Andrews-Garvan-Liang's Spt-crank for Marked Overpartitions

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ABSTRACT
In 2009, Bingmann, Lovejoy and Osburn have shown the generating function for $spt_2(n)$. In 2012, Andrews, Garvan, and Liang have defined the *sptcrank* in terms of partition pairs. In this article the number of smallest parts in the overpartitions of *n* with smallest part not overlined and even are discussed, and the vector partitions and *S*-partitions with 4 components, each a partition with certain restrictions are also discussed. The generating function for $spt_2(n)$, and the generating function for $M_S(m, n)$ are shown with a result in terms of modulo 3. This paper shows how to prove the Theorem 1, in terms of $M_S(m, n)$ with a numerical example, and shows how to prove the Theorem 2, with the help of *sptcrank* in terms of partition pairs. In 2014, Garvan and Jennings-Shaffer are capable to define the *sptcrank* for marked overpartitions. This paper also shows another result with the help of 15 SP_2-partition pairs of 8 and shows how to prove the Corollary with the help of 15 marked overpartitions of 8.

Key words: *crank*, non-negative, overpartitions, overlined, *sptcrank*, weight.

INTRODUCTION

In this paper we give some related definitions of $\overline{spt_2}(n)$, various product notations, vector partitions and $\overline{S}$-partitions, $M_{\overline{S}_2}(m,n)$, $M_{\overline{S_2}}(m,t,n)$, $\overline{S}_2(z,x)$, marked partition and $\overline{sptcrank}$ for marked overpartitions. We discuss the generating function for $\overline{spt}_2(n)$ and prove the Corollary 1 with the help of generating function to prove the Result 1 with the help of 3 vector partitions from $\overline{S}_2$ of 4. We prove the Theorem 1 with the help of various generating functions and prove the Corollary 2 with a special series $\overline{S}_2(z,x)$, when n =1 and prove the Theorem 2 with the help of $\overline{sptcrank}$ in terms of partition pairs (λ_1,λ_2) when $0 < s(\lambda_1) \le s(\lambda_2)$. We prove the Result 2 using the $\overline{crank}$ of partition pairs $\vec{\lambda} = (\lambda_1,\lambda_2)$ and prove the Corollary 3 and 4 with the help of marked overpartition of $3n$ and of $3n+1$ (when n = 2) respectively. Finally we analyze the Corollary 5 with the help of marked overpartitions of $5n+3$ when n =1.

Some Related Definitions

In this section we have described some definitions related to the article following (Garvan and Shaffer 2014).

$\overline{spt_2}(n)$ (Bringann et al. 2009): The number of smallest parts in the overpartitions of n with smallest part not overlined and even is denoted by $\overline{spt_2}(n)$ for example,

n			$\overline{spt}_2(n)$
1	:		0
2	:	$\dot{2}$	1
3	:		0
4	:	$\dot{4},\ \dot{2}+\dot{2}$	3
5	:	$\dot{3}+\dot{2},\ \bar{3}+\dot{2}$	2
...		… …	

From above we get;
$\overline{spt}_2(6) = 6,\ \overline{spt}_2(7) = 6, ...$

Product Notations

$$(x)_\infty = (1-x)(1-x^2)(1-x^3)...$$
$$(x^2;x^2)_\infty = (1-x^2)(1-x^4)...$$
$$(x)_k = (1-x)(1-x^2)(1-x^3)...(1-x^k)$$
$$(-x^5;x)_\infty = (1+x^5)(1+x^6)(1+x^7)...$$

Vector Partitions and $\overline{S}$ -Partitions

A vector partition can be done with 4 components each partition with certain restrictions (Bringann et al. 2013). Let, $\vec{V} = D \times P \times P \times D$, where D denote the set of all partitions into distinct parts, P denotes the set of all partitions. For a partition π, we let, $s(\pi)$ denotes the smallest part of π (with the convention that the empty partition has smallest part ∞), $\#(\pi)$ the number of parts in π, and $|\pi|$ the sum of the parts of π.

For $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4) \in \vec{V}$, we define the weight $\omega(\vec{\pi}) = (-1)^{\#(\pi_1)-1}$, the crank c $(\vec{\pi})$ =$\#(\pi_2)$–$\#(\pi_3)$, the norm $\left|\vec{\pi}\right| = |\pi_1| + |\pi_2| + |\pi_3| + |\pi_4|$.

We say $\vec{\pi}$ is a vector partition of *n* if $\vec{\pi} = n$. Let $\overline{S}$ denotes the subset of $\overline{V}$ and it is given by; $\overline{S} = \{(\pi_1, \pi_2, \pi_3, \pi_4) \in \vec{V}, 1 \le s(\pi_1) < \infty, s(\pi_1) \le$
$s(\pi_2), s(\pi_1) \le s(\pi_3), s(\pi_1) < s(\pi_4)\}$.

Let $\overline{S_2}$ denotes the subset of $\overline{S}$ with $s(\pi_1)$ even.

$M_{\overline{S}_2}(m,n)$: The number of vector partitions of *n* in $\overline{S}_2$ with crank *m* are counted according to the weight ω is exactly $M_{\overline{S}_2}(m,n)$.

$M_{\overline{S}_2}(m,t,n)$: The number of vector partitions of *n* in $\overline{S}_2$ with crank congruent to *m* modulo *t* are counted according to the weight ω is exactly $M_{\overline{S}_2}(m,t,n)$.

$\overline{S}_2$(z, *x*): The series $\overline{S}_2(z, x)$ is defined by the generating function for $M_{\overline{S}_2}(m,n)$.

i.e., $\overline{S}_2$(z, *x*)

$$= \sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}$$

$$= \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\overline{S_2}}(m,n) z^m x^n .$$

Marked Partition (Andrews et al. 2013): We define a marked partition as a pair (λ, k) where λ is a partition and *k* is an integer identifying one of its smallest parts i.e., k =1, 2, ..., $\nu(\lambda)$, where $\nu(\lambda)$ is the number of smallest parts of λ.

$\overline{sptcrank}$ for Marked overpartitions (Chen et al. 2013): We define a marked overpartitions of *n* as a pair (π, j) where π is an overpartition of *n* in which the smallest part is not overlined and even. It is clear that $\overline{spt_2}(n)$ = # of marked overpartitions (π, j) of *n*. For example, there are 3 marked overpartitions of 4, like:

(4,1), (2+2,1), and (2+2,2).

Then, $\overline{spt}_2(4) = 3$.

The Generating Function for $\overline{spt}_2$ (*N*)

The generating function (Bringann et al. 2013) for $\overline{spt}_2$ (*n*) is given by;

$$\sum_{n=1}^{\infty} \frac{x^{2n}(-x^{2n+1};x)_\infty}{(1-x^{2n})^2(x^{2n+1};x)_\infty}$$

$$= \frac{x^2(-x^3;x)_\infty}{(1-x^2)^2(x^3;x)_\infty} + \frac{x^4(-x^5;x)_\infty}{(1-x^4)^2(x^5;x)_\infty} + \ldots$$

$$= o.x + 1.x^2 + o.x^3 + 3.x^4 + 2.x^5 + 6.x^6 + \ldots$$

$$= \overline{spt}_2(1)x + \overline{spt}_2(2)x^2 + \overline{spt}_2(3)x^3 + \overline{spt}_2(4)x^4 + \overline{spt}_2(5)x^5 + \ldots.$$

$$= \sum_{n=1}^{\infty} \overline{spt}_2(n)x^n.$$

For convenience, define $\overline{spt}_2$ (1) = 0.

From above we get $\overline{spt}_2(3) = 0$, $\overline{spt}_2(6) = 6, \ldots$

i.e., $\overline{spt}_2(3.1) = 0 \equiv 0 \pmod 3$,

$\overline{spt}_2(3.2) = 6 \equiv 0 \pmod 3$, ...

We can conclude that $\overline{spt}_2(3n) \equiv 0 \pmod 3$.

We also get $\overline{spt}_2(4) = 3$, $\overline{spt}_2(7) = 6, \ldots$

i.e., $\overline{spt}_2(3+1) = 3 \equiv 0 \pmod 3$,

$\overline{spt}_2(3.2+1) = 6 \equiv 0 \pmod 3$, …

We can conclude that $\overline{spt}_2(3n+1) \equiv 0 \pmod 3$ (Bringann 2009). Again from above we get;

$\overline{spt}_2(3) = 0$, $\overline{spt}_2(8) = 15, \ldots$

i.e., $\overline{spt}_2(3) = 0 \equiv 0 \pmod 5$,

$\overline{spt}_2(5+3) = 15 \equiv 0 \pmod 5$, ...

We can conclude that $\overline{spt}_2(5n+3) \equiv 0 \pmod 5$.

Corollary 1: $\overline{spt}_2(n) = \sum_{m=-\infty}^{\infty} M_{\bar{S}_2}(m,n)$.

Proof: The generating function for $M_{\bar{S}_2}(m,n)$ is given by;

$$\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\overline{S}_2}(m,n)\ z^m x^n$$

$$=\sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}.$$

If $z = 1$, then,

$$\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\overline{S}_2}(m,n)\ x^n$$

$$=\sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty}{(x^{2n};x)_\infty(x^{2n};x)_\infty}$$

$$=\frac{x^2(-x^3;x)_\infty(x^3;x)_\infty}{(x^2;x)_\infty^2}+\frac{x^4(-x^5;x)_\infty(x^5;x)_\infty}{(x^4;x)^2_\infty}+\ldots$$

$$=\frac{x^2(-x^3;x)_\infty(1-x^3)(1-x^4)\ldots}{(1-x^2)^2(1-x^3)^2\ldots}+\frac{x^4(-x^5;x)_\infty(1-x^5)(1-x^6)\ldots}{(1-x^4)^2(1-x^5)^2\ldots}+\ldots$$

$$=\frac{x^2(-x^3;x)_\infty}{(1-x^2)^2(1-x^3)(1-x^4)\ldots}+\frac{x^4(-x^5;x)_\infty}{(1-x^4)^2(1-x^5)(1-x^6)\ldots}+\ldots$$

$$=\sum_{n=1}^{\infty} \frac{x^{2n}(-x^{2n+1};x)_\infty}{(1-x^{2n})^2(x^{2n+1};x)_\infty}$$

$$=\sum_{n=1}^{\infty} \overline{spt_2}(n)x^n.$$

i.e., $\sum_{n=1}^{\infty} \overline{spt_2}(n)x^n=\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\overline{S}_2}(m,n)\ x^n$.

Now equating the co-efficient of x^n from both sides we get;

$$\overline{spt_2}(n) = \sum_{m=-\infty}^{\infty} M_{\overline{S}_2}(m,n).$$

Hence the Corollary.

Result 1:

$$M_{\overline{S_2}}(0,3,4) = M_{\overline{S_2}}(1,3,4) = M_{\overline{S_2}}(2,3,4) = \frac{1}{3}\overline{spt_2}(4).$$

Proof: We prove the result with the help of examples. We see the vector partitions from $\overline{S_2}$ of 4 along with their weights and cranks and are given as follows:
Here we have used ϕ to indicate the empty partition. Thus we have,

$M_{\overline{S_2}}(0,3,4)=1,\quad M_{\overline{S}_2}(1,3,4)=1$,

$M_{\overline{S_2}}(2,3,4)=M_{\overline{S_2}}(-1,3,4)=1$

$\therefore M_{\overline{S_2}}(0,3,4)=M_{\overline{S_2}}(1,3,4)$

$= M_{\overline{S_2}}(2,3,4) = 1 = \frac{1}{3}.3 = \frac{1}{3}\overline{spt_2}(3)$.

Hence the Result.

Table 1

$\overline{S_2}$-vector partition $(\vec{\pi})$ of 4	Weight $\omega(\vec{\pi})$	Crank $(\vec{\pi})$	mod 3
$\vec{\pi}_1 = (4,\phi,\phi,\phi)$	1	0	0
$\vec{\pi}_2 = (2+2,\phi,\phi)$	1	1	1
$\vec{\pi}_3 = (2,\ \phi,\ 2,\phi)$	1	−1	2
	$\sum\omega(\vec{\pi}) = 3$		

Now from table 1 we get; $\sum\omega(\vec{\pi}) = 3$, i.e., $\sum_{k=0}^{2} M_{\overline{S_2}}(k,3,4) = 3$.

$\therefore \overline{spt_2}(4) = \sum_{k=0}^{2} M_{\overline{S_2}}(k,3,4) = \sum\omega(\vec{\pi}).$

Now we can define;

$$M_{\overline{S_2}}(k,t,n) = \sum_{m\equiv k(\bmod t)} M_{\overline{S_2}}(m,n)$$

and $\overline{spt_2}(n) = \sum_{m=-\infty}^{\infty} M_{\overline{S_2}}(m,n) = \sum_{k=0}^{t-1} M_{\overline{S_2}}(k,t,n).$

Theorem 1: The number of vector partitions of n in $\overline{S_2}$ with crank m counted according to the weight ω is non-negative, i.e., $M_{\overline{S_2}}(m,n) \geq 0.$

Proof: The generating function for $M_{\overline{S_2}}(m,n)$ is given by;

$$\sum_{n=1}^{\infty}\sum_{m=-\infty} M_{\overline{S_2}}(m,n)z^m x^n$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}.(x^{4n+2};x^2)_\infty .$$

[Since $\sum_{n=1}^{\infty} (x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty$

$= (x^3;x)_\infty(-x^3;x)_\infty + (x^5;x)_\infty(-x^5;x)_\infty + ...$

$= (1-x^3)(1-x^4)...(1+x^3)(1+x^4)... + (1-x^5)(1-x^6)...(1+x^5)... + ...$

$= (1-x^6)(1-x^8)... + (1-x^{10})(1-x^{12})... + (1-x^{14})... + ...$

$$=(x^6;x^2)_\infty+(x^{10};x^2)_\infty+...$$

$$=\sum_{n=1}^{\infty}(x^{4n+2};x^2)_\infty]$$

$$=\sum_{n=1}^{\infty}\frac{x^{2n}(x^{4n};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}.\frac{(x^{4n+2};x^2)_\infty}{(x^{4n};x)_\infty}$$

$$=\sum_{n=1}^{\infty}\frac{x^{2n}(x^{4n};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}.\frac{1}{(1-x^{4n})(x^{4n+1};x^2)_\infty}$$

[Since, $\sum_{n=1}^{\infty}\frac{(x^{4n+2};x^2)_\infty}{(x^{4n};x)_\infty}=\frac{(x^6;x^2)_\infty}{(x^4;x)_\infty}+\frac{(x^{10};x^2)_\infty}{(x^8;x)_\infty}+...$

$$=\frac{(1-x^6)(1-x^8)...}{(1-x^4)(1-x^5)(1-x^6)...}+\frac{(1-x^{10})(1-x^{12})...}{(1-x^8)(1-x^9)(1-x^{10})(1-x^{11})...}+...$$

$$=\frac{1}{(1-x^4)(1-x^5)(1-x^7)...}+\frac{1}{(1-x^8)(1-x^9)(1-x^{11})...}+...$$

$$=\sum_{n=1}^{\infty}\frac{1}{1-x^{4n}}.\frac{1}{(x^{4n+1};x^2)_\infty}]$$

$$=\sum_{n=1}^{\infty}x^{2n}\sum_{k=0}^{\infty}\frac{(z^{-1}x^{2n})^k}{(zx^{2n+k};x)_\infty(x)_k}.\frac{1}{(1-x^{4n})(x^{4n+1};x^2)_\infty}$$

[Since, $\sum_{n=1}^{\infty}\frac{x^{2n}(x^{4n};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}$

$=\sum_{n=1}^{\infty}x^{2n}\sum_{k=0}^{\infty}\frac{(z^{-1}x^{2n})^k}{(zx^{2n+k};x)_\infty(x)_k}]$. (by Berkovich and Garvan 2008)

We see that the coefficient of any power x in the right hand side is non-negative so the coefficient $M_{\bar{S}_2}(m,n)$ of $z^m x^n$ is non-negative, i.e., $M_{\overline{S_2}}(m,n)\geq 0$. Hence the Theorem.

Numerical example 1

The vector partitions from $\overline{S_2}$ of 5 along with their weights and cranks are given as follows:

Here we have used ϕ to indicate the empty partition. Thus we have;

$M_{\overline{S_2}}(0,5)=1-1=0$, $M_{\overline{S_2}}(1,5)=1$, and $M_{\overline{S_2}}(-1,5)=1$, i.e., $\sum_m M_{\overline{S_2}}(m,5)=2$,

i.e., every term is non-negative, i.e., $M_{\overline{S_2}}(m,n)\geq 0$.

So we can conclude that, $M_{\bar{S}_2}(m,n)\geq 0$.

Corollary 2: $\overline{S}_2(1,x) = \sum_{n=1}^{\infty} \overline{spt_2}(n)\, x^n$.

Proof: We get;

$$\overline{S}_2(z,x)=\sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}$$ (Andrews et al. 2012).

If z = 1, then we get;

$$\overline{S}_2(1,x)=\sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty}{(x^{2n};x)_\infty(x^{2n};x)_\infty}$$

$$= \frac{x^2(x^3;x)_\infty(-x^3;x)_\infty}{(x^2;x)_\infty^2}+\frac{x^4(-x^5;x)_\infty(x^5;x)_\infty}{(x^4;x)_\infty^2}+...$$

$$= \frac{x^2(-x^3;x)_\infty(1-x^3)(1-x^4)...}{(1-x^2)^2(1-x^3)^2...}+\frac{x^4(-x^5;x)_\infty(1-x^5)(1-x^6)...}{(1-x^4)^2(1-x^5)^2...}+...$$

$$= \frac{x^2(-x^3;x)_\infty}{(1-x^2)^2(1-x^3)...}+\frac{x^4(-x^5;x)_\infty}{(1-x^4)^2(1-x^5)...}+...$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}(-x^{2n+1};x)_\infty}{(1-x^{2n})^2(x^{2n+1};x)_\infty}$$

$$=\sum_{n=1}^{\infty}\overline{spt_2}(n)\, x^n .$$

i.e., $\overline{S}_2(1,x) = \sum_{n=1}^{\infty} \overline{spt_2}(n)\, \mathrm{x}^{\mathrm{n}}$. Hence the Corollary.

Theorem 2: $\overline{spt_2}(n)=\sum_{\substack{\overline{\lambda}\in \overline{SP}_2 \\ |\overline{\lambda}|=|\lambda_1|+|\lambda_2|=n}} 1$

Proof: First we define the $\overline{sptcrank}$ in terms of partition pairs,

$\overline{SP}=\{\overline{\lambda}=(\lambda_1,\lambda_2)\in P\times P: 0<s(\lambda_1)\le s(\lambda_2)$ and all parts of λ_2 that are $\ge 2s(\lambda_1)+1$ are odd$\}$.

Let $\overline{SP}_2$ be the set of $\vec{\lambda}=(\lambda_1,\lambda_2)\in\overline{SP}$ with $s(\lambda_1)$ even. The generating function for $\overline{spt}_2(n)$ is given by;

$$\sum_{n=1}^{\infty}\overline{spt}_2(n)x^n=\sum_{n=1}^{\infty} \frac{x^{2n}(-x^{2n+1};x)_\infty}{(1-x^{2n})^2(x^{2n+1};x)_\infty}$$

$$=\sum_{n=1}^{\infty} \frac{x^{2n}}{(1-x^{2n})^2(x^{2n+1};x)_\infty}(-x^{2n+1};x)_\infty$$

$$=\sum_{n=1}^{\infty} \frac{x^{2n}}{(1-x^{2n})^2(x^{2n+1};x)_\infty}.\frac{(x^{4n+2};x^2)_\infty}{(x^{2n+1};x)_\infty}$$

[Since, $\sum_{n=1}^{\infty}(-x^{2n+1};x)_\infty=(-x^3;x)_\infty+(-x^5;x)_\infty+...$

$$=(1+x^3)(1+x^4)\ldots+(1+x^5)(1+x^6)\ldots+(1+x^7)(1+x^8)\ldots+\ldots$$

$$=\frac{(1-x^6)(1-x^8)\ldots}{(1-x^3)(1-x^4)\ldots}+\frac{(1-x^{10})(1-x^{12})\ldots}{(1-x^5)(1-x^6)\ldots}+\frac{(1-x^{14})\ldots}{(1-x^7)\ldots}+\ldots$$

$$=\frac{(x^6;x^2)_\infty}{(x^3;x)_\infty}+\frac{(x^{10};x^2)_\infty}{(x^5;x)_\infty}+\ldots$$

$$=\sum_{n=1}^{\infty}\frac{(x^{4n+2};x^2)_\infty}{(x^{2n+1};x)_\infty}]$$

$$=\sum_{n=1}^{\infty}\frac{x^{2n}}{(1-x^{2n})^2(x^{2n+1};x)_\infty}.\frac{(x^{4n+2};x^2)_\infty}{(x^{2n+1};x)_\infty}$$

$$=\sum_{n=1}^{\infty}\frac{x^{2n}}{(x^{2n};x)_\infty}.\frac{1}{(1-x^{2n})}.\frac{(x^{4n+2};x^2)_\infty}{(x^{2n+1};x)_\infty}$$

[Since, $\sum_{n=1}^{\infty}\frac{x^{2n}}{(1-x^{2n})^2(x^{2n+1};x)_\infty}$

$$=\frac{x^2}{(1-x^2)^2(x^3;x)_\infty}+\frac{x^4}{(1-x^4)^2(x^5;x)_\infty}+\ldots$$

$$=\frac{x^2}{(1-x^2)^2(1-x^3)(1-x^4)\ldots}+\frac{x^4}{(1-x^4)^2(1-x^5)(1-x^6)\ldots}+\ldots$$

$$=\sum_{n=1}^{\infty}\frac{x^{2n}}{(x^{2n};x)_\infty}.\frac{1}{(1-x^{2n})}]$$

$$=\sum_{n=1}^{\infty}\frac{x^{2n}}{(x^{2n};x)_\infty}.\frac{1}{(1-x^{2n})(1-x^{2n+1})\ldots(1-x^{4n})(x^{4n+1};x^2)_\infty}$$

[Since, $\sum_{n=1}^{\infty}\frac{(x^{4n+2};x^2)_\infty}{(x^{2n+1};x)_\infty}=\frac{(x^6;x^2)_\infty}{(x^3;x)_\infty}+\frac{(x^{10};x^2)_\infty}{(x^5;x)_\infty}+\ldots$

$$=\frac{(1-x^6)(1-x^8)\ldots}{(1-x^3)(1-x^4)\ldots}+\frac{(1-x^{10})(1-x^{12})\ldots}{(1-x^5)(1-x^6)\ldots(1-x^{10})(1-x^{11})\ldots}+\ldots$$

$$=\frac{1}{(1-x^3)(1-x^4)(1-x^5)\ldots}+\frac{1}{(1-x^5)(1-x^6)\ldots(1-x^9)(1-x^{11})\ldots}+\ldots$$

$$=\sum_{n=1}^{\infty}\frac{1}{(1-x^{2n+1})\ldots.(1-x^{4n})(x^{4n+1};x^2)_\infty}]$$

$$=\sum_{n=1}^{\infty}\sum_{\substack{\lambda_1\in P\\ S(\lambda_1)=n}}x^{|\lambda_1|}\sum_{\substack{\lambda_2\in P\\ s(\lambda_2)\ge n}}x^{|\lambda_2|}$$

all parts in $\lambda_2\ge 2n+1$ are odd

$$=\sum_{n=1}^{\infty}\sum_{\substack{\bar\lambda\in \overline{SP_2}\\ |\bar\lambda|=|\lambda_1|+|\lambda_2|=n}}x^{|\bar\lambda|}.$$

Equating the co-efficient of x^n from both sides we get;

$\overline{spt_2}(n) = \sum_{\substack{\bar{\lambda} \in \overline{SP_2} \\ |\bar{\lambda}| = |\lambda_1| + |\lambda_2| = n}} 1$. Hence the Theorem.

Numerical Example 2

The overpartitions of 6 with smallest parts not overlined and even are 6, 4+2, $\bar{4}+2$, and 2+2+2. Consequently, the number of smallest parts in the overpartitions of 6 with smallest part not overlined and even is given by;

$\dot{6}$ $4+\dot{2}$, $\bar{4}+\dot{2}$, $\dot{2}+\dot{2}+\dot{2}$,

so that $\overline{spt}_2(6) = 6$ i.e., there are $6\ \overline{SP_2}$ -partition pairs of 6 like:
$(6,\phi)$, $(4+2,\phi)$, (2,4), $(2+2+2,\phi)$, $(2+2,2)$ and (2, 2+2).

Result 2:
$M_{\overline{S_2}}(0,5,8) = M_{\bar{S}_2}(1,5,8,) = M_{\bar{S}_2}(2,5,8,) =$

$M_{\bar{S}_2}(3,5,8) = M_{\bar{S}_2}(4,5,8) = 3 = \frac{1}{5}\ \overline{spt}_2(8)$.

Proof: We prove the result with the help of examples. We can define a $\overline{crank}$ of partition pairs $\vec{\lambda} = (\lambda_1, \lambda_2) \in \overline{SP}_2$.

For $\vec{\lambda} = (\lambda_1, \lambda_2) \in \overline{SP_2}$, we define, $k(\vec{\lambda}) =$ # of pairs j in λ_2 such that $s(\lambda_1) \le j \le 2\ s(\lambda_1) - 1$,

and also define; $\overline{crank}(\vec{\lambda}) = \begin{cases} (\#\text{of parts of } \lambda_1 \ge s(\lambda_1) + k) - k; \\ \text{if } k > 0 \\ (\#\text{of parts of } \lambda_1) - 1; \text{if } k = 0 \end{cases}$ where $k = k(\vec{\lambda})$.

We know that $\overline{spt_2}(8) = 15$. There are 15 $\overline{SP_2}$ -partition pairs of 8.

Table 2

$\overline{S_2}$ -vector partition $(\vec{\pi})$ of 5	Weight $\omega(\vec{\pi})$	Crank $c(\vec{\pi})$
$\vec{\pi}_1 = (3+2, \phi, \phi, \phi)$	-1	0
$\vec{\pi}_2 = (2, \phi, \phi, 3)$	1	0
$\vec{\pi}_3 = (2, 3, \phi, \phi)$	1	1
$\vec{\pi}_4 = (2, \phi, 3, \phi)$	1	-1
	$\sum \omega(\vec{\pi}) = 2$	

From the table 2 we get;

$M_{\overline{S}_2}(0,5,8) = M_{\overline{S}_2}(1,5,8,) = M_{\overline{S}_2}(2,5,8,) =$

$M_{\overline{S}_2}(3,5,8) = M_{\overline{S}_2}(4,5,8) = 3 = \frac{1}{5}\overline{spt}_2(8)$. Hence the Result.

Table 3

$\overline{SP_2}$ -partition pair of 8	k	$\overline{crank}$	(mod 5)
(3+2, 3)	1	0	0
(4+2, 2)	1	0	0
$(8, \phi)$	0	0	0
(2+2, 4)	0	1	1
(4+4, ϕ)	0	1	1
(6+2, ϕ)	0	1	1
(2, 2+2+2)	3	–3	2
(3+3+2, ϕ)	0	2	2
(4+2+2, ϕ)	0	2	2
(2, 3+3)	2	–2	3
(2+2, 2+2)	2	–2	3
(2+2+2+2, ϕ)	0	3	3
(2, 4+2)	1	–1	4
(4, 4)	1	–1	4
(2+2+2, 2)	1	–1	4

Now we will describe the $\overline{sptcrank}$ of a marked overpartition (Chen et al. 2013). To define the $\overline{sptcrank}$ of a marked overpartition we first need to define a function $k(m,n)$ for positive integers m, n such that $m \geq n+1$, we write $m = b2^j$, where b is odd and $j \geq o$. For a given odd integer b and a positive integer n we define $j_0 = j_0(b,n)$ to be the smallest non-negative integer j_0 such that $b2^{j_0} \geq n+1$.

We define; $k(m,n) = \begin{cases} 0, \text{if } b \geq 2n \\ 2^{j-j_0} \text{ if } b2^{j_0} < 2n \\ 0, \text{if } b2^{j_0} = 2n. \end{cases}$

Table 4

Marked overpartition (π, j) of 6	π_1	π_2	$\nu(\pi_1)$	$k((\pi_2, s(\pi_1))$	$\bar{k}$	$\overline{sptcrank}$	(mod 3)
(6 ,1)	6	ϕ	1	0	0	0	0
(4+2 ,1)	4+2	ϕ	1	0	0	1	1
$(\bar{4}+2,1)$	2	4	1	0	0	0	0
(2+2+2, 1)	2+2+2	ϕ	3	0	2	–2	1
(2+2+2, 2)	2+2+2	ϕ	3	0	1	–1	2
(2+2+2, 3)	2+2+2	ϕ	3	0	0	2	2

For a marked overpartitions (π, j) we let π_1 be the partition formed by the non-overlined parts of π, π_2 be the partition (into distinct parts) formed by the overlined parts of π so that $s(\pi_2) > s(\pi_1)$, we define $\bar{k}(\pi, i) = \nu(\pi_1) - j + k(\pi_2, s(\pi_1))$, where $\nu(\pi_1)$ is the number of smallest parts of π_1.

Now we can define;

$$\overline{sptcrank}(\pi, j) = \begin{cases} (\#\text{of parts of } \pi_1 \geq s(\pi_1) - \bar{k}), \\ \text{if } \bar{k} = \bar{k}(\pi, j) > 0 \\ (\#\text{of parts of } \pi_1) - 1; \\ \text{if } \bar{k} = \bar{k}(\pi, j) = 0. \end{cases}$$

Table 5

Marked overpartition (π, j) of 7	π_1	π_2	$\nu(\pi_1)$	$k((\pi_2, s(\pi_1))$	$\bar{k}$	$\overline{sptcrank}$	(mod 3)
(5+2, 1)	5+2	ϕ	1	0	0	1	1
$(\bar{5}+2,1)$	2	5	1	0	0	0	0
(3+2+2, 1)	3+2+2	ϕ	2	0	1	0	0
(3+2+2, 2)	3+2+2	ϕ	2	0	0	2	2
($\bar{3}$+2+2, 1)	2+2	3	2	1	2	–2	1
($\bar{3}$+2+2, 2)	2+2	3	2	1	1	–1	2

Table 6

Marked overpartition (π, j) of 8	π_1	π_2	$\nu(\pi_1)$	$k((\pi_2, s(\pi_1))$	$\bar{k}$	$\overline{sptcrank}$	(mod 5)
$(\bar{6}+2,1)$	2	6	1	2	2	–2	3
$(\bar{4}+2+2,1)$	2+2	4	2	0	1	–1	4
$(\bar{4}+2+2,2)$	2+2	4	2	0	0	1	1
$(\bar{3}+3+2,1)$	3+2	3	1	1	1	0	0
(2+2+2+2, 1)	2+2+2+2	ϕ	4	0	3	–3	2
(2+2+2+2, 2)	2+2+2+2	ϕ	4	0	2	–2	3
(2+2+2+2, 3)	2+2+2+2	ϕ	4	0	1	–1	4
(2+2+2+2, 4)	2+2+2+2	ϕ	4	0	0	3	3
(3+3+2, 1)	3+3+2	ϕ	1	0	0	2	2
(4+2+2, 1)	4+2+2	ϕ	1	0	1	0	0
(4+2+2, 2)	4+2+2	ϕ	2	0	0	2	2
(6+2, 1)	6+2	ϕ	1	0	0	1	1
(4+4, 1)	4+4	ϕ	2	0	1	–1	4
(4+4, 2)	4+4	ϕ	2	0	0	1	1
(8, 1)	8	ϕ	1	0	0	0	0

Corollary 3 (Lovejoy and Osburn 2009)**:** The residue of the $\overline{sptcrank}(\text{mod } 3)$ divides the marked overpartitions of 3*n* with the smallest part not overlined and even into 3 equal classes.

Proof: We prove the Corollary with the help of an example when *n* = 2. There are 6 marked overpartitions of 3*n* (when *n* = 2) with the smallest part not overlined and even so that, $\overline{spt}_2(6) = 6$.

We see that the residue of the $\overline{sptcrank}(\text{mod } 3)$ divides the marked overpartitions of 3*n* (when *n* = 2) with smallest part not overlined and even into 3 equal classes. Hence the Corollary.

Corollary 4: The residue of the $\overline{sptcrank}(\text{mod } 3)$ divides the marked overpartitions of 3*n*+1 with smallest part not overlined and even into 3 equal classes.

Proof: We prove the Corollary with the help of an example when *n* = 2. There are 6 marked overpartitions of 7 with the smallest part not overlined and even, so that $\overline{spt}_2(7) = 6$. We see that the residue of the $\overline{sptcrank}$ (mod 3) divides the marked overpartitions of 3*n*+1 (when *n* = 2) with smallest part not overlined and even. Hence the Corollary.

Corollary 5: The residue of the $\overline{sptcrank}(\text{mod } 5)$ divides the marked overpartitions of 5*n*+3 with smallest part not overlined and even into 5 equal classes.

Proof: We prove the Corollary with the help of example when n = 1. There are 15 marked overpartitions of $5n + 3$ (when n =1) with the smallest part not overlined and even so that $\overline{spt_2}(8) = 15$. We see that the residue of the divides the marked overpartitions of 8 with the smallest part not overlined and even into 5 equal classes. Hence the corollary.

CONCLUSION

In this study we have found the number of smallest parts in the overpartitions of n with the smallest part not overlined and even for n=1, 2, 3, 4 and 5. We have shown various relations $\overline{spt_2}(3n) \equiv 0 (\text{mod } 3)$, $\overline{spt_2}(3n+1) \equiv 0 (\text{mod } 3)$, $\overline{spt_2}(5n+3) \equiv 0$ (mod 5), $M_{\overline{S_2}}(0,3,4) = M_{\overline{S_2}}(1,3,4) = M_{\overline{S_2}}(2,3,4) = \frac{1}{3}\overline{spt_2}(4)$ and $M_{\overline{S_2}}(0,5,8) = M_{\overline{S_2}}(1,5,8) = M_{\overline{S_2}}(2,5,8)$ $= M_{\overline{S_2}}(3,5,8) = M_{\overline{S_2}}(4,5,8) = 3 = \frac{1}{5}\ \overline{spt_2}(8)$ with numerical examples respectively. We have verified the Theorem 1 when n = 5 and have verified the Theorem 2 when n = 6. We have verified the Corollary 3 with 6 marked overpartitions of 6 and have verified the Corollary 4 with 6 marked overpartitions of 7 and also have established the Corollary 5 with 15 marked overpartition of 8.

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