Nash-2 equilibrium: selective farsightedness under uncertain response

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Abstract

This paper provides an extended analysis of an equilibrium concept for non-cooperative games with boundedly rational players: a Nash-2 equilibrium. Players think one step ahead and account all profitable responses of player-specific subsets of opponents because of both the cognitive limitations to predict everyone's reaction and the inability to make more deep and certain prediction even about a narrow sample of agents. They cautiously reject improvements that might lead to poorest profit after some possible reasonable response. For n-person games we introduce a notion of reflection network consisting of direct competitors to express the idea of selective farsightedness. For almost every 2-person game with a complete reflection network, we prove the existence of Nash-2 equilibrium. Nash-2 equilibrium sets in the models of price and quantity competition, and in Tullock's rent-seeking model with 2 players are obtained. It is shown that such a farsighted behavior may provide a strategic support for tacit collusion.

Keywords: Iterated thinking, Improving deviation, Direct competitor, Heterogeneous farsightedness, Tacit collusion

JEL Classification C72, D03, D43, D70, L13

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1 Introduction

In a non-cooperative strategic environment making an individual choice is inevitably based on expectations of what the other players will do. For this reason, there is a great deal of uncertainty in making predictions of what the outcome of the game might be. The Nash equilibrium (NE) concept adopts the idea that each player unilaterally maximizes her own payoff under "expected" competitors' strategies. In a static context, this means that players are sophisticated enough to make correct predictions about the choices made by other participants while these beliefs are consistent in equilibrium.

This paper builds on the following alternative principle: *a player does not have certain beliefs about what her opponent is going to do. However, she supposes that any action by the opponent that raises the opponent's payoff is a priori possible in the absence of information about more specific predictions.* One more limitation concerns non-cooperative games with *many* players: it becomes more problematic for a player to forecast which one will immediately react to her deviation and what these reactions might be. To formalize the idea of predictive heterogeneity among competitors, I assume that every player recognizes some of their opponents are more important than others by being "direct competitors". Hence, to account for the whole set of profitable responses by direct competitors, I define an *improving* deviation for player i with respect to her set of direct competitors as a profitable strategy s_i such that this player is never made worse-off when any of her direct competitors j plays *any* of the profitable strategies against the strategy s_i chosen by player i, including their simultaneous but not coordinated reactions. In this paper, a player chooses an improving strategy because she understands that guessing what the opponents are going to do amounts to adopting a risky behavior.

In a two-person game where both players regard each other as direct competitors, a *Nash-2 equilibrium* is such that no player has a deviation which is profitable and at the same time improving. In other words, both players make an attempt to think ahead but the lack of information on the competitor's response forces them to be cautious. This means that they consciously reject improvements that *might* lead to poorer outcomes at the end of the game.

Not surprisingly, this approach leads to a multiplicity of possible outcomes which depend on everyone's choice of direct competitors. However, for several specific games, these various outcomes have natural and plausible interpretations. Interestingly enough, these outcomes can correspond to collusion or to intermediate solutions between the collusive and competitive outcomes. Predicting the opponent's behavior one step forward, even under a large range of possible responses, may play the role of implicit communication between players and foster tacit cooperation. As a result, players avoid making drastic decisions such as, say, undercutting in Bertrand or Hotelling competition [?], and thus get more beneficial outcomes.

It is worth noting that an alternative approach to the Nash equilibrium has been developed through an iterated strategic thinking process, which has been the starting point of a whole line of research [?]. In particular, multistep player's reasoning about opponents' possible responses have served as a motivation for the development of various concepts of bounded rationality. Related models of players' cognitive hierarchy, k-level of rationality, and smart players, are proposed in [?], [?] and [?]. Several experimental studies support the idea of k-level rationality ([?], [?]). The key point of hierarchical models is that each player assumes that others have a lower level of rationality than themselves. In other words, players of level-0 are strategically naive, while k-level players ($k > 0$) best respond to beliefs on how their opponents are distributed by lower levels of rationality. How players determine their opponents' levels of rationality remains an open question. Furthermore, these levels may evolve in the course of the game [?] and be the result of strategic choice based on comparison of the cost of reasoning with the benefits from an extra level of farsightedness [?].

The lowest, but higher than one-shot Nash, depth of prediction is *two*. In this case, a player takes into account opponents' best responses. Ever since [?], an example of such an approach is provided by the concept of *conjectural variations* (CV) used in oligopoly theory. This approach accounts for some posteriori interactions among firms by including reaction functions into firms' profit functions. Although using CV in static models as a substitute for dynamic modeling is generally plagued with several conceptual problems [?], the CV approach may be a good substitute for repeated oligopolistic games [?], [?]. In addition, matching beliefs on opponent's behavior with their actual best responses solves the problem of CV consistency [?]. The distinctive feature of the CV approach is that, before acting, a firm must guess what their competitors' possible reactions to their own decision will be. This can be viewed as a simple 2-level version of a general iterated strategic thinking process. The former is a firm's desire to maximize their own profits, while the latter is an attempt to predict opponents' responses, which in turn influences the final outcome. The multi-level generalization of the idea of reaction function is the concept of Markov perfect equilibrium [?].

To summarize, several solutions have been proposed to solve the problem of prediction and belief consistency: by ignoring any possible reactions (Nash approach: analysis of responses is transferred to fully dynamic setting), by imposing conditions that actual responses are the expected ones (CV approach), or by constructing models with hierarchy of possibilities to predict behavior of opponents (cognitive hierarchy models). A common feature of these methods is that they all select the most plausible reaction, which leads to relatively small sets of equilibria. These approaches are driven by players' confidence that they know the competitors' level of rationality. Alternatively, the concept of Nash-2 equilibrium allows players to be *unsure how sophisticated their opponents are.* I propose that if the agent doesn't know the opponents' iteration depth then they have no grounds to make an unambiguous prediction of other players behavior. The only reasonable guess is that the opponents will not act to their own detriment. Furthermore, because responses must be profitable, a Nash-2 equilibrium will guarantee each player an outcome that is at least as good as the maximin outcome. What is more, in many games Nash-2 equilibria yield everybody considerably higher payoffs than Nash equilibrium.

The main body of the paper is organized as follows. In Section 2, I begin with the general concept of *reflection network* to define the Nash-2 equilibrium in n-person non-cooperative games. In this network a link from player i to j means that, in the reasoning, player i regards player j as a direct competitor and accounts for the responses of player j. Since I focus on a non-cooperative framework, I assume that if player i has several direct competitors, these players do not coordinate their actions and may deviate simultaneously. Here I consider the setting where the reflection network is given exogenously before the game starts, as in many spatial competition models, and is not the result of strategic choices. I show how the structure of the network operates in the simplest examples of Bertrand competition and the Prisoner's dilemma, and provide Nash-2 solutions in this games for various types of reflection network.

In the remaining part of the paper I focus on 2-person games. In Section 3, I define the Nash-2 equilibrium for complete reflection network and examine the connections between the Nash-2 concept and alternative, but related, solution-concepts, such as sequentially stable set [?] and equilibrium in secure strategies [?].

Section 4 is devoted to the existence of Nash-2 equilibrium. I show that pure Nash-2 equilibria exist in a much wider class of two-person games than Nash equilibria. For any game with the bounded payoff functions, the Nash-2 equilibria can be obtained by a small perturbation of the payoffs. Section 5 discusses the multiplicity of Nash-2 equilibria. The problem of selecting a particular equilibrium profile can be solved in several ways. Each depends on the characteristics of the game including the initial conditions (status quo or a priory expectations). Instead of selecting a single outcome I construct a measure of outcome feasibility on the set of Nash-2 equilibria under the assumption that originally all game profiles are equiprobable. Several examples illustrate the relevance of this approach.

Section 6 provides a detailed analysis of some major economics models: quantity and price competition in oligopoly games and Tullock's rent-seeking model. In the classical Cournot model, the set of Nash-2 equilibria can be divided into two subclasses: the first one involves an extension of Stackelberg leadership equilibria; the other is formed by profiles associated with various degrees of toughness in competition. The results for Bertrand competition with imperfect substitutes and for Tullock contest suggest that the most favorable outcomes involve tacit collusion between players.

I conclude briefly linking the presented approach with a Bayesian view on the problem of decision making under uncertainty. Possible sources of inability to make a specific prediction about opponent's response which must be accounted in modeling long-time interaction are discussed. I also propose possible classes of problems where the concept on Nash-2 equilibrium might operate successfully.

2 Nash-2 equilibrium concept for n-person games: selective prediction of response

2.1 Reflection network, improving deviation, and Nash-2 equilibrium

Consider an n-person non-cooperative game in the normal form

$$G = (i \in I = \{1, \dots, n\}; \quad s_i \in S_i; \quad u_i : S_1 \times \dots \times S_n \to R),$$

where S_i is the set of pure strategies and u_i is the payoff function of player $i = 1, \dots, n$. In this paper I consider pure strategies and do not allow a mixed extension.

This is a one-shot game played in the following manner: players use 2-stage reasoning in analysis of strategy and outcomes and know that other players may also use 2-stage logic. Namely, when a player decides whether to do certain profitable deviation or not she supposes that *some other players* (direct competitors, strict definition is below) will learn (guess) this choice and may try in turn to improve in this new situation. However, any of these direct competitors has her own direct competitors whose future reactions she accounts for. So to predict the decision on the improving action accurately the initial player must reason on 3 steps: the first step is the initial profitability of deviation (the incentive to deviate), the second step is the accounting the possible reactions of direct competitors (caution farsighted element), the third stage is the estimation of plausibility of such reactions (refining), but *this stage can't be realised under the assumption of the same depth of reasoning 2*. Without the cognitive ability to conduct the third step the player must account multiple range of possible competitor reactions, even though this competitor is able to reduce the set of deviations since for them this "third" step is only the second one. The possible reaction of players who are not direct competitors is not accounted for.

The formal construction is the following. Let us define the **reflection network** g on the set of players I by the following rule:

- nodes are players i in I;
- link $g_{ij} = 1$ from player i to j exists if and only if player i takes into account possible profitable reactions (this will be formalized in Definition 1) of player j. Player j is called a *direct competitor* of player i.
- $g_{ij} = 0$, otherwise.

Denote by $N_i(g)$ the set of neighbours j of player i in the graph g, such that $g_{ij} = 1$.

It is worth mentioning that the term *direct and indirect competitor* is borrowed from spatial economics literature [**?**]. In spatial framework, direct competitors are those firms who potentially have some common share of buyers. They strategically compete with each other: a price shift in a firm immediately yields the change of the share of consumers who make the purchase in this store. This is caused by the demand structure with the rule of "mutually exclusive choice". Firms themselves behave in accordance with classical oligopoly theory and maximize under Nash logic. But now I propose to make one step further and suggest that the firms which fight for the same potential market *can be more sophisticated* and cautious with their direct competitors than with other firms in the industry. Leaving aside spatial consideration, this idea forms the basis of the equilibrium concept presented in this paper.

The profitable deviation of player i at profile $s = (s_i,\, s_{-i})$ is such a strategy that strictly increases the profit $u_i(s'_i, s_{-i})$ of player i under fixed strategies s_{-i} of all other players. However, if a player makes an attempt to predict the reaction of her direct competitors, then she need also to check if this deviation remains profitable in case of some of their responses.

Definition 1. A profitable deviation s'_i of player i at profile $s = (s_i,\, s_{-i})$ is *improving* with respect to g if for any subset $J \subseteq N_i(g)$ and any profitable deviation s'_j of every player $j \in J$ at intermediate profile (s'_i, s_{-i}) in case of simultaneous deviations of all players from J player i is not worse off, i.e.

$$u_i(s'_i, s'_J, s_{-iJ}) \geq u_i(s).$$

Note that in this definition we consider reacts with any number of direct competitors, including unilateral deviations. By analogy with most spatial economics papers, it is assumed that all players act independently (non-cooperatively), but they are able to deviate simultaneously, so that player i should take this possibility into consideration. In my opinion, such a selective farsightedness looks more plausible than total ignorance of reaction or perfect prediction of future behavior of *all* other competitors. In order to check whether the deviation is improving or not, the player needs to know only her own direct competitors, not necessarily the whole reflection network. This property can be useful for operating with equilibrium concept.

If $N_i(g) = \emptyset$, player i does not worry about any possible reactions, and so *every* profitable deviation is improving by definition. This situation will be referred to as a *fully myopic behavior*.

Definition 2. A strategy profile is a Nash-2 equilibrium in the game G with reflection network g if no player has an improving deviation at this profile with respect to g.

We will denote the set of Nash-2 equilibria by NE-2.

It is easy to see that every Nash equilibrium profile is also a Nash-2 equilibrium *irrespectively of architecture of the reflection network*. Moreover, in the case of an empty reflection network g they coincide by definition. It is only in this sense we may regard Nash equilibrium as fully myopic concept.

Remark 1. The origin of the reflection network is a special problem, but for simplicity in our setting it is assumed to be an exogenous parameter of the game. In particular, the appropriate network structure may be generated by spatial distribution of agents, or history of the interactions. One can consider the problem of strategic formation of the reflection network as an additional (first) stage in the proper game, but this requires clarifying the information on the rules and the cost of making the direct link.

Remark 2. One more burning question is a *detection* of the existing reflection network. I believe that this could be done by analyzing historical data and finding statistical regularities in the procedure of strategy changes.

It turns out that the set of Nash-2 equilibrium crucially depends on the topology of the reflection network. The more dense reflection network the game has, the richer Nash-2 equilibrium set might be. Avoiding potentially harmful perspective situations can provide necessary support for strategic cooperation: cautious players may appreciate their more high current profits and refuse myopic momentary improvements.

Now let us illustrate the idea how a tacit collusion might be supported by Nash-2 equilibrium with a non-empty reflection network. I will examine two classical models — Bertrand competition and Prisoner's dilemma — with n participants and provide solutions of them for various types of reflection network.

2.2 Bertrand competition with homogeneous product: either total cooperation, or zero profits

Consider the simplest model of price competition of n firms, concentrating at one point and producing homogeneous goods. Assume that they have equal marginal costs c, the demand is linear,

$$\pi_i(p_1, \ldots, p_n) = \begin{cases} (p_i - c)(1 - p_i)/K, & \text{if } p_i = \min\{p_j\}, \\ 0, & \text{if } p_i > p_j \text{ for some } j \neq i, \end{cases}$$

where K is the number of firms setting the minimum price p_i.

Nash solution yields zero-profit equilibrium, at least two firms price marginal costs and equally share the market. Nevertheless, in a case of non-trivial reflection network, equilibrium set occurs considerably wider, and for some structures of reflection network Bertrand paradox is resolved.

If each firm takes into account possible deviations of at least one other firm, or, in graph terminology, if for all nodes i in the network g their out-degree is greater or equal to 1 (see Figure **??**), then *any price level* $p_1 = p_2 = \ldots = p_n \geq c$ is also a Nash-2 equilibrium, together with Nash equilibrium prices.

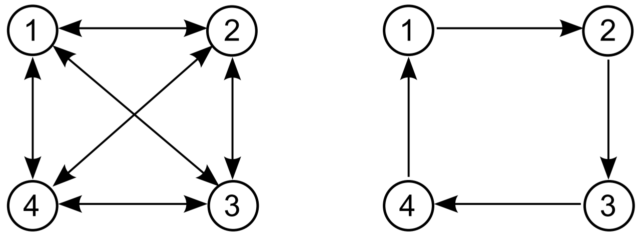


Figure 1: Complete and cycle reflection networks with 4 players, out-degree of every player is not less than 1

Indeed, without loss of generality, assume that firm 1 proposes a price $p_1 > p_i$, where $p_i = \min\{p_j\} > c$, and gets zero profit. Let $N_1(g) = \{k\}$. Then firm 1 can undercut firm i and set the price $p_i - \varepsilon$ with sufficiently small ε. This deviation from the strategy p_1 to $p_i - \varepsilon$ is profitable and now firm 1 gets the whole market. Moreover, it is also improving since the worst that can happen with firm 1 is that the firm k in turn undercuts it and firm 1 comes back to zero profit. So, any situation with $p_1 \neq p_i$ is not a Nash-2 equilibrium.

If *at least one* firm is fully myopic (see Figure **??**), then the unique Nash-2 equilibrium coincides with Nash solution.

This is due to the threshold structure of demand: every single infinitesimal decrease in price with respect to the common price level leads to immediate winning of the whole market. Thus, this model is extremely sensitive to such myopic deviations of any firm, and even one firm acting in a fully myopic way can break insecure tacit cooperation. There is no ability for cooperation among only several firms. However, in oligopoly markets with tough competition, because of homogeneous product, the assumption that each firm has at least one direct competitor appears to be not too incredible.

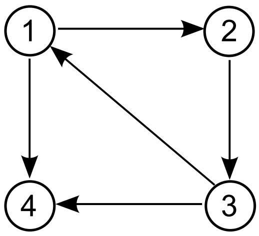


Figure 2: The reflection networks with 4 players, player 4 is fully myopic

I will go back to this example further in the paper, since it demonstrates in a very intuitive way the main features of the Nash-2 equilibrium concept.

2.3 Prisoner's dilemma: self-sufficient cooperation in a small group

Consider the model of n-player prisoner's dilemma as it is introduced in [?]. Each player has two possible strategies: to cooperate with the community or to defect. The utility function is

$$u_i = \begin{cases} bA/n - c, & \text{if player } i \text{ cooperates,} \\ bA/n, & \text{if player } i \text{ defects,} \end{cases}$$

where A is the number of cooperators in the game, each of them brings profit b to the society but pays the cost c. The total profit is equally divided to all n players irrespective of their real contribution. Unilateral defection is preferred for every player $c > \frac{b}{n}$, nevertheless, full cooperation is more preferable for everyone than common defection $b > c > 0$.

According to Nash's approach, cooperation is unlikely to emerge in the n-player prisoner's dilemma, and the same result is predicted by the evolutionary game theory [?].

Surprisingly, in the case of a non-empty reflection network cooperation is possible and the number of cooperators depends both on the architecture of the network and the relation between b and c.

Firstly note that for any player who defects unilateral switching to cooperation is never a profitable deviation. Let us find the conditions under which the reverse deviation is not improving.

Assume that initially A players (*altruists*) cooperate and any cooperator i reflects about n_i *other cooperators*. Their defection which is always profitable fails to be improving with respect to such a type of reflection network if

$$\frac{bA}{n} - c > \frac{b(A-1-n_i)}{n}, \quad \text{and} \quad \frac{bA}{n} - c > 0,$$

that yields

$$n_i > n^* = \frac{cn}{b} - 1, \quad A > \frac{cn}{b}.$$

This means that a player reflecting about a relatively small number of agents never cooperates. In Nash-2 equilibrium, any subset of players *with a sufficient number of links* with other cooperators (more than n^*) is able to cooperate while all others defect if the total number of cooperators is enough to provide positive profits for all of them. In particular, if these profits are close to zero then for cooperation we need a *complete reflection network* among cooperators. However, if cooperative strategy leads to material losses then nothing will force players to cooperate.

So, for supporting cooperative behavior it is important not only to provide a balance between the value of individual return and the cooperation cost, but also to *ensure close contacts between cooperators*. I believe that underlying idea, despite its simplicity, can be operative in explanation of functioning the social norms. Indeed, taking into consideration that your own defection might cause the defection of a great number of other individuals, especially learned in childhood, often prevents a person from antisocial activities.

Remark 3. The examples above demonstrate how significant is to take into account the agent reflection about possible behavior of the opponents. No matter what considerations (spatial or some others) underlie the reflection network, it fundamentally affects possible equilibria.

In the remaining part of the paper I will focus on 2-person games where every player regards her opponent as a direct competitor. I will discuss the connections of Nash-2 equilibrium with some existing concepts, clarify the problems of existence and multiplicity. Finally, I will analyze several models of competition with two agents: Cournot duopoly, general model of Bertrand competition with differentiated product, and Tullock's rent-seeking contest.

3 Nash-2 equilibrium for 2-person games: mutual restraint

3.1 Rethinking the definition in the light of security idea

Let us consider now a 2-person non-cooperative game G with players $i = 1, 2$ and *the complete reflection network* g. Further I will omit mentioning the reflection network. It was originally introduced in [?].

Checking whether the profitable deviation is improving or not, every player now takes into consideration all profitable responses of the opponent. The player refuses profitable deviations if there exists *at least one* profitable response of the opponent that leads to the worse situation than the initial one. This logic might look rather pessimistic, but I suppose that this is the only reasonable prediction that the player is able to do under lack of knowledge about the opponents' level of rationality (depth of thinking). Moreover, I do not add the idea of a possible *punishment* for deviation: nobody will behave in a way which is harmful for themselves. This is what distinguished given approach from the maximin principle.

So, the notion of improving deviation includes some intermediate scenario between the worst response (for the initial deviator) and the best response (for the follower) in the meaning of credible threat in sequential games with backward induction. In other words, the notion of a credible threat is maximally extended to *any possible profitable response*.

Such a reasoning allows to find an interesting connection between a bounded rationality with limited iteration thinking and a *security* in decision making. As an additional motivation for players' behavior, security arose for cooperative games in the concept of the bargaining set [?] based on the notion of threats and counter-threats. Later these ideas were imported into a non-cooperative setting: several papers introduce equilibrium concepts equivalent to Nash-2 equilibrium *for 2-player games* independently of our study: they are threatening-proof profile [?], equilibrium contained by counter-threats [?], and equilibrium in threats and counter-threats [?][1]. In turn, these concepts were proposed as a relaxation of so-called *equilibrium in secure strategies*. In order to explain the former concepts, one need to start with a formal idea of threat and the latter one.

3.2 Secure and risky profiles: equilibrium in secure strategies

Equilibrium in secure strategies [?] implies that a deviating player worries not only about her own first-stage benefit and profitability after opponents' responses, but also about the absence of harmful actions ("threats") of opponents.

[1]These are the same concepts. We will accurately refer to existing results during the further exposition in case of some intersection with ours.

Definition 3. A *threat* of player i to player $-i$ at strategy profile s is a strategy s_i' such that

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \quad \text{and} \quad u_{-i}(s_i', s_{-i}) < u_{-i}(s_i, s_{-i}).$$

The strategy profile s is said to *pose a threat* from player i to player $-i$. A strategy profile s is *secure* for player i if s poses no treats from other players to i.

Authors in [?] use a notion of secure deviation which coincides with improving deviation for 2-person game.

The definition of equilibrium in secure strategies looks as follows [?].

Definition 4. A strategy profile is an *equilibrium in secure strategies* (EinSS) if
i) it is secure,
ii) no player has a profitable and secure deviation.

In [?], [?], [?] authors mentioned the possibility of omitting the condition of security as an extension of equilibrium set and introduced the concept equivalent to Nash-2 equilibrium with several different names. However, the key point of this paper is that Nash-2 equilibrium is explained by the non-trivial farsightedness of agents which have limited information on the opponents' responses and, as a consequence of the logic, includes secure and non-secure situations.

It is worth mentioning that for the larger number of players definitions of secure and improving deviations are not equivalent and these equilibrium concepts do not coincide with the Nash-2 equilibrium. So we discuss here only connections for 2-person games.

The following relation for two-person games takes place [?]: *any Nash equilibrium is an equilibrium in secure strategies, and any equilibrium in secure strategies is a Nash-2 equilibrium*[2]. The converse is generally not true.

So, the set of Nash-2 equilibria can be naturally divided into two sets: secure profiles that form the set of equilibrium in secure strategies and risky outcomes containing threats. This division is driven by nature of profitable deviations which *may* exist at Nash-2 equilibrium (but they are not improving) and be of two types. The first kind is harmful for the opponent, while the second type is not. This can be regarded as various degrees of competition toughness among players.

Secure part corresponds to a tough competition where agents avoid any possible threats, even "non-credible". It often leads to the situations with low profits since players in such situations have nothing to lose. The paper [?] provides secure outcomes for Hotelling linear city model: EinSS is treated as dumping pricing for nearby firms.

The main feature of risky situations (NE-2 \ EinSS) is that agents have opportunities to harm one to another but they do not actualize these threats because of possible credible "sanctions". In a number of situations such a cautious behavior enables agents to hold on higher profits than in case when players also care about security. As it will be shown in Section 6, risky Nash-2 equilibrium might support *tacit collusion* in Cournot, Bertrand and Tullock's competitions.

3.3 Graph model of conflict resolution

In 1980th the idea of accounting ambiguous responses to unilateral improvements has been elaborated in the graph model of conflict resolution theory, which gives a methodology for analyzing real-world conflicts [?]. Authors proposed a new theory for non-cooperative games allowing players to make moves and chains of counter-moves with some limited horizon, and to carry out non-myopic calculations. Their analysis focused on 2×2 games and highlighted the importance of starting point, threat power, and abilities of players to think ahead for prediction of stable outcomes. Discussion in favor of applicability of this theory to modeling real-life situations presented in [?] and [?] is entirely suitable for the theory of Nash-2 equilibrium. In contrast with

[2] Authors formulated this result in terms of threatening-proof profile.

graph model approach accounting only ordinal preferences, Nash-2 equilibrium allows to make a description of stable situations for models with an infinite number of possible game situations.

The nearest to Nash-2 equilibrium concept within graph models of conflict resolution is the *sequential stability* [?], [?]. Let us reformulate the definition of a sequentially stable state here in terms of two-person games introduced in Section 2.1.

Definition 5. For two players $N = \{i, j\}$ and a conflict G, an outcome $s^{SEQ} \in S$ is *sequentially stable* for player i iff for every unilateral profitable deviation of player i to profile s_1 there exists a unilateral profitable deviation of player j from s_1 to s_2 such that $u_i(s^{SEQ}) \geq u_i(s_2)$. A state s^{SEQ} is *sequentially stable* for the conflict iff it is sequentially stable for both players.

Note that the deviation from from s_1 to s_2 is what is called *counter-threat* in [?].

This definition differs from the definition of Nash-2 equilibrium only in the *strictness* of the last inequality. Obviously, in 2-person games *if profile s is a Nash-2 equilibrium, then s is a sequentially stable state.*

Despite the similarity of these two concepts, the difference turns out to be highly significant for specific models. A striking example is the basic Bertrand model of price competition.

Example 1. Consider the model introduced in Section 2.2 with two firms. Firms' profits under prices $(p_1,\ p_2)$ are given by

$$\pi_i(p_1, p_2) = \begin{cases} (p_i - c)(1 - p_i), & if\, p_i < p_{-i}, \\ (p_i - c)(1 - p_i)/2, & if\, p_i = p_{-i}, \\ 0, & if\, p_i > p_{-i}, \end{cases} \qquad i = 1, 2$$

Nash-2 concept states that the equilibrium might be with any price level $p = p_1 = p_2 \in [c, 1]$. In particular, NE-2 includes the collusive (monopoly) price level $p = \frac{1+c}{2}$.

On the contrary, if one accepts the sequential stability as a solution concept, then *any* profile in Bertrand duopoly is sequentially stable. In asymmetric profiles, where firm i sets the price p_i higher than firm $-i$ and gets zero profit, any attempt to increase this profit leads to counter undercutting and a return to the initial profit level. This immediately means that the situation with $p_i > p_{-i}$ is also a sequentially stable.

This example demonstrates the crucial importance of allowing players to deviate from the initial state even if there is *a slight* possibility to come back to the initial profit.

3.4 Explicit and tacit collusion

As it has been already stated in Introduction, Nash-2 equilibrium often gives a suitable explanation for the phenomenon of a tacit collusion between two players. Naturally, the question on the relation of Nash-2 equilibrium and explicit collusion (cooperative behavior) arises. Let us start with simple example.

Example 2. Consider the following game which does not have pure Nash equilibrium.

	L	C	R
T	(1.5,3)	(0,0)	(4.5,0)
B	(2,0)	(0,1)	(2.5,2)

(B,C) is a Nash-2 equilibrium and an equilibrium in secure strategies. (T,L) and (B,R) are Nash-2 equilibria, but not equilibria in secure strategies.

In this example three profiles (T,L),(T,R), and (B,R) can be chosen during cooperation, but only (T,L) and (B,R) are supported by non-cooperative concept of Nash-2 equilibrium. Note that all they are risky outcomes.

Generally, in a two-person game, if an explicitly collusive outcome is a Nash-2 equilibrium then it is typically in NE-2 \ EinSS, or more strictly:

Theorem 1. *If a collusion outcome is not a Nash equilibrium, then it is not a secure profile.*

Proof. Let $(s_1^c, s_2^c) = \arg\max_{s_1,s_2}(u_1(s_1,s_2) + u_2(s_1,s_2))$ be a collusive outcome. Assume that it is secure. It means that it poses no threats from one player to another. The two cases are possible: there no profitable deviations, or for any profitable deviation $s_1^c \to s_i'$ of player i her opponent is not worse off $u_{-i}(s_i', s_{-i}^c) \geq u_{-i}(s_i, s_{-i})$.

In the first case we deal with Nash equilibrium. In the second case $u_i(s_i', s_{-i}^c) + u_{-i}(s_i', s_{-i}^c) > u_i(s_1^c, s_2^c) + u_{-i}(s_1^c, s_2^c)$. This contradicts to (s_1^c, s_2^c) be a collusive outcome. □

4 Existence and multiplicity of Nash-2 equilibrium in 2-person games

4.1 Existence for finite 2-person games: improving path and non-improving cycle

An important advantage of Nash-2 equilibrium concept is that it exists in most games and fails to exist only in "degenerate" cases. This result was firstly mentioned in [?], but without strict grounding. Let us start with finite games and present this idea formally. Recall the notion of improving path as a sequence of alternate improving deviations and non-improving cycle.

Definition 6. A path of profiles $\{(s_i^t, s_{-i}^t)\}_{t=1,\ldots,T}$ is called an *improving path* of length T if each its arc $(s_i^t, s_{-i}^t) \to (s_i^{t+1}, s_{-i}^{t+1}) = (s_i^{t+1}, s_{-i}^t)$ is an improving deviation from s_i^t to s_i^{t+1} for some player i. This path is called a *non-improving cycle* if it is closed: $(s_i^1, s_{-i}^1) = (s_i^T, s_{-i}^T)$, the minimum of such T is called a *length of cycle*.

Using this notion, I state the criterion for an absence of Nash-2 equilibrium in two-person *finite* games.

Proposition 1. *A finite 2-person game in normal form does not have the Nash-2 equilibrium if and only if it contains at least one non-improving cycle of finite length, and there is a finite improving path from any profile to some profile in a non-improving cycle.*

Proof. Assume that no profile is a Nash-2 equilibrium, then from any profile an improving deviation exists at least for one player. Without loss of generality, assume that player 1 deviates at odd steps while player 2 deviates at even ones. For any improving path starting from (s_1^1, s_2^1) the following inequalities hold

$$u_1(s_1^{2t+1}, s_2^{2t+1}) \geq u_1(s_1^{2t+3}, s_2^{2t+3}), \quad t = 0, 1, \ldots,$$

$$u_1(s_1^{2t+1}, s_2^{2t+1}) > u_1(s_1^{2t+2}, s_2^{2t+2}), \quad t = 0, 1, \ldots,$$
$$u_2(s_1^{2t}, s_2^{2t}) \geq u_2(s_1^{2t+2}, s_2^{2t+2}), \quad t = 1, \ldots,$$
$$u_2(s_1^{2t}, s_2^{2t}) > u_2(s_1^{2t+1}, s_2^{2t+1}), \quad t = 1, \ldots.$$

Since the game is finite, at some moment $\Theta < \infty$ (and not exceeding the number of possible game profiles) this path necessarily begins to reach the same situations again and forms a cycle of length $T \leq \Theta$. Moreover, it is necessary and sufficient for this that all non-strict inequalities above *become equalities* for all profiles forming the non-improving cycle. □

It is important to observe that non-improving cycles have a special form: all nodes where player 1 deviates should have *the same* payoff for this player

$$u_1(s_1^{2t+1}, s_2^{2t+1}) = u_1(s_1^{2t+3}, s_2^{2t+3})\ \forall t,$$

the same is true for even nodes and player 2: $(u_2(s_1^{2t}, s_2^{2t}) = u_2(s_1^{2t+2}, s_2^{2t+2}))$.

Corollary 1. *Whenever a game does not have a Nash-2 equilibrium, any perturbation of payoffs that breaks at least one equality for payoffs in non-improving cycle, yields NE-2 existence.*

Every 2-person game with n strategies for player 1 and m strategies for player 2 can be associated with a point in R^{2nm} (ordered payoffs for each pair of strategies are coordinates of this point). So, we can define a measure on the set of games as a measure of corresponding subset in the Euclidean space.

The minimum length of non-improving cycle is four, and at least two equalities on payoffs should take place for a game which does not have a Nash-2 equilibrium. So, the dimension of the subset of all such games does not exceed $2nm - 2$, and this subset has measure 0 in R^{2nm}. So, the following theorem holds.

Theorem 2. *Nash-2 equilibrium exists in almost every 2-person finite game.*

Note that Theorem **??** demonstrates the existence of Nash-2 equilibrium but not the optimal algorithm of finding it in an arbitrary game.

4.2 Existence for 2-person games with infinite number of strategies: costly deviation

The logic underlying discrete games can be easily extended to the case of the infinite number of strategies. Loosely speaking, one need the boundedness of utility function and some conditions ensuring the sequence of utilities in an improving path to grows up to the limit value not too slowly.

One way is to define an ε-equilibrium and to claim the existence of ε-equilibrium for 2-person games with some conditions on the limit of utilities. This is realized in [**?**, Propositions 2 and 8].

I develop other approach for games with infinite strategy sets (continuous or discontinuous 2-person games). The only reason why the logic of Section 4.1 may fail is that if players are permitted to use *hardly different* strategies they may ensure very slow but infinite growth of profits. In order to exclude such a possibility, I consider the games in which a deviation is costly. Assume now that player have to pay some fixed cost $d \geq 0$ for any unilateral changing of chosen strategy, we call d a *cost of deviation.*

Then, definitions of profitable and improving deviations for 2-person games with positive cost of deviation d can be rewritten as following.

Definition 7. A deviation s_i' of player i at profile $s = (s_i, s_{-i})$ is *d-profitable* if $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) + d$.

Definition 8. A d-profitable deviation s_i' of player i at profile $s = (s_i, s_{-i})$ is *improving* if for any d-profitable deviation s_{-i}' of the opponent at intermediate profile (s_i', s_{-i}) player i is not worse off:

$$u_i(s_i', s_{-i}') \geq u_i(s_i, s_{-i}) + d.$$

Note that this definition coincides with definition 1 if $d = 0$. The definition of Nash-2 equilibrium remains the same.

An introducing arbitrary $d > 0$ guarantees that the game does not contains any non-improving cycle. Similarly to Theorem 2, the following theorem holds.

Theorem 3. *Nash-2 equilibrium exists in every 2-person game with strictly positive cost of deviation and utility functions bounded from above.*

Really, starting from arbitrary profile, profits in the improving path should grow at least by d at each step. But if the utility functions are bounded, then this sequence will necessarily stop at Nash-2 equilibrium profile.

It is to be stressed that the continuity of utilities or compactness of action sets are not required. Moreover, for most games Nash-2 equilibrium exists even in case of zero cost of deviations.

Examples in Section 6 explicitly demonstrate this.

5 Selection among multiple equilibria profiles: measure of feasibility

The reverse side of the existence is the multiplicity of predicted outcomes. This problem can be resolved in several ways in dependence of a concrete game framework.

In the case of tough competition between firms, one can choose an equilibrium in secure strategies as the most attractive. For instance, in Hotelling linear city model EinSS concept provides the unique equilibrium corresponding to dumping pricing [**?**].

The alternative approach is to choose the collusive outcome, like in Bertrand or Cournot model, or, at least, a Pareto efficient profile in the set of Nash-2 profiles.

If players join the game *successively*, one after another, then Nash-2 equilibrium is an operative explanation why the Stackelberg leadership outcome remains stable. So, in games with such a prehistory, Stackelberg equilibrium can be selected as a specific Nash-2 profile. An example is the Cournot duopoly.

One more way of solving the selection problem is not to choose the unique outcome, but to introduce *the measure* on the set of Nash-2 equilibria that reflects the probability with which every certain equilibrium can be realized. This can be done in different ways, and I present here one of the simplest.

Suppose that originally players randomly get into any game profile s with equal probabilities $\nu_0(s) = \frac{\mu(s)}{\mu(S_1 \times S_2)}$, where $\mu(A)$ is a measure of the set A. If the profile s is not a Nash-2 equilibrium, then, in general, an improving path from this profile to some Nash-2 profile exists. Denote the subset of NE-2 that can be achieved from profile s by some improving path by NE-2_s. For simplicity I assume that when player learns the whole range of reachable from s Nash-2 profiles, she chooses each of them also with equal probabilities. (Naturally, more complicated method is to assign a probability proportional to the number of improving paths from s to concrete Nash-2 equilibrium.) So, the final probability of each Nash-2 profile to be realised is

$$\nu(s) = \frac{\mu(s)}{\mu(S_1 \times S_2)} + \sum_{\tilde{s}: s \in \text{NE-}2_{\tilde{s}}} \frac{\mu(\tilde{s})}{\mu(\text{NE-}2_{\tilde{s}})\mu(S_1 \times S_2)}, \quad \forall s \in \text{NE-2}.$$

These probabilities form the *measure of feasibility* on the set NE-2. What actually happens is the redistribution of initial probabilities in accordance with improving *dynamic* procedure.

If a Nash-2 profile s is not reachable from any point of $S_1 \times S_2 \setminus \{s\}$, we will call it isolated and $\nu(s) = \nu_0(s)$.

For the sake of visualization in the case of discrete action sets, let us construct a directed graph Γ by the following rule. The nodes of Γ are game profiles. The directed link from node s to node s' exists if there is an improving path from s to s' and there are no improving paths starting at s'.

In this graph the nodes s with zero outdegree $deg^+(s) = 0$ are Nash-2 equilibria. The links demonstrate how not Nash-2 profiles transmit their initial probabilities to Nash-2 profiles by improving paths. Here for all $s \in$ NE-2, the number of profiles from which an improving path to s

exists equals to the indegree $deg^-(s)$ of s in Γ. In particular, if $\forall s \in \Gamma$ not more than one Nash-2 profile can be reached, or $deg^+(s) \leq 1$, then

$$\nu(s) = \frac{1}{|S_1| \cdot |S_2|} \left(1 + deg^-(s)\right), \quad \forall s \in \text{NE-2},$$

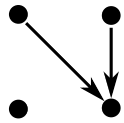
$|A|$ is the number of elements in the set A.

Let us give several examples.

Example 3.

In this example two situations are Nash-2 equilibria, players get zero profits in both. Indeed, strategy profile (B,R) is a NE and Nash-2 equilibrium, and profile (B,L) is a Nash-2 equilibrium, but not a NE.

	L	R
T	1	-1
B	0	0



The graph Γ is shown on the right. As one can see, (B,L) is an isolated Nash-2 profile, thus $\nu(B, L) = 1/4$.

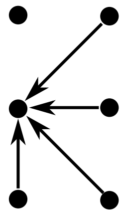
$deg^-(B, R) = 2$. Thereby, $\nu(B, R) = \frac{1}{4}(1 + 2) = 3/4$.

The probability of NE to be realized is considerably greater than for the another profile.

Example 4.

In this example Nash equilibrium fails to exist.

	L	R
T	(2/3, 1/3)	(-1, 2)
C	(1/2, 1/2)	(1, 0)
B	(1, 0)	(0, 1)



NE-2 set consists of two strategy profiles (C,L) and (T,L) with profits (1/2, 1/2) and (2/3, 1/3), respectively. (T,L) is an isolated Nash-2 equilibrium, thus $\nu(T, L) = 1/6$. $deg^-(C, L) = 4$. Thereby, $\nu(C, L) = \frac{1}{6}(1 + 4) = 5/6$.

Hence, though at first sight two Nash-2 profiles are similar, it is much more plausible that (C,L) will occur.

Example 5 (Bertrand model with homogeneous product).

Consider the simplest model of price competition, as in Example 1. In this case there is an improving path from each profile (p_1, p_2), $p_1 \neq p_2$, $p_1, p_2 \in [c, 1]$, to every Nash-2 profile (p, p) with $p \in [c, \min(p_1, p_2)]$. Figure 3 reflects the structure of possible improving paths in this game.

Technically, in the formula for $v(s)$ the sum should be rewritten in terms of integration, buy here we omit standard details. Explicit calculations yield (see Figure 4)

$$\nu(p, p) = \frac{2}{1 - c} \left(\ln \frac{1 - c}{p - c} - \frac{1 - p}{1 - c}\right), \quad \forall p \in [c, 1].$$

One can think about this measure function in the sense that the probability to come into an ϵ-neighbourhood of the prices (p, p) is $\int_{p-\varepsilon}^{p+\varepsilon} \nu(x)dx$. Note that the probability of low prices close to marginal cost is appreciably greater than that for high prices. It is caused by the high attractiveness of undercutting an opponent. However, the collusion price level $(\frac{1+c}{2}, \frac{1+c}{2})$ also has a positive measure of feasibility.

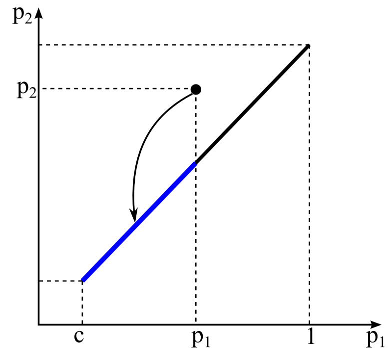


Figure 3: The structure of improving paths in Bertrand model

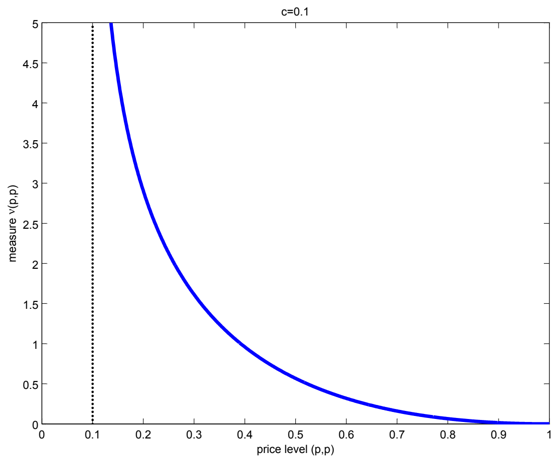


Figure 4: The measure of feasibility on the set NE-2 in Bertrand model with $c = 0.1$

6 Nash-2 equilibrium in duopolies and Tullock rent-seeking contest

Let us turn now to some applications of Nash-2 equilibrium concept to well-known microeconomics models. The section starts with Cournot duopoly with homogeneous product, linear demand, and equal marginal costs, and demonstrate in terms of Nash-2 equilibrium whether the possibility for collusion or more strong competition exists or not. Then the model of price competition between two firms producing imperfect substitutes is examined. Finally, this section contains the computational solution of rent-seeking game (Tullock contest) and outlines the difference between secure-but-strong-competitive and risky-but-collusive outcomes. For Tullock's model I demonstrate that risky Nash-2 profiles are typically more efficient than Nash equilibrium.

6.1 Cournot duopoly

Consider a model of competition between two firms producing q_1 and q_2 units of homogeneous product, respectively, with equal constant marginal costs c per unit. Assume that the equilibrium price $p(Q)$ is a linear decreasing function $p(Q) = 1 - Q$ of total output $Q = q_1 + q_2$. The profit function of the firm $i = 1, 2$ is

$$\pi_i(q_1, q_2) = q_i \cdot (p(Q) - c) = q_i(1 - q_1 - q_2 - c).$$

In Nash equilibrium, firms produce by one third of maximal total output which ensures positive prices on the market

$$q_1^* = q_2^* = \frac{1-c}{3}, \qquad \pi_1^* = \pi_2^* = \left(\frac{1-c}{3}\right)^2.$$

Theorem 4. *Nash-2 equilibria* (q_1, q_2) *are of two kinds:*
a) they belong to the set

$$\left\{\left(b;\ \frac{1-c-b}{2}\right) \cup \left(\frac{1-c-b}{2};\ b\right) \mid \quad b \in \left[\frac{1-c}{3};\ 1-c\right)\right\}.$$

b) they are

$$q_1 = q_2 \in (0,\ (1-c)/3)$$

including collusive outcome $(1-c)/4$, $(1-c)/4$.

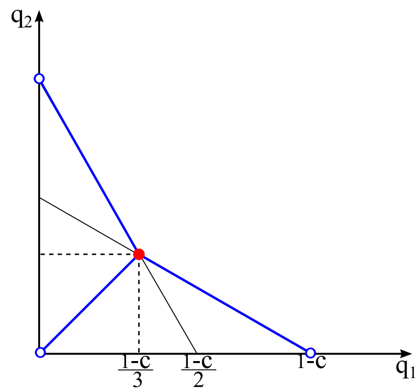


Figure 5: Bold point is NE, NE-2. Bold lines are NE-2.

One can easily check that the equilibrium subset satisfying condition (a) consists of secure profiles. For $b \in \left(\frac{1-c}{3};\ \frac{2(1-c)}{3}\right)$ such situations are fruitful for the firm that overproduces, and maximum is reached at $b = \frac{1-c}{2}$, and for any b they are bad for the another firm (in comparison with Nash equilibrium profits). Special cases of secure Nash-2 equilibria are Stackelberg outcomes $\left(\frac{1-c}{2};\ \frac{1-c}{4}\right)$ if the firm 1 is a leader, and $\left(\frac{1-c}{4};\ \frac{1-c}{2}\right)$ if the firm 2 is a leader. So, EinSS in Cournot model can be treated as an *extension of Stackelberg equilibrium*.

The set NE-2 \ EinSS (condition (b)) includes collusive outcome. Hence, in Cournot duopoly collusion is strategically (tacitly) supported by the concept of Nash-2 equilibrium. The profiles with $q_1 = q_2 \in \left(\frac{1-c}{4},\ \frac{1-c}{3}\right)$ cover all intermediate situations between a classic Nash competition and a cooperative behavior.

Proof. Reaction functions of both firms are

$$r_1(q_2) = (1-c-q_2)/2, \qquad r_2(q_1) = (1-c-q_1)/2.$$

Note that any decreasing of production of any player is profitable for the opponent. This immediately yields that if for one player decreasing her price is profitable, then other player (if she is not at her best response) has an improving deviation. Therefore, such situations are not Nash-2 equilibria.

If one player (for definiteness, firm 1) plays exactly her best response on the firm's 2 output, while the firm 2 produces more than it best response level, then such a situation is a Nash-2 equilibrium. Indeed, firm 1 has not a profitable deviation, and any profitable deviation of firm 2 decreases the output: $q_2 \to q_2 - \varepsilon$, for some $\varepsilon > 0$. If firm 2 deviates, then firm 1 acquires the profitable deviation $q_1 \to q_1 + \varepsilon - \delta$ with enough small $0 < \delta < \varepsilon$, such that the deviation $q_2 - \varepsilon$ is not improving for firm 2.

Now turn out to the case when both firms produce less than their best response level: $q_1 \leq (1-c-q_2)/2$ and $q_2 \leq (1-c-q_1)/2$.

Assume first that $q_1 > q_2$ (the symmetric case is similar). Then firm 2 has the improving deviation deviation from q_2 to $1 - c - q_1 - q_2 - \varepsilon$ with $0 < \varepsilon < q_1 - q_2$. After this q_1 becomes greater than new best response level and any profitable deviation of firm 1 decreases q_1 which guarantees that the initial deviation of firm 2 is improving.

The last possible situation is $q_1 = q_2 = q$. Let us show that the profile (q, q), $q \le (1-c)/3$, is a Nash-2 equilibrium. Carry out the reasoning for firm 1. Any profitable deviation of firm 1 has a form $q_1 \to q + \varepsilon$ with $0 < \varepsilon < 1 - c - 3q$. After this firm 2 has the profitable deviation from q_2 to $1 - c - 2q - \varepsilon - \delta$ which leads to breaking the improving inequality for firm 1, if $0 < \delta < \frac{q}{q+\varepsilon}(1 - c - 3q - \varepsilon)$. □

Nash-2 equilibrium provides a number of regimes with various degree of toughness from competitive till collusive. An explanation which outcome will actually be observed can be given on the base of the oligopolistic equilibrium [**?**], which suitably generalizes conjectural variation approach by introducing an extra coefficient of competitive toughness.

6.2 Bertrand competition with differentiated product

Consider more general model of price competition between two firms producing *imperfect substitutes* with marginal costs equal c_1 and c_2, respectively. The coefficient of substitution is $\gamma \in [0, \infty)$. Firms' demands are given by

$$q_1 = 1 - p_1 - \gamma(p_1 - p_2), \quad q_2 = 1 - p_2 - \gamma(p_2 - p_1).$$

The firms' profits are

$$\pi_1(p_1, p_2) = (p_1 - c_1)(1 - p_1 - \gamma(p_1 - p_2)).$$
$$\pi_2(p_1, p_2) = (p_2 - c_2)(1 - p_2 - \gamma(p_2 - p_1)).$$

The case of $\gamma = 0$ corresponds to the monopoly. When $\gamma \to \infty$ the product becomes more and more homogeneous.

In Nash equilibrium, prices are equal to

$$p_1^* = \frac{2 + 3\gamma + 2(1+\gamma)^2 c_1 + \gamma(1+\gamma)c_2}{(2+3\gamma)(2+\gamma)}$$

$$p_2^* = \frac{2 + 3\gamma + 2(1+\gamma)^2 c_2 + \gamma(1+\gamma)c_1}{(2+3\gamma)(2+\gamma)},$$

if $p_1^* \ge c_1$, $p_2^* \ge c_2$.

If marginal costs are equal $c_1 = c_2 = c$, then $p_1^* = p_2^* = \frac{1+(1+\gamma)c}{2+\gamma} > c$. As $\gamma \to \infty$, one faces the classical Bertrand paradox.

Let us describe the conditions which the set of Nash-2 profiles (p_1, p_2) meets.

Note firstly that two following conditions mean that the markup and the demand in an equilibrium should be non-negative

$$p_1 \ge c_1, \quad p_2 \ge c_2, \qquad a)$$

$$q_1(p_1, p_2) \ge 0, \quad q_2(p_1, p_2) \ge 0. \qquad b)$$

The next condition states that only prices exceeding the best response level can be a Nash-2 equilibrium:

$$p_1 \ge \frac{1 + \gamma p_2 + c_1(1+\gamma)}{2(1+\gamma)}, \quad p_2 \ge \frac{1 + \gamma p_1 + c_2(1+\gamma)}{2(1+\gamma)}. \qquad c)$$

One more claim is that in a Nash-2 equilibrium firms get not less than their maximin benefits

$$\pi_1(p_1, p_2) \ge \frac{(1 - c_1(1+\gamma))^2}{4(1+\gamma)}, \quad \pi_2(p_1, p_2) \ge \frac{(1 - c_2(1+\gamma))^2}{4(1+\gamma)}. \qquad d)$$

The last conditions directly state the absence of improving deviations

$$\begin{array}{l} \left(\frac{1-c_1}{2} - \frac{\gamma(1+\gamma)(p_2-c_2)}{2(1+2\gamma)}\right)\left(\frac{1+2\gamma+\gamma^2 c_2-(1+\gamma)^2 c_1}{2(1+\gamma)} + \frac{3}{2}(p_2-c_2)\right) \leq \pi_1(p_1,p_2), \\ \left(\frac{1-c_2}{2} - \frac{\gamma(1+\gamma)(p_1-c_1)}{2(1+2\gamma)}\right)\left(\frac{1+2\gamma+\gamma^2 c_1-(1+\gamma)^2 c_2}{2(1+\gamma)} + \frac{3}{2}(p_1-c_1)\right) \leq \pi_2(p_1,p_2). \end{array} \qquad e)$$

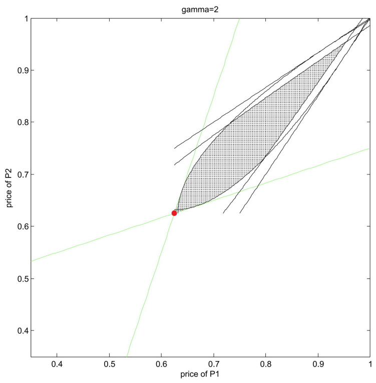


Figure 6: $c_1 = c_2 = 0.5$, $\gamma = 2$. Bold point is NE, EinSS, NE-2. Shaded area is NE-2.

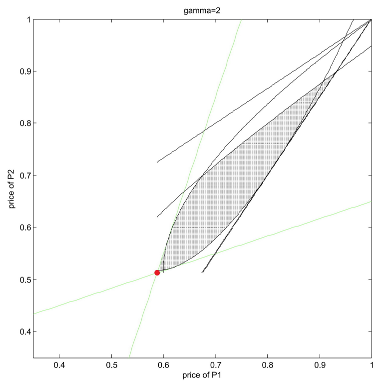


Figure 7: $c_1 = 0.5$, $c_2 = 0.3$, $\gamma = 2$. Bold point is NE, EinSS, NE-2. Shaded area is NE-2.

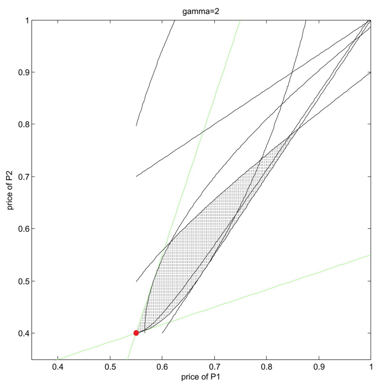


Figure 8: $c_1 = 0.5$, $c_2 = 0.1$, $\gamma = 2$. Bold point is NE, EinSS, NE-2. Shaded area is NE-2.

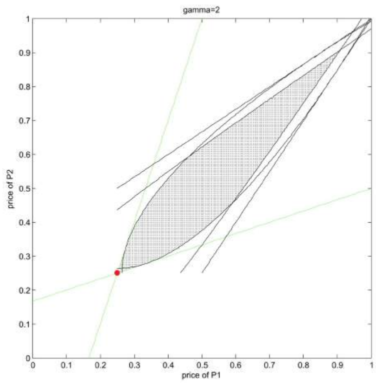


Figure 9: $c_1 = c_2 = 0$, $\gamma = 2$. Bold point is NE, EinSS, NE-2. Shaded area is NE-2.

In the proof of these sufficient conditions, let us start with the observation that, in contrast to Cournot duopoly, any increase in price of any firm is profitable for the opponent. From this fact it follows that if one firm assign a price less than best response level then it or its opponent have an improving deviation. It provides condition (c).

Condition (d) immediately follows from the fact that if a firm gets less than maximin value, then it has an improving deviation to the strategy which ensures it.

Now look at the residual area and establish which situations are Nash-2 equilibria. In this area firms propose prices more than at the best response level (condition (c)). Let us look on the situation by firm 1. Any profitable deviation of firm 1 decreases the price p_1 up to some $\tilde{p}_1^\varepsilon = p_1 - \varepsilon$ with $\varepsilon \in \left(0;\, 2\left(p_1 - \frac{1+\gamma p_2 + c_1(1+\gamma)}{2(1+\gamma)}\right)\right)$. The most harmful response of firm 2 is maximal decreasing the price: $p_2 \to \tilde{p}_2^\varepsilon = 2 \cdot \frac{1+\gamma(p_1-\varepsilon)+c_2(1+\gamma)}{2(1+\gamma)} - p_2 + \delta$ with $\delta = +0$.

If (p_1, p_2) is a Nash-2 profile, then for any ε firm 1 should get worse: $\pi_1(\tilde{p}_1^\varepsilon, \tilde{p}_2^\varepsilon) < \pi_1(p_1,p_2)$, or, equivalently, $\max_\varepsilon \pi_1(\tilde{p}_1^\varepsilon, \tilde{p}_2^\varepsilon) < \pi_1(p_1,p_2)$.

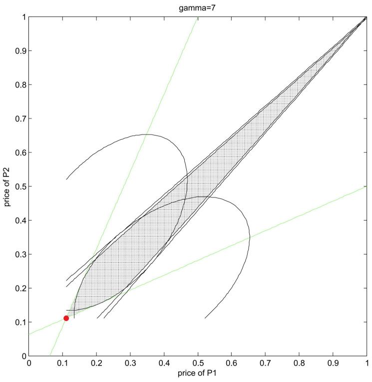


Figure 10: $c_1 = c_2 = 0$, $\gamma = 7$. Bold point is NE, EinSS, NE-2. Shaded area is NE-2.

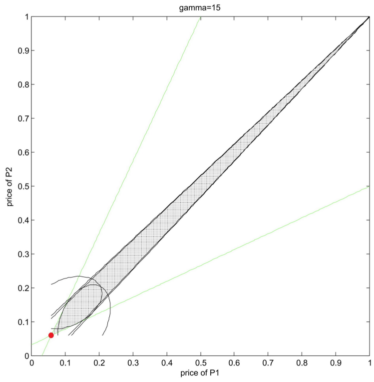


Figure 11: $c_1 = c_2 = 0$, $\gamma = 15$. Bold point is NE, EinSS, NE-2. Shaded area is NE-2.

Explicit calculation of this maximum provides condition (e).

As one can observe, the set of Nash-2 equilibria becomes more asymmetric as the difference between marginal costs increases (see Figures 6 – 8). On the other hand, it becomes narrower and elongate as $\gamma \to \infty$ (see Figures 9 – 11).

Note that in the case of $c_1 = c_2 = c$, the collusion profile $p_1 = p_2 = (1 + c)/2$ is in the Nash-2 set. Nevertheless, some other outcomes on the Pareto frontier of the set of Nash-2 equilibrium profits exist.

6.3 Tullock contest

In a rent-seeking modeling, most papers focus on the manipulation efforts of firms to gain monopolistic advantage on the market. Tullock contest [**?**] is a widespread way to examine the processes of political lobbying for government benefits or subsidies, or to impose regulations on competitors in order to increase a market share.

In the classical Tullock contest setting, the efforts $x = (x_1, x_2)$ of the participants affect the probability of winning the resource R in accordance with the following success function $p_i(x_i, x_{-i})$

$$p_i(x_i, x_{-i}) = \frac{x_i^\alpha}{x_i^\alpha + x_{-i}^\alpha}, \quad x \neq 0, i = 1, 2.$$

If $x = (0,\, 0)$ then $p_i = p_{-i} = 1/2$.

The payoff function of each player is

$$u_i(x_i, x_{-i}) = Rp_i(x_i, x_{-i}) - x_i.$$

Without loss of generality, assume $R = 1$, $x_i \in [0,\, 1]$.

Players' behavior essentially depends on the value α. It can be treated as a sensitivity of the utility function to increasing the effort.

When $\alpha \leq 2$, Nash equilibrium exists and equilibrium efforts are equal to $\alpha/4$. In [**?**] the equilibrium in secure strategies in Tullock model was obtained *for all* α, and it was shown that for $\alpha > 1$ it is not unique.

Here I present the computer solution (using simple MatLab computations based on the definition of improving deviation) for the whole set of Nash-2 equilibria (see Fig. 12 – 14).

Note that all equilibria in secure strategies, and in particular Nash equilibrium, are Pareto dominated by *some* risky Nash-2 profiles (see Fig. 15)

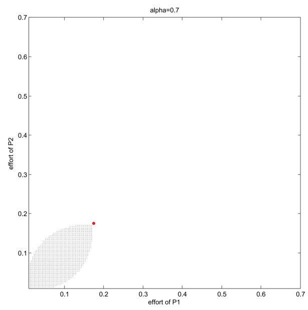


Figure 12: $\alpha = 0.7$. Bold point is NE, EinSS, NE-2. Shaded area is NE-2.

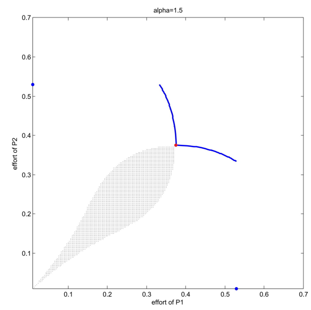


Figure 13: $\alpha = 1.5$. Bold central point is NE, EinSS, NE-2. Bold curve and points on the axes are EinSS, NE-2. Shaded area is NE-2.

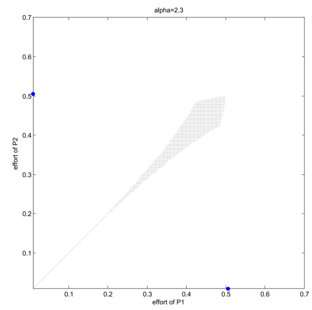


Figure 14: $\alpha = 2.3$. Bold points are EinSS, NE-2. Shaded area is NE-2.

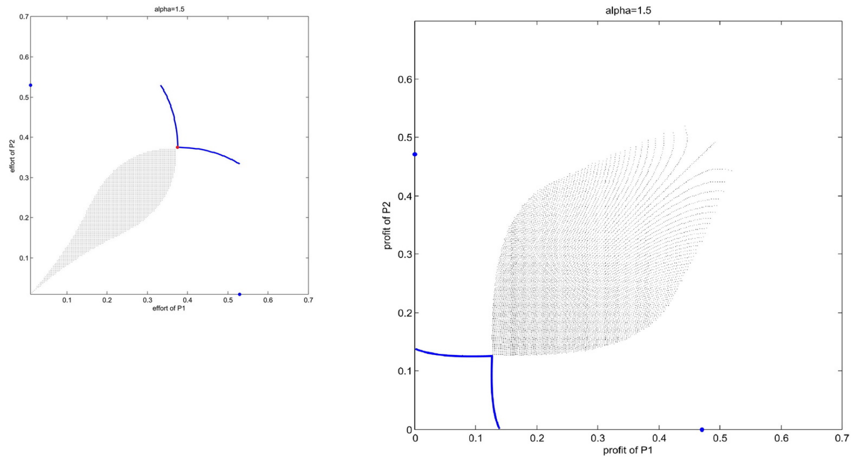


Figure 15: $\alpha = 1.5$. Efforts and profits. Bold curves and points on the axes are profits at equilibrium in secure strategies (and Nash-2 equilibrium), shaded area is the set of profits at risky Nash-2 profiles.

The set NE-2$\setminus$EinSS seems to be intuitively clear: farsighted players in some sense are engaged in a tacit collaboration and make smaller efforts to reach the same probability of obtaining the resource. This is what I mean by tacit collusion.

Remark 4 (on the efficiency of equilibrium obtained). The *rent dissipation* is often used as a measure of solution efficiency in rent-seeking games. If $R = 1$, it is equal to the *sum of agent efforts* in equilibrium. If $\alpha \leq 2$, the rent dissipation in Nash equilibrium is equal to $\alpha/2$. The paper [**?**] shows that when $\alpha > 2$ and the strategy space is continuous, full rent dissipation occurs in a symmetric mixed-strategy equilibrium. It follows from our simulations that there is a wide range of risky Nash-2 equilibria that are *more efficient* than Nash pure or mixed strategy equilibrium for any α. To be exact, for $\alpha \leq 2$ any risky Nash-2 equilibrium together with *some* secure Nash-2 equilibrium (for accurate characterization of secure part see [**?**]) are more efficient. For $\alpha > 2$ some part of risky Nash-2 equilibria (for which $x_1 + x_2 < 1$) and only "monopolistic" (when only one player makes positive efforts) secure Nash-2 equilibria are more efficient. However, it is to be noted that sometimes zero efforts for one participant of the contest may be not allowed by rules of the contest (for instance, if in this case the contest may be declared invalid). Then only risky Nash-2 profiles ensure smaller rent dissipation than mixed-strategy Nash equilibrium.

7 Conclusion

This paper is closely related to the analysis of rational expectations and equilibrium under *uncertainty*. A Bayesian approach requires a definite probabilities of possible game scenarios which may be updated during the game is played. However, process of assignment of initial expectations about agent rationality lies beyond most papers using Bayesian technique, may use some invalid data or non-strict procedures, and so might lead to disagreement between theoretical predictions and empirical evidence. Here I presented an alternative way to handle games with lack of information about possible responses: to account all reactions which agree the main principle of utility increasing without accurate estimation of probabilities for all possible scenarios.

Moreover, my belief is that some aspects of rationality exceed standard utility function expression of a game, namely, a "quasi-social" structure of links among non-anonymous competitors. This factor together with initial point of consideration and natural limitations of iteration thinking depth leads to the equilibrium concept providing multiple predictions. This multiplicity seems to be a natural expression of a great variety of real-life agents' behavior.

One more source of uncertainty is a problem of appropriate timing because of a lack of information on the duration of interaction and intermediate moments of opponent's update of strategies. Using n-stage games is sometimes a rather restrictive way to treat such situations. In order to incorporate the idea of *potentially* long-time interaction, the two-stage simplification is developed, by analogy with conjectural variation approach.

Nash-2 equilibrium can be useful for analyzing oligopolies with a few agents familiar with each other. Moreover, in oligopoly models with large number of agents it also could provide interesting results by examining heterogeneous reflection networks. For example, this concept allows to explain higher price dispersion in online sales, than models ignoring this structure.

Certainly, a lot of related issues must be clarified and checked in future research. The most interesting part concerns games with many players and detecting some patterns of reflection networks in real-world data.

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