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# Bingmann-Lovejoy-Osburn's generating function in the overpartitions

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## Abstract

In 2009, Bingmann, Lovejoy and Osburn defined the generating function for  $spt(n)$ . In 2012, Andrews, Garvan and Liang defined the  $sptcrank$  in terms of partition pairs. In this article the number of smallest parts in the overpartitions of  $n$  with smallest part not *overlined* is discussed, and the vector partitions and  $\bar{5}$ -partitions with 4 components, each a partition with certain restrictions are also discussed. The generating function for  $spt(n)$ , and the generating function for  $M_s(m, n)$  are shown with a result in terms of modulo 3. This paper shows how to prove the Theorem 1 in terms of  $M_s(m, n)$  with a numerical example, and shows how to prove the Theorem 2 with the help of  $sptcrank$  in terms of partition pairs. In 2014, Garvan and Jennings-Shaffer are able to define the  $sptcrank$  for marked overpartitions. This paper also shows another result with the help of 6  $SP$ -partition pairs of 3 and shows how to prove the Corollary with the help of 42 marked overpartitions of 6.

**Keywords:** Components, Congruent, Crank, Non-Negative, Overpartitions, Overlined, Weight.

## 1. Introductions

In this paper we give some related definitions of  $\overline{spt}(n)$ , various product notations, vector partitions and  $\bar{s}$ -partitions,  $M_{\bar{s}}(m, n)$ ,  $M_{\bar{s}}(m, t, n)$ , marked partition and  $\overline{sptcrank}$  for marked overpartitions. We discuss the generating function for  $\overline{spt}(n)$  and prove the Corollary 1 with the help of generating function for  $M_{\bar{s}}(m, n)$ . We prove the Result 1 with the help of 8 vector partitions from  $\bar{s}$  of 3. We prove the Theorem 1 with the help of various generating functions and establish the Corollary 2 with a special series  $\bar{s}(z, x)$ , when  $n=1$ , and prove the Theorem 2: This is;

$$\overline{spt}(n) = \sum_{\substack{\vec{\lambda} \in SP \\ |\vec{\lambda}| = |\lambda_1| + |\lambda_2| = n}} 1,$$

with a numerical example. We establish the Result 2 using the  $\overline{crank}$  of partition pairs  $\vec{\lambda} = (\lambda_1, \lambda_2)$  and analyze the Corollary 3 with the help of 42 marked overpartitions of 6.

## 2. Some Related Definitions

Here we introduce some definitions following (Lovejoy and Osburn 2009).

$\overline{spt}(n)$  (Bringmann et al. 2009): The number of smallest parts in the overpartitions of  $n$  with smallest part not overlined is denoted by  $\overline{spt}(n)$  for example;

$n$	$\overline{spt}(n)$
1: $\dot{1}$	1
2: $\dot{2}, \dot{1} + \dot{1}$	3
3: $\dot{3}, \dot{2} + \dot{1}, \overline{2} + \dot{1}, \dot{1} + \dot{1} + \dot{1}$	6
4: $\dot{4}, \dot{3} + \dot{1}, \dot{3} + \dot{1}, \dot{2} + \dot{2}, \dot{2} + \dot{1} + \dot{1}, \overline{2} + \dot{1} + \dot{1} + \dot{1}, \dot{1} + \dot{1} + \dot{1} + \dot{1}$	13
...	...

Hence we get;

$$\overline{spt}(5) = 22, \overline{spt}(6) = 42, \dots$$

### 2.1 Product Notation

$$(x)_\infty = (1-x)(1-x^2)(1-x^3)\dots$$

$$(x^2; x^2)_\infty = (1-x^2)(1-x^4)\dots$$

$$(x)_k = (1-x)(1-x^2)(1-x^3)\dots(1-x^k)$$

$$(-x^5; x)_\infty = (1+x^5)(1+x^6)(1+x^7)\dots$$

### 2.2 Vector Partitions and $\vec{S}$ -Partitions

A vector partition can be done with 4 components each partition with certain restrictions (Bringmann et al. 2010). Let  $\vec{V} = D \times P \times P \times D$  where  $D$  denotes the set of all partitions into distinct parts,  $P$  denotes the set of all partitions. For a partition  $\pi$ , we let  $s(\pi)$  denote the smallest part of  $\pi$  (with the convention that the empty partition has smallest part  $\infty$ ),  $\#(\pi)$  the number of parts in  $\pi$ , and  $|\pi|$  the sum of the parts of  $\pi$ .

For  $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4) \in \vec{V}$ , we define the weight  $\omega(\vec{\pi}) = (-1)^{\#(\pi_1)-1}$ , the crank  $c(\vec{\pi}) = \#(\pi_2) - \#(\pi_3)$ , the norm  $|\vec{\pi}| = |\pi_1| + |\pi_2| + |\pi_3| + |\pi_4|$ .

We say  $\vec{\pi}$  is a vector partition of  $n$  if  $|\vec{\pi}| = n$ . Let  $\vec{S}$  denote the subset of  $\vec{V}$  and it is given by  $\vec{S} = \{(\pi_1, \pi_2, \pi_3, \pi_4) \in \vec{V}, 1 \leq s(\pi_1) < \infty, s(\pi_1) \leq s(\pi_2), s(\pi_1) \leq s(\pi_3), s(\pi_1) < s(\pi_4)\}$ .

$M_{\bar{s}}(m,n)$ : The number of vector partitions of  $n$  in  $\bar{s}$  with crank  $m$  counted according to the weight  $\omega$  is exactly  $M_{\bar{s}}(m,n)$ .

$M_{\bar{s}}(m,t,n)$ : The number of vector partitions of  $n$  in  $\bar{s}$  with crank congruent to  $m$  modulo  $t$  counted according to the weight  $\omega$  is exactly  $M_{\bar{s}}(m,t,n)$ .

Marked Partition (Andrews et al. 2013): We define a marked partition as a pair  $(\lambda, k)$  where  $\lambda$  is a partition and  $k$  is an integer identifying one of its smallest parts i.e.,  $k = 1, 2, \dots, \nu(\lambda)$ , where  $\nu(\lambda)$  is the number of smallest parts of  $\lambda$ .

$\overline{sptcrank}$  for Marked overpartitions (Chen et al. 2013): We define a marked overpartitions of  $n$  as a pair  $(\pi, j)$  where  $\pi$  is an over partition of  $n$  in which the smallest parts is not overlined and  $j$  is an integer  $1 \leq j \leq \nu(\pi)$ , where  $\nu(\pi)$  is the number of smallest parts to  $\pi$ . It is clear that  $\overline{spt}(n) = \#$  of marked overpartitions  $(\pi, j)$  of  $n$ . For example, there are 3 marked overpartitions of 2 like;

$(2,1), (1+1,1), (1+1,2)$ , so that  $\overline{spt}(2) = 3$ .

Again there are 6 marked overpartitions of 3 like;

$(3,1), (2+1,1), (\bar{2}+1,1), (1+1+1,1), (1+1+1,2)$  and  $(1+1+1,3)$ , so that  $\overline{spt}(3) = 6$ .

### 3. The Generating Function for $\overline{spt}_2(n)$

The generating function (Bringmann et al. 2010) for  $\overline{spt}(n)$  is given by;

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{x^n (-x^{n+1}; x)_{\infty}}{(1-x^n)^2 (x^{n+1}; x)_{\infty}} \\ &= \frac{x (-x^2; x)_{\infty}}{(1-x)^2 (x^2; x)_{\infty}} + \frac{x^2 (-x^3; x)_{\infty}}{(1-x^2)^2 (x^3; x)_{\infty}} + \dots \\ &= \frac{x(1+x^2)(1+x^3)\dots}{(1-x)^2(1-x^2)(1-x^3)\dots} + \frac{x^2(1+x^3)(1+x^4)\dots}{(1-x^2)^2(1-x^3)(1-x^4)\dots} + \dots \\ &= 1.x + 3.x^2 + 6.x^3 + 13.x^4 + 22.x^5 + 42.x^6 + \dots \\ &= \overline{spt}(1)x + \overline{spt}(2)x^2 + \overline{spt}(3)x^3 + \overline{spt}(4)x^4 + \dots \\ &= \sum_{n=1}^{\infty} \overline{spt}(n)x^n. \end{aligned}$$

From above we get;  $\overline{spt}(3) = 6, \overline{spt}(6) = 42, \dots$  i.e.,  $\overline{spt}(3.1) = 6 \equiv 0 \pmod{3}, \overline{spt}(3.2) = 42 \equiv 0 \pmod{3}, \dots$

We can conclude that;

$$\overline{spt}(3n) \equiv 0 \pmod{3}, \text{ for } n \geq 0 [4].$$

**Corollary 1:**  $\overline{spt}(n) = \sum_{m=-\infty}^{\infty} M_{\overline{S}}(m, n)$ .

**Proof:** The generating function for  $M_{\overline{S}}(m, n)$  is given by;

$$\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\overline{S}}(m, n) z^m x^n = \sum_{n=1}^{\infty} \frac{x^n (x^{n+1}; x)_{\infty} (-x^{n+1}; x)_{\infty}}{(zx^n; x)_{\infty} (z^{-1}x^n; x)_{\infty}}.$$

If  $z = 1$ , then we get;

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\overline{S}}(m, n) x^n &= \sum_{n=1}^{\infty} \frac{x^n (x^{n+1}; x)_{\infty} (-x^{n+1}; x)_{\infty}}{(x^n; x)_{\infty} (x^n; x)_{\infty}} \\ &= \frac{x(x^2; x)_{\infty} (-x^2; x)_{\infty}}{(x; x)_{\infty}^2} + \frac{x^2(x^3; x)_{\infty} (-x^4; x)_{\infty}}{(x^2; x)_{\infty}^2} + \dots \\ &= \frac{x(-x^2; x)_{\infty} (1-x^2)(1-x^3)\dots}{(1-x)^2(1-x^2)^2(1-x^3)^2\dots} + \frac{x^2(-x^3; x)_{\infty} (1-x^3)(1-x^4)\dots}{(1-x^2)^2(1-x^3)^2(1-x^4)^2\dots} + \dots \\ &= \frac{x(-x^2; x)_{\infty}}{(1-x)^2(1-x^2)(1-x^3)\dots} + \frac{x^2(-x^3; x)_{\infty}}{(1-x^2)^2(1-x^3)(1-x^4)\dots} + \dots \\ &= \frac{x(-x^2; x)_{\infty}}{(1-x)^2(x^2; x)_{\infty}} + \frac{x^2(-x^3; x)_{\infty}}{(1-x^2)^2(x^3; x)_{\infty}} + \dots \\ &= \sum_{n=1}^{\infty} \frac{x^n (-x^{n+1}; x)_{\infty}}{(1-x^n)^2 (x^{n+1}; x)_{\infty}} = \sum_{n=1}^{\infty} \overline{spt}(n) x^n. \end{aligned}$$

$$\text{i.e., } \sum_{n=1}^{\infty} \overline{spt}(n) x^n = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\overline{S}}(m, n) x^n.$$

Now equating the co-efficient of  $x^n$  from both sides we get;

$$\overline{spt}(n) = \sum_{m=-\infty}^{\infty} M_{\overline{S}}(m, n). \text{ Hence the Corollary.}$$

**Result 1:**  $M_{\overline{S}}(0, 3, 3) = M_{\overline{S}}(1, 3, 3) = M_{\overline{S}}(2, 3, 3) = \frac{1}{3} \overline{spt}(3)$ .

**Proof:** We prove the result with an example. We see the vector partitions from  $\overline{S}$  of 3 along with their weights and cranks are given as in table 1:

Here we have used  $\phi$  to indicate the empty partition. Thus we have,

$$M_{\overline{S}}(0, 3, 3) = +1 + 1 - 1 + 1 = 2, \quad M_{\overline{S}}(1, 3, 3) = M_{\overline{S}}(-2, 3, 3) = 1 + 1 = 2,$$

$$M_{\overline{S}}(2, 3, 3) = M_{\overline{S}}(-1, 3, 3) = 1 + 1 = 2.$$

$$\therefore M_{\overline{S}}(0, 3, 3) = M_{\overline{S}}(1, 3, 3) = M_{\overline{S}}(2, 3, 3) = 2 = \frac{1}{3} \cdot 6 = \frac{1}{3} \overline{spt}(3). \text{ Hence the Result.}$$

**Table 1:** The vector partitions from  $\bar{s}$  of 3 along with their weights and cranks.

$\bar{s}$ -vector partition $\vec{\pi}$ of 3	Weight $\omega(\vec{\pi})$	Crank $c_{(\vec{\pi})}$
$\vec{\pi}_1 = (1, \phi, \phi, 2)$	+ 1	0
$\vec{\pi}_2 = (1, \phi, 1+1, \phi)$	+1	-2
$\vec{\pi}_3 = (1, 1+1, \phi, \phi)$	+1	2
$\vec{\pi}_4 = (1, 1, 1, \phi)$	+1	0
$\vec{\pi}_5 = (1, \phi, 2, \phi)$	+1	-1
$\vec{\pi}_6 = (1, 2, \phi, \phi)$	+1	1
$\vec{\pi}_7 = (1+2, \phi, \phi, \phi)$	-1	0
$\vec{\pi}_8 = (3, \phi, \phi, \phi)$	+1	0
	$\sum \omega(\vec{\pi}) = 6$	

Now from above table we get;

$$\sum \omega(\vec{\pi}) = 6$$

$$\sum_{k=0}^2 M_{\bar{s}}(k, 3, 3) = 6$$

$$\therefore \overline{spt}(3) = \sum_{k=0}^2 M_{\bar{s}}(k, 3, 3) = \sum \omega(\vec{\pi}).$$

We define,  $M_{\bar{s}}(k, t, m) = \sum_{m \equiv k \pmod{t}} M_{\bar{s}}(m, n)$

$$\text{and } \overline{spt}(n) = \sum_{m=-\infty}^{\infty} M_{\bar{s}}(m, n) = \sum_{k=0}^{t-1} M_{\bar{s}}(k, t, n).$$

**Theorem 1:** The number of vector partitions of  $n$  in  $\bar{s}$  with crank  $m$  counted according to the weight  $\omega$  is non-negative, i.e.,  $M_{\bar{s}}(m, n) \geq 0$ .

**Proof:** The generating function for  $M_{\bar{s}}(m, n)$  is given by;

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_m M_{\bar{s}}(m, n) z^m x^n \\ &= \sum_{n=1}^{\infty} \frac{x^n (x^{n+1}; x)_{\infty} (-x^{n+1}; x)_{\infty}}{(zx^n; x)_{\infty} (z^{-1}x^n; x)_{\infty}} \\ &= \sum_{n=1}^{\infty} \frac{x^n}{(zx^n; x)_{\infty} (z^{-1}x^n; x)_{\infty}} \cdot (x^{n+1}; x)_{\infty} (-x^{n+1}; x)_{\infty} \\ &= \sum_{n=1}^{\infty} \frac{x^n (x^{2n+2}; x^2)_{\infty}}{(zx^n; x)_{\infty} (z^{-1}x^n; x)_{\infty}} \end{aligned}$$

[Since,  $\sum_{n=1}^{\infty} (x^{n+1}; x)_{\infty} (-x^{n+1}; x)_{\infty}$

$$= (x^2; x)_\infty (-x^2; x)_\infty + (x^3; x)_\infty (-x^3; x)_\infty + (x^4; x)_\infty (-x^4; x)_\infty + \dots$$

$$= (1-x^2)(1-x^3)\dots(1+x^2)(1+x^3)\dots + (1-x^3)(1-x^4)\dots(1+x^3)(1+x^4)\dots$$

$$= (1-x^4)(1-x^6)\dots + (1-x^6)(1-x^8) + \dots = \sum_{n=1}^{\infty} (x^{2n+2}; x^2)_\infty]$$

$$= \sum_{n=1}^{\infty} \frac{x^n (x^{2n}; x)_\infty}{(zx^n; x)_\infty (z^{-1}x^n; x)_\infty} \cdot \frac{(x^{2n+2}; x^2)}{(x^{2n}; x)_\infty}$$

$$= \sum_{n=1}^{\infty} \frac{x^n (x^{2n}; x)_\infty}{(zx^n; x)_\infty (z^{-1}x^n; x)_\infty} \cdot \frac{1}{(1-x^{2n})(x^{2n+1}; x^2)_\infty}$$

$$[\text{Since, } \sum_{n=1}^{\infty} \frac{(x^{2n+2}; x^2)_\infty}{(x^{2n}; x)_\infty} = \frac{(x^4; x^2)_\infty}{(x^2; x)_\infty} + \frac{(x^6; x^2)_\infty}{(x^4; x)_\infty} + \dots$$

$$= \frac{(1-x^4)(1-x^6)\dots}{(1-x^2)(1-x^3)(1-x^4)\dots} + \frac{(1-x^6)(1-x^8)\dots}{(1-x^4)(1-x^5)\dots}$$

$$= \frac{1}{(1-x^2)} \cdot \frac{1}{(1-x^3)(1-x^5)\dots} + \frac{1}{(1-x^4)} \cdot \frac{1}{(1-x^5)(1-x^7)\dots} + \dots$$

$$= \sum_{n=1}^{\infty} \frac{1}{1-x^{2n}} \cdot \frac{1}{(x^{2n+1}; x^2)_\infty}]$$

$$= \sum_{n=1}^{\infty} \frac{x^n (x^{2n}; x)_\infty}{(zx^n; x)_\infty (z^{-1}x^n; x)_\infty} \cdot \frac{1}{(1-x^{2n})(x^{2n+1}; x^2)_\infty}$$

$$= \sum_{n=1}^{\infty} x^n \sum_{k=0}^{\infty} \frac{(z^{-1}x^n)^k}{(zx^{n+k}; x)_\infty (x)_k} \cdot \frac{1}{(1-x^{2n})(x^{2n+1}; x^2)_\infty}$$

$$[\text{since } = \sum_{n=1}^{\infty} \frac{x^n (x^{2n}; x)_\infty}{(zx^n; x)_\infty (z^{-1}x^n; x)_\infty} = \sum_{n=1}^{\infty} x^n \sum_{k=0}^{\infty} \frac{(z^{-1}x^n)^k}{(zx^{n+k}; x)_\infty (x)_k}] \text{ (by [3])}.$$

We see that the co-efficient of any power  $x$  in right hand side is non-negative so the co-efficient  $M_{\bar{S}}(m, n)$  of  $z^m x^n$  is non-negative, i.e.,  $M_{\bar{S}}(m, n) \geq 0$ . Hence the Theorem.

**Numerical Example 1:** The vector partitions from  $\bar{S}$  of 4 along with their weights and cranks are given as in table 2:

Here we have used  $\phi$  to indicate the empty partition. Thus we have;

$$M_{\bar{S}}(0, 4) = 3, M_{\bar{S}}(1, 4) = 3, M_{\bar{S}}(-1, 4) = 3, M_{\bar{S}}(2, 4) = 1, M_{\bar{S}}(-2, 4) = 1, M_{\bar{S}}(3, 4) = 1, \text{ and } M_{\bar{S}}(-3, 4) = 1.$$

$$\therefore \sum_m M_{\bar{S}}(m, 4) = 13, \text{ i.e., every term is non-negative.}$$

$$\therefore M_{\bar{S}}(m, 4) \geq 0. \text{ But we have already found that } \sum_m M_{\bar{S}}(m, 3) = 6,$$

$$\text{i.e., every term is non-negative. } \therefore M_{\bar{S}}(m, 3) \geq 0.$$

So we can conclude that;  $M_{\bar{S}}(m, n) \geq 0$ .

**Table 2:** The vector partitions from  $\bar{S}$  of 4 along with their weights and cranks.

$\bar{S}$ -vector partition $\vec{\pi}$ of 4	Weight $\omega(\vec{\pi})$	Crank $c(\vec{\pi})$
$\vec{\pi}_1 = (4, \phi, \phi, \phi)$	+ 1	0
$\vec{\pi}_2 = (3+1, \phi, \phi, \phi)$	-1	0
$\vec{\pi}_3 = (1, 3, \phi, \phi)$	+1	1
$\vec{\pi}_4 = (1, \phi, 3, \phi)$	+1	-1
$\vec{\pi}_5 = (1, \phi, \phi, 3)$	+1	0
$\vec{\pi}_6 = (2, 2, \phi, \phi)$	+1	1
$\vec{\pi}_7 = (2, \phi, 2, \phi)$	+1	-1
$\vec{\pi}_8 = (1+2, 1, \phi, \phi)$	-1	1
$\vec{\pi}_9 = (1+2, \phi, 1, \phi)$	-1	-1
$\vec{\pi}_{10} = (1, 1, 2, \phi)$	+1	0
$\vec{\pi}_{11} = (1, 2, 1, \phi)$	+1	0
$\vec{\pi}_{12} = (1, 1, \phi, 2)$	+1	1
$\vec{\pi}_{13} = (1, \phi, 1, 2)$	+1	-1
$\vec{\pi}_{14} = (1, 1+2, \phi, \phi)$	+1	2
$\vec{\pi}_{15} = (1, \phi, 1+2, \phi)$	+1	-2
$\vec{\pi}_{16} = (1, 1+1+1, \phi, \phi)$	+1	3
$\vec{\pi}_{17} = (1, \phi, 1+1+1, \phi)$	+1	-3
$\vec{\pi}_{18} = (1, 1+1, 1, \phi)$	+1	1
$\vec{\pi}_{19} = (1, 1, 1+1, \phi)$	+1	-1
	$\sum \omega(\vec{\pi}) = 13$	

**Corollary 2:**  $\bar{S}(1, x) = \sum_{n=1}^{\infty} \overline{spt}(n) x^n$ .

**Proof:** We get  $\bar{S}(z, x) = \sum_{n=1}^{\infty} \frac{x^n (-x^{n+1}; x)_{\infty} (x^{n+1}; x)_{\infty}}{(zx^n; x)_{\infty} (z^{-1}x^n; x)_{\infty}}$  ([2]).

If  $z = 1$ , then we get;

$$\begin{aligned} \bar{S}(1, x) &= \sum_{n=1}^{\infty} \frac{x^n (-x^{n+1}; x)_{\infty} (x^{n+1}; x)_{\infty}}{(x^n; x)_{\infty} (x^n; x)_{\infty}} \\ &= \frac{x(-x^2; x)_{\infty} (x^2; x)_{\infty}}{(x; x)_{\infty}^2} + \frac{x^2(-x^3; x)_{\infty} (x^3; x)_{\infty}}{(x^2; x)_{\infty}^2} + \dots \end{aligned}$$



$$\begin{aligned}
&= \frac{x(-x^2; x)_\infty (1-x^2)(1-x^3)(1-x^4)\dots}{(1-x)^2(1-x^2)^2(1-x^3)^2\dots} + \\
&\quad \frac{x^2(-x^3; x)_\infty (1-x^3)(1-x^4)\dots}{(1-x^2)^2(1-x^3)^2(1-x^4)^2\dots} + \dots \\
&= \frac{x(-x^2; x)_\infty}{(1-x)^2(1-x^2)(1-x^3)\dots} + \frac{x^2(-x^3; x)_\infty}{(1-x^2)^2(1-x^3)(1-x^4)\dots} + \dots \\
&= \sum_{n=1}^{\infty} \frac{x^n(-x^{n+1}; x)_\infty}{(1-x^n)^2(x^{n+1}; x)_\infty} \\
&= \sum_{n=1}^{\infty} \overline{spt}(n)x^n, \\
&\text{i.e., } \overline{s}(1, x) = \sum_{n=1}^{\infty} \overline{spt}(n)x^n. \text{ Hence the Corollary.}
\end{aligned}$$

**Theorem 2:**  $\overline{spt}(n) = \sum_{\substack{\bar{\lambda} \in SP \\ |\bar{\lambda}| = |\lambda_1| + |\lambda_2| = n}} 1$  .

**Proof:** First we define the  $\overline{sptcrank}$  in terms of partition pairs,

$$\overline{SP} = \{\bar{\lambda} = (\lambda_1, \lambda_2) \in P \times P : 1 < s(\lambda_1) \leq s(\lambda_2) \text{ and all parts of } \lambda_2 \text{ that are } \geq 2s(\lambda_1) + 1 \text{ are odd}\}.$$

The generating function for  $\overline{spt}(n)$  is given by;

$$\begin{aligned}
\sum_{n=1}^{\infty} \overline{spt}(n)x^n &= \sum_{n=1}^{\infty} \frac{x^n(-x^{n+1}; x)_\infty}{(1-x^n)^2(x^{n+1}; x)_\infty} \\
&= \sum_{n=1}^{\infty} \frac{x^n}{(1-x^n)^2(x^{n+1}; x)_\infty} (-x^{n+1}; x)_\infty \\
&= \sum_{n=1}^{\infty} \frac{x^n}{(1-x^n)^2(x^{n+1}; x)_\infty} \cdot \frac{(x^{2n+2}; x^2)_\infty}{(x^{n+1}; x)_\infty}
\end{aligned}$$

[Since,  $\sum_{n=1}^{\infty} (-x^{n+1}; x)_\infty = (-x^2; x)_\infty + (-x^3; x)_\infty + \dots$

$$\begin{aligned}
&= (1+x^2)(1+x^3)\dots + (1+x^3)(1+x^4)\dots + (1+x^4)\dots \\
&= \frac{(1-x^4)(1-x^6)\dots}{(1-x^2)(1-x^3)(1-x^4)\dots} + \frac{(1-x^6)(1-x^8)\dots}{(1-x^3)(1-x^4)\dots} + \dots \\
&= \frac{(x^4; x^2)_\infty}{(x^2; x)_\infty} + \frac{(x^6; x^2)_\infty}{(x^3; x)_\infty} + \dots \\
&= \sum_{n=1}^{\infty} \frac{(x^{2n+2}; x^2)_\infty}{(x^{n+1}; x)_\infty} ] \\
&= \sum_{n=1}^{\infty} \frac{x^n}{(x^n; x)_\infty (1-x^n)} \cdot \frac{(x^{2n+2}; x^2)_\infty}{(x^{n+1}; x)_\infty}
\end{aligned}$$

$$\begin{aligned}
 & [\text{Since, } \sum_{n=1}^{\infty} \frac{x^n}{(1-x^n)^2(x^{n+1};x)_{\infty}} \\
 &= \frac{x}{(1-x)^2(x^2;x)_{\infty}} + \frac{x^2}{(1-x^2)^2(x^3;x)_{\infty}} + \dots \\
 &= \frac{x}{(1-x)^2(1-x^2)(1-x^3)\dots} + \frac{x^2}{(1-x^2)(1-x^3)(1-x^4)\dots} + \dots \\
 &= \frac{x}{(1-x)(x;x)_{\infty}} + \frac{x^2}{(1-x^2)(x^2;x)_{\infty}} + \dots \\
 &= \sum_{n=1}^{\infty} \frac{x^n}{(1-x^n)(x^n;x)_{\infty}} ] \\
 &= \sum_{n=1}^{\infty} \frac{x^n}{(x^n;x)_{\infty}} \cdot \frac{1}{(1-x^n)} \cdot \frac{(x^{2n+2};x^2)_{\infty}}{(x^{n+1};x)_{\infty}} \\
 &= \sum_{n=1}^{\infty} \frac{x^n}{(x^n;x)_{\infty}} \cdot \frac{1}{(1-x^n)} \cdot \frac{1}{(1-x^{n+1})\dots(1-x^{2n})(x^{2n+1};x^2)_{\infty}}
 \end{aligned}$$

$$\begin{aligned}
 & [\text{Since, } \sum_{n=1}^{\infty} \frac{(x^{2n+2};x^2)_{\infty}}{(x^{n+1};x)_{\infty}} = \frac{(x^4;x^2)_{\infty}}{(x^2;x)_{\infty}} + \frac{(x^6;x^2)_{\infty}}{(x^3;x)_{\infty}} + \dots \\
 &= \frac{(1-x^4)(1-x^6)\dots}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)\dots} + \frac{(1-x^6)(1-x^4)\dots}{(1-x^3)(1-x^4)(1-x^5)\dots} + \dots \\
 &= \frac{1}{(1-x^2)(1-x^3)(1-x^5)(1-x^7)\dots} + \frac{1}{(1-x^3)(1-x^4)(1-x^5)\dots} + \dots \\
 &= \sum_{n=1}^{\infty} \frac{1}{(1-x^{n+1})\dots(1-x^{2n})(x^{2n+1};x^2)_{\infty}} ] \\
 &= \sum_{n=1}^{\infty} \frac{x^n}{(x^n;x)_{\infty}} \cdot \frac{1}{(1-x^n)(1-x^{n+1})\dots(1-x^{2n})(x^{2n+1};x^2)_{\infty}} \\
 &= \sum_{n=1}^{\infty} \sum_{\substack{\lambda_1 \in P \\ s(\lambda_1)=n}} x^{|\lambda_1|} \sum_{\substack{\lambda_2 \in P \\ s(\lambda_2) \geq n}} x^{|\lambda_2|}, \text{ all parts in } \lambda_2 \geq 2n+1 \text{ are odd} \\
 &= \sum_{n=1}^{\infty} \sum_{\substack{\bar{\lambda} \in SP \\ |\bar{\lambda}|=|\lambda_1|+|\lambda_2|=n}} x^{|\bar{\lambda}|} .
 \end{aligned}$$

Equating the co-efficient of  $x^n$  from both sides we get;  
 $\overline{spt}(n) = \sum_{\substack{\bar{\lambda} \in SP \\ |\bar{\lambda}|=|\lambda_1|+|\lambda_2|=n}} 1$  . Hence the Theorem.

**Numerical Example 2:** The overpartitions of 3 with smallest parts not overlined are; 3, 2+1,  $\bar{2}+1$ , 1+1+1. Consequently, the number of smallest parts in the overpartitions of 3 with smallest part not overlined is

given by;  $\dot{3}, 2+\dot{1}, \bar{2}+\dot{1}, \dot{1}+\dot{1}+\dot{1}$  so that  $\overline{spt}(3) = 6$ , i.e., there are 6  $\overline{SP}$ -partition pairs of 3 like;  $(3, \phi), (2+1, \phi), (1+1+1, \phi), (1+1, 1), (1, 1+1)$  and  $(1, 2)$ .

**Result 2:**  $M_{\overline{s}}(0, 3, 3) = M_{\overline{s}}(1, 3, 3) = M_{\overline{s}}(2, 3, 3) = \frac{\overline{spt}(3)}{3}$ .

**Proof:** First we define a  $\overline{crank}$  of partitions pairs  $\vec{\lambda} = (\lambda_1, \lambda_2) \in \overline{SP}$ .

For  $\vec{\lambda} = (\lambda_1, \lambda_2) \in \overline{SP}$  we define;

$k(\vec{\lambda}) = \#$  of pairs  $j$  in  $\lambda_2$  such that  $s(\lambda_1) \leq j \leq 2s(\lambda_1) - 1$ , and also define;

$$\overline{crank}(\vec{\lambda}) = \begin{cases} (\# \text{ of parts of } \lambda_1 \geq s(\lambda_1) + k) - k; & \text{if } k > 0 \\ (\# \text{ of parts of } \lambda_1) - 1; & \text{if } k = 0 \end{cases}$$

where  $k = k(\vec{\lambda})$ .

**Table 3:**  $\overline{SP}$ -partition pairs of 3 with  $\overline{crank}$ .

$\overline{SP}$ -partition pair $\vec{\lambda} = (\lambda_1, \lambda_2)$	$k$	$\overline{crank}$	(mod3)
$(3, \phi)$	0	0	0
$(2+1, \phi)$	0	1	1
$(1+1+1, \phi)$	0	2	2
$(1+1, 1)$	1	-1	2
$(1, 1+1)$	2	-2	1
$(1, 2)$	0	0	0

From the table 3 we get;

$$M_{\overline{s}}(0, 3, 3) = M_{\overline{s}}(1, 3, 3) = M_{\overline{s}}(2, 3, 3) = 2 = \frac{1}{3}6 = \frac{1}{3}\overline{spt}(3). \text{ Hence the Result.}$$

#### 4. The $\overline{sptcrank}$ of a Marked Overpartition

We want to describe the  $\overline{sptcrank}$  of a marked overpartition [6]. To define the  $\overline{sptcrank}$  of a marked overpartition we first need to define a function  $k(m, n)$  for positive integers  $m$  and  $n$  such that  $m \geq n+1$  we write  $m = b2^j$ , where  $b$  is odd and  $j \geq 0$ . For a given odd integer  $b$  and a positive integer  $n$  we define  $j_0 = j_0(b, n)$  to be the smallest nonnegative integer  $j_0$  such that  $b2^{j_0} \geq n+1$ .

**Table 4:** Marked overpartitions of 6 with  $\overline{sptcrank} \pmod{3}$ .

Marked overpartition $(\pi, j)$ of 6	$\pi_1$	$\pi_2$	$v(\pi_1)$	$k((\pi_2, s(\pi_1)))$	$\bar{k}$	$\overline{sptcrank}$	(mod3)
(6,1)	6	$\phi$	1	0	0	0	0
( $\bar{5}$ +1,1)	1	5	1	0	0	0	0
( $\bar{4}$ +2,1)	2	4	1	0	0	0	0
(4+1+1,1)	4+1+1	$\phi$	2	0	1	0	0
( $\bar{3}$ + $\bar{2}$ +1,1)	1	3+2	1	0	0	0	0
(3+1+1+1,2)	3+1+1+1	$\phi$	3	0	1	0	0
(3+1+1+1,3)	3+1+1+1	$\phi$	3	0	0	3	0
(2+2+1+1,2)	2+2+1+1	$\phi$	2	0	0	3	0
( $\bar{2}$ +2+1+1,2)	2+1+1	2	2	0	1	0	0
(2+1+1+1+1,1)	2+1+1+1+1	$\phi$	4	0	3	-3	0
(2+1+1+1+1,3)	2+1+1+1+1	$\phi$	4	0	1	0	0
( $\bar{2}$ +1+1+1+1,1)	1+1+1+1	2	4	0	3	-3	0
( $\bar{2}$ +1+1+1+1,4)	1+1+1+1	2	4	0	0	-3	0
(1+1+1+1+1+1,3)	1+1+1+1+1+1	$\phi$	6	0	3	-3	0
(5+1,1)	5+1	$\phi$	1	0	0	1	1
(4+2,1)	4+2	$\phi$	1	0	0	1	1
( $\bar{4}$ +1+1,2)	1+1	4	2	0	0	1	1
(3+3,2)	3+3	$\phi$	2	0	0	1	1
( $\bar{3}$ +2+1,1)	2+1	3	1	0	0	1	1
(3+ $\bar{2}$ +1,1)	3+1	2	1	0	0	1	1
( $\bar{3}$ +1+1+1,1)	1+1+1	3	3	0	2	-2	1
(2+2+2,1)	2+2+2	$\phi$	3	0	2	-2	1
(2+2+1+1,1)	2+2+1+1	$\phi$	2	0	1	1	1
(2+1+1+1+1,2)	2+1+1+1+1	$\phi$	4	0	2	-2	1
(2+1+1+1+1,4)	2+1+1+1+1	$\phi$	4	0	0	4	1
( $\bar{2}$ +1+1+1+1,2)	1+1+1+1	2	4	0	2	-2	1
(1+1+1+1+1+1,1)	1+1+1+1+1+1	$\phi$	6	0	5	-5	1
(1+1+1+1+1+1,4)	1+1+1+1+1+1	$\phi$	6	0	2	-2	1
(4+1+1,2)	4+1+1	$\phi$	2	0	0	2	2
( $\bar{4}$ +1+1,1)	1+1	4	2	0	1	-1	2
(3+3,1)	3+3	$\phi$	2	0	1	-1	2
(3+2+1,1)	3+2+1	$\phi$	1	0	0	2	2
(3+1+1+1,1)	3+1+1+1	$\phi$	3	0	2	-1	2
( $\bar{3}$ +1+1+1,2)	1+1+1	3	3	0	1	-1	2
( $\bar{3}$ +1+1+1,3)	1+1+1	3	3	0	0	2	2
(2+2+2,2)	2+2+2	$\phi$	3	0	1	-1	2
(2+2+2,3)	2+2+2	$\phi$	3	0	0	2	2
( $\bar{2}$ +2+1+1,2)	2+1+1	2	2	0	0	2	2
( $\bar{2}$ +1+1+1+1,3)	1+1+1+1	2	4	0	1	-1	2
(1+1+1+1+1+1,2)	(1+1+1+1+1+1)	$\phi$	6	0	4	-4	2
(1+1+1+1+1+1,5)	(1+1+1+1+1+1)	$\phi$	6	0	1	-1	2
(1+1+1+1+1+1,6)	(1+1+1+1+1+1)	$\phi$	6	0	0	5	2

$$\text{We define; } k(m, n) = \begin{cases} 0, & \text{if } b \geq 2n \\ 2^{j-j_0} & \text{if } b2^{j_0} < 2n \\ 0, & \text{if } b2^{j_0} = 2n. \end{cases}$$

For a marked overpartitions  $(\pi, j)$  we let  $\pi_1$  be the partition formed by the non-overlined parts of  $\pi$ ,  $\pi_2$  be the partition (into distinct parts) formed by the overlined parts of  $\pi$  so that  $s(\pi_2) > s(\pi_1)$ , we define  $\bar{k}(\pi, i) = \nu(\pi_1) - j + k(\pi_2, s(\pi_1))$ , where  $\nu(\pi_1)$  is the number of smallest parts of  $\pi_1$ . Now we can define;

$$\overline{sptcrank}(\pi, j) = \begin{cases} (\# \text{of parts of } \pi_1 \geq s(\pi_1) - \bar{k}), & \text{if } \bar{k} = \bar{k}(\pi, j) > 0 \\ (\# \text{of parts of } \pi_1) - 1; & \text{if } \bar{k} = \bar{k}(\pi, j) = 0. \end{cases}$$

**Corollary 3[8]:** The residue of the  $\overline{sptcrank} \pmod{3}$  divides the marked overpartitions of  $3n$  into 3 equal classes.

**Proof:** We prove the corollary with the help of an example when  $n = 2$ . There are 42 marked overpartitions of 6 so that  $\overline{spt}(6) = 42$ .

We see that the residue of the  $\overline{sptcrank} \pmod{3}$  divides the marked overpartitions of  $3n$  into 3 equal classes. Hence the Corollary.

## 5. Conclusion

In this study we have found the number of smallest parts in the overpartitions of  $n$  with smallest part not overlined for  $n=1, 2, 3, 4$ . We have shown the relation  $\overline{spt}(3n) \equiv 0 \pmod{3}$  for  $n = 1, 2, \dots$  and have shown the result 1 with the help of vector partitions from  $\bar{s}$  of 3 along with their weights and cranks. We have verified the Theorem 1 when  $n = 4$  and have verified the Theorem 2 when  $n = 3$ . We have established the Corollary 3 with 42 marked overpartitions of 6.

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