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Abstract

The method of Lagrange multipliers is a very useful and powerful technique in multivariable calculus. In this paper interpretation of Lagrange multipliers is given by showing their positive values. Three models on optimization are given with detailed mathematical calculations. The Implicit Function Theorem is important for solving a system of non-linear equations for the dependent variables and calculating partial derivatives of these variables with respect to the independent variables. In this paper an attempt has been made to optimize economic models subject to a budget constraint, using Lagrange multipliers technique, as well as, using necessary and sufficient conditions for optimal value.

Keywords: Lagrange multipliers, optimization, comparative static analysis, necessary and sufficient conditions

1. INTRODUCTION

The method of Lagrange multipliers is a very useful and powerful technique in multivariable calculus and has been used to facilitate the determination of necessary conditions; normally, this method was considered as device for transferring a constrained problem to a higher dimensional unconstrained problem (Moolio et al. 2009, Islam et al. 2010, 2011). Baxley and Moorhouse (1984) considered implicit functions with assumed characteristic qualitative features and provided illustration of an example, generating meaningful economic behavior. This approach and formulation may enable one to view optimization problems in economics from a somewhat wider perspective.

We examine a set of related examples to highlight the following features (Baxley and Moorhouse 1984, Mohajan 2012):

- To begin with, functions are not explicitly given but they have some assumed characteristic features, which are meaningful for and give insight into economic behavior. Later, explicit functions are considered to clarify the characteristics.

- Assuming, for example, that a firm wishes to minimize the cost of producing a given output, one may want to know how changes in the input prices will affect the situation. So the problem is not: "find the minimum", but, "assuming the minimum is obtained, what consequences can be deduced."
- The Lagrange multipliers λ or λ_i, $i=(1,...,m)$ for some $m>1$, as indicated, have usually been used as a device. In economic problems, as we shall see, the Lagrange multipliers can be interpreted as rates of change of optimal values relative to some parameters.
- In these considerations and discussions, the Implicit Function Theorem is important for solving a system of non-linear equations for the endogenous (dependent) variables and calculating partial derivatives of these variables with respect to the exogenous (independent) variables.

In section 2 we illustrate three examples on optimization, namely Model (A), Model (B) and Model (C) following Baxley and Moorhouse (1984). Section 3 is developed by mathematical techniques to explain the models and necessary conditions for optimal values. Sufficient conditions for implicit functions are given in section 4.

2. THREE EXAMPLES ON OPTIMIZATION

Assume that an individual consumes two commodities *x* and *y*; the amounts he purchases in the market place are *X* and *Y* kg respectively. He keeps a certain quantity *L* of his leisure time *l* to himself, when he is not earning. We observe that the larger the value of *L*, implicitly, the less his money income, and vice-versa (Baxley and Moorhouse 1984). Let P_1 and P_2 be the prices of per unit of *x* and *y* respectively, let *T* be the total time period available, so that *L* is the leisure time per period with $0 \le L \le T$. The time during which the individual works, i.e., earns, is therefore $(T-L)$ per period. Let his wage per unit time be *w*, so that his total income is $(T-L)w$. Since he spends all his income for purchasing the two commodities, the budget constraint is as follows:

$$(T-L)w = XP_1 + YP_2 . \tag{1}$$

The utility *U* of the individual is given by a utility function *u* unique to him, as a function of *X*, *Y* and *L*;

$$U = u(X,Y,L). \tag{2}$$

We now impose certain general and reasonable conditions on the function *u* as follows (where a subscript denotes partial derivative with respect to the subscript):

$$u_X > 0, \qquad u_Y > 0, \qquad u_L > 0, \tag{3a}$$

$$u_{XX} < 0, \qquad u_{YY} < 0, \qquad u_{LL} < 0, \tag{3b}$$

$$u_{XL} > 0, \qquad u_{YL} > 0, \tag{3c}$$

$$\text{either} \quad u_{XY} > 0 \quad \text{or} \quad u_{XY} < 0 \quad \text{or} \quad u_{XY} = 0 . \tag{3d}$$

The inequalities in (3a) are the so called marginal utilities, which indicate that higher levels of consumption of the commodities and more leisure time per period increase utility. The conditions in (3b) of course reflect the "law of diminishing marginal utility". The inequalities (3c) display that the satisfaction of consuming *x* or

y is enhanced by having more leisure time. The three conditions (3d) represent the circumstance, respectively, that x and y are: (i) substitutes, (ii) complements, or (iii) unrelated. Baxley and Moorhouse (1984) have given by assuming and easily understandable instances of the inequalities involving second order derivatives. For (3b): "the third coke one drinks within an hour does not quench one's thirst as much as the second coke"; for (3c): "it takes time to *enjoy things*"; for (3d): (i) substitutes "tea and coffee", (ii) compliments such as "bun and burger", and finally, (iii) "mathematics lessons and jellybeans", are unrelated.

We now formulate the maximization problem for the utility function u given by (2) in terms of a single Lagrange multiplier λ, by defining the Lagrange function as follows (Mohajan et al. 2013):

$$v\left(X,Y,L,\lambda \right)=u\left(X,Y,L \right)+\lambda\left(\left(T-L\right)w-XP_1-YP_2\right). \tag{4}$$

Maximization of utility occurs for values X^*,Y^*,L^*,λ^* of X, Y, L, λ that must satisfy the following equations:

$$v_\lambda=\left(T-L\right)w-XP_1-YP_2=0, \tag{5a}$$

$$v_X=u_X-\lambda P_1=0, \tag{5b}$$

$$v_Y=u_Y-\lambda P_2=0, \tag{5c}$$

$$v_L=u_L-\lambda w=0. \tag{5d}$$

Here X, Y, L are chosen by the individuals whereas P_1, P_2, w are determined by market conditions, changing from time to time, to values that are beyond the individual's influence or control. Hence X, Y, L will referred to as endogenous variables, and P_1, P_2, w as exogenous variables. If P_1 were to increase; with P_2, w remaining fixed, one might expect the consumer to decrease X and increase Y or decrease L, so that the additional income or saving finances the acquisition of the more expensive x. Mathematically we can write twelve partial derivatives as follows:

$$\begin{bmatrix} X_{P_1} & Y_{P_1} & L_{P_1} & \lambda_{P_1} \\ X_{P_2} & Y_{P_2} & L_{P_2} & \lambda_{P_2} \\ X_w & Y_w & L_w & \lambda_w \end{bmatrix}. \tag{6}$$

These twelve partial derivatives are called the *comparative statics* of the problem (Chiang 1984). Now we introduce three explicit models (A), (B), and (C) as follows (Mohajan 2012):

2.1 Model (A): $u_{XY}>0$

Consider the function u is given by;

$$u\left(X,Y,L \right)=u_0X^aY^bL^c, \tag{7}$$

where u_0, a, b, c are constants. Taking partial derivatives, we get,

$$u_X=u_0aX^{a-1}Y^bL^c,\ \ u_Y=u_0bX^aY^{b-1}L^c,\ \ u_L=u_0cX^aY^bL^{c-1} \tag{8a}$$

$$u_{XX}=u_0a\left(a-1\right)X^{a-2}Y^bL^c,\ u_{YY}=u_0b\left(b-1\right)X^aY^{b-2}L^c,\ u_{LL}=u_0c\left(c-1\right)X^aY^bL^{c-2} \tag{8b}$$

$$u_{XL} = u_0 acX^{a-1}Y^b L^{c-1}, \quad u_{YL} = u_0 bcX^a Y^{b-1} L^{c-1}, \quad u_{XY} = u_0 abX^{a-1}Y^{b-1}L^c. \tag{8c}$$

If we now assume the constants *a*, *b* and *c* to satisfy the following inequalities:

$$0 < a < 1, \ \ 0 < b < 1, \ 0 < c < 1 \tag{9}$$

and assume *X*, *Y*, *L* to be positive, as is required by the nature of the problem, we readily see that the conditions (3a,b,c) are satisfied, and also the first condition in (3d).

2.2 Model (B): $u_{XY} < 0$

Consider the function *u* is given by,

$$u(X,Y,L) = u_0\left(A\left(1 - e^{-aX-bY}\right) + CXYe^{-f(L)}\right), \tag{10}$$

where u_0, *a*, *b*, *A*, *C* are positive constants, and $f(L)$ is a function of *L* and is given by;

$$f(L) = c(L_0 + L)^{-1}, \tag{10a}$$

with c, L_0 positive constants which are distinct to those of Model (A). Taking partial derivatives of (10) we get;

$$u_X = u_0\left(Aae^{-aX-bY} + CYe^{-f(L)}\right), \tag{11a}$$
$$u_Y = u_0\left(Abe^{-aX-bY} + CXe^{-f(L)}\right), \tag{11b}$$
$$u_L = u_0 Cc(L_0 + L)^{-2} XYe^{-f(L)}, \tag{11c}$$
$$\text{with } f'(L) = df/dL = -c(L_0 + L)^{-2}. \tag{11d}$$

Taking the second partial derivatives of (11) we get,

$$u_{XX} = -u_0 a^2 Ae^{-aX-bY}, \ u_{YY} = -u_0 b^2 Ae^{-aX-bY}, u_{LL} = -u_0 CcXY\left(f''(L) - \left(f'(L)\right)^2\right)e^{-f(L)} \tag{12a}$$
$$u_{XL} = -u_0 Cf'(L)Ye^{-f(L)}, u_{YL} = -u_0 Cf'(L)Xe^{-f(L)}, \ u_{XY} = -u_0\left(-abAe^{-aX-bY} + Ce^{-f(L)}\right) \tag{12b}$$

with $f''(L) = 2c\,(L_0 + L)^{-3}$, so that;

$$\left(f''(L) - \left(f'(L)\right)^2\right) = 2c\,(L_0 + L)^{-3} - c^2(L_0 + L)^{-4}. \tag{13}$$

Since u_{XX} and u_{YY} given in (12a), are clearly negative, as required by (3b). Again u_{LL}, given in (12a), to be negative, the quantity on the right hand side of (13) must be positive, so that;

$$2(L_0 + L) > c,$$

which is satisfied by all positive *L* if we choose L_0, c so that $2L_0 > c$; this we assume to be the case. Hence all the conditions of (3a,b) are satisfied, as can be verified from (11a–d) and (12a). Consider now u_{XY} is given by the last relation in (12b). We shall see that the constants or parameters *A*, *C*, *a*, *b* and *c* can be chosen so that $u_{XY} < 0$ is satisfied.

2.3 Model (C): $u_{XY}=0$

It is similar to Model (A), but u consists of two parts as follows:

$$u(X,Y,L)=u_1X^aL^c+u_2Y^bL^c, \tag{14}$$

where $u_1,u_2;a,b,c$ are new constants. Taking partial derivatives of (14) we get;

$$u_X=u_1aX^{a-1}L^c,\ \ u_Y=u_2bY^{b-1}L^c,\ \ u_L=c\left(u_1X^a+u_2Y^b\right)L^{c-1}, \tag{15a}$$

$$u_{XX}=u_1a(a-1)X^{a-2}L^c,\ u_{YY}=u_2b(b-1)Y^{b-2}L^c,\ \ u_{LL}=c(c-1)\left(u_1X^a+u_2Y^b\right)L^{c-2}, \tag{15b}$$

$$u_{XL}=u_1acX^{a-1}L^{c-1},\ \ u_{YL}=u_2bcY^{b-1}L^{c-1},\ u_{XY}=0. \tag{15c}$$

Hence, (3a,b,c) and the last relation in (3d) are satisfied, if we choose u_1, u_2 to be positive and a, b, c to satisfy (9) of Model (A).

3. Mathematical Discussions of the Models

Consider the four equations (5a–d) in seven variables X, Y, L, λ,P_1,P_2,w. We solve for X, Y, L,λ in terms of P_1,P_2,w and denote the solution as follows (Moolio et al. 2009, Islam et al. 2010):

$$X^*(P_1,P_2,w),\ \ Y^*(P_1,P_2,w),\ \ L^*(P_1,P_2,w),\ \ \lambda^*(P_1,P_2,w), \tag{16}$$

and set,

$$U=u\left(X^*,Y^*,L^*\right)=\tilde{u}\left(P_1,P_2,w\right). \tag{17}$$

If the left hand sides of (5a–d) are assumed to be continuously differentiable, then by the implicit function (will be discussed later) X^*,Y^*,L^*,λ^* will all continuously differentiable functions of P_1,P_2,w provided the following Jacobian matrix is non-singular at $\left(X^*,Y^*,L^*\right)$:

$$H=\begin{bmatrix} o & -P_1 & -P_2 & -w \\ -P_1 & u_{XX} & u_{XY} & u_{XL} \\ -P_2 & u_{YX} & u_{YY} & u_{YL} \\ -w & u_{LX} & u_{LY} & u_{LL} \end{bmatrix}. \tag{18}$$

Omitting 'star' from (17) and (18), and using chain rule we get,

$$\frac{\partial U^*}{\partial w}=u_X\frac{\partial X}{\partial w}+u_Y\frac{\partial Y}{\partial w}+u_L\frac{\partial L}{\partial w}=\lambda\left(P_1\frac{\partial X}{\partial w}+P_2\frac{\partial Y}{\partial w}+w\frac{\partial L}{\partial w}\right). \tag{19}$$

From (5a) we get,

$$wT=XP_1+YP_2+wL, \tag{20}$$

so that taking partial derivative we get,

$$T=P_1\frac{\partial X}{\partial w}+P_2\frac{\partial Y}{\partial w}+w\frac{\partial L}{\partial w}+L. \tag{21}$$

Using (21) in (19) we get,

$$\lambda = \frac{1}{(T-L)}\frac{\partial U^*}{\partial w}. \tag{22}$$

We have mentioned that $(T-L)$ is the period of work, so that λ could be constructed as the marginal utility of *w*, the wage rate, per unit time. Let, $w(T-L)$, the money earned by the individual, be denoted by *B*;

$$B = w(T-L). \tag{23}$$

Since *L* is a function of P_1, P_2, w; so that (23) can be written as,

$$B = f\left(P_1, P_2, w\right) \equiv Tw - L\left(P_1, P_2, w\right)w. \tag{24}$$

Solution of (24) is as follows:

$$w = g\left(P_1, P_2, B\right), \text{ say,} \tag{25}$$

and we express U^* in terms of P_1, P_2, B;

$$U^* = \tilde{u}\left(P_1, P_2, B\right) = \tilde{u}\left(P_1, P_2, g\left(P_1, P_2, B\right)\right) = \tilde{\tilde{u}}\left(P_1, P_2, B\right). \tag{26}$$

Taking partial derivative with respect to *B* we obtain;

$$\frac{\partial U^*}{\partial B} = \frac{\partial}{\partial B}\tilde{\tilde{u}}\left(P_1, P_2, B\right) = \frac{\partial U^*}{\partial w}\frac{\partial w}{\partial B} = \frac{\partial}{\partial B}\tilde{u}\left(P_1, P_2, B\right)\frac{\partial w}{\partial B}. \tag{27}$$

To find a convenient expression for $\frac{\partial w}{\partial B}$, we consider for a moment P_1, P_2, w, B as independent variables and define a function of these four variables as follows:

$$h\left(P_1, P_2, w, B\right) = f\left(P_1, P_2, w\right) - B, \tag{28}$$

$$dh = \frac{\partial h}{\partial P_1}dP_1 + \frac{\partial h}{\partial P_2}dP_2 + \frac{\partial h}{\partial w}dw + \frac{\partial h}{\partial B}dB. \tag{29}$$

We set $dP_1 = 0 = dP_2$, $dh = 0$ in (29) that is, we hold P_1, P_2 constant and confine P_1, P_2, w, B to the 'surface' (24), i.e., consider *B* to be given by (24), (or (25)). Now (29) becomes;

$$\frac{dw}{dB} = -\frac{\partial h/\partial B}{\partial h/\partial w} = -\frac{h_B}{h_w}. \tag{30}$$

Since P_1, P_2 are constants we can write, $dw/dB = \partial w/\partial B$, also from (28), $h_w = f_w$, $h_B = -1$, so that ,

$$\frac{\partial w}{\partial B} = -\frac{h_B\left(P_1, P_2, w, B\right)}{h_w\left(P_1, P_2, w, B\right)} = \frac{1}{f_w\left(P_1, P_2, w\right)}, \tag{31}$$

where $f_w\left(P_1, P_2, w\right) \neq 0$. From (23) and (24) we get,

$$f\left(P_1, P_2, w\right) = w\left(T - L\right),$$

and taking the partial derivative with respect to w we get;

$$f_w\left(P_1, P_2, w\right) = T - L + w\frac{\partial}{\partial w}\left(T - L\right) = \left(T - L\right)\left\{1 + \frac{w}{T - L}\frac{\partial}{\partial w}\left(T - L\right)\right\} = \left(T - L\right)\left(1 + \varepsilon_S\right) \tag{32}$$

where $\varepsilon_S = \frac{w}{T - L}\frac{\partial}{\partial w}\left(T - L\right)$ is the individual's elasticity of labor supply. The quantity ε_S can be interpreted as the ratio of a fractional change in work time to that in wage rate. Using (27), (31) and (32) we get;

$$\frac{\partial U^*}{\partial B} = \frac{1}{\left(T - L\right)\left(1 + \varepsilon_S\right)}\frac{\partial U^*}{\partial w}, \tag{33}$$

where $\varepsilon_S \neq -1$ and so that the expression for λ is given by (22) now becomes,

$$\lambda = \left(1 + \varepsilon_S\right)\frac{\partial U^*}{\partial B}. \tag{34}$$

From (34) we see that the Lagrange multiplier λ is proportional to the marginal utility of income, the proportionality being the elasticity of labor supply plus unity; λ equals the marginal utility of income if $\varepsilon_S = 0$, i.e., if there is no supply response to change in wage rate.

Now we consider P_1, P_2, w are all positive in the Jacobian matrix (18). We see from (3b,c), and whichever choice is made in (3d), it is not clear that the determinant of H will be non-singular, since some of the terms in the expansion will be positive and others negative. Baxley and Moorhouse (1984) say that there is a 'widespread economic folklore' which assumes H to be negative. These authors also say, that at this point 'the economist deeply wishes that the sufficient conditions be necessary'. They state the two conditions for this as follows (for a relative maximum to occur at a solution X^*, Y^*, L^*, λ^*):

i. the determinant of Jacobian Matrix H, is given in (18), is negative,

ii. the determinant of the Hessian matrix,

$$\begin{bmatrix} 0 & -P_1 & -P_2 \\ -P_1 & u_{XX} & u_{XY} \\ -P_2 & u_{YX} & u_{YY} \end{bmatrix}, \tag{35}$$

is positive.

In models (A) and (C) the parameters can be chosen so that properties i) and ii) of (35), important as they are, can also be satisfied. In model (A), with the use of (8b,c), the determinant of the Jacobian matrix (18) can be written as follows (Mohajan 2012):

$$\begin{vmatrix} 0 & -P_1 & -P_2 & -w \\ -P_1 & u_0 a(a-1)X^{a-2}Y^bL^c & u_0 abX^{a-1}Y^{b-1}L^c & u_0 acX^{a-1}Y^bL^{c-1} \\ -P_2 & u_0 abX^{a-1}Y^{b-1}L^c & u_0 b(b-1)X^aY^{b-2}L^c & u_0 bcX^aY^{b-1}L^{c-1} \\ -w & u_0 acX^{a-1}Y^bL^{c-1} & u_0 bcX^aY^{b-1}L^{c-1} & u_0 c(c-1)X^aY^bL^{c-2} \end{vmatrix} \tag{36}$$

After expanding and simplifying we get,

$$\hat{H} = |H| = u_0^2 \begin{Bmatrix} bc(b+c-1)X^{2a}Y^{2b-2}L^{2c-2}P_1^2 + ac(a+c-1)X^{2a-2}Y^{2b}L^{2c-2}P_2^2 \\ + ab(a+b-1)X^{2a-2}Y^{2b-2}L^{2c}w^2 \\ -2abc\left(X^{2a-2}Y^{2b-1}L^{2c-1}P_2w + X^{2a-1}Y^{2b-2}L^{2c-1}P_1\,w + X^{2a-1}Y^{2b-1}L^{2c-2}P_1P_2\right) \end{Bmatrix} \tag{37}$$

It looks as if for various sets of values of the constants a, b, c in the allowed range (9), this expression could be positive or negative. We consider $a=b=c$; $0<a<1$. Then we can write (37) as follows:

$$\hat{H} = u_0^2a^2X^{2a-2}Y^{2a-2}L^{2a-2}\left\{(2a-1)\left(P_1^2X^2 + P_2^2Y^2 + w^2L^2\right) - 2a\left(P_1P_2XY + P_1wXL + P_2wYL\right)\right\} \tag{38}$$

Now for this model the determinant of the Hessian matrix (35) can be written as follows (Mohajan et al. 2013):

$$\mathrm{H}' = \begin{vmatrix} 0 & -P_1 & -P_2 \\ -P_1 & u_0a(a-1)X^{a-2}Y^bL^c & u_0abX^{a-1}Y^{b-1}L^c \\ -P_2 & u_0abX^{a-1}Y^{b-1}L^c & u_0b(b-1)X^aY^{b-2}L^c \end{vmatrix}, \tag{39}$$

for general values of a, b and c. After expanding and simplifying we get,

$$H' = u_0X^{a-2}Y^{b-2}L^c\left\{b(1-b)P_1^2X^2 + a(1-a)P_2^2Y^2 + 2abP_1P_2XY\right\}. \tag{40}$$

Now we choose in the range of (9) $a=b=c$, to obtain the following expression for H':

$$H' = u_0aX^{a-2}Y^{a-2}L^a\left\{(1-a)\left(P_1^2X^2 + P_2^2Y^2\right) + 2aP_1P_2XY\right\}. \tag{41}$$

Now we show that there are at least two values of 'a' in the range $0<a<1$ such that the determinant of the Jacobian matrix $\hat{H}$ given by (36) is negative definite while the determinant of the Hessian matrix H' given by (39) is positive definite, as required by the sufficient conditions i) and ii) of (35). First we choose $a=\frac{1}{2}$, then we get,

$$\hat{H} = -\frac{1}{4}u_0^2(XYL)^{-1}\left(P_1P_2XY + P_1wXL + P_2wYL\right), \tag{41a}$$

$$H' = \frac{1}{4}u_0(XY)^{-\frac{3}{2}}L^{\frac{1}{2}}\left(P_1X + P_2Y\right)^2. \tag{41b}$$

Now we set $a=\frac{1}{3}$, then we get,

$$\hat{H}=-\frac{1}{27}u_0^2\left(XYL\right)^{-\frac{4}{3}}\left(P_1X+P_2Y+wL\right)^2, \tag{42a}$$

$$H'=\frac{2}{9}u_0\left(XY\right)^{-\frac{5}{3}}L^{\frac{1}{3}}\left(P_1^2X^2+P_2^2Y^2+P_1P_2XY\right). \tag{42b}$$

In each case given above $\hat{H}$ is negative definite and H' is positive definite, supporting the 'widespread economic folklore' and the 'economist's deep wish'! Now for the Model (C) we will show that similar situation holds, where $u_{XX}=0$.

In Model (C) the determinant of the Jacobian matrix (18) can be written as follows (Mohajan et al. 2013):

$$\begin{vmatrix} 0 & -P_1 & -P_2 & -w \\ -P_1 & u_1a\left(a-1\right)X^{a-2}L^c & 0 & u_1acX^{a-1}L^{c-1} \\ -P_2 & 0 & u_2b\left(b-1\right)Y^{b-2}L^c & u_2bcY^{b-1}L^{c-1} \\ -w & u_1acX^{a-1}L^{c-1} & u_2bcY^{b-1}L^{c-1} & c\left(c-1\right)\left(u_1X^a+u_2Y^b\right)L^{c-2} \end{vmatrix} \tag{43}$$

Similarly, as before after expanding and simplifying we get,

$$\hat{H}=\left\{\begin{array}{l} u_2bc\left[\left(b+c-1\right)u_2Y^{2b-2}+\left(b+c-bc-1\right)u_1X^aY^{b-2}\right]L^{2c-2}P_1^2 \\ +u_1ac\left[\left(a+c-1\right)u_1X^{2a-2}+\left(a+c-ac-1\right)u_2X^{a-2}Y^b\right]L^{2c-2}P_2^2 \\ +u_1u_2ab\left(1-a\right)\left(b-1\right)X^{a-2}Y^{b-2}L^{2c}w^2 \\ -2u_1u_2\left[abc^2X^{a-1}Y^{b-1}L^{2c-2}P_1P_2+a\left(1-a\right)X^{a-2}Y^{b-1}L^{2c-1}P_2w+b\left(1-b\right)X^{a-1}Y^{b-2}L^{2c-1}P_1w\right] \end{array}\right\} \tag{44}$$

We set, $a=b=c=\frac{1}{2}$, so that for Model (C) inequalities in (9) are satisfied. Hence (44) can be written as:

$$\hat{H}=-\frac{1}{16}u_1u_2\left(XYL\right)^{-\frac{3}{2}}L^{-1}\left(P_1X+P_2Y+wL\right)^2, \tag{45}$$

which is negative definite, as required. The determinant for the Hessian matrix (35) for Model (C) can be written as;

$$\mathrm{H}'=\begin{vmatrix} 0 & -P_1 & -P_2 \\ -P_1 & u_1a\left(a-1\right)X^{a-2}L^c & 0 \\ -P_2 & 0 & u_2b\left(b-1\right)Y^{b-2}L^c \end{vmatrix} \tag{46}$$

Similarly, as before after expanding and simplifying we get,

$$H' = \left\{u_2 b\,(1-b)Y^{b-2}\,P_1^2 + u_1 a\,(1-a)X^{a-2}P_2^2\right\}L^c \tag{47}$$

which is positive definite for allowed values of *a*, *b* and *c*. Hence for suitable values of the parameters, Model (C) also satisfies sufficient conditions i) and ii) of (35).

4. Sufficient Conditions for Implicit Functions

We consider (5a–d) incorporating necessary conditions for an extremum, and examine the sufficiency conditions for a solution X^*, Y^*, L^*, λ^* to be a maximum (or minimum). Again we follow closely the discussion of this matter given by Baxley and Moorhouse (1984), but we make the calculations more explicit so that the novice or the economist not sufficiently familiar with mathematical concepts and manipulations can follow the steps relatively easily.

Since *u* is a function of the endogenous variables *X*, *Y*, *L*, the functions u_X, u_Y, u_L also depend on the same variables:

$$u_X = u_X(X,Y,L), \qquad u_Y = u_Y(X,Y,L), \qquad u_L = u_L(X,Y,L) \tag{47a}$$

We denote the left hand sides of (5a–d) by the four components of a vector **F**, which all depend on $\lambda, X, Y, L, P_1, P_2, w,$ which may be regarded as points in a 7-dimensional Euclidean space, R^7. Hence,

$$\mathbf{F} = (F_1, F_2, F_3, F_4), \quad F_i = F_i(\lambda, X, Y, L, P_1, P_2, w) = 0\,;\ i = 1, 2, 3, 4, \tag{48}$$

the latter representing the four equations (5a–d). Hence **F** is a four-vector valued function taking values in R^4 and defined for points in R^7. The solution of (48) be,

$$\begin{bmatrix} \lambda \\ X \\ Y \\ L \end{bmatrix} = \mathbf{G}\,(P_1, P_2, w), \tag{49}$$

where $\mathbf{G} = (G_1, G_2, G_3, G_4)$, being a four vector valued function of P_1, P_2, w. The Jacobian matrix for **G**, J_G is given by;

$$J_G = \begin{bmatrix} \frac{\partial \lambda}{\partial P_1} & \frac{\partial \lambda}{\partial P_2} & \frac{\partial \lambda}{\partial w} \\ \frac{\partial X}{\partial P_1} & \frac{\partial X}{\partial P_2} & \frac{\partial X}{\partial w} \\ \frac{\partial Y}{\partial P_1} & \frac{\partial Y}{\partial P_2} & \frac{\partial Y}{\partial w} \end{bmatrix}. \tag{50}$$

Assuming the solution λ, *X*, *Y*, *L* to be given as functions of P_1, P_2, w as in (49), we write (5a–d) explicitly (with the use of (48)) as follows (Mohajan et al. 2013):

$$w\{T-L(P_1,P_2,w)\}-X(P_1,P_2,w)P_1-Y(P_1,P_2,w)P_2=0, \tag{51a}$$

$$u_X\{X(P_1,P_2,w),Y(P_1,P_2,w),L(P_1,P_2,w)\}-\lambda(P_1,P_2,w)P_1=0, \tag{51b}$$

$$u_Y\{X(P_1,P_2,w),Y(P_1,P_2,w),L(P_1,P_2,w)\}-\lambda(P_1,P_2,w)P_2=0, \tag{51c}$$

$$u_L\{X(P_1,P_2,w),Y(P_1,P_2,w),L(P_1,P_2,w)\}-\lambda(P_1,P_2,w)w=0. \tag{51d}$$

Appling the first partial derivatives with respect to P_1, P_2, w respectively, of (51a), to get the following three equations:

$$w\frac{\partial L}{\partial P_1}+\frac{\partial X}{\partial P_1}P_1+\frac{\partial Y}{\partial P_1}P_2=-X\ , \tag{52a}$$

$$w\frac{\partial L}{\partial P_2}+\frac{\partial X}{\partial P_2}P_1+\frac{\partial Y}{\partial P_2}P_2=-Y\ , \tag{52b}$$

$$w\frac{\partial L}{\partial w}+\frac{\partial X}{\partial w}P_1+\frac{\partial Y}{\partial w}P_2=T-L\,. \tag{52c}$$

Similarly taking the second order partial derivatives of (51b–d) we get as follows:

$$u_{XX}\frac{\partial X}{\partial P_1}+u_{XY}\frac{\partial Y}{\partial P_1}+u_{XL}\frac{\partial L}{\partial P_1}-\frac{\partial \lambda}{\partial P_1}P_1=\lambda\ , \tag{53a}$$

$$u_{XX}\frac{\partial X}{\partial P_2}+u_{XY}\frac{\partial Y}{\partial P_2}+u_{XL}\frac{\partial L}{\partial P_2}-\frac{\partial \lambda}{\partial P_2}P_1=0\,, \tag{53b}$$

$$u_{XX}\frac{\partial X}{\partial w}+u_{XY}\frac{\partial Y}{\partial w}+u_{XL}\frac{\partial L}{\partial w}-\frac{\partial \lambda}{\partial w}P_1=0\,. \tag{53c}$$

$$u_{YX}\frac{\partial X}{\partial P_1}+u_{YY}\frac{\partial Y}{\partial P_1}+u_{YL}\frac{\partial L}{\partial P_1}-\frac{\partial \lambda}{\partial P_1}P_2=0\,, \tag{54a}$$

$$u_{YX}\frac{\partial X}{\partial P_2}+u_{YY}\frac{\partial Y}{\partial P_2}+u_{YL}\frac{\partial L}{\partial P_2}-\frac{\partial \lambda}{\partial P_2}P_2=\lambda\ , \tag{54b}$$

$$u_{YX}\frac{\partial X}{\partial w}+u_{YY}\frac{\partial Y}{\partial w}+u_{YL}\frac{\partial L}{\partial w}-\frac{\partial \lambda}{\partial w}P_2=0\,. \tag{54c}$$

$$u_{LX}\frac{\partial X}{\partial P_1}+u_{LY}\frac{\partial Y}{\partial P_1}+u_{LL}\frac{\partial L}{\partial P_1}-\frac{\partial \lambda}{\partial P_1}P_1=0\,, \tag{55a}$$

$$u_{LX}\frac{\partial X}{\partial P_2}+u_{LY}\frac{\partial Y}{\partial P_2}+u_{LL}\frac{\partial L}{\partial P_2}-\frac{\partial \lambda}{\partial P_2}w=0\,, \tag{55b}$$

$$u_{LX}\frac{\partial X}{\partial w}+u_{LY}\frac{\partial Y}{\partial w}+u_{LL}\frac{\partial L}{\partial w}-\frac{\partial \lambda}{\partial w}w=\lambda\ . \tag{55c}$$

Assuming that the Jacobian matrix H is given by (18), and the Jacobian matrix for $\mathbf{G}$, J_G, is given by (50), four sets of equations (52) to (55) can be written as follows:

$$J_G = -H^{-1}\begin{bmatrix} -X & -Y & T-L \\ -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}. \tag{56}$$

According to the rules of matrices, we get;

$H^{-1} = \dfrac{1}{\det H} C^T$, where $C = (C_{ij})$, the matrix of cofactors of H and T for transpose. From (18) and (56) we get,

$$\frac{\partial X}{\partial P_1} = -\frac{1}{\det H}\left(-XC_{12} - \lambda C_{22}\right), \tag{57}$$

where C_{12}, C_{22} are given by;

$$C_{12} = P_1\left\{u_{YY}u_{LL} - (u_{LY})^2\right\} - P_2\left\{u_{XY}u_{LL} - u_{LY}u_{XL}\right\} + w\left\{u_{XY}u_{LL} - u_{YY}u_{XL}\right\},$$
(58a)

$$C_{22} = -P_1^2 u_{LL} - w^2 u_{YY} + 2wP_2 u_{LY}. \tag{58b}$$

We confine ourselves to Models (A) and (C), so that, $u_{XY} > 0$, or $u_{XY} = 0$. With $u_{XY} > 0$, and conditions (3b,c), the second and third terms in C_{12} are positive.

The determinant of the Jacobian matrix (18) is given as follows:

$$\det H = -P_1^2\begin{vmatrix} u_{YY} & u_{YL} \\ u_{LY} & u_{LL} \end{vmatrix} - P_2^2\begin{vmatrix} u_{XX} & u_{XL} \\ u_{LX} & u_{LL} \end{vmatrix} - w^2\begin{vmatrix} u_{XX} & u_{XY} \\ u_{YX} & u_{YY} \end{vmatrix} + 2P_1P_2\begin{vmatrix} u_{XY} & u_{XL} \\ u_{LY} & u_{LL} \end{vmatrix} - 2P_1 w\begin{vmatrix} u_{XY} & u_{XL} \\ u_{YY} & u_{YL} \end{vmatrix} + 2P_2 w\begin{vmatrix} u_{XX} & u_{XY} \\ u_{LX} & u_{LY} \end{vmatrix}.$$

Let us assume that;

$$\det\begin{bmatrix} u_{LL} & u_{LY} \\ u_{YL} & u_{YY} \end{bmatrix} = u_{LL}u_{YY} - (u_{LY})^2 \geq 0, \tag{59}$$

"as economist generally do", say Baxley and Moorhouse (1984). Then, $C_{12} > 0$; from (3b,c), $C_{22} > 0$. Thus, $\dfrac{\partial X}{\partial P_1} < 0$, that is, if the price of x increases, then the amount of x, given by X, decreases, which is reasonable.

For Model (A) we have, from (8b,c), after some calculations,

$$u_{LL}u_{YY} - (u_{LY})^2 = u_0^2 bc(1-b-c)X^{2a}Y^{2b-2}L^{2c-2}, \tag{60}$$

which is positive if $b + c < 1$, which is valid for suitable choice of b, c, consistent with (9).

For Model (C) we have, from (15b,c), after some manipulation,

$$u_{LL}u_{YY} - (u_{LY})^2 = bc(1-b)(1-c)u_1u_2X^aY^{b-2}L^{2c-2} + u_2^2 bc(1-b-c)Y^{2b-2}L^{2c-2}. \tag{61}$$

Assuming as before $u_{XX} \geq 0$, (3b,c) lead to C_{44} as well as the first two terms in C_{14} being positive. The assumption ("with the economist") that;

$$\det\begin{bmatrix} u_{XX} & u_{XY} \\ u_{YX} & u_{YY} \end{bmatrix} = u_{XX}u_{YY} - \left(u_{XY}\right)^2 \geq 0, \tag{62}$$

then implies that C_{14} is positive. We pause to examine (62) for models (A) and (C). For the format we get,

$$u_{XX}u_{YY} - \left(u_{XY}\right)^2 = u_0^2 ab\left(1-a-b\right)X^{2a-2}Y^{2b-2}L^{2c}, \tag{63a}$$

which is positive definite if $a+b<1$; a, b can be chosen to satisfy this inequality, consistent with (9). For Model (C), since $u_{XX}=0$, (62) follows trivially from (3b). Having established that both C_{14} and C_{44} are positive, it is clear from (61) that this implies that the sign of $\partial L/\partial w$ is ambiguous; "this is itself a very interesting result" (Baxley and Moorhouse 1984).

We have det*H* is negative, as has been established in some specific cases above; we confine ourselves to such cases. Thus the factor, $-\left(\det H\right)^{-1}$ is positive. We find that;

$$\frac{\partial L}{\partial w} = -\frac{1}{\det H}\left(-\lambda C_{44} + \left(T-L\right)C_{14}\right), \tag{64}$$

where

$$C_{14} = P_1\left\{u_{YX}u_{LY} - u_{LX}u_{YY}\right\} - P_2\left\{u_{XX}u_{LY} - u_{LX}u_{XY}\right\} + w\left\{u_{XX}u_{YY} - \left(u_{XY}\right)^2\right\},$$

$$C_{44} = -P_1^2 u_{XX} - P_2^2 u_{YY} + 2P_1P_2u_{XY}.$$

Of the two terms in the brackets, $-\lambda C_{44}$ is negative and $\left(T-L\right)C_{14}$ is positive. Recall that L is the leisure time so that $\left(T-L\right)$ is the work period. If the wage increases, one effect is an urge to work longer, that is, decreases the leisure time. This contributes a negative component to $\partial L/\partial w$; for this reason the term, $-\lambda C_{44}$ is referred to as the *substitution effect* of an increase in *w*. Another effect of a wage increase is for the individual to resort to more leisure time to enjoy the extra goods he can purchase; this term ($\left(T-L\right)C_{14}$) is called the *income effect*. Thus a wage increase can give rise to both income and substitution effects which have opposite influences (Mohajan et al. 2013).

Now we consider two other ways of looking at $\partial L/\partial w$ which is given by (57). Again in (22), λ can be interpreted as the marginal utility of *w* per unit time. Hence (64) can be written as;

$$\frac{\partial L}{\partial w} = -\frac{1}{\det H}\left(T-L\right)^{-1}\left(-u_w C_{44} + \left(T-L\right)^2 C_{14}\right). \tag{64a}$$

We see from (64a) that if u_w is small or if $\left(T-L\right)$ is large, i.e., the individual spends longer work time, then $\partial L/\partial w > 0$, so that the income effect dominates. On the other hand if u_w is large or if $\left(T-L\right)$ is small, then $\partial L/\partial w < 0$, so the substitution effect is more important.

Consider in the present context, the elasticity of labor supply ε_S, then (64) can be written as;

$$\varepsilon_S = \frac{w}{(T-L)}\frac{\partial(T-L)}{\partial w} = \frac{1}{\det H}\frac{w}{(T-L)^2}\left(-u_w C_{44} + (T-L)^2 C_{14}\right). \tag{65}$$

If we assume $T \neq L$, that is, the individual does the some work, then $\varepsilon_S = 0$, if and only if,

$$u_w C_{44} = (T-L)^2 C_{14}. \tag{66}$$

This implies that the income effect equals the substitution effect. The properties of the comparative statics $\partial X/\partial P_1$ and $\partial L/\partial w$ derived here may be of interest in wider contexts. One such circumstance is that of income tax. An increase of after tax wages is often carried out with a view to stimulate further work. We see from the above analysis that the tax cut may lead to both substitution and income effect, so that the desired response may not be forthcoming. Normally, policy makers, perhaps more often than not, discuss these matters in intuitive, qualitative and verbal terms (Mohajan 2012).

From the expansion of the determinant of the matrix (18), by considering goods *x* and *y* weakly dependent with respect to *u*, we can write,

$$\begin{vmatrix} u_{XX} & u_{XY} \\ u_{YX} & u_{YY} \end{vmatrix} = \left(u_{XX}u_{YY} - (u_{XY})^2\right) \geq 0. \tag{67}$$

As mentioned, this condition has been shown to hold for Modes (A) and (C). Let us assume that the pairs (x, y), (x, l), (y, l), are weakly dependent, and $u_{XY} \geq 0$. Then it is readily verified, with the use of (3b,c) and others previous results, that $\det(H) < 0$. The case $u_{XY} < 0$, as mentioned earlier, will be considered on another occasion.

Now consider the property of weak dependence, Baxley and Moorhouse (1984) say that "widespread economic folklore has held that"

$$|u_{XY}| \leq |u_{XX}|,\ |u_{XY}| \leq |u_{YY}|, \tag{68}$$

would hold. The idea is that if $u_{XY} = 0$, the inequalities are trivial, while if, e.g., $X = Y$, these becomes equations and other cases are intermediate. We examine (68) for model (A). Using (8a,b) we get,

$$|bX| \leq |(a-1)Y| \quad \text{and} \quad |aY| \leq |(b-1)X|. \tag{69}$$

One may choose to restrict *a*, *b*, *X*, *Y* (recall that $a > 0$, $b < 1$) so that (69) is satisfied. In particular, in the three cases (i) $X = \frac{1}{2}Y$, (ii) $\frac{1}{2}X = Y$, (iii) $X = Y$, we get from (69):

$$\text{i)} \quad \frac{1}{2}b \leq (1-a),\ a \leq \frac{1}{2}(1-b); \tag{70a}$$

$$\text{ii)} \quad b \leq \frac{1}{2}(1-a), \quad \frac{1}{2}a \leq (1-b). \tag{70b}$$

If we set $X = Y$, it might appear from (69), that the inequalities become equations. However, as the following simplified form of the utility function (7) shows, this is not necessarily the case. For simplicity here we ignore the leisure time *L*;

$$u'(X,Y) = X^a Y^b \text{, with } \quad u'_{XX} = a(a-1)X^{a-2}Y^b, \; u'_{XY} = abX^{a-1}Y^{b-1}, \quad u'_{YY} = b(b-1)X^a Y^{b-2},$$

so that we get,

$$u'_{XX}\big|_{X=Y} = a(a-1)X^{a+b-2}, \; u'_{XY}\big|_{X=Y} = abX^{a+b-2}, \; u'_{YY}\big|_{X=Y} = b(b-1)X^{a+b-2}.$$

Thus the inequalities do not straightway become equations. The more relevant inequality in the present context is (67), which follows from (68), but the converse is not true. That the later is somewhere unrealistic can be seen from the following example:

For some particular situation, let $u_{XY} = 0.1$, $u_{XX} = -1$. We change units of *Y* so it changes to $Y' = \frac{1}{12}Y$ (i.e., *Y* is measured in inches and Y' in feet). At this situation we get;

$$u_{XY'} = u_{XY}\frac{dY}{dY'} = 12\, u_{XY} = 1.2.$$

Hence $|u_{XY'}| \le |u_{XX}|$ is not satisfied after a change of units. However, it can be verified that the determinant (67) does not change sign when units are changed. If we measure the unit in yard, then the inequality is satisfied (Mohajan et al. 2013).

5. Conclusions

In this study we have established that the value of the Lagrange multiplier is positive and sometimes it indicates shadow price. We have used necessary and sufficient conditions to obtain optimal value in each case. With the help of comparative static analysis and application of Implicit Function Theorem, we mathematically have shown the behavior of the firm (including explicit examples).

6. References

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