Andrews-Garvan-Liang’s Spt-crank for Marked Overpartitions

Bhattacharjee, Nil and Das, Sabuj and Mohajan, Haradhan

Assistant Professor, Premier University, Chittagong, Bangladesh

13 October 2014

Online at https://mpra.ub.uni-muenchen.de/83047/
MPRA Paper No. 83047, posted 29 Dec 2017 17:37 UTC
Andrews-Garvan-Liang’s spt crank for Marked Overpartitions

Nil Ratan Bhattacharjee
Department of Mathematics, University of Chittagong, Bangladesh

Sabuj Das
Senior Lecturer, Department of Mathematics
Raozan University College, Bangladesh
E-mail: sabujdas.ctg@gmail.com

Haradhan Kumar Mohajan
Premier University, Chittagong, Bangladesh
E-mail: haradhan1971@gmail.com

ABSTRACT
In 2009, Bingmann, Lovejoy and Osburn have shown the generating function for spt_2(n). In 2012, Andrews, Garvan, and Liang have defined the spt crank in terms of partition pairs. In this article the number of smallest parts in the overpartitions of n with smallest part not overlined and even are discussed, and the vector partitions and S-partitions with 4 components, each a partition with certain restrictions are also discussed. The generating function for spt_2(n), and the generating function for M_S(m, n) are shown with a result in terms of modulo 3. This paper shows how to prove the Theorem 1, in terms of M_S(m, n) with a numerical example, and shows how to prove the Theorem 2, with the help of spt crank in terms of partition pairs. In 2014, Garvan and Jennings-Shaffer are capable to define the spt crank for marked overpartitions. This paper also shows another result with the help of 15 SP_2-partition pairs of 8 and shows how to prove the Corollary with the help of 15 marked overpartitions of 8.

Key words: crank, non-negative, overpartitions, overlined, spt crank, weight.

INTRODUCTION
In this paper we give some related definitions of spt_2(n), various product notations, vector partitions and S-partitions, M approve (m, n), M_3(Z, t, n), S(Z, x), marked partition and spt crank for marked overpartitions. We discuss the generating function for spt_2(n) and prove the Corollary 1 with the help of generating function to prove the Result 1 with the help of 3 vector partitions from S_2 of 4. We prove the Theorem 1 with the help of various generating functions and prove the Corollary 2 with a special series S(Z, x), when n =1 and prove the Theorem 2 with the help of spt crank in terms of partition pairs (λ_1, λ_2) when 0 < s(λ_1) ≤ s(λ_2). We prove the Result 2 using the crank of partition pairs λ = (λ_1, λ_2) and
prove the Corollary 3 and 4 with the help of marked overpartition of $3n$ and of $3n+1$ (when $n = 2$) respectively. Finally we analyze the Corollary 5 with the help of marked overpartitions of $5n+3$ when $n = 1$.

**Some Related Definitions**

In this section we have described some definitions related to the article following (Garvan and Shaffer 2014).

$	ext{spt}_2(n)$ (Bringann et al. 2009): The number of smallest parts in the overpartitions of $n$ with smallest part not overlined and even is denoted by $\text{spt}_2(n)$ for example,

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\text{spt}_2(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

From above we get; $\text{spt}_2(6) = 6, \text{spt}_2(7) = 6, ...$

**Product Notations**

$$(x)_\infty = (1-x)(1-x^2)(1-x^3)...$$
$$(x^2;x^2)_\infty = (1-x^2)(1-x^4)...$$
$$(x)_k = (1-x)(1-x^2)(1-x^3)...(1-x^k)$$
$$(-x^5;x)_\infty = (1+x^5)(1+x^6)(1+x^7)...$$

**Vector Partitions and $\overline{5}$-Partitions**

A vector partition can be done with 4 components each partition with certain restrictions (Bringann et al. 2013). Let, $\overline{V} = D \times P \times P \times D$, where $D$ denote the set of all partitions into distinct parts, $P$ denotes the set of all partitions. For a partition $\pi$, we let, $s(\pi)$ denotes the
smallest part of $\pi$ (with the convention that the empty partition has smallest part $\infty$), $\#(\pi)$ the number of parts in $\pi$, and $|\pi|$ the sum of the parts of $\pi$.

For $\pi = (\pi_1, \pi_2, \pi_3, \pi_4) \in \mathcal{V}$, we define the weight $\omega(\pi) = (-1)^{\#(\pi) - 1}$, the crank $c(\pi) = \#(\pi_2) - \#(\pi_3)$, the norm $|\pi| = |\pi_1| + |\pi_2| + |\pi_3| + |\pi_4|$.

We say $\pi$ is a vector partition of $n$ if $\pi = n$. Let $\mathcal{V}$ denotes the subset of $\mathcal{V}$ and it is given by: 

$$\mathcal{V} = \{(\pi_1, \pi_2, \pi_3, \pi_4) \in \mathcal{V}, 1 \leq s(\pi_1) < \infty, s(\pi_1) \leq s(\pi_2), s(\pi_1) \leq s(\pi_3), s(\pi_1) < s(\pi_4)\}.$$ 

Let $\mathcal{S}_2$ denotes the subset of $\mathcal{V}$ with $s(\pi_1)$ even.

$M_{\mathcal{S}_2}(m, n)$: The number of vector partitions of $n$ in $\mathcal{S}_2$ with crank $m$ are counted according to the weight $\omega$ is exactly $M_{\mathcal{S}_2}(m, n)$.

$M_{\mathcal{S}_2}(m, t, n)$: The number of vector partitions of $n$ in $\mathcal{S}_2$ with crank congruent to $m$ modulo $t$ are counted according to the weight $\omega$ is exactly $M_{\mathcal{S}_2}(m, t, n)$.

$\mathcal{S}_2(z, x)$: The series $\mathcal{S}_2(z, x)$ is defined by the generating function for $M_{\mathcal{S}_2}(m, n)$.

i.e., $\mathcal{S}_2(z, x) = \sum_{m=\infty}^{n} \sum_{n=\infty}^{\infty} M_{\mathcal{S}_2}(m, n)z^m x^n$.

Marked Partition (Andrews et al. 2013): We define a marked partition as a pair $(\lambda, k)$ where $\lambda$ is a partition and $k$ is an integer identifying one of its smallest parts i.e., $k = 1, 2, ..., \nu(\lambda)$, where $\nu(\lambda)$ is the number of smallest parts of $\lambda$.

$\text{spt} \cap \text{crank}$ for Marked overpartitions (Chen et al. 2013): We define a marked overpartition of $n$ as a pair $(\pi, j)$ where $\pi$ is an overpartition of $n$ in which the smallest part is not overlined and even. It is clear that $\text{spt}_2(n) = \#$ of marked overpartitions $(\pi, j)$ of $n$. For example, there are 3 marked overpartitions of 4, like:

$(4,1), (2+2,1)$, and $(2+2,2)$. 
Then, \( spt_2(4) = 3 \).

**The Generating Function for \( spt_2(N) \)**

The generating function (Bringann et al. 2013) for \( spt_2(n) \) is given by;

\[
\sum_{n=1}^{\infty} \frac{x^{2n}(-x^{2n+1};x)_\infty}{(1-x^2)^2(x^{2n+1};x)_\infty} \\
= \frac{x^2(-x^3;x)_\infty}{(1-x^2)^2(x^3;x)_\infty} + \frac{x^4(-x^5;x)_\infty}{(1-x^2)^2(x^5;x)_\infty} + \ldots
\]

\[= o.x + 1.x^2 + o.x^3 + 3.x^4 + 2.x^5 + 6.x^6 + \ldots\]

\[= spt_2(1)x + spt_2(2)x^2 + spt_2(3)x^3 + spt_2(4)x^4 + spt_2(5)x^5 + \ldots\]

\[= \sum_{n=1}^{\infty} spt_2(n)x^n.\]

For convenience, define \( spt_2(1) = 0 \).

From above we get \( spt_2(3) = 0 \), \( spt_2(6) = 6 \), ...  

i.e., \( spt_2(3.1) = 0 \equiv 0 \pmod{3} \),  
\( spt_2(3.2) = 6 \equiv 0 \pmod{3} \), ...

We can conclude that \( spt_2(3n) \equiv 0 \pmod{3} \).

We also get \( spt_2(4) = 3 \), \( spt_2(7) = 6 \), ...  

i.e., \( spt_2(3 + 1) = 3 \equiv 0 \pmod{3} \),  
\( spt_2(3.2 + 1) = 6 \equiv 0 \pmod{3} \), ...

We can conclude that \( spt_2(3n + 1) \equiv 0 \pmod{3} \) (Bringann 2009). Again from above we get;  
\( spt_2(3) = 0 \), \( spt_2(8) = 15 \), ...  

i.e., \( spt_2(3) = 0 \equiv 0 \pmod{5} \),  
\( spt_2(5 + 3) = 15 \equiv 0 \pmod{5} \), ...

We can conclude that \( spt_2(5n + 3) \equiv 0 \pmod{5} \).

**Corollary 1:** \( spt_2(n) = \sum_{m=-\infty}^{\infty} M_{s2}(m,n) \).

**Proof:** The generating function for \( M_{s2}(m,n) \) is given by;
\[
\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\mathcal{S}_2} (m, n) \ z^n x^n
\]
\[
= \sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1}; x)_\infty (-x^{2n+1}; x)_\infty}{(z x^{2n}; x)_\infty (z^{-1} x^{2n}; x)_\infty}.
\]

If \( z = 1 \), then,
\[
\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\mathcal{S}_2} (m, n) \ x^n
\]
\[
= \sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1}; x)_\infty (-x^{2n+1}; x)_\infty}{(x^{2n}; x)_\infty (x^{2n}; x)_\infty} + \frac{x^{4}(x^{4}; x)_\infty (x^{5}; x)_\infty}{(x^{4}; x)_\infty} + \ldots
\]
\[
= \frac{x^{2}(x^{2}; x)_\infty (1-x^3)(1-x^4)\ldots}{(1-x^3)^2 (1-x^4)^2 \ldots} + \frac{x^{4}(x^{4}; x)_\infty (1-x^5)(1-x^6)\ldots}{(1-x^4)^2 (1-x^6)^2 \ldots} + \ldots
\]
\[
= \frac{x^{2}(x^{2}; x)_\infty}{(1-x^3)^2 (1-x^4)^2 \ldots} + \frac{x^{4}(x^{4}; x)_\infty}{(1-x^4)^2 (1-x^6)^2 \ldots} + \ldots
\]
\[
= \sum_{n=1}^{\infty} \frac{x^{2n}(-x^{2n+1}; x)_\infty}{(1-x^{2n+1}; x)_\infty}
\]
\[
= \sum_{n=1}^{\infty} \overline{spt}_2 (n) x^n.
\]

i.e., \( \sum_{n=1}^{\infty} \overline{spt}_2 (n) x^n = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\mathcal{S}_2} (m, n) \ x^n \).

Now equating the co-efficient of \( x^n \) from both sides we get;
\[
\overline{spt}_2 (n) = \sum_{m=-\infty}^{\infty} M_{\mathcal{S}_2} (m, n).
\]

Hence the Corollary.

**Result 1:**
\[
M_{\mathcal{S}_2} (0,3,4) = M_{\mathcal{S}_2} (1,3,4) = M_{\mathcal{S}_2} (2,3,4) = \frac{1}{3} \overline{spt}_2 (4).
\]

**Proof:** We prove the result with the help of examples. We see the vector partitions from \( \mathcal{S}_2 \) of 4 along with their weights and cranks and are given as follows:

Here we have used \( \phi \) to indicate the empty partition. Thus we have,
\[
M_{\mathcal{S}_2} (0,3,4) = 1, \quad M_{\mathcal{S}_2} (1,3,4) = 1 \,
\]
\[
M_{\mathcal{S}_2} (2,3,4) = M_{\mathcal{S}_2} (1,-3,4) = 1
\]
\[
\therefore M_{\mathcal{S}_2} (0,3,4) = M_{\mathcal{S}_2} (1,3,4)
\]
\[ M_{S_2}(2,3,4) = 1 = \frac{1}{3} \cdot 3 = \frac{1}{3} \text{sp}_{S_2}(3) \].

Hence the Result.

### Table 1

<table>
<thead>
<tr>
<th>$\vec{S}_2$-vector partition ($\vec{\pi}$) of 4</th>
<th>Weight ($\omega(\vec{\pi})$)</th>
<th>Crank ($\phi(\vec{\pi})$)</th>
<th>mod 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vec{\pi}_1 = (4, \phi, \phi, \phi)$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\vec{\pi}_2 = (2 + 2, \phi, \phi)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\vec{\pi}_3 = (2, \phi, 2, \phi)$</td>
<td>1</td>
<td>-1</td>
<td>2</td>
</tr>
</tbody>
</table>

$\sum \omega(\vec{\pi}) = 3$

Now from table 1 we get; \( \sum \omega(\vec{\pi}) = 3 \), i.e., \( \sum_{k=0}^{3} M_{S_2}(k,3,4) = 3 \).

\[ \therefore \text{sp}_{S_2}(4) = \sum_{k=0}^{3} M_{S_2}(k,3,4) = \sum \omega(\vec{\pi}). \]

Now we can define;

\[ M_{S_2}(k,t,n) = \sum_{m=k \text{ (mod } t)} M_{S_2}(m,n) \]

and \( \text{sp}_{S_2}(n) = \sum_{m=-\infty}^{\infty} M_{S_2}(m,n) = \sum_{k=0}^{\infty} M_{S_2}(k,t,n) \).

**Theorem 1:** The number of vector partitions of \( n \) in \( S_2 \) with crank \( m \) counted according to the weight \( \omega \) is non-negative, i.e., \( M_{S_2}(m,n) \geq 0 \).

**Proof:** The generating function for \( M_{S_2}(m,n) \) is given by;

\[
\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{S_2}(m,n)z^m x^n
\]

\[
= \sum_{n=1}^{\infty} x^{2n} (x^{2n+1};x)_\infty (-x^{2n+1};x)_\infty
\]

\[
= \sum_{n=1}^{\infty} \frac{x^{2n}}{(zx^{2n};x)_\infty} (z^{-1} x^{2n};x)_\infty
\]

\[
= \sum_{n=1}^{\infty} (x^{2n+1};x)_\infty (-x^{2n+1};x)_\infty
\]

[Since \( \sum_{n=1}^{\infty} (x^{2n+1};x)_\infty (-x^{2n+1};x)_\infty \)

\[
= (x^2;x)_\infty - x (x^3;x)_\infty + (x^4;x)_\infty - x (x^5;x)_\infty + ... = (1-x^3)(1-x^4)...(1+x^3)(1+x^4)... + (1-x^3)(1-x^6)...(1+x^4)... + ...
\]

\[
= (1-x^6)(1-x^8)... + (1-x^{10})(1-x^{12})... + (1-x^{14})... + ...
\]
\[= (x^6; x^2)_{\infty} + (x^{10}; x^2)_{\infty} + \ldots\]
\[= \sum_{n=1}^{\infty} (x^{4n+2}; x^2)_{\infty}\]
\[= \sum_{n=1}^{\infty} \frac{x^{2n}(x^{4n}; x)_{\infty}}{(z^{2n}; x)_{\infty}(z^{-\frac{1}{2}}x^{2n}; x)_{\infty}} \cdot \frac{(x^{4n+2}; x^2)_{\infty}}{(x^4; x)_{\infty}}\]
\[= \sum_{n=1}^{\infty} \frac{x^{2n}(x^{4n}; x)_{\infty}}{(z^{2n}; x)_{\infty}(z^{-\frac{1}{2}}x^{2n}; x)_{\infty}} \cdot \frac{1}{(1-x^{4n})(x^{4n+2}; x^2)_{\infty}}\]

[Since, \[\sum_{n=1}^{\infty} \frac{(x^{4n+2}; x^2)_{\infty}}{(x^{4n}; x)_{\infty}} = \frac{(x^6; x^2)_{\infty}}{(x^4; x)_{\infty}} + \frac{(x^{10}; x^2)_{\infty}}{(x^4; x)_{\infty}} + \ldots\]
\[= \frac{(1-x^6)(1-x^{10})\ldots}{(1-x^4)(1-x^8)(1-x^{12})\ldots} + \frac{(1-x^{10})(1-x^{12})\ldots}{(1-x^8)(1-x^{10})(1-x^{14})\ldots} + \ldots\]
\[= \frac{1}{(1-x^4)(1-x^8)(1-x^{12})\ldots} + \frac{1}{(1-x^8)(1-x^{10})(1-x^{14})\ldots} + \ldots\]
\[= \sum_{n=1}^{\infty} \frac{1}{1-x^{4n}} \cdot (x^{4n+1}; x^2)\]
\[= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(z^{-1}x^{2n})^k}{(z^{2n+k}; x)_{\infty}(x)_{\infty}} \cdot \frac{1}{(1-x^{4n})(x^{4n+2}; x^2)_{\infty}}\]

[Since, \[\sum_{n=1}^{\infty} \frac{x^{2n}(x^{4n}; x)_{\infty}}{(z^{2n}; x)_{\infty}(z^{-\frac{1}{2}}x^{2n}; x)_{\infty}}\]
\[= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(z^{-1}x^{2n})^k}{(z^{2n+k}; x)_{\infty}(x)_{\infty}} \cdot \frac{1}{(1-x^{4n})(x^{4n+2}; x^2)_{\infty}}\] (by Berkovich and Garvan 2008)

We see that the coefficient of any power \(x\) in the right hand side is non-negative so the coefficient \(M_{\overline{2}}(m, n)\) of \(z^nx^n\) is non-negative, i.e., \(M_{\overline{2}}(m, n) \geq 0\). Hence the Theorem.

**Numerical example 1**

The vector partitions from \(\overline{S}_2\) of 5 along with their weights and cranks are given as follows:

Here we have used \(\phi\) to indicate the empty partition. Thus we have:

\(M_{\overline{2}}(0, 5) = 1 - 1 = 0, M_{\overline{2}}(1, 5) = 1, \) and \(M_{\overline{2}}(-1, 5) = 1, \) i.e., \(\sum M_{\overline{2}}(m, 5) = 2,\)

i.e., every term is non-negative, i.e., \(M_{\overline{2}}(m, n) \geq 0\).

So we can conclude that, \(M_{\overline{2}}(m, n) \geq 0.\)
Corollary 2: \( \tilde{S}_2(1, x) = \sum_{n=1}^{\infty} \text{spt}_2(n) x^n \).

**Proof:** We get;
\[
\tilde{S}_2(z, x) = \sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1}; x)_n}{(x^{2n}; x)_n} (-1)^n (-x^{2n+1}; x)_n
\]

If \( z = 1 \), then we get;
\[
\tilde{S}_2(1, x) = \sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1}; x)_n}{(x^{2n}; x)_n} \frac{(-1)^n x^{2n+1}}{(x^{2n}; x)_n} \frac{x^n}{(x^{2n}; x)_n} + \ldots
\]
\[
= \frac{x^2(x^3; x)_n}{(x^2; x)_n} \frac{(-x^3; x)_n}{(x^2; x)_n} + \frac{x^4(x^5; x)_n}{(x^4; x)_n} \frac{(x^5; x)_n}{(x^4; x)_n} + \ldots
\]
\[
= \frac{x^2(-x^3; x)_o}{(1-x^2)^2(1-x^4)} + \frac{x^4(-x^5; x)_o}{(1-x^4)^2(1-x^8)} + \ldots
\]
\[
= \sum_{n=1}^{\infty} \frac{x^{2n}(-x^{2n+1}; x)_o}{(1-x^{2n})^2(x^{2n+1}; x)_o}
\]
\[
= \sum_{n=1}^{\infty} \text{spt}_2(n) x^n .
\]
i.e., \( \tilde{S}_2(1, x) = \sum_{n=1}^{\infty} \text{spt}_2(n) x^n \). Hence the Corollary.

Theorem 2: \( \overline{\text{spt}}_2(n) = \sum_{\overset{\lambda \in \overline{SP}_2}{|\lambda| = n}} 1 \)

**Proof:** First we define the \( \text{sptcrank} \) in terms of partition pairs,
\( \overline{SP} = \{ \overline{\lambda} = (\lambda_1, \lambda_2) \in P \times P : 0 < s(\lambda_1) \leq s(\lambda_2) \} \) and all parts of \( \lambda_2 \) that are \( \geq 2s(\lambda_1) + 1 \) are odd.

Let \( \overline{SP}_2 \) be the set of \( \lambda = (\lambda_1, \lambda_2) \in \overline{SP} \) with \( s(\lambda_1) \) even. The generating function for \( \overline{\text{spt}}_2(n) \) is given by;
\[
\sum_{n=1}^{\infty} \overline{\text{spt}}_2(n) x^n = \sum_{n=1}^{\infty} \frac{x^{2n}(-x^{2n+1}; x)_o}{(1-x^{2n})^2(x^{2n+1}; x)_o}
\]
\[
= \sum_{n=1}^{\infty} \frac{x^{2n}}{(1-x^{2n})^2(x^{2n+1}; x)_o} \frac{(-x^{2n+1}; x)_o}{(x^{2n+1}; x)_o}
\]
\[
= \sum_{n=1}^{\infty} \frac{x^{2n}}{(1-x^{2n})^2(x^{2n+1}; x)_o} \frac{x^{4n+2n^2}}{(x^{2n+1}; x)_o} \frac{x^2}{(x^{2n+1}; x)_o}
\]
[Since, \( \sum_{n=4}^{\infty} (-x^{2n+1}; x)_o = (-x^3; x)_o + (-x^5; x)_o + \ldots \)
\[(1 + x^3)(1 + x^4) + (1 + x^5)(1 + x^6) + (1 + x^7)(1 + x^8) + \ldots = (1 - x^6)(1 - x^8) + (1 - x^{10})(1 - x^{12}) + (1 - x^{14}) + \ldots\]
\[= \frac{(x^6; x^2)_{\infty}}{(x^3; x)_{\infty}} + \frac{(x^{10}; x^2)_{\infty}}{(x^5; x)_{\infty}} + \ldots\]
\[= \sum_{n=1}^{\infty} \frac{(x^{4n+2}; x^2)_{\infty}}{(x^{2n+1}; x)_{\infty}}\]

\[\frac{x^{2n}}{(1 - x^{2n+1}; x)_{\infty}} = \frac{(x^{4n+2}; x^2)_{\infty}}{(x^{2n+1}; x)_{\infty}}\]
\[= \sum_{n=1}^{\infty} \frac{x^{2n}}{(x^{2n}; x)_{\infty}} \cdot \frac{1}{(1 - x^{2n})}\]
\[= \sum_{n=1}^{\infty} \frac{x^{2n}}{(x^{2n}; x)_{\infty}} \cdot \frac{1}{(1 - x^{2n})}\]
\[= \frac{(x^{4n+2}; x^2)_{\infty}}{(x^{2n+1}; x)_{\infty}} = \frac{(x^6; x^2)_{\infty}}{(x^3; x)_{\infty}} + \frac{(x^{10}; x^2)_{\infty}}{(x^5; x)_{\infty}} + \ldots\]
\[= \frac{(1 - x^6)(1 - x^8)}{(1 - x^3)(1 - x^4)} + \frac{(1 - x^{10})(1 - x^{12})}{(1 - x^5)(1 - x^6)} + \ldots\]
\[= \frac{1}{(1 - x^3)(1 - x^4)(1 - x^5)} + \frac{1}{(1 - x^5)(1 - x^6)(1 - x^9)} + \ldots\]
\[= \sum_{n=1}^{\infty} \frac{1}{(1 - x^{2n+1}) \ldots (1 - x^{2n}) (x^{4n+1}; x^2)_{\infty}}\]
\[= \sum_{n=1}^{\infty} \sum_{\lambda_2 \geq 2n + 1 \text{ odd}} \lambda_2 \alpha_{\lambda_2} \sum_{\lambda_1 \leq n} \lambda_1 \beta_{\lambda_1}\]

all parts in \( \lambda_2 \geq 2n + 1 \) are odd

\[\sum_{n=1}^{\infty} \frac{\lambda_2 \alpha_{\lambda_2}}{\lambda_1 \beta_{\lambda_1}} \sum_{\lambda_1 \leq n} \lambda_1 \beta_{\lambda_1}\]

Equating the co-efficient of \( x^n \) from both sides we get;
\[ \overline{spt_2}(n) = \sum_{\lambda \in \overline{SP}_2} 1 \] . Hence the Theorem.

**Numerical Example 2**

The overpartitions of 6 with smallest parts not overlined and even are 6, 4+2, \( \bar{4} + 2 \), and 2+2+2. Consequently, the number of smallest parts in the overpartitions of 6 with smallest part not overlined and even is given by:

\[ 6, 4 + 2, \bar{4} + 2, 2 + 2 + 2, \]

so that \( \overline{spt_2}(6) = 6 \) i.e., there are \( 6 \overline{SP}_2 \)-partition pairs of 6 like:

\( (6, \phi), (4 + 2, \phi), (2, 4), (2 + 2 + 2, \phi), (2, 2, 2) \) and \( (2, 2, 2) \).

**Result 2:**

\[ M_{\overline{SP}_2}(0,5,8) = M_{\overline{SP}_2}(1,5,8,) = M_{\overline{SP}_2}(2,5,8,) = \]

\[ M_{\overline{SP}_2}(3,5,8) = M_{\overline{SP}_2}(4,5,8) = 3 = \frac{1}{5} \overline{spt_2}(8). \]

**Proof:** We prove the result with the help of examples. We can define a \( \text{crank} \) of partition pairs \( \lambda = (\lambda_1, \lambda_2) \in \overline{SP}_2. \)

For \( \lambda = (\lambda_1, \lambda_2) \in \overline{SP}_2 \), we define, \( k(\lambda) = \# \) of pairs \( j \) in \( \lambda_2 \) such that \( s(\lambda_1) \leq j \leq 2 s(\lambda_1) - 1 \),

and also define:\( \text{crank}(\lambda) = \begin{cases} \# \text{of parts of } \lambda_1 \geq s(\lambda_1) + k - k & \text{if } k > 0 \\ \# \text{of parts of } \lambda_1 - 1 & \text{if } k = 0 \end{cases} \) \( \text{where } k = k(\lambda). \)

We know that \( \overline{spt_2}(8) = 15 \). There are 15 \( \overline{SP}_2 \)-partition pairs of 8.

**Table 2**

<table>
<thead>
<tr>
<th>( \overrightarrow{S_2} )-vector partition of 5 ( (\overrightarrow{\pi}) )</th>
<th>Weight ( \omega(\pi) )</th>
<th>Crank ( \text{crank}_c(\pi) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \overrightarrow{\pi}_1 = (3 + 2, \phi, \phi, \phi) )</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>( \overrightarrow{\pi}_2 = (2, \phi, \phi, 3) )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \overrightarrow{\pi}_3 = (2, 3, \phi, \phi) )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \overrightarrow{\pi}_4 = (2, \phi, 3, \phi) )</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

\[ \sum \omega(\pi) = 2 \]
From the table 2 we get;

\[ M_{\overline{\mathcal{S}}_2}(0,5,8) = M_{\overline{\mathcal{S}}_2}(1,5,8,.) = M_{\overline{\mathcal{S}}_2}(2,5,8,.) = \]

\[ M_{\overline{\mathcal{S}}_2}(3,5,8) = M_{\overline{\mathcal{S}}_2}(4,5,8) = 3 = \frac{1}{5} \overline{spt}_2(8). \] Hence the Result.

**Table 3**

<table>
<thead>
<tr>
<th>(3P_k)-partition pair of 8</th>
<th>(k)</th>
<th>(\overline{\text{crank}})</th>
<th>(mod 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((3+2,3))</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>((4+2,2))</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>((8,\phi))</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>((2+2,4))</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>((4+4,\phi))</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>((6+2,\phi))</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>((2,2+2+2))</td>
<td>3</td>
<td>-3</td>
<td>2</td>
</tr>
<tr>
<td>((3+3+2,\phi))</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>((4+2+2,\phi))</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>((2,3+3))</td>
<td>2</td>
<td>-2</td>
<td>3</td>
</tr>
<tr>
<td>((2+2,2+2))</td>
<td>2</td>
<td>-2</td>
<td>3</td>
</tr>
<tr>
<td>((2+2+2+2,\phi))</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>((2,4+2))</td>
<td>1</td>
<td>-1</td>
<td>4</td>
</tr>
<tr>
<td>((4,4))</td>
<td>1</td>
<td>-1</td>
<td>4</td>
</tr>
<tr>
<td>((2+2+2,2))</td>
<td>1</td>
<td>-1</td>
<td>4</td>
</tr>
</tbody>
</table>

Now we will describe the \(\overline{spt_{crank}}\) of a marked overpartition (Chen et al. 2013). To define the \(\overline{spt_{crank}}\) of a marked overpartition we first need to define a function \(k(m,n)\) for positive integers \(m, n\) such that \(m \geq n + 1\), we write \(m = b2^j\), where \(b\) is odd and \(j \geq 0\). For a given odd integer \(b\) and a positive integer \(n\) we define \(j_0 = j_0(b,n)\) to be the smallest non-negative integer \(j_0\) such that \(b2^{j_0} \geq n + 1\).

We define; \(k(m,n) = \)  
\[
\begin{cases} 
0, & \text{if } b \geq 2n \\
2^{j_0} & \text{if } b2^{j_0} < 2n \\
0, & \text{if } b2^{j_0} = 2n.
\end{cases}
\]
Table 4

<table>
<thead>
<tr>
<th>Marked overpartition ((\pi, j)) of 6</th>
<th>(\pi_1)</th>
<th>(\pi_2)</th>
<th>(\nu(\pi_1))</th>
<th>(k((\pi_2, s(\pi_1))))</th>
<th>(\bar{k})</th>
<th>sptcrank</th>
<th>((\text{mod } 3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((6, 1))</td>
<td>6</td>
<td>(\phi)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>((4+2, 1))</td>
<td>4+2</td>
<td>(\phi)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>((4+1, 1))</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>((2+2+2, 1))</td>
<td>2+2+2</td>
<td>(\phi)</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>–2</td>
<td>1</td>
</tr>
<tr>
<td>((2+2+2, 2))</td>
<td>2+2+2</td>
<td>(\phi)</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>–1</td>
<td>2</td>
</tr>
<tr>
<td>((2+2+2, 3))</td>
<td>2+2+2</td>
<td>(\phi)</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

For a marked overpartitions \((\pi, j)\) we let \(\pi_1\) be the partition formed by the non-overlined parts of \(\pi\), \(\pi_2\) be the partition (into distinct parts) formed by the overlined parts of \(\pi\) so that \(s(\pi_j) > s(\pi_i)\), we define \(\bar{k}(\pi, i) = i + k(\pi_2, s(\pi_1))\), where \(\nu(\pi_1)\) is the number of smallest parts of \(\pi_1\).

Now we can define:

\[
\text{sptcrank}(\pi, j) = \begin{cases} 
\text{# of parts of } \pi_i \geq s(\pi_1) - \bar{k}, & \text{if } \bar{k} = \bar{k}(\pi, j) > 0 \\
\text{# of parts of } \pi_i - 1, & \text{if } \bar{k} = \bar{k}(\pi, j) = 0.
\end{cases}
\]

Table 5

<table>
<thead>
<tr>
<th>Marked overpartition ((\pi, j)) of 7</th>
<th>(\pi_1)</th>
<th>(\pi_2)</th>
<th>(\nu(\pi_1))</th>
<th>(k((\pi_2, s(\pi_1))))</th>
<th>(\bar{k})</th>
<th>sptcrank</th>
<th>((\text{mod } 3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((5+2, 1))</td>
<td>5+2</td>
<td>(\phi)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>((5+2, 1))</td>
<td>5</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>((3+2+2, 1))</td>
<td>3+2+2</td>
<td>(\phi)</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>((3+2+2, 2))</td>
<td>3+2+2</td>
<td>(\phi)</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>((3+2+2, 1))</td>
<td>2+2</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>–2</td>
<td>1</td>
</tr>
<tr>
<td>((3+2+2, 2))</td>
<td>2+2</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>–1</td>
<td>2</td>
</tr>
</tbody>
</table>
Table 6

<table>
<thead>
<tr>
<th>Marked overpartition ((\pi, j)) of (8)</th>
<th>(\pi_1)</th>
<th>(\pi_2)</th>
<th>(\nu(\pi_1))</th>
<th>(k((\pi_2, s(\pi_1)))</th>
<th>(\text{sprank})</th>
<th>((\text{mod } 5))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((6 + 2, 1))</td>
<td>2</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>(-2)</td>
</tr>
<tr>
<td>((4 + 2 + 2, 1))</td>
<td>2 + 2</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>(-1)</td>
</tr>
<tr>
<td>((4 + 2 + 2, 2))</td>
<td>2 + 2</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>((3 + 3 + 2, 1))</td>
<td>3 + 2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>((2 + 2 + 2 + 2, 1))</td>
<td>2 + 2 + 2 + 2</td>
<td>(\phi)</td>
<td>4</td>
<td>0</td>
<td>3</td>
<td>(-3)</td>
</tr>
<tr>
<td>((2 + 2 + 2 + 2, 2))</td>
<td>2 + 2 + 2 + 2</td>
<td>(\phi)</td>
<td>4</td>
<td>0</td>
<td>2</td>
<td>(-2)</td>
</tr>
<tr>
<td>((2 + 2 + 2 + 2, 3))</td>
<td>2 + 2 + 2 + 2</td>
<td>(\phi)</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>(-1)</td>
</tr>
<tr>
<td>((2 + 2 + 2 + 2, 4))</td>
<td>2 + 2 + 2 + 2</td>
<td>(\phi)</td>
<td>4</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>((3 + 3 + 2, 1))</td>
<td>3 + 3 + 2</td>
<td>(\phi)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>((4 + 2 + 2, 1))</td>
<td>4 + 2 + 2</td>
<td>(\phi)</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>((4 + 2 + 2, 2))</td>
<td>4 + 2 + 2</td>
<td>(\phi)</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>((6 + 2, 1))</td>
<td>6 + 2</td>
<td>(\phi)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>((4 + 4, 1))</td>
<td>4 + 4</td>
<td>(\phi)</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>(-1)</td>
</tr>
<tr>
<td>((4 + 4, 2))</td>
<td>4 + 4</td>
<td>(\phi)</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>((8, 1))</td>
<td>8</td>
<td>(\phi)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Corollary 3** (Lovejoy and Osburn 2009): The residue of the \(\text{sprank}(\text{mod } 3)\) divides the marked overpartitions of \(3n\) with the smallest part not overlined and even into 3 equal classes.  
**Proof:** We prove the Corollary with the help of an example when \(n = 2\). There are 6 marked overpartitions of \(3n\) (when \(n = 2\)) with the smallest part not overlined and even so that, \(\text{spt}_2(6) = 6\).

We see that the residue of the \(\text{sprank}(\text{mod } 3)\) divides the marked overpartitions of \(3n\) (when \(n = 2\)) with smallest part not overlined and even into 3 equal classes. Hence the Corollary.

**Corollary 4:** The residue of the \(\text{sprank}(\text{mod } 3)\) divides the marked overpartitions of \(3n+1\) with smallest part not overlined and even into 3 equal classes.  
**Proof:** We prove the Corollary with the help of an example when \(n = 2\). There are 6 marked overpartitions of 7 with the smallest part not overlined and even, so that \(\text{spt}_2(7) = 6\). We see that the residue of the \(\text{sprank}(\text{mod } 3)\) divides the marked overpartitions of \(3n+1\) (when \(n = 2\)) with smallest part not overlined and even. Hence the Corollary.

**Corollary 5:** The residue of the \(\text{sprank}(\text{mod } 5)\) divides the marked overpartitions of \(5n+3\) with smallest part not overlined and even into 5 equal classes.
Proof: We prove the Corollary with the help of example when \( n = 1 \). There are 15 marked overpartitions of \( 5n + 3 \) (when \( n = 1 \)) with the smallest part not overlined and even so that \( spt_2(8) = 15 \). We see that the residue of the divides the marked overpartitions of 8 with the smallest part not overlined and even into 5 equal classes. Hence the corollary.

CONCLUSION

In this study we have found the number of smallest parts in the overpartitions of \( n \) with the smallest part not overlined and even for \( n = 1, 2, 3, 4 \) and 5. We have shown various relations \( spt_2(3n) \equiv 0(\text{mod } 3), \ spt_2(3n + 1) \equiv 0(\text{mod } 3), \ spt_2(5n + 3) \equiv 0(\text{mod } 5), \)

\[
M_{s_2}(0,3,4) = M_{s_2}(1,3,4) = M_{s_2}(2,3,4) = \frac{1}{3} spt_2(4) \quad \text{and} \quad M_{s_2}(0,5,8) = M_{s_2}(1,5,8) = M_{s_2}(2,5,8)
\]

\[
= M_{s_2}(3,5,8) = M_{s_2}(4,5,8) = 3 = \frac{1}{5} spt_2(8) \quad \text{with numerical examples respectively.}
\]

We have verified the Theorem 1 when \( n = 5 \) and have verified the Theorem 2 when \( n = 6 \). We have verified the Corollary 3 with 6 marked overpartitions of 6 and have verified the Corollary 4 with 6 marked overpartitions of 7 and also have established the Corollary 5 with 15 marked overpartition of 8.

REFERENCES


