Development of Partition Functions of Ramanujan's Works

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Development of Partition Functions of Ramanujan's Works

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Abstract

In 1986, Dyson defined the rank of a partition as the largest part of a partition π minus the number of parts of π. In 1988, Garvan discussed the theta series in x like $A(x)$, $B(x)$, $C(x)$, $D(x)$ and also discussed Jacobi's triple product Identity (1829). Both of the authors have worked on Ramanujan's seminal works "Ramanujan's Lost Notebooks". This paper proves the Theorem 1 with the help of Dyson's rank conjectures $N(0,5,5n+1)$, $N(2,5, 5n+1)$ and proves the Theorem 2 with the help of Garvan's theta series and Dyson's rank conjectures $N(1,5, 5n+2)$, $N(2,5, 5n+2)$, respectively. An attempt has been taken here to the development of the Ramanujan's works with the contributions of Dyson and Garvan. Definitions and simple mathematical calculations are presented here to make the paper easier to the common readers.

Keywords: Congruent, Jacobi's Triple product, Modulo, Ramanujan's Lost Notebook, Theta series.

1. Introduction

In this paper we give some related definitions of $P(n)$, Rank of partitions, $N(m,n)$, $N(m,t,n)$, Z, $(x)_\infty$, $(zx)_\infty$, $(x^n)_m$, $(x^k;x^5)_m$, $\beta_1(n)$ and $\beta_2(n)$. We state the relations [4] to establish the Garvan's theta series in x and establish the identity which in used in proving the Theorems and also establish Dyson's various rank conjectures with the help of Ramanujan's identities. Garvan [4] stated the terms $\beta_1(n)$ and $\beta_2(n)$ with certain conditions. We also generate the generating functions for $\beta_1(n)$ and $\beta_2(n)$ [1]. We prove the Theorem 1 in terms of $\beta_1(n)$ with the help of Ramanujan's identities and prove the Theorem 2 in terms of $\beta_2(n)$ with the help of Dyson's rank conjectures and Ramanujan's identities.

2. Some Related Definitions

$P(n)$ [5] : The number of partitions of n like;

4, 3+1, 2+2, 2+1+1, 1+1+1+1 $\therefore$ $P(4) = 5$.

Rank of partition: The largest part of a partition π minus the number of parts of π (a partition of n).

$N(m,n)$: The number of partitions of n with rank m.

Dyson's rank conjectures are defined as follows [2]:

$N(m,t,n)$: The number of partition of n with rank congruent to m modulo t.

Z: The set of complex numbers.

Product notations [6]:

$(x)_\infty$: The product of infinite factors;

$(x)_\infty = (1-x)(1-x^2)(1-x^3)\ldots\infty$.

$(zx)_\infty$: The product of infinite factors;

$(zx)_\infty = (1-zx)(1-zx^2)(1-zx^3)\ldots\infty$.

$(x^n)_m$: The product of m factors;

$(x^n)_m = (1-x^n)(1-x^{n+1})(1-x^{n+2})\ldots(1-x^{n+m-1})$.

$(x^k;x^5)_m$: The product of m factors;

$(x^k;x^5)_m = (1-x^k)(1-x^{k+5})(1-x^{k+10})\ldots(1-x^{k+(m-1)5})$

$\beta_1(n)$: The number of partitions of n into 1's and parts congruent to 0 or -1 modulo 5 with the largest part $\equiv 0(\mathrm{mod}\,5) \le 5$ times the number of 1's $\le$ the smallest part $\equiv -1(\mathrm{mod}\,5)$.

$\beta_2(n)$: The number of partitions of n into 2's and parts congruent to 0 or -2 modulo 5 with the largest part $\equiv 0(\mathrm{mod}\,5) \le 5$ times the number of 2's $\le$ the smallest part $\equiv -2(\mathrm{mod}\,5)$.

3. The Relations from Ramanujan's Lost Notebook [1]

$$F(x) = \frac{(1-x)(1-x^2)(1-x^3)\ldots\infty}{(1-2x\cos\frac{2n\pi}{5}+x^2)(1-2x^2\cos\frac{2n\pi}{5}+x^4)\ldots\infty}$$

$$f'(x) = 1 + \frac{x}{1-2x\cos\frac{2n\pi}{5}+x^2} + \frac{x^4}{(1-2x\cos\frac{2n\pi}{5}+x^2)(1-2x^2\cos\frac{2n\pi}{5}+x^4)} + \ldots\infty$$

$$, n = 1 \text{ or } 2.$$

$$F(x^{\frac{1}{5}}) = A(x) - 4x^{\frac{1}{5}}\cos\frac{2n\pi}{5}B(x) + 2x^{\frac{2}{5}}\cos\frac{4n\pi}{5}C(x) - 2x^{\frac{3}{5}}\cos\frac{2n\pi}{5}D(x). \qquad (1)$$

$$f'(x^{\frac{1}{5}}) = \left\{A(x) - 4\sin^2\frac{2n\pi}{5}\Phi(x)\right\} + x^{\frac{1}{5}}B(x) + 2x^{\frac{2}{5}}\cos\frac{2n\pi}{5}C(x) - 2x^{\frac{3}{5}}\cos\frac{2n\pi}{5}\left\{D(x) + 4\sin^2\frac{2n\pi}{5}.\frac{\psi(x)}{x}\right\}. \qquad (2)$$

Garvan's theta series in x [4] are;

$$A(x) = \frac{1-x^2-x^3+x^9+\ldots\infty}{(1-x)^2(1-x^4)^2(1-x^6)^2\ldots\infty},$$

$$B(x) = \frac{(1-x^5)(1-x^{10})(1-x^{15})\ldots\infty}{(1-x)(1-x^4)(1-x^6)\ldots\infty},$$

$$C(x) = \frac{(1-x^5)(1-x^{10})(1-x^{15})\ldots\infty}{(1-x^2)(1-x^3)(1-x^7)\ldots\infty},$$

$$D(x) = \frac{1-x-x^4+x^7+\ldots\infty}{(1-x^2)^2(1-x^3)^2(1-x^7)^2\ldots\infty},$$

$$\phi(x) = -1 + \left\{\frac{1}{1-x} + \frac{x^5}{(1-x)(1-x^4)(1-x^6)} + \frac{x^{20}}{(1-x)(1-x^4)(1-x^6)(1-x^9)(1-x^{11})} + \ldots\infty\right\}.$$

But we get;

$$A(x^5) - 4x\cos\frac{2\pi}{5}B(x^5) + 2x^2\cos\frac{4\pi}{5}C(x^5) - 2x^3\cos\frac{2\pi}{5}D(x^5)$$

$$= 1 - 4x\cos^2\frac{2\pi}{5} + 2x^2\cos\frac{4\pi}{5} - 2x^3\cos\frac{2\pi}{5} + 2x^5 -$$

$$4x^6\cos^2\frac{2\pi}{5}+2x^8\cos\frac{2\pi}{5}-x^{10}+\ldots\infty$$

$$\Psi(x)=-1+\left\{\frac{1}{1-x^2}+\frac{x^5}{(1-x^2)(1-x^3)(1-x^7)}+\frac{x^{20}}{(1-x^2)(1-x^3)(1-x^7)(1-x^8)(1-x^{12})}+\ldots\infty\right\}$$

.

Now,

$$\frac{x}{1-x}+\frac{x^3}{(1-x^2)(1-x^3)}+\frac{x^5}{(1-x^3)(1-x^4)(1-x^5)}+\ldots\infty$$

$$=3\phi(x)+1-A(x).$$

And,

$$\frac{x}{1-x}+\frac{x^2}{(1-x^2)(1-x^3)}+\frac{x^3}{(1-x^3)(1-x^4)(1-x^5)}+\ldots\infty$$

$$=3\Psi(x)+xD(x).$$

We assume without loss of generality that $n = 1$. Let $\zeta=\exp^{\frac{2\pi i}{5}}$, then we may write the definitions of $F(x)$ and $f'(x)$ as;

$$F(x)=\frac{(x)_\infty}{(\zeta x)_\infty(\zeta^{-1}x)_\infty},$$

and,

$$f'(x)=1+\sum_{n=1}^{\infty}\frac{x^{n^2}}{(1-\zeta x)(1-\zeta^{-1}x)\ldots(1-\zeta x^n)(1-\zeta^{-1}x^n)}$$

$$=\sum_{n=1}^{\infty}\frac{x^{n^2}}{(\zeta x)_n(\zeta^{-1}x)_n},$$

where we have used the relations;

$$(a)_0=1,$$

$(a)_n=(1-a)(1-ax)\ldots(1-ax^{n-1})$, for $n\geq 1$

and,

$$(a)_\infty=\lim_{n\to\infty}(a)_n=\prod_{n=1}^{\infty}(1-ax^{n-1}).$$

After replacing x by x^5 we see that (1) and (2) are identities for $F(x)$ and $f'(x)$. We note that the numerators in the definitions of $A(x)$, $B(x)$, $C(x)$, and $D(x)$ are theta series in x and hence may be written as infinite products using Jacobi's triple product identity [3];

$$\prod_{n=1}^{\infty}(1-zx^n)(1-z^{-1}x^{n-1})(1-x^n)$$

$$=\prod_{n=-\infty}^{\infty}(-1)^n z^n x^{\frac{n(n+1)}{2}} \qquad (3)$$

$$=\ldots+z^{-2}x-z^{-1}+1-zx+z^2x^3-\ldots\infty,$$

where $z\neq 0$ and $|x|<1$.

Replacing x by x^5 and z by x^{-3} we get from (3);

$$\prod_{n=1}^{\infty}(1-x^{5n-3})(1-x^{5n-2})(1-x^{5n})$$

$$=\ldots+x^{11}+1-x^2+x^9-\ldots\infty$$

$$=1-x^2-x^3+x^9+x^{11}-\ldots\infty.$$

Again replacing x by x^5 and z by x^{-3} equation (3) becomes;

$$\prod_{n=1}^{\infty}(1-x^{5n-4})(1-x^{5n-1})(1-x^{5n})$$

$$=\ldots+x^{13}-x^4+1-x+x^7-\ldots\infty$$

$$=1-x-x^4+x^7+x^{13}-\ldots\infty.$$

In fact we have;

$$A(x)=\prod_{n=1}^{\infty}\frac{\left(1-x^{5n-3}\right)\left(1-x^{5n-2}\right)\left(1-x^{5n}\right)}{\left(1-x^{5n-4}\right)^{2}\left(1-x^{5n-1}\right)^{2}},$$

$$B(x)=\prod_{n=1}^{\infty}\frac{\left(1-x^{5n}\right)}{\left(1-x^{5n-4}\right)\left(1-x^{5n-1}\right)},$$

$$C(x)=\prod_{n=1}^{\infty}\frac{\left(1-x^{5n}\right)}{\left(1-x^{5n-3}\right)\left(1-x^{5n-2}\right)},$$

$$D(x)=\prod_{n=1}^{\infty}\frac{\left(1-x^{5n-4}\right)\left(1-x^{5n-1}\right)\left(1-x^{5n}\right)}{\left(1-x^{5n-3}\right)^{2}\left(1-x^{5n-2}\right)^{2}}.$$

Rank of a Partition

The rank of a partition is defined as the largest part minus the number of parts. Thus the partition; 6 + 5 + 2 + 1 + 1 + 1 + 1 of 17 has rank, 6 –7 = –1 and the conjugated partition, 7 + 3 + 2 + 2 + 2 + 1 has rank, 7– 6 = 1, i.e., the rank of a partition and that of the conjugate partition differ only in sign. The rank of a partition of 5 belongs to any one of the residues (mod 5) and we have exactly 5 residues. There is similar result for all partitions of 7 leading to (mod 7).

The generating function for $N(m, n)$ is of the form [3];

$$\sum_{n=1}^{\infty}(-1)^{n-1}x^{\frac{n}{2}(3n-1)+|m|n}\left(1-x^{n}\right)\prod_{j=1}^{\infty}\left(1-x^{j}\right)^{-1}$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\left\{x^{\frac{n}{2}(3n+2+|m|-1)}-x^{\frac{n}{2}(3n+2|m|+1)}\right\}\sum_{k=0}^{\infty}P(k)x^{k}$$

$$=\left(x^{|m|+1}+0.x^{|m|+2}+x^{|m|+3}+\ldots\infty\right)-\left(x^{2|m|+5}+x^{2|m|+6}+\ldots\infty\right)$$

$$=\sum_{n=0}^{\infty}N(m,n)x^{n}.$$

The generating function for $N(m,t,n)$ is of the form;

$$\sum_{\substack{n=-\infty\\ n\neq 1}}^{\infty}(-1)^{n}x^{\frac{n}{2}(3n+1)}\frac{\left(x^{mn}+x^{n(t-m)}\right)}{1-x^{tn}}\prod_{j=1}^{\infty}\left(1-x^{j}\right)^{-1}$$

$$=\sum_{\substack{n=-\infty\\ n\neq 1}}^{\infty}(-1)^{n}x^{\frac{n}{2}(3n+1)}\left(x^{mn}+x^{n(t-m)}\right)\times\left(1+x^{tn}+x^{2tn}+\ldots\infty\right)\sum_{k=0}^{\infty}P(k)x^{k}$$

$$=\sum_{n=0}^{\infty}N(m,t,n)x^{n};$$

which shows that all the coefficients of x^{-n} (where n is any positive integer) are zero.

Now we define the generating function;

$r_a(d)$ for $N(a,t,tn+d)$

where $r_a(d)=r_a(d,t)=\prod_{n=0}^{\infty}N(a,t,tn+d)x^{n}$,

and

$$r_{a,b}(d)=r_{a,b}(d,t)=r_a(d)-r_b(d)$$

$$=\prod_{n=0}^{\infty}\left\{N(a,t,tn+d)-N(b,t,tn+d)\right\}x^{n}.$$

The generating function $\phi(x)$ [7] is of the form;

$$\phi(x)=-1+\left\{\frac{1}{1-x}+\frac{x^{5}}{(1-x)\left(1-x^{4}\right)\left(1-x^{6}\right)}+\frac{x^{20}}{(1-x)\left(1-x^{4}\right)\left(1-x^{6}\right)\left(1-x^{9}\right)\left(1-x^{11}\right)}+\ldots\infty\right\},$$

$$=-1+\left(1+x+x^{2}+\ldots\infty\right)+x^{5}\left(1+x+x^{2}+\ldots\infty\right)\times\left(1+x^{4}+\ldots\infty\right)\left(1+x^{6}+\ldots\infty\right)+\ldots\infty$$

$$=x+x^{2}+x^{3}+x^{4}+2x^{5}+2x^{6}+2x^{7}+2x^{8}+\ldots\infty$$

$$=\sum_{n=0}^{\infty}\{N(1,5,5n)-N(2,5,5n)\}x^n$$

$$=r_{1,2}(0).$$

The generating function $A(x)$ is defined as;

$$A(x)=\frac{1-x^2-x^3+x^9+\ldots\infty}{(1-x)^2(1-x^4)^2(1-x^6)^2\ldots\infty}$$

$$=(1-x^2-x^3+x^9+\ldots\infty)(1+2x+3x^2+\ldots\infty)\times$$

$$(1+2x^4+3x^8+\ldots\infty)\ldots\infty$$

$$=1+2x+2x^2+x^3+2x^4+\ldots\infty$$

$$=1+\sum_{n=0}^{\infty}\{N(0,5,5n)-N(2,5,5n)+$$

$$2N(1,5,5n)-2N(2,5,5n)\}x^n$$

$$=1+\sum_{n=0}^{\infty}\{N(0,5,5n)-N(2,5,5n)\}x^n+$$

$$2\sum_{n=0}^{\infty}\{N(1,5,5n)-N(2,5,5n)\}x^n$$

$$=1+r_{0,2}(0)+2r_{1,2}(0).$$

The generating function is of the form;

$$\prod_{n=1}^{\infty}\frac{1-x^{5n}}{(1-x^{5n-4})(1-x^{5n-1})}$$

$$=\prod_{n=1}^{\infty}(1-x^{5n})(1+x^{5n-4}+\ldots\infty)(1+x^{5n-1}+\ldots\infty)$$

$$=(1-0)+(3-2)x+(12-11)x^2+x^3+2x^4+\ldots\infty$$

$$=\sum_{n=0}^{\infty}\{N(0,5,5n+1)-N(2,5,5n+1)\}x^n$$

$$=r_{0,2}(1).$$

The generating function is of the form;

$$\prod_{n=1}^{\infty}\frac{1-x^{5n}}{(1-x^{5n-3})(1-x^{5n-2})}$$

$$=\prod_{n=1}^{\infty}(1-x^{5n})(1+x^{5n-3}+x^{10n-6}+\ldots\infty)$$

$$=(1-0)+(3-3)x+(16-15)x^2+\ldots\infty$$

$$=\sum_{n=0}^{\infty}\{N(0,5,5n+2)-N(2,5,5n+2)\}x^n\times$$

$$(1+x^{5n-2}+x^{10n-4}+\ldots\infty)$$

$$=r_{1,2}(2).$$

The generating function $\Psi(x)$ is of the form;

$$\Psi(x)=-1+\left\{\frac{1}{1-x^2}+\frac{x^5}{(1-x^2)(1-x^3)(1-x^7)}+\right.$$

$$\left.\frac{x^{20}}{(1-x^2)(1-x^3)(1-x^7)(1-x^8)(1-x^{12})}+\ldots\infty\right\}$$

$$=-1+(1+x^2+x^4+\ldots\infty)+x^5(1+x^2+\ldots\infty)$$

$$(1+x^3+x^6+\ldots\infty)(1+x^7+\ldots\infty)+\ldots\infty$$

$$=x^2+x^4+x^6+x^7+2x^8+x^9+2x^{10}+\ldots\infty.$$

Hence,

$$\frac{\Psi(x)}{x}=x+x^3+x^4+x^5+x^6+2x^7+x^8+2x^9+\ldots\infty$$

$$=\sum_{n=0}^{\infty}\{N(2,5,5n+3)-N(0,5,5n+3)\}x^n$$

$$=r_{2,0}(3)$$

and, $r_{0,2}(3) = -\frac{\Psi(x)}{x}$.

The generating function $D(x)$ is of the form;

$$D(x) = \frac{1 - x - x^4 + x^7 + \ldots\infty}{(1-x^2)^2(1-x^3)^2(1-x^7)^2\ldots\infty}$$

$$= (1 - x - x^4 + x^7 + \ldots\infty)(1 + 2x^2 + 3x^4 + \ldots\infty)\times (1 + 2x^3 + \ldots\infty)(1 + 2x^7 + \ldots\infty)\ \ldots\infty$$

$$= 1 - x + 2x^2 + 0.x^3 + \ldots\infty$$

$$= \sum_{n=0}^{\infty} \{N(0,5,5n+3) - N(1,5,5n+3) + N(0,5,5n+3) - N(2,5,5n+3)\} x^n$$

$$= \sum_{n=0}^{\infty} \{N(0,5,5n+3) - N(1,5,5n+3)\} x^n + \sum_{n=0}^{\infty} \{N(0,5,5n+3) - N(2,5,5n+3)\} x^n$$

$$= r_{0,1}(3) + r_{0,2}(3)$$

4. The Generating Functions for $\beta_1(n)$ and $\beta_2(n)$

First we shall establish the following identity, which is used in proving the Theorems. If a and t are both real numbers with $|a| < 1$ and $|t| < 1$, we have;

$$\frac{(at)_\infty}{(t)_\infty} = \frac{(1-at)(1-atx)(1-atx^2)\ldots\infty}{(1-t)(1-tx)(1-tx^2)\ldots\infty}$$

$$= \{(1-at)(1-atx)\ \ldots\infty\}(1 + t + t^2 + \ldots\infty)\times (1 + tx + t^2x^2 + \ldots\infty)(1 + t^4x^4 + \ldots\infty)\ \ldots\infty$$

$$= 1 + t\{(1 + x + x^2 + \ldots\infty) - a(1 + x + x^2 + \ldots\infty)\} + t^2\{(1 + x + 2x^2 + 2x^3 + \ldots\infty) - a(1 + 2x + 3x^2 + \ldots\infty) + a^2(x + x^2 + 2x^3 + 2x^4 + \ldots\infty) + \ldots\infty$$

$$= 1 + (1-a)t(1 + x + x^2 + \ldots\infty) + (1-a)(1-ax)t^2(1 + x + 2x^2 + 2x^3 + \ldots\infty) + \ldots\infty$$

$$= 1 + \frac{(1-a)t}{1-x} + \frac{(1-a)(1-ax)t^2}{(1-x)(1-x^2)} + \frac{(1-a)(1-ax)(1-ax^2)t^3}{(1-x)(1-x^2)(1-x^3)} + \ldots\infty$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n t^n}{(x)_n}$$

i.e., $$\frac{(at)_\infty}{(t)_\infty} = \sum_{n=0}^{\infty} \frac{(a)_n t^n}{(x)_n}. \qquad (4)$$

The generating function for $\beta_1(n)$ is defined as;

$$\sum_{n=0}^{\infty} \frac{x^n}{(x^5;x^5)_n (x^{5n+4};x^5)_\infty}$$

$$= \frac{1}{(1-x^4)(1-x^9)(1-x^{14})\ldots\infty} + \frac{1}{(1-x^5)(1-x^9)(1-x^{14})\ldots\infty} + \frac{x^2}{(1-x^5)(1-x^{10})(1-x^{14})\ldots\infty} + \ldots\infty$$

$$= 1 + x + x^2 + x^3 + 2x^4 + x^5 + 2x^6 + 3x^8 + \ldots\infty$$

$$= \sum_{n=0}^{\infty} \beta_1(n)\, x^n, \qquad (5)$$

where we have assumed $\beta_1(0)=1$.

The generating function for $\beta_2(n)$ is defined as;

$$\sum_{n=0}^{\infty}\frac{x^{2n}}{\left(x^5;x^5\right)_n\left(x^{5n+3};x^5\right)_\infty}$$

$$=\frac{1}{\left(1-x^3\right)\left(1-x^8\right)\left(1-x^{13}\right)\ldots\infty}+$$

$$\frac{x^2}{\left(1-x^5\right)\left(1-x^8\right)\left(1-x^{13}\right)\ldots\infty}+$$

$$\frac{x^4}{\left(1-x^5\right)\left(1-x^{10}\right)\left(1-x^{13}\right)\ldots\infty}+\ldots\infty$$

$$=1+x^2+x^3+x^4+2x^6+x^7+2x^8+\ldots\infty$$

$$=\sum_{n=0}^{\infty}\beta_1(n)\,x^n\,, \qquad (6)$$

where we have assumed, $\beta_1(0)=1$.

Here we give two Theorems, which are related to the terms $\beta_1(n)$ and $\beta_2(n)$ respectively.

Theorem1: $N(0,5,5n+1)=\beta_1(n)+N(2,5,5n+1)$, where $\beta_1(n)$ is the number of partitions of *n* into 1's and parts congruent to 0 or –1 modulo 5 with the largest part $\equiv 0(\mathrm{mod}\,5)\leq 5$ times the number of 1's ≤ the smallest part $\equiv -1(\mathrm{mod}\,5)$.

Proof: From (4) by replacing $\left(z^{-1}x\right)$ for *a* and *z* for *t* we have;

$$\frac{(x)_\infty}{(z)_\infty\left(z^{-1}x\right)_\infty}=\frac{1}{\left(z^{-1}x\right)_\infty}\sum_{n=0}^{\infty}\frac{\left(z^{-1}x\right)_n z^n}{(x)_n}\,,$$

where $|z|<1$ but $z\neq 0$

$$=\frac{1}{\left(1-z^{-1}x\right)\left(1-z^{-1}x^2\right)\ldots\infty}\left[1+\frac{\left(1-z^{-1}x\right)z}{(1-x)}+\frac{\left(1-z^{-1}x\right)\left(1-z^{-1}x^2\right)z^2}{(1-x)\left(1-x^2\right)}+\ldots\infty\right]$$

$$=\frac{1}{\left(1-z^{-1}x\right)\left(1-z^{-1}x^2\right)\ldots\infty}+\frac{z}{(1-x)\left(1-z^{-1}x^2\right)\ldots\infty}$$

$$+\frac{z^2}{(1-x)\left(1-x^2\right)\left(1-z^{-1}x^3\right)\ldots\infty}+\ldots\infty$$

$$=\frac{(1-x)\left(1-x^2\right)\ldots\infty}{(1-z)(1-zx)\ldots\left(1-z^{-1}x\right)\left(1-z^{-1}x^2\right)\ldots\infty}\,.$$

Replacing *x* by x^5 and *z* by *x*, we obtain;

$$\frac{1}{\left(1-x^4\right)\left(1-x^9\right)\ldots\infty}+\frac{x}{\left(1-x^5\right)\left(1-x^9\right)\left(1-x^{14}\right)\ldots\infty}+$$

$$\frac{x^2}{\left(1-x^5\right)\left(1-x^{10}\right)\left(1-x^{14}\right)\ldots\infty}+\ldots\infty$$

$$=\frac{\left(1-x^5\right)\left(1-x^{10}\right)\ldots\infty}{\left\{(1-x)\left(1-x^6\right)\ldots\infty\right\}\left(1-x^4\right)\left(1-x^9\right)\ldots\infty}\,.$$

Hence,

$$\sum_{n=0}^{\infty}\frac{x^n}{\left(x^5;x^5\right)_n\left(x^{5n+4};x^5\right)_\infty}=\frac{\left(x^5;x^5\right)_\infty}{\left(x;x^5\right)_\infty\left(x^4;x^5\right)_\infty}$$

$$\therefore\sum_{n=0}^{\infty}\beta_1(n)x^n=r_{0,2}(1),\ \text{by above;}$$

$$=\sum_{n=0}^{\infty}\left\{N(0,5,5n+1)-N(2,5,5n+1)\right\}x^n\,.$$

Equating the coefficient of x^n on both sides, we get;

$\beta_1(n)=N(0,5,5n+1)-N(2,5,5n+1)$.
Hence the Theorem.

Example 1: $N(0,5,11)=12$, $N(2,5,11)=11$, $\beta_1(2)=1$, with the relevant partition is 1 + 1.

$\therefore N(0, 5, 11) = \beta_1(2) + N(2, 5, 11)$.

Theorem 2: $N(1,5,5n+1) = \beta_2(n) + N(2,5,5n+2)$, where $\beta_2(n)$ is the number of partitions of n into 2's and parts congruent to 0 or – 2 modulo 5 with the largest part $\equiv 0 (\mathrm{mod}\, 5) \le 5$ times the number of 2's ≤ the smallest part $\equiv -2 (\mathrm{mod}\, 5)$.

Proof: From (4) by replacing $(z^{-1}x)$ for a, and z for t we have;

$$\frac{(x)_\infty}{(z)_\infty (z^{-1}x)_\infty} = \frac{1}{(z^{-1}x)_\infty} \sum_{n=0}^{\infty} \frac{(z^{-1}x)_n z^n}{(x)_n},$$

where $|z| < 1$ but $z \ne 0$

$$= \frac{1}{(1-z^{-1}x)(1-z^{-1}x^2)...\infty}\left[1 + \frac{(1-z^{-1}x)z}{(1-x)} + \frac{(1-z^{-1}x)(1-z^{-1}x^2)z^2}{(1-x)(1-x^2)} + ...\infty\right]$$

$$= \frac{1}{(1-z^{-1}x)(1-z^{-1}x^2)...\infty} + \frac{z}{(1-x)(1-z^{-1}x^2)...\infty}$$

$$+ \frac{z^2}{(1-x)(1-x^2)(1-z^{-1}x^3)...\infty} + ...\infty$$

$$= \frac{(1-x)(1-x^2)...\infty}{(1-z)(1-zx)...(1-z^{-1}x)(1-z^{-1}x^2)...\infty}.$$

After replacing x by x^5, and z by x^2, we get;

$$\frac{1}{(1-x^4)(1-x^9)...\infty} + \frac{x}{(1-x^5)(1-x^9)(1-x^{14})...\infty} +$$

$$\frac{x^2}{(1-x^5)(1-x^{10})(1-x^{14})...\infty} + ...\infty.$$

We get by replacing x by x^5 and z by x^2;

$$\frac{1}{(1-x^3)(1-x^8)...\infty} + \frac{x^2}{(1-x^5)(1-x^8)(1-x^{13})...\infty} +$$

$$\frac{x^4}{(1-x^5)(1-x^{10})(1-x^{13})...\infty} + ...\infty$$

$$= \frac{(1-x^5)(1-x^{10})(1-x^{15})...\infty}{\{(1-x^2)(1-x^7)...\infty\}\{(1-x^3)(1-x^8)...\infty\}}.$$

Hence,

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(x^5;x^5)_n (x^{5n+3};x^5)_\infty} = \frac{(x^5;x^5)_\infty}{(x^2;x^5)_\infty (x^3;x^5)_\infty}$$

$$\therefore \sum_{n=0}^{\infty} \beta_2(n) x^n = r_{1,2}(2), \text{ by above;}$$

$$= \sum_{n=0}^{\infty} \{N(1,5,5n+2) - N(2,5,5n+2)\} x^n.$$

Equating the coefficient of x^n on both sides, we get;

$$\beta_2(n) = N(1,5,5n+2) - N(2,5,5n+2)$$

$$N(1,5,5n+2) = \beta_2(n) + N(2,5,5n+2).$$

Hence the Theorem.

Example 2: $N(1,5,12) = 16$, $N(2, 5, 12) = 15$, $\beta_2(2) = 1$, with the relevant partition is 2.

$\therefore N(1, 5, 12) = \beta_2(2) + N(2, 5, 12)$

5. Conclusion

In this study we have discussed the relations collected from Ramanujan's Lost Notebook VI. We have established Garvan's theta series with the help of Ramanujan's identities. We have proved the Theorem 1 in terms of $\beta_1(n)$ and have verified the Theorem with a numerical example. Finally we have established the Theorem 2 in terms of $\beta_2(n)$ and have verified the Theorem with a numerical example when n =2

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