepsilon}(\varepsilon - x)f(-x)dx + 2\varepsilon\int_{0}^{+\infty} f(-x)dx =$$
$$= \int_{0}^{+\infty} x[f(x)+f(-x)]dx + 2\int_{0}^{\varepsilon}(\varepsilon - x)f(-x)dx + 2\varepsilon\int_{0}^{m} f(-x)dx + \varepsilon \; .$$

The sought distance from the expectation of the reflected auto-image to the nearest finite-end boundary of the auto-image interval (that is to the point of reflection) is

$$\mu_{AID}(\varepsilon) - \varepsilon = \int_{0}^{+\infty} x[f(x)+f(-x)]dx + 2\int_{0}^{\varepsilon}(\varepsilon - x)f(-x)dx + 2\varepsilon\int_{0}^{m} f(-x)dx \; .$$

Difference

The difference between the initial and shifted expectation of the auto-image distribution is

$$[\mu_{AID}(\varepsilon)-\varepsilon]-[\mu_{AID}(0)-0]=$$
$$=\int_0^{+\infty} x[f(x)+f(-x)]dx+2\int_0^{\varepsilon}(\varepsilon-x)f(-x)dx+2\varepsilon\int_0^{m}f(-x)dx-$$
$$-\int_0^{+\infty} x[f(x)+f(-x)]dx=$$
$$=2\int_0^{\varepsilon}(\varepsilon-x)f(-x)dx+2\varepsilon\int_0^{m}f(-x)dx$$

If m is sufficiently large (that is the median of the MFD is located sufficiently far leftward zero) to

$$\int_0^{m}f(-x)dx\approx\frac{1}{2},$$

be true, then the difference is approximately equal to ε. That is the expectation of the AID is practically motionless, and the point of reflection is shifted away from the expectation of the AID by the same displacement ε. This is similar to the above case, when f_1 and f_2 of the MFD move away from the mirror.

As for the main goal of the section, the both above integrals are non-negative. Hence the difference is non-negative as well.

Therefore, for the left location of the median, the minimal distance from the expectation $\mu_{AID} \equiv \mu_{Im} \equiv \mu_{Image}$ of the auto-image distribution to the point of reflection is attained at $\varepsilon = m$.

Therefore, both for the left and right locations of the median, the minimal distance from the expectation μ_{AID} of the auto-image distribution to the point of reflection $\pi_{Rfl} \equiv \pi_{Reflection}$ is attained at $\pi_{Rfl} = m$.