Axiomatizations of the proportional Shapley value

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Abstract

We provide new axiomatic characterizations of the proportional Shapley value, a weighted value with the worths of the singletons as weights. This value satisfies anonymity and therefore symmetry just as the Shapley value and has characterizations which are proportional counterparts to the famous characterizations of the Shapley value in Shapley (1953b), Myerson (1980) and Young (1985a). If the stand alone worths are plausible weights the proportional Shapley value is a convincing alternative to the Shapley value for example in cost allocation. We introduce two new axioms, called proportionality and player splitting respectively. Each of which gives a main difference between the proportional Shapley value and the Shapley value.

Keywords Cost allocation · Dividends · Proportional Shapley value · (Weighted) Shapley value · Proportionality · Player splitting

1. Introduction

In contrast to Thomas (1969, 1974), who assert that all cost allocation methods are arbitrary and no one allocation scheme can be defended against all others, we have on the one hand a large group of economists which prefers traditional cost accounting practices on the other hand a small group which prefers cost allocation based on solutions to cooperative games with transferable utility dominated by the Shapley value, e. g. Shubik (1962), Spinetto (1975), Roth and Verrecchia (1979), Young (1985a), Young (1985b) and Leng and Parlar (2009). Moriarity (1975) states

> "A proposal for a new allocation procedure can be justified only on the basis of the advantages of the proposed method over existing methods."

An empirical study in Barton (1988) shows a dramatic preference for the proportional solution, called Moriarity's method (Moriarity 1975), also known as proportional rule, compared to the nucleolus or the Shapley value. Banker (1981) use an axiomatic approach.

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He worked out some shortcomings of the Shapley Value in cost allocation, especially the additivity axiom is considered questionable. It makes the allocation sensitive to the way cost centres are used or organized. Banker shows in an example that the allocations can differ significantly if two cost centres are merged and considered as a single entry. His own proposal for an axiomatization for cost allocation contains a splitting axiom instead of additivity. It turns out that the unique value of his axiomatization is identical with the proportional rule (Moriarity 1975).

In contrast there are some other authors who stress the disadvantages of the proportional rule and suggest the Shapley value. For example, in Amer et al. (2007) is criticized the restricted domain of the proportional rule, the lack of additivity, a doubly discriminatory level and that it does not take into account most of the marginal contributions.

The last point of criticism is avoided by the proper Shapley values (Brink et al. 2015) or the proportional value, developed by Ortmann (2000) respectively Feldman (1999) simultaneously. Feldman also suggests his value to cost allocation and gives a short overview over proportional cost allocation and pointed to Gangolly (1981). Gangolly introduced there a new cost allocation scheme, denoted as "Independent Cost Proportional Scheme (ICPS)". He used in this scheme for each given coalition function $\overline{v}$ a (proportional) weighted Shapley value, where the weights are the worths of the singletons $\overline{v}(\{i\})$ for every player i. Yet a general formalisation as a TU-value and an axiomatic characterization was still missing. Independently there was a "rediscovery" of this value by Besner (2016) and Béal et al. (2017). Both denote their non linear TU-value "proportional Shapley value" and give an axiomatization in spirit to the axiomatization of the weighted Shapley values with weighted balanced contributions (Myerson 1980; Hart and Mas-Colell 1989) and point out that the proportional Shapley value inherits many of the properties of the weighted Shapley values. In Besner (2016) there are some extensions of the proportional Shapley value, for example to graphs and level structures. Béal et al. (2017) introduce a potential and give some comparable axiomatizations to the Shapley value and economic applications.

The aim of this paper is to establish the proportional Shapley value as an application-relevant allocation scheme, where there are asymmetries that are included exclusively in the underlying game and not in exogenous weights. So this value is symmetric and has many analog axiomatizations to the Shapley value. Two new axioms give a main difference to the Shapley value. The first, proportionality, is an proportional counterpart to symmetry: The payoffs to two weakly dependent players, that means the marginal contributions of one of these players to any coalition which contains only one of both players, is only his singleton worth, are proportional to the singleton worths of each other. Nowak and Radzik (1995) give a similar axiom, called ω-mutual dependence, for the weighted Shapley values. By the second axiom, player splitting, related to Banker (1981), where a player is splitted into two new players, the payoff to unconcerned players does not change. In Radzik (2012) is a similar idea in the opposite direction for the weighted Shapley values by his amalgamating payoffs axiom where players who build a partnership (Kalai and Samet 1987) amalgamate to a new player.

The paper is organized as follows. Section 2 contains some preliminaries. In section 3 we give an motivating example and show there is a inconsistency of the Shapley value in this case. Section 4 presents a short overview of known and simple results for the proportional Shapley value. As the main result we give in section 5 axiomatizations, which are close to Shapley's original Shapley (1953b) and to Young (1985a). Section 6 gives a short conclusion. An appendix (section 7) provides all the proofs, some related lemmas and shows logical independence of the axioms used for characterization.

2. Preliminaries

We denote by $\mathbb{R}$ the real numbers, by $\mathbb{Q}_{++}$ all positive rational numbers and by $\mathbb{R}^N_{++} = \{x \in \mathbb{R}^N : x_i > 0 \text{ for all } i \in N\}$ the set of all vectors $x \in \mathbb{R}^N$ where all coordinates x_i are positive. Let $\mathfrak{U}$ a infinite set, the universe of all players, and denote by $\mathcal{N}$ the set of all finite subsets of $\mathfrak{U}$. A cooperative game with transferable utility (**TU-game**) is a pair (N, v) consisting of a set of players $N \in \mathcal{N}$ and a **coalition function** $v \in \{f : 2^N \to \mathbb{R} | f(\emptyset) = 0\}$, where 2^N is the power set of N. We refer to a TU-game also only by v. The subsets $S \subseteq N$ are called **coalitions** and $v(S)$ is called the **worth** of coalition S.

The set of all TU-games with player set N is denoted by $\mathcal{G}^N$ and the set of all TU-games on N where the worths of all singletons are all positive by $\mathcal{G}^N_0 = \{v \in \mathcal{G}^N \colon v(\{i\}) > 0 \text{ for all } i \in N\}$. If the worths of the singletons can only be all positive rational we mark this set by $\mathcal{G}^N_{0_\mathbb{Q}} = \{v \in \mathcal{G}^N \colon v(\{i\}) \in \mathbb{Q}_{++} \text{ for all } i \in N\}$.

A **TU-value** on $\mathcal{G}^N$ (respectively on subdomains of $\mathcal{G}^N$) is an operator φ, which assigns any $v \in \mathcal{G}^N$ (respectively v is an element of a subdomain of $\mathcal{G}^N$) a payoff vector $\varphi(N, v) \in \mathbb{R}^N$ or short $\varphi(v)$ for all $N \in \mathcal{N}$, with the meaning that $\varphi_i(v)$ is the payoff to player i in the TU-game v.

Let $N \in \mathcal{N}$, $v \in \mathcal{G}^N$ and $S \subseteq N$. We denote by (S, v) the **restriction** of (N, v) to the player set S. The **Harsanyi dividends** $\Delta_v(S)$ (Harsanyi 1959) are defined inductively by

$$\Delta_v(S) = \begin{cases} v(S) - \sum_{R \subsetneq S} \Delta_v(R) & \text{if } |S| \geq 1, \text{ and} \\ 0 & \text{if } S = \emptyset. \end{cases} \tag{1}$$

Another well-known formula of the dividends is given for all $S \subseteq N$, $S \neq \emptyset$, by

$$\Delta_v(S) = \sum_{R \subseteq S} (-1)^{|S|-|R|} v(R). \tag{2}$$

The **marginal contribution** $MC_i^v(S)$ of player $i \in N$ to $S \subseteq N \backslash \{i\}$ is given by $MC_i^v(S) := v(S \cup \{i\}) - v(S)$. We call a coalition $S \subseteq N$ **active** in v if $\Delta_v(S) \neq 0$. Player $i \in N$ is a **dummy player** if $v(S \cup \{i\}) = v(S) + v(\{i\})$, $i \notin S$, $S \subseteq N$, or, equivalent as a well-known fact, if $\Delta_v(S \cup \{i\}) = 0$ for all $S \subseteq N \backslash \{i\}$, $S \neq \emptyset$. If in addition $v(\{i\}) = 0$ then i is called a **null player**; players $i, j \in N$ are called **symmetric** in v, if $v(S \cup \{i\}) = v(S \cup \{j\})$, or, equivalent also as a well-known fact, $\Delta_v(S \cup \{i\}) = \Delta_v(S \cup \{j\})$, for all coalitions $S \subseteq N \backslash \{i, j\}$.

In many applications the assumption of symmetry of the players is not realistic, e. g. if the bargaining power or the amount of used venture capital are different. Shapley (1953a) introduced for this case the (positive) weighted Shapley values[1]: For all $N \in \mathcal{N}$, $v \in \mathcal{G}^N$

[1] We desist from possibly null weights as in Shapley (1953a) or Kalai and Samet (1987).

and a vector $\omega \in \mathbb{R}^N_{++}$ of positive weights ω_i for all $i \in N$, the (positively) **weighted Shapley Value** Sh^ω (Shapley 1953a) is defined by

$$Sh_i^\omega(v) = \sum_{\substack{S\subseteq N,\\ S\ni i}} \frac{\omega_i}{\sum_{j\in S}\omega_j}\Delta_v(S) \text{ for all } i \in N.$$

The next value distributes the dividends equally among all players in a coalition: For all $N \in \mathcal{N}$, $v \in \mathcal{G}^N$, the **Shapley value** Sh (Shapley 1953b) is defined by

$$Sh_i(v) = \sum_{\substack{S\subseteq N,\\ S\ni i}} \frac{\Delta_v(S)}{|S|} \text{ for all } i \in N.$$

We see that the Shapley value is a weighted Shapley value where all weights are equal. Our following value distributes the dividends proportionally to the singleton worths among all players in a coalition: For all $N \in \mathcal{N}$, $v \in \mathcal{G}_0^N$[2] the **proportional Shapley Value** Sh^p (Besner 2016; Béal et al. 2017; Gangolly 1981) is given by

$$Sh_i^p(v) = \sum_{\substack{S\subseteq N,\\ S\ni i}} \frac{v(\{i\})}{\sum_{j\in S} v(\{j\})}\Delta_v(S) \text{ for all } i \in N \tag{3}$$

and coincides therefore with the Shapley value if all singletons have the same worth.

We refer to the following standard axioms:

Efficiency, E. For all $N \in \mathcal{N}$, $v \in \mathcal{G}^N$, we have $\sum_{i\in N}\varphi_i(v) = v(N)$.

Null player. For all $N \in \mathcal{N}$, $v \in \mathcal{G}^N$ and $i \in N$ such that i is a null player in v, we have $\varphi_i(v) = 0$.

Dummy, D. For all $N \in \mathcal{N}$, $v \in \mathcal{G}^N$ and $i \in N$ such that i is a dummy player in v, we have $\varphi_i(v) = v(\{i\})$.

Homogeneity, H (of degree 1)**.** For all $N \in \mathcal{N}$, $v \in \mathcal{G}^N$, $i \in N$ and scalars $\alpha \in \mathbb{R}$, we have $\varphi_i(\alpha v) = \alpha\varphi_i(v)$.

Marginality, M. For all $N \in \mathcal{N}$, $v, v' \in \mathcal{G}^N$ and $i \in N$ such that $MC_i^v(S) = MC_i^{v'}(S)$ for all $S \subseteq N\backslash\{i\}$, we have $\varphi_i(v) = \varphi_i(v')$.

Monotonicity (with respect to the grand coalition)**.** For all $N \in \mathcal{N}$, $v, v' \in \mathcal{G}^N$ and $i \in N$ such that

$$v(S)\begin{cases} = v'(S), & \text{for } S \subsetneq N,\\ \geq v'(S), & \text{for } S = N,\end{cases}$$

we have $\varphi_i(v) \geq \varphi_i(v')$.

Additivity, A. For all $N \in \mathcal{N}$, $v, v' \in \mathcal{G}^N$, we have $\varphi_i(v) + \varphi_i(v') = \varphi_i(v + v')$.

Anonymity, AN. For all $N \in \mathcal{N}$, $v \in \mathcal{G}^N$, $i \in N$ and all bijections $\pi\colon N \to N$, where $v_\pi \in \mathcal{G}^N$ is defined by $v_\pi(S) = v \circ \pi^{-1}(S)$ for all $S \subseteq N$, $S \neq \emptyset$, we have $\varphi_{\pi(i)}(v_\pi) = \varphi_i(v)$.

[2] Béal et al. (2017) also allow $v(\{i\}) < 0$ for all $i \in N$. In accordance with the weights used in the weighted Shapley values we desist from this possibility.

Symmetry, S. For all $N \in \mathcal{N}$, $v \in \mathcal{G}^N$, $i, j \in N$ such that i and j are symmetric in v, we have $\varphi_i(v) = \varphi_j(v)$.

Balanced contributions. For all $N \in \mathcal{N}$, $v \in \mathcal{G}^N$ and for all $S \subseteq N$, $i, j \in S$, $i \neq j$, we have

$$\varphi_i(S, v) - \varphi_i(S \backslash \{j\}, v) = \varphi_j(S, v) - \varphi_j(S \backslash \{i\}, v).$$

Weighted balanced contributions, WBC. For all $N \in \mathcal{N}$, $v \in \mathcal{G}^N$ and for all $S \subseteq N$, $i, j \in S$, $i \neq j$, and a vector $\omega \in \mathbb{R}^N_{++}$ with positive weights ω_i for all $i \in N$, we have

$$\frac{\varphi_i(S, v) - \varphi_i(S \backslash \{j\}, v)}{\omega_i} = \frac{\varphi_j(S, v) - \varphi_j(S \backslash \{i\}, v)}{\omega_j}.$$

3. Example

3.1. Situation 1

Assume that three districts of a city, district A, B and C, wish to get a motorway ring. In figure 1 the lengths of the motorway sections are given in kilometres. The offer of the

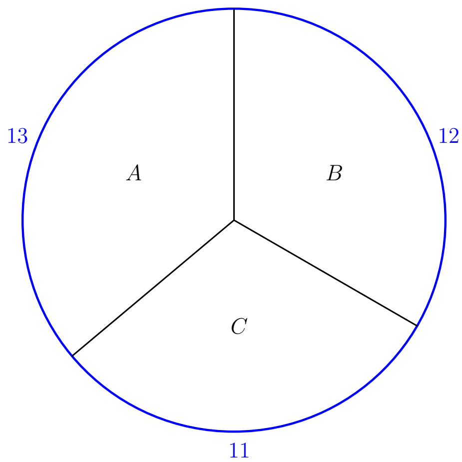

Figure 1: City with three districts and a motorway ring (lengths in km).

road-building company with the most favourable prices is given by

$$p(\ell) = \begin{cases} 100\ell, & \text{for } 0 \leq \ell < 20, \\ 100\ell - 100, & \text{for } 20 \leq \ell < 32, \\ 100\ell - 200, & \text{for } 32 \leq \ell, \end{cases}$$

where ℓ is the length in kilometres and the prices are given in millions monetary units. To share the building costs we can establish a cost game v on $N = \{1, 2, 3\}$ with players $1 := A$, $2 := B$, $3 := C$, where the worth of a coalition $S \subseteq N$ is the cost of the coalition

S (in millions monetary units). We get

$$v(\{1\}) = 1300, \quad v(2) = 1200, \quad v(3) = 1100,$$
$$v(\{1,2\}) = 2400, \quad v(\{1,3\}) = 2300, \quad v(\{2,3\}) = 2200,$$
$$v(\{1,2,3\}) = 3400.$$

The three districts have the problem, how to share the costs in the game v.

3.2. Situation 2

We modify our example in situation 1 to situation 2 (figure 2). District C is splitted in

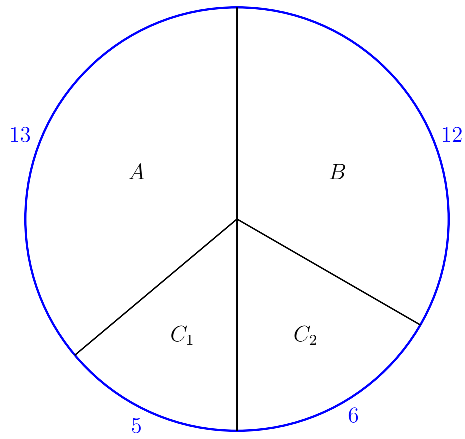

Figure 2: City with four districts and a motorway ring (lengths in km).

two districts C_1 and C_2 and we get a new coalition function v' on $N = \{1, 2, 3_1, 3_2\}$ with players $1 := A$, $2 := B$, $3_1 := C_1$, $3_2 := C_2$, given by

$$v'(\{1\}) = 1300, \quad v'(2) = 1200, \quad v'(3_1) = 500,$$
$$v'(3_2) = 600, \quad v'(\{1,2\}) = 2400, \quad v'(\{1,3_1\}) = 1800,$$
$$v'(\{1,3_2\}) = 1900, \quad v'(\{2,3_1\}) = 1700, \quad v'(\{2,3_2\}) = 1800,$$
$$v'(\{3_1,3_2\}) = 1100 \quad v'(\{1,2,3_1\}) = 2900, \quad v'(\{1,2,3_2\}) = 3000,$$
$$v'(\{1,3_1,3_2\}) = 2300, \quad v'(\{2,3_1,3_2\}) = 2200, \quad v'(\{1,2,3_1,3_2\}) = 3400.$$

Here is a special kind of "dependency" between player 3_1 and 3_2:

- The sum of the singleton worths from player 3_1 and 3_2 in v' is equal to the worth of player 3 in game v.
- The marginal contributions of player 3_1 or player 3_2 to any coalition which does not contain the respective other player are only the singleton worths of this players.
- All coalitions which concludes both players 3_1 and 3_2 have the same worth in v' as the related coalitions in v which content the player 3.
- Coalitions which are the same in v and v' have the same worth in v and v'.

In the sum in this example there is no effect by splitting player 3 into two new players 3_1 and 3_2 to the other players. Hence we call a value consistent for splitting in a game v, if in a corresponding game v', fulfilling the same conditions as here (we will formulate this conditions in def. 5.6), the payoff to not splitted players does not change.

We get with the Shapley value in game v

$$Sh_1(v) = 1233.33, \quad Sh_2(v) = 1133.33, \quad Sh_3(v) = 1033.33$$

and in game v'

$$Sh_1(v') = 1241.67, \quad Sh_2(v') = 1141.67, \quad Sh_{3_1}(v') = 458.33, \quad Sh_{3_2}(v') = 558.33.$$

The total cost saving of cooperating is in each game 200 millions. District C saves in game v 66.67 millions (one third of the total saving) and district C_1 and district C_2 save together in game v' 83,34 millions (42% of the total saving), although district C owns only 31% of the length of the motorway. So there is additional to the inconsistency also a discriminatory level of players which have a greater share of costs.

On the contrary we get with the proportional Shapley value in game v

$$Sh^p_1(v) = 1229.94, \quad Sh^p_2(v) = 1133.16, \quad Sh^p_3(v) = 1036.90$$

and in game v'

$$Sh^p_1(v') = 1229.94, \quad Sh^p_2(v') = 1133.16, \quad Sh^p_{3_1}(v') = 471.32, \quad Sh^p_{3_2}(v') = 565.58.$$

District A and B have in both games the same costs and district C saves the same as districts C_1 and C_2 together, 32% of the total cost saving, and the proportional Shapley value is consistent for splitting in our sense, what we will prove in general in section 5.

4. Known and simple results

If we use for each given game $\overline{v} \in \mathcal{G}_0^N$ weights $w_i(\overline{v}) := \overline{v}(\{i\})$ for all $i \in N$, then the proportional Shapley value Sh^p is for this coalition function $\overline{v}$ (and corresponding to its subgames) identical to the weighted Shapley value $Sh^w(\overline{v})$ and for cost games to the "Independent Cost Proportional Scheme (ICPS)" (Gangolly 1981). So it is clear (the proof is immediate and omitted), that well-known results for a fixed coalition function of a weighted Shapley value hold also for the proportional Shapley value (Besner 2016; Béal et al. 2017). E. g. the following properties of the proportional Shapley value are inherited:

- **efficiency**, **dummy**, **homogeneity** and **monotonicity**.

A main difference between all weighted Shapley values, with exception of the Shapley value, and the proportional Shapley value make

- **symmetry** and **anonymity**. For the proof, see appendix 7.2.1.

The next axiom is analog to weighted balanced contributions (Myerson 1980).

Proportional balanced contributions, PBC (Besner 2016; Béal et al. 2017)**.** For all $N \in \mathcal{N}$, $v \in \mathcal{G}_0^N$, all $S \subseteq N$, all players $i, j \in S$, $i \neq j$, we have

$$\frac{\varphi_i(S,v) - \varphi_i(S\backslash\{j\},v)}{v(\{i\})} = \frac{\varphi_j(S,v) - \varphi_j(S\backslash\{i\},v)}{v(\{j\})}.$$

In Myerson (1980) is given an axiomatization of the Shapley value by efficiency and balanced contributions. Hart and Mas-Colell (1989) could show that the values, uniquely characterized by efficiency and weighted balanced contributions in Myerson (1980), are the weighted Shapley values. Analogue we have

Theorem 4.1 (Besner 2016; Béal et al. 2017)**.** *Let $N \in \mathcal{N}$ and $v \in \mathcal{G}_0^N$. Sh^p is the unique TU-Value that satisfies **E** and **PBC**.*

Béal et al. (2017) have presented a lot other interesting axioms and axiomatizations, which are satisfied by the proportional Shapley value, like weak consistency, weak linearity and admittance of a potential. We don't use this here and refer to the authors.

5. Axiomatizations

This main part of our paper provides new axioms which are satisfied by the proportional Shapley value and lead to new axiomatic characterizations. So the proportional Shapley value has a counterpart not only to the famous characterization of the Shapley value in Myerson (1980) but also to the classics in Shapley (1953b) and Young (1985a).

In the case of using subdomains, we require an axiom to hold when all games belong to this subdomain. All proves, related lemmas and the logical independence of the axioms used for characterization are relegated to the appendix (section 7).

5.1. A characterization similar to Shapley

In our example, situation 3.2, the new splitted players are in a kind of dependency: Each new player has to all coalitions where the other new player is not a member only a marginal contribution of his singleton worth. Nowak and Radzik (1995) has used for axiomatizations of the weighted Shapley values, also in the spirit of Shapley (1953b) and Young (1985a), an axiom called ω-mutual dependence. There are, in different to our following definition, the singleton worths of dependent players zero.

Definition 5.1. *Players $i, j \in N$ are called **weakly dependent** in v, $v \in \mathcal{G}^N$, if $v(S \cup \{k\}) = v(S) + v(\{k\})$, $k \in \{i, j\}$, for all coalitions $S \subseteq N\backslash\{i, j\}$.*

This definition has the interpretation that a player is only interested to join a coalition which contents weakly dependent players, if all weakly dependent players are in the joined coalition. So all weakly dependent players are in mutually dependency.

Lemma 5.2. *Players $i, j \in N$ are weakly dependent in v, $v \in \mathcal{G}^N$, iff $\Delta_v(S \cup \{k\}) = 0$, $k \in \{i, j\}$, for all $S \subseteq N\backslash\{i, j\}$, $S \neq \emptyset$.*

For the proof, see appendix 7.2.2.

With definition 5.1 for weakly dependent players we get a related axiom to symmetry which is familiar to ω-mutual dependence in Nowak and Radzik (1995):

Proportionality, P. For all $v \in \mathcal{G}_0^N$, $i, j \in N$ such that i and j are weakly dependent in v, we have

$$\frac{\varphi_i(v)}{v(\{i\})} = \frac{\varphi_j(v)}{v(\{j\})}.$$

In Shapley (1953b) are formulated desirable properties for an TU-value by his well-known three axioms, which can be represented by the four axioms efficiency, null player, symmetry and additivity, where null player can be replaced by dummy. But it is not appropriate to claim additivity in the case of a proportional value, because additivity is not even satisfied in the two player case (see Ortmann 2000, def. 1.3.). So we use an axiom of additivity where in each game the stand-alone worths of all players are in the same proportion.

Weak additivity, WA. For all $N \in \mathcal{N}$, $v, w \in \mathcal{G}^N$, $w(\{i\}) = c \cdot v(\{i\})$ for all $i \in N$, $c > 0$, we have

$$\varphi_i(v) + \varphi_i(w) = \varphi_i(v + w).$$

It follows a characterization close to the original in Shapley (1953b).

Theorem 5.3. *Let $N \in \mathcal{N}$ and $v \in \mathcal{G}_0^N$. Sh^p is the unique TU-Value that satisfies **E**, **D**, **P** and **WA**.*

For the proof, see appendix 7.2.3.

5.2. A characterization similar to Young

One of the most elegant characterizations of the Shapley value is suggested by Young (1985a). In addition to efficiency and symmetry there is used marginality[3]. To characterize the proportional Shapley value we weaken marginality:

Weak marginality, WM. For all $N \in \mathcal{N}$, $v, w \in \mathcal{G}^N$, $w(\{j\}) = v(\{j\})$ for all $j \in N$, and $i \in N$ such that $MC_i^v(S) = MC_i^w(S)$ for all $S \subseteq N \backslash \{i\}$, we have

$$\varphi_i(v) = \varphi_i(w).$$

By this axiom, if the stand-alone worths of all players are given, the payoff to a player depends only on his own marginal contributions. Chun (1989) offers another appealing axiomatization of the Shapley value, using efficiency, symmetry and coalitional strategic equivalence, itself a generalization of strategic equivalence from von Neumann and Morgenstern (1944). But in Casajus and Huettner (2008) is shown that coalitional strategic equivalence and marginality are equivalent.

Our next axiom weakens coalitional strategic equivalence.

[3]Originally Young used an axiom called strong monotonicity. Chun (1989) named the essential part of this axiom for the proof of the uniqueness marginality.

Weak coalitional strategic equivalence, WCSE. For all $N \in \mathcal{N}$, $v, w \in \mathcal{G}^N$ such that for any coalition $R \subseteq N$, $|R| \geq 2$, $c \in \mathbb{R}$ and all coalitions $S \subseteq N$,

$$v(S) = \begin{cases} w(S) + c, & \text{if } S \supseteq R, \\ w(S), & \text{if } S \not\supseteq R, \end{cases} \tag{4}$$

$$\text{we have } \varphi_i(v) = \varphi_i(w) \quad \text{for all } i \in N \backslash R.$$

If the members of a coalition R are improving their cooperation and take this improvement into all supersets of R, the payoff to all non-members of R does not change. Unlike our axiom, in coalitional strategic equivalence from Chun there are also admitted singletons for the coalition R, in strategic equivalence from Neumann and Morgenstern only singletons.

We show, analog to Casajus and Huettner (2008):

Proposition 5.4. ***WM*** *is equivalent to* ***WCSE****.*

For the proof, see appendix 7.2.4.

Theorem 5.5. *Let $N \in \mathcal{N}$ and $v \in \mathcal{G}_0^N$. Sh^p is the unique TU-Value that satisfies* ***E****,* ***P*** *and* ***WM/WCSE*** *(equivalent by proposition 5.4).*

For the proof, see appendix 7.2.5.

5.3. Player Splitting

In many applications it is not desired that the payoff to players changes, if another player splits into several new players, which together have only the same input to the game as the original splitting player, like in situation 2 (subsection 3.2) in our example. We define a corresponding game where a player of the original game is "splitted" in two new players:

Definition 5.6. *Let $N, N^j \in \mathcal{N}$, $(N, v) \in \mathcal{G}^N$, $(N^j, v^j) \in \mathcal{G}^{N^j}$, $j \in N$, $k, \ell \in N^j$, $k, \ell \notin N$, $N^j = (N \backslash \{j\}) \cup \{k, \ell\}$. The game (N^j, v^j) is called a* ***corresponding splitted player game*** *to (N, v) if for all $S \subseteq N \backslash \{j\}$*

- $v^j(\{k\}) + v^j(\{\ell\}) = v(\{j\})$,
- $v^j(S \cup \{i\}) = v^j(S) + v^j(\{i\})$, $i \in \{k, \ell\}$,
- $v^j(S \cup \{k, l\}) = v(S \cup \{j\})$ *and*
- $v^j(S) = v(S)$.

It should be observed that players k, ℓ are weakly dependent in the game v^j.

Banker (1981) notes, that an allocation scheme should not be sensitive to the way cost centres are used or organized. For this kind of games the Shapley value is not the right choice, because it does not satisfy the following axiom:

Player splitting, PS. For all $N \in \mathcal{N}$, $(N, v) \in \mathcal{G}^N$, $j \in N$ and a corresponding splitted player game $(N^j, v^j) \in \mathcal{G}^{N^j}$ to (N, v), we have

$$\varphi_i(N, v) = \varphi_i(N^j, v^j) \text{ for all } i \in N \backslash \{j\}.$$

A suitable application of this axiom would be, that a player, who was participating to an online game full-time, now is participating part-time under cover-names to the game, where the sum of her time in part-time activities is equal to the original time in full-time activities. She participates with the same productivity in all original coalitions, but now under all her cover-names at the same time in total. In all other coalitions she has only a pro forma membership. This means that her marginal contributions to this coalitions in each case are only her singleton worths or, more specifically, all part-time players are weakly dependent. In such a situation the payoff to the other players should not change.

Remark 5.7. *Let $N \in \mathcal{N}$, $(N, v) \in \mathcal{G}^N$, $j \in N$ and $(N^j, v^j) \in \mathcal{G}^{N^j}$ a corresponding splitted player game to (N, v). If φ is a TU-value that satisfies **E** and **PS**, then we have*

$$\varphi_j(N, v) = \varphi_k(N^j, v^j) + \varphi_\ell(N^j, v^j).$$

This remark is related to the amalgamating payoffs axiom in Radzik (2012) which is there used to characterize the weighted Shapley values.

Remark 5.8. *In def. 5.6 the game (N, v) could also be considered as a **corresponding merged player game** to (N^j, v^j) and so also **PS** as a **player merging** axiom. But we expressly point out that player $j \in N$ and players $k, \ell \in N^j$ can be completely independent apart from the given properties in def. 5.6 unlike amalgamating payoffs in Radzik (2012).*

It transpires that splitting fits best with the proportional Shapley value.

Proposition 5.9. *Sh^p satisfies **PS**.*

For the proof, see appendix 7.2.6.

The following lemma shows dependence on symmetry for efficient values which satisfy player splitting.

Lemma 5.10. *Let $N \in \mathcal{N}$ and $v \in \mathcal{G}_0^N$. If a TU-Value φ satisfies **E** and **PS** then φ satisfies also **S**.*

For the proof, see appendix 7.2.7.

We have another interesting lemma which uses lemma 5.10 in the proof.

Lemma 5.11. *Let $N \in \mathcal{N}$ and $v \in \mathcal{G}_{0_{\mathbb{Q}}}^N$. If a TU-Value φ satisfies **E** and **PS** then φ satisfies also proportionality **P**.*

For the proof, see appendix 7.2.8.

Thus we get if the worths of all singletons are positive rational similar to Young (1985a):

Corollary 5.12. *Let $N \in \mathcal{N}$ and $v \in \mathcal{G}_{0_{\mathbb{Q}}}^N$. Sh^p is the unique TU-Value that satisfies **E**, **PS** and **WM/WCSE** (equivalent by proposition 5.4).*

The proof follows immediately by proposition 5.9 and lemma 5.11 from theorem 5.5.

We have another characterization with splitting, similar to Shapley (1953b).

Corollary 5.13. *Let $N \in \mathcal{N}$ and $v \in \mathcal{G}_{0_{\mathbb{Q}}}^N$. Sh^p is the unique TU-Value that satisfies **E**, **D**, **PS** and **WA**.*

Obviously follows the proof by proposition 5.9 and lemma 5.11 from theorem 5.3.

Remark 5.14. *Lemma 5.11 holds for $v \in \mathcal{G}_0^N$ if we require continuity of the TU-value in $v(\{i\})$ for all $v \in \mathcal{G}_0^N$ and all $i \in N$ in an additional axiom. So also corollary 5.12 and corollary 5.13 are valid for $v \in \mathcal{G}_0^N$ if there is in each case an additional continuity axiom.*

6. Conclusion

In this paper we have shown, that in games where the stand alone worths of the singletons are reasonable weights, the proportional Shapley value is a powerful tool to share benefits of cooperation due to convincing axioms, similar to the Shapley value. It is especially suitable for games where we don't want that the payoffs to uninvolved players are changing, if another player is splitting into two new players which together have the same input in the game as the single player before. In such games the Shapley value completely fails and this could be one of the main difficulties, why there is significant resistance to the use of the Shapley value in cost allocation in practice.

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7. Appendix

7.1. Additional lemmas and a remark, used in the proofs

Remark 7.1. *We can consider the collection of all TU-games $v \in \mathcal{G}^N, N \in \mathcal{N},\ n := |N|$, as a vector space $\mathbb{R}^{2^n-1}$. Each game v is represented by a vector $\overrightarrow{v} \in \mathbb{R}^{2^n-1}$, where the entries in the 2^n-1 coordinates of the 2^n-1 coalitions $S \subseteq N$, $S \neq \emptyset$, get the worth $v(S)$ of the respective coalition S. Hence there exists for every game v a vector $\overrightarrow{\Delta_v} \in \mathbb{R}^{2^N-1}$, which is corresponding to the vector $\overrightarrow{v}$, where the entries of the coordinates get the dividends of the respective coalitions. By (1) we get with $v, v_1, v_2 \in \mathcal{G}^N$*

$$\begin{aligned}
& & \overrightarrow{\Delta_v} &= \overrightarrow{\Delta_{v_1}} + \overrightarrow{\Delta_{v_2}} \\
\Leftrightarrow & & \Delta_v(S) &= \Delta_{v_1}(S) + \Delta_{v_2}(S) \text{ for all } S \subseteq N \\
\Leftrightarrow & & v(S) - \sum_{R \subsetneq S} \Delta_v(R) &= v_1(S) - \sum_{R \subsetneq S} \Delta_{v_1}(R) + v_2(S) - \sum_{R \subsetneq S} \Delta_{v_2}(R) \text{ for all } S \subseteq N \\
\Leftrightarrow & & v &= v_1 + v_2.
\end{aligned}$$

Lemma 7.2. *Statement (4) in **WCSE** can be replaced equivalently by*

$$\Delta_v(S) = \begin{cases} \Delta_w(R) + c, & \text{if } S = R, \\ \Delta_w(S), & \text{otherwise.} \end{cases}$$

Proof. Let the notation and the preconditions as in **WCSE**. By (2), if $v(S) = w(S)$ for all $S \not\supseteq R$, we have $\Delta_v(S) = \Delta_w(S)$ for all such S and vice versa. Hence, by (1), $v(R) = w(R) + c$ is equivalent to $\Delta_v(R) = \Delta_w(R) + c$. By induction on $s := |S|$ we show now $v(S) = w(S) + c \ \Leftrightarrow\ \Delta_v(S) = \Delta_w(S)$ for all proper supersets $S \supsetneq R$.

Initialisation: Let $S \supsetneq R$ and $s = |R|+1$. R is the only proper subset of S where there is a difference of the related dividends in both coalition functions and we obtain

$$v(S) = w(S) + c \underset{(1)}{\Leftrightarrow} v(S) - \sum_{T \subsetneq S} \Delta_v(T) = w(S) + c - \sum_{\substack{T \subsetneq S, \\ T \neq R}} \Delta_w(T) - (\Delta_w(R) + c)$$
$$\underset{(1)}{\Leftrightarrow} \Delta_v(S) = \Delta_w(S).$$

Induction step: Assume equivalence holds for $s' = s-1$, $|R|+1 \leq s' \leq n-1$ (IH). Then by (IH) R is again the only proper subset of S with not equal related dividends in v and w and we get by (1) $v(S) = w(S)+c \;\Leftrightarrow\; \Delta_v(S) = \Delta_w(S)$ as before and lemma 7.2 is shown. □

Lemma 7.3 (Casajus and Huettner 2008)**.** *If $i \in N$ and $v, w \in \mathcal{G}^N$, then $MC_i^v(S) = MC_i^w(S)$ for all $S \subseteq N \backslash \{i\}$ iff $\Delta_v(S \cup \{i\}) = \Delta_w(S \cup \{i\})$ for all $S \subseteq N \backslash \{i\}$.*

7.2. Proofs

7.2.1. Proof of symmetry/anonymity

It is well-known that **AN** implies **S**. We have

$$Sh_i^p(v) = \sum_{\substack{S \subseteq N, \\ S \ni i}} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Delta_v(S) = \sum_{\substack{S \subseteq N, \\ S \ni i}} \frac{v \circ \pi^{-1}\big(\pi(i)\big)}{\sum_{j \in S} v \circ \pi^{-1}\big(\pi(j)\big)} \Delta_{v \circ \pi^{-1}}\big(\pi(S)\big)$$
$$= \sum_{\substack{\pi(S) \subseteq N, \\ \pi(S) \ni \pi(i)}} \frac{v_\pi\big(\pi(i)\big)}{\sum_{\pi(j) \in \pi(S)} v_\pi\big(\pi(j)\big)} \Delta_{v_\pi}\big(\pi(S)\big) = Sh^p_{\pi(i)}(v_\pi) \text{ for all } i \in N \text{ and } v \in \mathcal{G}_0^N$$

and so **AN** and therefore also **S** is satisfied. □

7.2.2. Proof of lemma 5.2

Let $i, j \in N$ and $v \in \mathcal{G}^N$. If $S = \emptyset$ we have $v(S \cup \{k\}) = v(S) + v(\{k\})$. We show by induction on the size $s := |S|$ of all coalitions $S \subseteq N \backslash \{i,j\}$, $S \neq \emptyset$,

$$v(S \cup \{k\}) = v(S) + v(\{k\}) \quad \Leftrightarrow \quad \Delta_v(S \cup \{k\}) = 0.$$

Initialisation: Let $s = 1$. For $k \in \{i, j\}$ we have

$$\begin{aligned} v(S \cup \{k\}) &= v(S) + v(\{k\}) \\ \underset{(1)}{\Leftrightarrow} \quad \Delta_v(S \cup \{k\}) + \Delta_v(S) + \Delta_v(\{k\}) &= \Delta_v(S) + \Delta_v(\{k\}) \\ \Leftrightarrow \quad \Delta_v(S \cup \{k\}) &= 0. \end{aligned}$$

Induction step: Assume that equivalence and equality in the first and last line of the system above hold for all coalitions S' with $s' \geq 1$ (IH) and let $s = s'+1$ and $k \in \{i, j\}$.

We get

$$\begin{array}{rrcl} & v(S \cup \{k\}) & = & v(S) + v(\{k\}) \\ \underset{(1)}{\Leftrightarrow} & \Delta_v(S \cup \{k\}) + \sum\limits_{R \subsetneq (S \cup \{k\})} \Delta_v(R) & = & \sum\limits_{R \subseteq S} \Delta_v(R) + \Delta_v(\{k\}) \\ \underset{(IH)}{\Leftrightarrow} & \Delta_v(S \cup \{k\}) + \Delta_v(\{k\}) + \sum\limits_{R \subseteq S} \Delta_v(R) & = & \sum\limits_{R \subseteq S} \Delta_v(R) + \Delta_v(\{k\}) \\ \Leftrightarrow & \Delta_v(S \cup \{k\}) & = & 0. \end{array}$$

□

7.2.3. Proof of theorem 5.3

I. Existence: By section 4 Sh^p satisfies **E** and **D**.

- **P**: Let $v \in \mathcal{G}_0^N$ and $i, j \in N$ such that i and j are weakly dependent in v. We have

$$\begin{aligned} Sh_i^p(v) &= \sum_{\substack{S \subseteq N, \\ S \ni i}} \frac{v(\{i\})}{\sum_{k \in S} v(\{k\})} \Delta_v(S) \underset{5.2}{=} v(\{i\}) + \sum_{\substack{S \subseteq N, \\ \{i,j\} \subseteq S}} \frac{v(\{i\})}{\sum_{k \in S} v(\{k\})} \Delta_v(S) \\ &= \frac{v(\{i\})}{v(\{j\})} v(\{j\}) + \frac{v(\{i\})}{v(\{j\})} \sum_{\substack{S \subseteq N, \\ \{i,j\} \subseteq S}} \frac{v(\{j\})}{\sum_{k \in S} v(\{k\})} \Delta_v(S) \\ &\underset{5.2}{=} \frac{v(\{i\})}{v(\{j\})} \sum_{\substack{S \subseteq N, \\ S \ni j}} \frac{v(\{j\})}{\sum_{k \in S} v(\{k\})} \Delta_v(S) = \frac{v(\{i\})}{v(\{j\})} Sh_j^p(v). \end{aligned}$$

- **WA**: Let $v, w \in \mathcal{G}_0^N$ with $w(\{i\}) = c \cdot v(\{i\})$ for all $i \in N$, $c > 0$. We have

$$\begin{aligned} Sh_i^p(v) + Sh_i^p(w) &\underset{(3)}{=} \sum_{\substack{S \subseteq N, \\ S \ni i}} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Delta_v(S) + \sum_{\substack{S \subseteq N, \\ S \ni i}} \frac{w(\{i\})}{\sum_{j \in S} w(\{j\})} \Delta_w(S) \\ &= \sum_{\substack{S \subseteq N, \\ S \ni i}} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Delta_v(S) + \sum_{\substack{S \subseteq N, \\ S \ni i}} \frac{c \cdot v(\{i\})}{\sum_{j \in S} c \cdot v(\{j\})} \Delta_w(S) \\ &= \sum_{\substack{S \subseteq N, \\ S \ni i}} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \left[\Delta_v(S) + \Delta_w(S)\right] \\ &= \sum_{\substack{S \subseteq N, \\ S \ni i}} \frac{(1+c) v(\{i\})}{\sum_{j \in S} (1+c) v(\{j\})} \left[\Delta_v(S) + \Delta_w(S)\right] \\ &= \sum_{\substack{S \subseteq N, \\ S \ni i}} \frac{v(\{i\}) + w(\{i\})}{\sum_{j \in S} \left[v(\{j\}) + w(\{j\})\right]} \Delta_{v+w}(S) = Sh_i^p(v + w). \end{aligned}$$

II. Uniqueness: Let $N \in \mathcal{N}$, $n := |N|$, $v \in \mathcal{G}_0^N$ and φ a TU-value which satisfies all axioms of theorem 5.3. To prove uniqueness, we will show that φ equals Sh^p.

For $n = 1$ φ equals Sh^p by **E**. Let now $n \geq 2$. For each coalition $S \subseteq N$, $S \neq \emptyset$, we define, corresponding to remark 7.1, a TU-game $v_S \in \mathcal{G}_0^N$ through a vector $\overrightarrow{v_S} \in \mathbb{R}^{2^n - 1}$ by assigning the coordinates of the related vector $\overrightarrow{\Delta_{v_S}} \in \mathbb{R}^{2^n - 1}$ in the entry of a coalition

$R \subseteq N$, $R \neq \emptyset$, the dividend

$$\Delta_{v_S}(R) := \begin{cases} \frac{v(\{j\})}{2^n - 1}, \text{ if } R = \{j\} \text{ for all } j \in N, \\ \Delta_v(S), \text{ if } R = S, \ |S| \geq 2, \\ 0, \text{ otherwise.} \end{cases}$$

So each vector $\overrightarrow{v_S} \in \mathbb{R}^{2^n - 1}$ gets in the coordinates of coalitions $R \subseteq N$, $R \neq \emptyset$, the entry

$$v_S(R) = \begin{cases} \Delta_v(S) + \sum_{j \in R} \frac{v(\{j\})}{2^n - 1}, \text{ if } R \supseteq S, \ |S| \geq 2, \\ \sum_{j \in R} \frac{v(\{j\})}{2^n - 1}, \ else. \end{cases} \tag{5}$$

$$\text{We have } \overrightarrow{\Delta_v} = \sum_{\substack{S \subseteq N, \\ S \neq \emptyset}} \overrightarrow{\Delta_{v_S}} \text{ and so by remark 7.1 } v = \sum_{\substack{S \subseteq N, \\ S \neq \emptyset}} v_S.$$

By **D** we obtain

$$\varphi_i(v_S) = \begin{cases} v_S(\{i\}) = \frac{v(\{i\})}{2^n - 1} \text{ for all } i \in N \text{ and } |S| = 1, \text{ and} \\ v_S(\{i\}) = \frac{v(\{i\})}{2^n - 1} \text{ for all } i \in N, \, i \notin S, \, |S| \geq 2. \end{cases} \tag{6}$$

By lemma 5.2 all players $i \in S$, $|S| \geq 2$, are pairwise weakly dependent in v_S. We get for an arbitrary $i \in S$, $|S| \geq 2$, and by $v_S(N) \underset{(5)}{=} \Delta_v(S) + \sum_{j \in N} \frac{v(\{j\})}{2^n - 1}$

$$\begin{aligned} \sum_{j \in S} \varphi_j(v_S) &\underset{(\mathbf{P})}{=} \sum_{j \in S} \frac{v_S(\{j\})}{v_S(\{i\})} \varphi_i(v_S) = \sum_{j \in S} \frac{v(\{j\})}{v(\{i\})} \varphi_i(v_S) \\ &\underset{(\mathbf{E})}{=} v_S(N) - \sum_{j \in N \setminus S} \varphi_j(v_S) \underset{(6)}{=} \Delta_v(S) + \sum_{j \in S} \frac{v(\{j\})}{2^n - 1} \\ \Leftrightarrow \quad \varphi_i(v_S) &= \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Delta_v(S) + \frac{v(\{i\})}{2^n - 1}. \end{aligned} \tag{7}$$

So we have by (3), (6) and (7) for all $S \subseteq N$, $S \neq \emptyset$,

$$\varphi_i(v_S) = Sh_i^p(v_S) \text{ for all } i \in N.$$

Sh^p and φ satisfy **WA** and it follows

$$\varphi_i(v) = Sh_i^p(v) \text{ for all } i \in N.$$

□

7.2.4. Proof of proposition 5.4

$\Rightarrow$: We show **WM** implies **WCSE**: Let v and w two TU-games satisfying the hypotheses of **WCSE**, i. e. for a coalition $R \subseteq N$, $|R| \geq 2$, $c \in \mathbb{R}$, we have

$$v(S) = \begin{cases} w(S) + c, \text{ if } S \supseteq R, \\ w(S), \text{ if } S \not\supseteq R, \end{cases}$$

and let φ a value which obeys **WM**. By lemma 7.2 we have

$$\Delta_v(S) = \begin{cases} \Delta_w(R) + c, & \text{if } S = R, \\ \Delta_w(S), & \text{otherwise.} \end{cases}$$

Thus we have for all $i \in N \backslash R \quad \Delta_v(S \cup \{i\}) = \Delta_w(S \cup \{i\})$ for all $S \subseteq N \backslash \{i\}$. By lemma 7.3 follows $MC_i^v(S) = MC_i^w(S)$ for all $S \subseteq N \backslash \{i\}$. So we can use **WM** and get $\varphi_i(v) = \varphi_i(w)$ for all $i \in N \backslash R$ and **WCSE** is satisfied.
$\Leftarrow$: We show **WCSE** implies **WM**: Let $N \in \mathcal{N}$, $i \in N$, $v, w \in \mathcal{G}^N$ two coalition functions satisfying the hypothesis of **WM**, i.e. $MC_i^v(S) = MC_i^w(S)$ for all $S \subseteq N \backslash \{i\}$ and $w(\{k\}) = v(\{k\})$ for all $k \in N$ and φ a value satisfying **WCSE**. Then by lemma 7.3 we have $\Delta_v(T) = \Delta_w(T)$ for all $T \subseteq N, T \ni i$. Let $\mathcal{R} = \{R_j \subseteq N : \Delta_v(R_j) \neq \Delta_w(R_j)\}$ an indexed set of all subsets of N with different dividends in v and w, $1 \leq j \leq |\mathcal{R}|$. We define inductively a sequence of coalition functions w_j, $0 \leq j \leq |\mathcal{R}|$, by $w_j := w$ if $j = 0$, and, if $1 \leq j \leq |\mathcal{R}|$,

$$\Delta_{w_j}(S) := \begin{cases} \Delta_{w_{j-1}}(R_j) + \left[\Delta_v(R_j) - \Delta_{w_{j-1}}(R_j)\right], & \text{if } S = R_j, \\ \Delta_{w_{j-1}}(S), & \text{if } S \subseteq N,\ S \neq R_j. \end{cases}$$

Then we have $w_{|\mathcal{R}|} = v$ and, by lemma 7.2 and **WCSE,** we get $\varphi_i(w_j) = \varphi_i(w_{j-1})$ for all j, $1 \leq j \leq |\mathcal{R}|$, and therefore $\varphi_i(v) = \varphi_i(w)$ and **WM** is satisfied. $\square$

7.2.5. Proof of theorem 5.5

I. Existence: By theorem 5.3 Sh^p satisfies **E** and **P**.

- **WCSE**: By lemma 7.2 we have for v and a coalition R from **WCSE**

$$\Delta_v(S) = \begin{cases} \Delta_w(R) + c, & \text{if } S = R, \\ \Delta_w(S), & \text{else} \end{cases}$$

and get thus for all $i \in N \backslash R$

$$Sh_i^p(v) = \sum_{\substack{S \subseteq N, \\ S \ni i}} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Delta_v(S) = \sum_{\substack{S \subseteq N, \\ S \ni i}} \frac{w(\{i\})}{\sum_{j \in S} w(\{j\})} \Delta_w(S) = Sh_i^p(w).$$

II. Uniqueness: Let $N \in \mathcal{N}$, $n := |N|$, $v \in \mathcal{G}_0^N$ and φ a TU-value which satisfies all axioms of theorem 5.5. We will show that φ satisfies eq. (3).

For $n = 1$ eq. (3) is satisfied by **E**.

Let $n \geq 2$. We use induction on the size $r := |\{R \subseteq N : R \text{ is active in } v \text{ and } |R| \geq 2\}|$.

Initialisation: Let $r = 0$. By lemma 5.2 all players $i, j \in N$ are pairwise weakly dependent in v. We get for an arbitrary $i \in N$

$$\sum_{j \in N} \varphi_j(v) \underset{(\mathbf{P})}{=} \sum_{j \in N} \frac{v(\{j\})}{v(\{i\})} \varphi_i(v) \underset{(\mathbf{E})}{=} v(N).$$

With $v(N) = \sum_{j \in N} v(\{j\})$ follows $\varphi_i(v) = v(\{i\})$ and eq. (3) holds to φ if r = 0.

Induction step: Assume that eq. (3) holds to φ if $r \geq 0$, r arbitrary, (IH) and let exactly $r+1$ coalitions $Q_k \subseteq N$, $|Q_k| \geq 2$, $1 \leq k \leq r+1$, active in v. Let Q the intersection of all such coalitions Q_k

$$Q = \bigcap_{1 \leq k \leq r+1} Q_k.$$

We distinguish two cases: *a:* $i \in N \backslash Q$ and *b:* $i \in Q$.

a: Each player $i \in N \backslash Q$ is a member of at most r active coalitions Q_k, $|Q_k| \geq 2$, and v gets at least one active coalition R_i, $|R_i| \geq 2$, $i \notin R_i$. Hence exists a coalition function $w_i \in \mathcal{G}_0^N$, where all coalitions get the same dividend in w_i as in v, except the coalition R_i, which get the dividend $\Delta_{w_i}(R_i) = 0$, and there is existing a scalar $c \in \mathbb{R}$, $c \neq 0$, with

$$\Delta_v(S) = \begin{cases} \Delta_{w_i}(R_i) + c, & \text{if } S = R_i, \\ \Delta_{w_i}(S), & \text{else.} \end{cases}$$

By lemma 7.2 and **WCSE** we get $\varphi_i(v) = \varphi_i(w_i)$ with $i \in N \backslash R_i$ and because there exists for all $i \in N \backslash Q$ a such R_i we get $\varphi_i(v) = \varphi_i(w_i)$ for all $i \in N \backslash Q$. All coalition functions w_i get at most r active coalitions with at least two players and by (IH) and eq. (3) follows

$$\varphi_i(v) = Sh_i^p(v) \text{ for all } i \in N \backslash Q. \tag{8}$$

b: Each player $j \in Q$ is a member of all $r+1$ active coalitions $Q_k \subseteq N$, $|Q_k| \geq 2$, $1 \leq k \leq r+1$, and therefore, by lemma 5.2, all players $j \in Q$ are weakly dependent. By **P** and **E** of φ and Sh^p we get for an arbitrary $i \in Q$

$$\sum_{j \in Q} \varphi_j(v) \underset{\textbf{(P)}}{=} \sum_{j \in Q} \frac{v(\{j\})}{v(\{i\})} \varphi_i(v) \underset{\substack{\textbf{(E)} \\ (8)}}{=} v(N) - \sum_{j \in N \backslash Q} Sh_j^p(v) \underset{\textbf{(E)}}{=} \sum_{j \in Q} Sh_j^p(v) \underset{\textbf{(P)}}{=} \sum_{j \in Q} \frac{v(\{j\})}{v(\{i\})} Sh_i^p(v)$$

$\Leftrightarrow \quad \varphi_i(v) = Sh_i^p(v)$ and together with *I.* the proof is complete. $\square$

7.2.6. Proof of proposition 5.9

Let $v \in \mathcal{G}_0^N$, $j \in N$ and $(N^j, v^j) \in \mathcal{G}^{N^j}$ a corresponding splitted player game to (N, v). We point out that we have for all $S \subseteq N \backslash \{j\}$, $S \neq \emptyset$, $\Delta_{v^j}(S) = \Delta_v(S)$, $\Delta_{v^j}(S \cup \{k, l\}) = \Delta_v(S \cup \{j\})$ and $\Delta_{v^j}(S \cup \{k\}) = \Delta_{v^j}(S \cup \{\ell\}) = 0$. Then we get for all $i \in N \backslash \{j\}$

$$\begin{aligned} Sh_i^p(N, v) &= \sum_{\substack{R \subseteq N, \\ R \ni i}} \frac{v(\{i\})}{\sum_{m \in R} v(\{m\})} \Delta_v(R) \\ &= \sum_{\substack{S \subseteq N \backslash \{j\}, \\ S \ni i}} \frac{v(\{i\})}{\sum_{m \in S} v(\{m\})} \Delta_v(S) + \sum_{\substack{S \subseteq N \backslash \{j\}, \\ S \ni i}} \frac{v(\{i\})}{\sum_{m \in S \cup \{j\}} v(\{m\})} \Delta_v(S \cup \{j\}) \\ &= \sum_{\substack{S \subseteq N^j \backslash \{k, \ell\}, \\ S \ni i}} \frac{v^j(\{i\})}{\sum_{m \in S} v^j(\{m\})} \Delta_{v^j}(S) + \sum_{\substack{S \subseteq N^j \backslash \{k, \ell\}, \\ S \ni i}} \frac{v^j(\{i\})}{\sum_{m \in S \cup \{k, \ell\}} v^j(\{m\})} \Delta_{v^j}(S \cup \{k, \ell\}) \\ &= \sum_{\substack{R \subseteq N^j, \\ R \ni i}} \frac{v^j(\{i\})}{\sum_{m \in R} v^j(\{m\})} \Delta_{v^j}(R) = Sh_i^p(N^j, v^j). \end{aligned}$$

$\square$

7.2.7. Proof of lemma 5.10

Let $N = \{1, 2, ..., n\}$, $|N| \geq 2$, $v \in \mathcal{G}_0^N$, φ a TU-value which satisfies **E** and **PS** for all $v \in \mathcal{G}_0^N$ and, w.l.o.g., player 1 and player 2 symmetric in v. If we split player 1 according to **PS** into two new players, player $n+1$ and player $n+2$, $N^1 = \{2, 3, ..., n, n+1, n+2\}$, we have

$$\varphi_2(N^1, v^1) = \varphi_2(N, v), \tag{9}$$

and, if we split player 2 according to **PS** into the same players as before, player $n+1$ and player $n+2$, instead, $N^2 = \{1, 3, 4, ..., n, n+1, n+2\}$, we have

$$\varphi_1(N^2, v^2) = \varphi_1(N, v), \tag{10}$$

where we choose $v^2(\{n+1\}) := v^1(\{n+1\})$ and $v^2(\{n+2\}) := v^1(\{n+2\})$.

In the same manner we split now in game (N^1, v^1) player 2 into two new players, player $n+3$ and player $n+4$, and analogous in game (N^2, v^2) player 1 into the same players as before, player $n+3$ and player $n+4$, and choose $v^{2^1}(\{n+3\}) := v^{1^2}(\{n+3\})$ and $v^{2^1}(\{n+4\}) := v^{1^2}(\{n+4\})$. We have $N^{1^2} = N^{2^1} = \{3, 4, ..., n, n+1, n+2, n+3, n+4\}$ and $v^{1^2} = v^{2^1}$ and get by **E**, according to remark 5.7,

$$\begin{aligned}
\varphi_{n+3}\left(N^{1^2}, v^{1^2}\right) + \varphi_{n+4}\left(N^{1^2}, v^{1^2}\right) &= \varphi_2(N^1, v^1) \underset{(9)}{=} \varphi_2(N, v),\\
\varphi_{n+3}\left(N^{2^1}, v^{2^1}\right) + \varphi_{n+4}\left(N^{2^1}, v^{2^1}\right) &= \varphi_1(N^2, v^2) \underset{(10)}{=} \varphi_1(N, v)
\end{aligned}$$

and hence $\varphi_1(N, v) = \varphi_2(N, v)$ and **S** is shown. □

7.2.8. Proof of lemma 5.11

Let $N \in \mathcal{N}$, $|N| \geq 2$, $v \in \mathcal{G}_{0_\mathbb{Q}}^N$ a TU-game and, w.l.o.g., player $i, j \in N$ weakly dependent in v and let φ a TU-value, which satisfies **E** and **PS** for all $v \in \mathcal{G}_{0_\mathbb{Q}}^N$ and therefore, by lemma 5.10, also **S**. Due to $v(\{i\}), v(\{j\}) \in \mathbb{Q}_{++}$ the worths of singletons $v(\{k\})$, $k \in \{i, j\}$, can be written as a fraction

$$v(\{k\}) = \frac{p_k}{q_k} \quad \text{with} \quad p_k, q_k \in \mathbb{N}.$$

We choose a main denominator q of these two fractions by $q := q_i q_j$. With $z_i := p_i q_j$ and $z_j := p_j q_i$ we get

$$v(\{i\}) = \frac{z_i}{q} \quad \text{and} \quad v(\{j\}) = \frac{z_j}{q}. \tag{11}$$

Now we define a player set N' and a coalition function v' by "splitting" each player $k \in \{i, j\}$ into z_k players k_1 to k_{z_k} so that we have $N' = (N\backslash\{i, j\}) \cup \{i_m : 1 \leq m \leq z_i\} \cup \{j_m : 1 \leq m \leq z_j\}$. Each player $k_m \in N'\backslash(N\backslash\{i, j\})$, $1 \leq m \leq z_k$, get a singleton worth $v'(\{k_m\}) = \frac{1}{q}$ for $k \in \{i, j\}$, synonymous with

$$v'(\{\ell\}) = \frac{1}{q} \text{ for all } \ell \in N'\backslash(N\backslash\{i, j\}),$$

where $|N'\backslash(N\backslash\{i,j\})| = z_i + z_j$ and $v(\{k\}) = \sum_{1\le m\le z_k} v'(\{k_m\})$, $k \in \{i,j\}$. We define $v'(R') := v(R)$ for all $R' = R\backslash\{i,j\} \cup N'\backslash(N\backslash\{i,j\})$, $R \subseteq N$, $\{i,j\} \subseteq R$ and $v'(S) := v(S)$ for all $S \subseteq N'$ with $S \subseteq N$. All other coalitions $T \subseteq N'$ are defined as not active in v'.

Applying splitting 5.3 (repeatedly) to v, φ and the two players $i, j \in N$ we can get the coalition function v' defined just before and by remark 5.7 we have

$$\varphi_k(N,v) = \sum_{1\le m\le z_k} \varphi_{k_m}(N',v') \text{ for } k \in \{i,j\}. \tag{12}$$

All players $\ell \in N'\backslash(N\backslash\{i,j\})$ are symmetric in v' and hence follows by **S**

$$\varphi_\ell(N',v') = \frac{\varphi_i(N,v) + \varphi_j(N,v)}{z_i + z_j} \text{ for } \ell \in N'\backslash(N\backslash\{i,j\}).$$

We get

$$\varphi_k(N,v) \underset{(12)}{=} \sum_{1\le m\le z_k} \varphi_{k_m}(N',v') = \frac{z_k}{z_i+z_j}\left[\varphi_i(N,v) + \varphi_j(N,v)\right] \text{ for } k \in \{i,j\}.$$

It follows

$$\varphi_i(N,v) = \frac{z_i}{z_j}\varphi_j(N,v) \underset{(11)}{=} \frac{v(\{i\})}{v(\{j\})}\varphi_j(N,v)$$

and **P** is shown. □

7.3. Logical independence

Finally, we want to show the independence of the axioms used in the characterizations.

Remark 7.4. *Let $v \in \mathcal{G}_0^N$, $N \in \mathcal{N}$. The axioms in theorem 5.3/corollary 5.13 are logically independent:*

- ***E**: The TU-value φ defined by*

$$\varphi_i(v) = v(\{i\}) + 2 \cdot \sum_{\substack{S\subseteq N,\\ S\ni i,\, S\neq\{i\}}} \frac{v(\{i\})}{\sum_{j\in S} v(\{j\})}\Delta_v(S) \quad \text{for all } i \in N$$

  *satisfies **D**, **P/PS** and **WA** but not **E**.*

- ***D**: The **proportional rule** π (Moriarity 1975), given by*

$$\pi_i(v) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) \quad \text{for all } i \in N \tag{13}$$

  *satisfies **E**, **P/PS** and **WA**, but not **D**.*

- ***P/PS**: Sh satisfies **E**, **D** and **WA** but not **P/PS**.*

- ***WA***: *The TU-value φ defined for all $i \in N$ by*

$$\varphi_i(v) = \begin{cases} v(\{i\}), & \text{if } i \text{ is a dummy player,} \\ \dfrac{v(\{i\})}{\sum_{\substack{j \in N, \\ j \text{ is no dummy}}} v(\{j\})} \Big[v(N) - \sum_{\substack{j \in N, \\ j \text{ is a dummy}}} v(\{j\}) \Big], & \text{else,} \end{cases}$$

*satisfies **E**, **D** and **P/PS** but not **WA**.*

Remark 7.5. *Let $v \in \mathcal{G}_0^N$, $N \in \mathcal{N}$. The axioms in theorem 5.5/corollary 5.12 are logically independent:*

- ***E***: *The TU-value φ defined for all $i \in N$ by*

$$\varphi_i(v) = \begin{cases} 0, & \text{if } |N| = 1, \\ Sh_i^p(v), & \text{else,} \end{cases}$$

*satisfies **P/PS** and **WCSE** but not **E**.*

- ***P/PS***: *Sh satisfies **E** and **WCSE** but not **P/PS**.*
- ***WCSE***: *The proportional rule π (eq. (13)) satisfies **E** and **P/PS** but not **WCSE**.*

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