MPRA

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18 June 2017

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Abstract
The de Sitter universe is the second model of the universe just after the publications of the Einstein's static and closed model. In 1917, Wilhelm de Sitter has developed this model which is a maximally symmetric solution of the Einstein field equation with zero density. The geometry of the de Sitter universe is theoretically more complicated than that of the Einstein universe. The model does not contain matter or radiation. But, it predicts that there is a redshift. This article tries to describe the de Sitter model in some detail but easier mathematical calculations. In this study an attempt has been taken to provide a brief discussion of de Sitter model to the common readers.

Keywords: de Sitter space-time, empty matter universe, redshifts.

1. Introduction

The de Sitter space-time universe is one of the earliest solutions to Einstein's equation of motion for gravity. It is a matter free and fully regular solution of Einstein's equation with positive cosmological constant $\Lambda = \dfrac{3}{S^2}$, where S is the scale factor. After the publications of the Einstein's static and closed universe model a second static model of the static and closed but, empty of matter universe was suggested by Dutch astronomer Wilhelm de Sitter (1872–1934). As the de Sitter space-time is a maximally symmetric solution of the Einstein field equation with zero density, it has little value for describing our own universe (i.e., an unrealistic model). In 1917, he developed a model of a finite, static cosmos of spherical spatial geometry whose radius was directly related to the density of matter (Einstein 1917). The de Sitter space-time universe has the topology $R^1 \times S^3$ where, R^1 represents time and S^3 represents 3-dimensional space. It is simply-connected and the maximally symmetric Lorentzian manifold with constant positive curvature (Barata et al. 2017). Wilhelm de Sitter derived his solution as a variation on Einstein's theme; rather than taking only the spatial sections to be closed spheres, he treated the entire space-time manifold M as a closed space. It has some special properties, such as, it is an expanding universe, and it possesses a cosmological horizon (de Sitter 1917a). The de Sitter model of the universe was an exciting, because he discussed the relation between gravity and inertia, and also on Ernst Mach's hypothesis that the inertia of one body is caused by the presence of all others (Bartelmann 2012). The equations of motion for the scale factor of the universe derived from Einstein's theory of gravity always required the size of our universe to be dynamically changing. Einstein was very uncomfortable with the notion of a dynamic universe as at that time no evidence indicated an expanding or contracting universe (Hofbaur 2014). The

objection on de Sitter was brought by Einstein, because it was matter free. On the other hand, de Sitter model had redshifts (Lemaître 1925).

The original solution of de Sitter was written in time-independent, static coordinates, which leads to the following two phenomena: i) time flows with different speed at different points in space, and ii) there exists an 'event' horizon, where the time flow becomes infinitely slow (Duerbeck and Seitter 2001).

The de Sitter solution is one possible model for the basic geometry of space-time when local irregularities are smoothed out. It provides, amongst other things, for a cosmological redshift. In his model, the stars and nebulae (the word *nebulae* was first introduced by French astronomer, Charles Messier, in 1781) are just *test particles* that do not curve the universe (de Sitter 1917b). In relativistic cosmology the de Sitter universe is perhaps the simplest model that allows for the possibility of a galactic redshift. It is curved but homogeneous and isotropic in space-time (Inomata 1968). The expanding half of de Sitter space-time is approximately the exponentially expanding universe in the inflationary model and our universe may be approaching the de Sitter space-time asymptotically. Hence, de Sitter space-time can be a model for our universe in its early expanding phase, and in its far future (Hawking and Ellis 1973). Also de Sitter's cosmology is provoked a vigorous discussion among theoreticians about the appropriate interpretation about the universe.

The current observations indicate that the universe is expanding in an accelerated rate, and may approach de Sitter space asymptotically (Perlmutter 1999). The boundaries of global de Sitter space are the spacelike hypersurfaces given at future and past infinity $I^{\pm}$, where it has been suggested that a reputed holographic theory might reside (Strominger 2001). The discovery of a nonzero cosmological constant suggests that our universe asymptotes to a de Sitter space-time in the infinite future (I^+). But, the correlations on I^+ cannot be measured by any physical experiment because; all points on I^+ are causally disconnected (Anninos et al. 2012).

The de Sitter space-time universe is geodesically complete, i.e., the affine parameter t of any geodesic $t \to \gamma(t)$ passing through an arbitrary point p in de Sitter space-time universe can be extended to reach arbitrary values. Hence, it is free from singularity (Barata et al. 2017).

2. General Relativity in Brief

In 25 November 1915, famous physicist Albert Einstein submitted his completed General Theory of Relativity to the Prussian Academy of Sciences (Einstein 1917). General Relativity models the physical universe as a 4-dimensional space-time manifold (M, g). The universe is not simply a random collection of irregular distributed matter, but it is a single entity, all parts of which are connected (Joshi 2008). Tensors are geometric objects defined on a manifold M, which remain invariant under the change of coordinates. Tensor analysis is a generalization of vector analysis. The tensor formulation became very popular when Albert Einstein used it as an essential tool to present his General Theory of Relativity (Mohajan 2017a).

A contravariant vector A^μ $\left(\mu = 0,1,2,3\right)$ and a covariant vector (one form) A_μ in coordinates x^μ to x'^μ transform as;

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu, \qquad A'_\mu = \frac{\partial x'^\nu}{\partial x^\mu} A_\nu. \tag{1}$$

The second rank tensors are defined as;

$$A^\nu_\mu = g^{\nu\sigma} A_{\sigma\mu}, \quad A^\mu_\nu = g^{\mu\sigma} A_{\sigma\nu}, \quad A^{\mu\nu} = g^{\mu\sigma} g^{\nu\lambda} A_{\sigma\lambda}.$$

The tensors A^ν_μ, A^μ_ν, $A_{\mu\nu}$ or $A^{\mu\nu}$ are considered as the representations of the same geometric object because, these are uniquely associated tensors. Similarly, a mixed tensor of rank three can be transformed as,

$$A'^\mu_{\nu\lambda} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\gamma}{\partial x'^\lambda} A^\alpha_{\beta\gamma} \tag{2}$$

where we have used summation convention. The distance between two infinitesimally separated points in the space-time (Riemannian space) is called a metric, and is defined as;

$$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu \tag{3}$$

where $\det\left(g_{\mu\nu}(x)\right) \neq 0$ and $g_{\mu\nu} = g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)$ is a covariant tensor of rank two. Here dx^μ and dx^ν are contravariant vectors and ds^2 is invariant for arbitrary choice of vectors dx^μ and dx^ν. The contravariant fundamental tensor $g^{\mu\nu}$ is defined as, $g^{\mu\nu}$ = (cofactor of $g_{\mu\nu}$ in g) $\div$ g, where g is determinant of $g_{\mu\nu}$. The tensors $g^{\mu\nu}$ and $g_{\mu\nu}$ are symmetric; $g^{\mu\nu}$ is reciprocal of $g_{\mu\nu}$ and is called the conjugate metric tensor of rank two (Mohajan 2017a).

If $A_{\mu\nu}$ is symmetric then (Carroll 2004);

$$A_{\mu\nu} = A_{\nu\mu} \tag{4}$$

and if $A_{\mu\nu}$ is anti-symmetric then;

$$A_{\mu\nu} = -A_{\nu\mu}. \tag{5}$$

Hence, for a tensor with components $A_{\mu\nu}$, its symmetric and anti-symmetric parts are written respectively as;

$$A_{(\mu\nu)} = \frac{1}{2!}\left(A_{\mu\nu} + A_{\nu\mu}\right), \tag{6}$$

$$A_{[\mu\nu]} = \frac{1}{2!}\left(A_{\mu\nu} - A_{\nu\mu}\right). \tag{7}$$

Then, A is called symmetric if $A_{(\mu\nu)} = A_{\nu\mu}$, and it is called anti-symmetric if $A_{[\mu\nu]} = -A_{\nu\mu}$; we can express a four rank anti-symmetric tensor as;

$$A^\mu_{[\nu\lambda\sigma]} = \frac{1}{3!}\left(A^\mu_{\nu\lambda\sigma} + A^\mu_{\lambda\sigma\nu} + A^\mu_{\sigma\nu\lambda} - A^\mu_{\lambda\nu\sigma} - A^\mu_{\sigma\lambda\nu} - A^\mu_{\nu\sigma\lambda}\right). \tag{8}$$

The Kronecker delta is defined by;

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$$g_{\mu\nu}g^{\nu\alpha} = g_\mu^\alpha = \delta_\mu^\alpha = \begin{cases} 1 \text{ if } \alpha = \mu \text{ (no summation)} \\ 0 \text{ if } \alpha \neq \mu. \end{cases} \quad (9)$$

Christoffel symbols of the first and second kind are defined respectively by;

$$[\nu\lambda,\mu] = \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\mu\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\mu} \right), \quad \text{where } [\nu\lambda,\mu] = [\lambda\nu,\mu]. \quad (10)$$

$$\Gamma_{\nu\lambda}^\sigma = \frac{1}{2} g^{\sigma\mu}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\mu\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\mu} \right), \text{ where } \Gamma_{\nu\lambda}^\sigma = \Gamma_{\lambda\nu}^\sigma. \quad (11)$$

The covariant differentiations of vectors are defined as;

$$A_{;\nu}^\mu = A_{,\nu}^\mu + \Gamma_{\nu\lambda}^\mu A^\lambda \quad (12)$$

$$A_{\mu;\nu} = A_{\mu,\nu} - \Gamma_{\mu\nu}^\lambda A_\lambda \quad (13)$$

where semi-colon (;) denotes the covariant differentiation, and comma (,) denotes the partial differentiation. By (11) we can write;

$$A_{\mu;\nu;\sigma} - A_{\mu;\sigma;\nu} = R_{\mu\nu\sigma}^\alpha A_\alpha, \quad (14)$$

where $R_{\mu\nu\sigma}^\alpha = \Gamma_{\mu\sigma;\nu}^\alpha - \Gamma_{\mu\nu;\sigma}^\alpha + \Gamma_{\beta\nu}^\alpha \Gamma_{\mu\sigma}^\beta - \Gamma_{\beta\sigma}^\alpha \Gamma_{\mu\nu}^\beta \quad (15)$

is a tensor of rank four, and called Riemann curvature tensor. From (13) and (14) we observe that the curvature tensor components are expressed in terms of the metric tensor and its second derivatives. From (15) we get (Mohajan 2017b);

$$R_{\mu\nu\sigma}^\alpha + R_{\nu\sigma\mu}^\alpha + R_{\sigma\mu\nu}^\alpha = 0. \quad (16)$$

Taking inner product of both sides of (15) with $g_{\rho\alpha}$ one gets covariant curvature tensor,

$$R_{\rho\mu\nu\sigma} = \frac{1}{2}\left(\frac{\partial^2 g_{\rho\sigma}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\nu}}{\partial x^\sigma \partial x^\rho} - \frac{\partial^2 g_{\mu\sigma}}{\partial x^\nu \partial x^\rho} - \frac{\partial^2 g_{\rho\nu}}{\partial x^\sigma \partial x^\mu} \right) + g_{\alpha\lambda}\left(\Gamma_{\mu\nu}^\alpha \Gamma_{\rho\sigma}^\lambda - \Gamma_{\mu\omega}^\alpha \Gamma_{\rho\nu}^\lambda \right). \quad (17)$$

At the pole of geodesic, coordinates system both kinds of Christoffel symbols vanish, but not their derivatives, hence, (15) takes the form,

$$R_{\mu\nu\sigma}^\alpha = \Gamma_{\mu\sigma;\nu}^\alpha - \Gamma_{\mu\nu;\sigma}^\alpha.$$

Taking covariant derivative with respect to ρ at the pole we get,

$$R_{\mu\nu\sigma;\rho}^\alpha = \frac{\partial^2 \Gamma_{\mu\sigma}^\lambda}{\partial x^\rho \partial x^\nu} - \frac{\partial^2 \Gamma_{\mu\nu}^\lambda}{\partial x^\rho \partial x^\sigma}. \quad (18)$$

From (18) we get the Bianchi identity as;

$$R_{\mu\nu\sigma,\rho}^\lambda + R_{\mu\sigma\rho,\nu}^\lambda + R_{\mu\rho\nu,\sigma}^\lambda = 0. \quad (19)$$

This holds for all coordinates systems. Contraction of curvature tensor (15) gives Ricci tensor;

$$R_{\mu\nu} = g^{\lambda\sigma} R_{\lambda\mu\sigma\nu}. \quad (20)$$

Further contraction of (20) gives Ricci scalar;

$$\hat{R} = g^{\lambda\sigma} R_{\lambda\sigma}. \quad (21)$$

The Ricci curvature scalar $\hat{R}$ has the property that it depends only on $g_{\lambda\sigma}$ and on their derivatives only up to the second order, and $\hat{R}$ is linear in the second derivatives of the metric components. From which one gets Einstein tensor as;

$$G_v^\mu = R_v^\mu - \frac{1}{2}\delta_v^\mu R \tag{22}$$

where $div\left(G_v^\mu\right) = G_{v;\mu}^\mu = 0$. The space-time manifold (M,g) is said to have a flat connection if and only if;

$$R_{v\lambda\sigma}^\mu = 0. \tag{23}$$

This is necessary and sufficient condition for a vector at a point p to remain unaltered after parallel transported along an arbitrary closed curve through p. This is because all such curves can be shrunk to zero, in which case the space-time is simply connected (Mohajan 2017b).

In Euclidean 3-dimensional space the path of shortest distance between two fixed points is a straight line. But, the path of extremum (maximum or minimum) distance between any two points in Riemannian space is called the geodesic. Let, $\gamma(t): R \to M$ be a C^1-curve in M. If T is a $C^r (r \geq 0)$ tensor field on M then the covariant derivative of T along $\gamma(t)$ is defined as;

$$\frac{DT}{\partial t} = T_{c...d;e}^{a...b} X^e. \tag{24}$$

Here, X is a tangent vector to $\gamma(t)$. Then, γ is a geodesic if the tangent vector to γ is parallel along it. In a Riemannian manifold with a positive definite metric geodesic gives the curves of shortest distance between two points p and q. The arc length between these two points on a curve $x^\mu = x^\mu(t)$ is given by;

$$S = \int_p^q g_{\mu\nu} u^\mu u^\nu, \text{ where } u^\mu = X^\mu = \frac{dx^\mu}{dt}. \tag{25}$$

In a space-time with a Lorentzian metric the non-spacelike geodesics maximize the distance between the points defined by (25). If there is a timelike geodesic between these points p and q there is no shortest distance geodesics between these points because, introducing null geodesic pieces, one could join these points by curves of arbitrary small lengths. On the other hand, any maximal length curve between p and q must necessarily be timelike geodesic. If the distance between any two points on a geodesic is zero, then the geodesic is said to be null geodesic.

A connection ∇ at a point $p \in M$ is a rule which assigns to each vector field at p a different operator ∇_X which maps an arbitrary C^r vector field Y at p into a vector field $\nabla_X Y$ such that above conditions are satisfied. Now, if X denotes the tangent vector field along γ, then it is required that $\nabla_X X$ is proportional to X, i.e., there exists a function f such that (Joshi 1996);

$$\nabla_X X = fX,$$
$$\left(X_{;\nu}^\mu X^\nu\right)e_\mu = fXe_\mu, \tag{26}$$

which holds for all e_μ. Hence, we can write (26) along the curve as;

$$X_{;\nu}^\mu X^\nu = fX^\nu. \tag{27}$$

If $f = 0$ then equation for geodesic is;

$$X^{\mu}_{;\nu}X^{\nu} = 0. \tag{28}$$

Let, $\{x^{\mu}\}$ is the local coordinate system, then $X^{\mu} = \dfrac{dx^{\mu}}{dt} = u^{\mu}$ are the components of the tangent vector to the geodesic. Here the parameter t is the affine parameter along γ, and such a situation γ is called the affinely parametrized geodesic. Now, the geodesic equation can be written as;

$$\frac{du^{\mu}}{dt} + \Gamma^{\mu}_{\nu\lambda}u^{\mu}u^{\nu} = 0. \tag{29}$$

The energy-momentum tensor $T^{\mu\nu}$ is defined as;

$$T^{\mu\nu} = \rho_0 u^{\mu}u^{\nu} \tag{30}$$

where ρ_0 is the proper density of matter, and if there is no pressure. The component T^{00} of the energy-momentum tensor (30) is defined by,

$$T^{00} = \rho_0 \frac{dx^0}{d\tau}\frac{dx^0}{d\tau}. \tag{31}$$

A perfect fluid is characterized by pressure $p = p(x^{\mu})$, then energy-momentum tensor is given by (Carroll 2004);

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + pg^{\mu\nu}. \tag{32}$$

The principle of local conservation of energy and momentum states that;

$$T^{\mu\nu}_{;\nu} = 0. \tag{33}$$

According to the Newton's law of gravitation, the field equation in the presence of matter is;

$$\nabla^2\phi = 4\pi G\rho \tag{34}$$

where ϕ is the gravitational potential, ρ is the scalar density of matter, $G = 6.673\times10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$ is the Newton's gravitational constant.

If classical equation (34) is generalized for the relative theory of gravitation, then this must be expressed as a tensor equation satisfying following conditions (Harvey and Schucking 2000);
    i)   the tensor equation should not contain derivatives of $g_{\mu\nu}$ higher than the second order,
    ii)   it must be linear in the second differential coefficients, and
    iii)   its covariant divergence must vanish identically.

The most appropriate tensor of the form required is the Einstein's tensor which is given by (20); then Einstein's field equation can be written as;

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T^{\mu\nu} \tag{35}$$

where $G = 6.673\times10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$ is the Newtonian gravitational constant and $c = 10^8$ m/s is the velocity of light. Einstein has introduced a cosmological constant $\Lambda(\approx 0)$ for static universe solutions as (Einstein 1915);

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}T^{\mu\nu}. \tag{36}$$

In relativistic unit $G = c = 1$, that is to say, we translate our units according to, 1 s = 3×10^{10} cm = 4×10^{38} g (Penrose 1970). Hence, in relativistic units, for $\Lambda = 0$, (36) becomes;

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu} \ . \tag{37}$$

Equation (36) can be expressed as a statement about the relative acceleration of very close test free falling particles. It is clear that divergence of both sides of (36) and (37) is zero. For empty space $T_{\mu\nu} = 0$ then, $R_{\mu\nu} = \Lambda g_{\mu\nu}$, and hence;

$$R_{\mu\nu} = 0 \ \text{for} \ \Lambda = 0 \tag{38}$$

which is Einstein's law of gravitation for empty space.

3. de Sitter's Universe

In mathematical physics the de Sitter space-time can be defined as a sub-manifold of a generalized Minkoski space of one higher dimension, that is, space-time, of a sphere in ordinary, Euclidean space. More recently it has been considered as the setting for Special Relativity rather than using Minkowski space, since a group contraction reduces the isometry group of de Sitter space to the Poincaré group, allowing a unification of the space-time translation subgroup and Lorentz transformation subgroup of the Poincaré group into a simple group rather than a semi-simple group (figure 1). This alternate formulation of special relativity is called de Sitter Relativity (Wikipedia 2017).
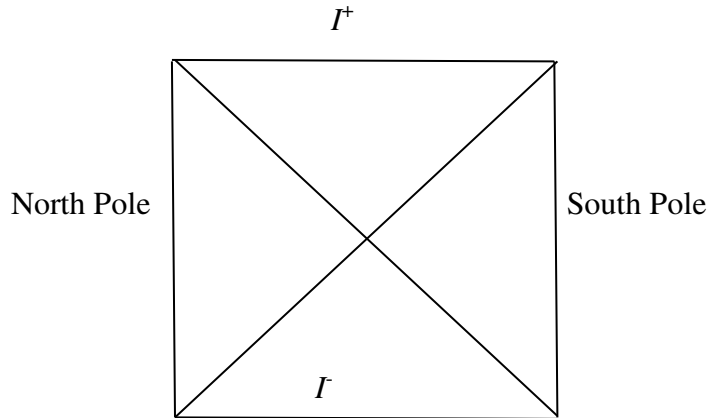

Figure 1: Penrose diagram of de Sitter space.

Einstein's static model of the universe is based on the following assumptions (Einstein 1917):
  i) The universe is static, i.e., in a proper coordinate system matter is at rest, and the proper pressure P_0 and proper density ρ_0 are the same everywhere.
  ii) The universe is isotropic, i.e., all the spatial directions are equivalent.
  iii) The universe is homogeneous, i.e., no part of the universe can be distinguished from the other. Matter is distributed homogeneously on large scales, and does not show large-scale motions.
  iv) For small values of r, the line element takes the form of special relativity of flat space-time, since local gravitational field can be neglected for small space-time.

The most general static, isotropic and homogeneous universe has the familiar form (Tolman 1939);

$$ds^2 = -e^\lambda dr^2 - r^2 d\Omega^2 + e^\nu \, dt^2 \qquad (39)$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$. Here, (r,θ,ϕ) are spherical polar coordinates, and $\nu = \nu(r)$, $\lambda = \lambda(r)$. For the universe containing perfect fluid the pressure P_0 and proper density ρ_0 are determined by the field equation (36) which are given by (Tolman 1939);

$$8\pi P_0 = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda \qquad (40a)$$

$$8\pi \rho_0 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda \qquad (40b)$$

$$\frac{dP_0}{dr} = -\frac{1}{2}(P_0 + \rho_0)\nu' \qquad (40c)$$

where ν', λ' represent the differentiation with respect to r.

By the assumption (i) $\dfrac{dP_0}{dr} = 0$, then (40c) gives;

$$(P_0 + \rho_0)\nu' = 0. \qquad (41)$$

Equation (41) satisfies any one of the following properties (Tolman 1969):

$$\nu' = 0 \qquad (42a)$$
$$P_0 + \rho_0 = 0 \qquad (42b)$$
$$P_0 + \rho_0 = 0 \text{ and } \nu' = 0. \qquad (42c)$$

The de Sitter's cosmological model is based on the possibilities (42b) (de Sitter 1917a);

$$P_0 + \rho_0 = 0. \qquad (43)$$

Now, adding (40a) and (40b) we get;

$$8\pi(P_0 + \rho_0) = e^{-\lambda}\left(\frac{\nu' + \lambda'}{r}\right). \qquad (44)$$

Using (43) in (44) we get;

$$\lambda' = -\nu'. \qquad (45)$$

Integrating (45) we get;

$$\lambda = -\nu + c_1. \qquad (46)$$

According to the assumption (iv) $\lambda = 0 = \nu$ at $r = 0$, we get, $c_1 = 0$, then (46) gives;

$$\lambda = -\nu. \qquad (47)$$

Using (40b) we get (Tolman 1969);

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) = 8\pi \rho_0 - \frac{1}{r^2} + \Lambda$$

$$e^{-\lambda}(\lambda' r - 1) = (8\pi \rho_0 + \Lambda)r^2 - 1$$

$$\frac{d}{dr}(re^{-\lambda}) = 1 - (8\pi \rho_0 + \Lambda)r^2. \qquad (48)$$

Integrating (48) we get;

$$re^{-\lambda} = r - (8\pi\rho_0 + \Lambda)\frac{r^3}{3} + c_2. \tag{49}$$

Using $v = 0 = \lambda$ at $r = 0$ in (49) we get, $c_2 = 0$;

$$re^{-\lambda} = r - (8\pi\rho_0 + \Lambda)\frac{r^3}{3},$$

$$e^{-\lambda} = 1 - (8\pi\rho_0 + \Lambda)\frac{r^2}{3} = 1 - \frac{r^2}{S_0^2} \tag{50}$$

where $\dfrac{1}{S_0^2} = \dfrac{1}{3}(8\pi\rho_0 + \Lambda)$. Hence,

$$e^{-\lambda} = e^{v} = 1 - \frac{r^2}{S_0^2}. \tag{51}$$

So that the line element (39) becomes;

$$ds^2 = -\left(1 - \frac{r^2}{S_0^2}\right)^{-1} dr^2 - r^2 d\Omega^2 + \left(1 - \frac{r^2}{S_0^2}\right) dt^2. \tag{52}$$

This line element is called de Sitter line element for static, isotropic and homogeneous universe.

4. Geometrical Description of de Sitter Universe

The geometry of the de Sitter universe is theoretically more complicated than that of the Einstein universe. Since, not only the three space coordinates, but also the time coordinate is included into the curvature of space-time. It has a higher symmetry than the space-time in Einstein's universe (Duerbeck and Seitter 2001).

Let us put, $r = S_0 \sin\chi$ then (52) becomes (Lord 1973),

$$ds^2 = -S_0^2 (d\chi^2 + \sin^2\chi d\Omega^2) + \cos^2\chi dt^2. \tag{53}$$

Now let us put,

$$\alpha = r \sin\theta\cos\phi$$
$$\beta = r \sin\theta\sin\phi$$
$$\gamma = r \cos\theta$$

$$\delta + \in = S_0 e^{t/R_0}\sqrt{\left(1 - \frac{r^2}{S_0^2}\right)} \tag{54}$$

$$\delta - \in = S_0 e^{-t/R_0}\sqrt{\left(1 - \frac{r^2}{S_0^2}\right)}.$$

Hence, (53) becomes (Schmidt 1993),

$$d\alpha^2 + d\beta^2 + d\gamma^2 = dr^2 + r^2 d\Omega^2.$$

Now we can write,

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$$d(\delta+\in)=e^{t/R_0}\sqrt{\left(1-\frac{r^2}{S_0^2}\right)}dt-\frac{re^{t/S_0}}{S_0\sqrt{\left(1-\frac{r^2}{S_0^2}\right)}}dr$$

$$d(\delta-\in)=-e^{-t/S_0}\sqrt{\left(1-\frac{r^2}{S_0^2}\right)}dt-\frac{re^{-t/S_0}}{S_0\sqrt{\left(1-\frac{r^2}{S_0^2}\right)}}dr \qquad (55)$$

$$d(\delta+\in)d(\delta-\in)=d\delta^2-d\in^2=\frac{r^2dr^2}{S_0^2\left(1-\frac{r^2}{S_0^2}\right)}-\left(1-\frac{r^2}{S_0^2}\right)dt^2 .$$

Hence, line element (52) reduces to (Penrose 1968),
$$ds^2=-d\alpha^2-d\beta^2-d\gamma^2-d\delta^2-d\in^2 . \qquad (56)$$

Let us put,
$z_1=i\alpha$, $z_2=i\beta$, $z_3=i\gamma$, $z_4=i\delta$, $z_5=\in$, then, we can write,

$$z_1^2+z_2^2+z_3^2+z_4^2+z_5^2=-\alpha^2-\beta^2-\gamma^2-\delta^2+\in^2=(iS_0)^2 \qquad (57)$$
$$dz_1^2+dz_2^2+dz_3^2+dz_4^2+dz_5^2=-d\alpha^2-d\beta^2-d\gamma^2-d\delta^2+d\in^2=(iS_0)^2 .$$

Hence, line element (56) takes the hyperboloid form (Hannabuss 1971);
$$ds^2=dz_1^2+dz_2^2+dz_3^2+dz_4^2+dz_5^2 . \qquad (58)$$

In 5-dimensional Euclidean space, the space-time of constant curvature appears as a 4-dimensional one-shell hyperboloid, which extends for each observer from minus infinity to plus infinity (Duerbeck and Seitter 2001).

Again, let us consider,

$$r'=\frac{r}{\sqrt{\left(1-\frac{r^2}{S_0^2}\right)}}e^{-t/S_0}$$

$$t'=t+\frac{1}{2}S_0\log\left(1-\frac{r^2}{S_0^2}\right) . \qquad (59)$$

Using (59) the line element (52) takes the form;
$$ds^2=-e^{-2t/S_0}\left(dr'^2-r'^2d\Omega^2\right)+dt'^2 . \qquad (60)$$

Substituting $k=\frac{1}{S_0}$ and dropping the primes we get (Hofbaur 2014);
$$ds^2=-e^{-2kt}\left(dr^2-r^2d\Omega^2\right)+dt^2 . \qquad (61)$$

5. Pressure and Density in de Sitter Universe

From (42b) we get;

$$P_0 + \rho_0 = 0.$$

Since, the proper density ρ_0 can either be zero or positive, i.e., $\rho_0 \geq 0$, hence, $P_0 = 0 = \rho_0$ (Tolman 1939). This means that the de Sitter universe is completely empty (i.e., $T_{\mu\nu} = 0$). It contains neither matter nor radiation, that is, it contains neither galaxies nor observers (de Sitter 1917a). Therefore, it is incompatible with Mach's principle (Inomata 1968). de Sitter assumed the galaxies as 'probe particles' whose masses are too small to influence the curvature of space-time.

6. Motion of a Test Particle in de Sitter Universe

The motion of a test particle is given by geodesic equation (29). The survival Γ's are as follows:

$$\Gamma^1_{11} = \frac{\lambda'}{2}, \; \Gamma^1_{22} = -re^{-\lambda}, \; \Gamma^1_{33} = -re^{-\lambda}\sin^2\theta, \; \Gamma^1_{44} = \frac{1}{2}v'e^{v-\lambda}, \; \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \; \Gamma^2_{33} = -\sin\theta\cos\theta,$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \; \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta, \; \Gamma^4_{14} = \Gamma^4_{41} = \frac{v'}{2}.$$

For $r = 1$ we get from (29),

$$\frac{d^2x^1}{ds^2} + \Gamma^1_{11}\left(\frac{dx^1}{ds}\right)^2 + \Gamma^1_{22}\left(\frac{dx^2}{ds}\right)^2 + \Gamma^1_{33}\left(\frac{dx^3}{ds}\right)^2 + \Gamma^1_{44}\left(\frac{dx^4}{ds}\right)^2 = 0$$

$$\frac{d^2r}{ds^2} + \frac{\lambda'}{2}\left(\frac{dr}{ds}\right)^2 - re^{-\lambda}\left(\frac{d\theta}{ds}\right)^2 - r^2\sin^2\theta e^{-\lambda}\left(\frac{d\phi}{ds}\right)^2 + \frac{1}{2}v'e^{v-\lambda}\left(\frac{dt}{ds}\right)^2 = 0. \tag{62}$$

For $r = 2$ we get from (29),

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0. \tag{63}$$

For $r = 3$ we get from (29),

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} + 2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0. \tag{64}$$

For $r = 4$ we get from (29),

$$\frac{d^2t}{ds^2} + v'\frac{dr}{ds}\frac{dt}{ds} = 0. \tag{65}$$

Let us choose the initial motion to be in the plane $\theta = \frac{\pi}{2}$ then,

$$\sin\theta = 1, \; \cos\theta = 0 = \frac{d\theta}{ds}. \tag{66}$$

From (62) we get; $\frac{d^2\theta}{ds^2} = 0$, which implies that the particle will move in the plane $\theta = \frac{\pi}{2}$ continuously. Using (66), equations (63), (64), and (65) become;

$$\frac{d^2r}{ds^2} + \frac{\lambda'}{2}\left(\frac{dr}{ds}\right)^2 - re^{-\lambda}\left(\frac{d\theta}{ds}\right)^2 + \frac{1}{2}v'e^{v-\lambda}\left(\frac{dt}{ds}\right)^2 = 0. \tag{67}$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} = 0. \tag{68}$$

$$\frac{d^2t}{ds^2} + v'\frac{dr}{ds}\frac{dt}{ds} = 0. \tag{69}$$

From (68) we get;

$$\frac{d}{ds}\left(r^2\frac{d\phi}{ds}\right) = 0. \tag{70}$$

Integrating (70) we get;

$$r^2\frac{d\phi}{ds} = h \Rightarrow \frac{d\phi}{ds} = \frac{h}{r^2}. \tag{71}$$

From (69) we get;

$$\frac{d}{ds}\left(e^v\frac{dt}{ds}\right) = 0. \tag{72}$$

Integrating (72) we get;

$$e^v\frac{dt}{ds} = k \qquad \Rightarrow \frac{dt}{ds} = ke^{-v} = k\left(1 - \frac{r^2}{R_0^2}\right)^{-1}. \tag{73}$$

Using (66) in (39) we get,

$$e^\lambda\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2 - e^v\left(\frac{dt}{ds}\right)^2 + 1 = 0. \tag{74}$$

Using (72) and (73) in (74) we get,

$$e^\lambda\left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2} - e^v k^2\left(1 - \frac{r^2}{S_0^2}\right)^{-2} + 1 = 0$$

$$\left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2}\left(1 - \frac{r^2}{S_0^2}\right) - k^2 + \left(1 - \frac{r^2}{S_0^2}\right) = 0$$

$$\left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2}\left(1 - \frac{r^2}{S_0^2}\right) - k^2 + \left(1 - \frac{r^2}{S_0^2}\right) = 0$$

$$\frac{dr}{ds} = \left(k^2 - 1 + \frac{r^2}{S_0^2} - \frac{h^2}{r^2} + \frac{h^2}{S_0^2}\right)^{1/2}. \tag{75}$$

From (75) we get,

$$\left(\frac{h}{r^2}\frac{dr}{d\phi}\right)^2 = k^2 - 1 + \frac{r^2}{S_0^2} - \frac{h^2}{r^2} + \frac{h^2}{S_0^2}. \tag{76}$$

Let us put, $u = \frac{1}{r}$, so that, $\frac{du}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi}$ then (76) we get,

$$\left(\frac{du}{d\phi}\right)^2 = \frac{k^2 - 1}{h^2} + \frac{1}{u^2 h^2 S_0^2} - u^2 + \frac{1}{S_0^2}.$$

Differentiating we get,

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$$\frac{d^2u}{d\phi^2} + u = -\frac{1}{u^3 h^2 S_0^2}. \tag{77}$$

Equation (77) represents the orbital equation of the particle in de Sitter universe (Mohajan 2017a).

7. Velocity and Acceleration in de Sitter Universe

From (73) and (75) we get,

$$\frac{dr}{dt} = \frac{dr}{ds}\cdot\frac{ds}{dt} = \frac{1}{k}\left(1 - \frac{r^2}{S_0^2}\right)\left(k^2 - 1 + \frac{r^2}{S_0^2} - \frac{h^2}{r^2} + \frac{h^2}{S_0^2}\right)^{\frac{1}{2}}. \tag{78}$$

From (71) and (73) we get,

$$\frac{d\phi}{dt} = \frac{d\phi}{ds}\cdot\frac{ds}{dt} = \frac{h}{kr^2}\left(1 - \frac{r^2}{S_0^2}\right). \tag{79}$$

The radial velocity $\frac{dr}{dt} = 0$ when either $r = S_0$ or $k^2 - 1 + \frac{r^2}{S_0^2} - \frac{h^2}{r^2} + \frac{h^2}{S_0^2} = 0$. $\tag{80}$

The velocity component $\frac{d\phi}{dt} = 0$ if $r = S_0$, which is called the parent horizon of the universe.

Now differentiating (78) and (79) we get,

$$\frac{d^2r}{dt^2} = -\frac{2r}{kS_0^2}\frac{dr}{dt}\left(k^2 - 1 + \frac{r^2}{S_0^2} - \frac{h^2}{r^2} + \frac{h^2}{S_0^2}\right)^{\frac{1}{2}}$$

$$+ \frac{1}{2k}\left(1 - \frac{r^2}{S_0^2}\right)\left(k^2 - 1 + \frac{r^2}{S_0^2} - \frac{h^2}{r^2} + \frac{h^2}{S_0^2}\right)^{-\frac{1}{2}}\left(\frac{2r}{S_0^2} + \frac{2h^2}{r^3}\right)\frac{dr}{dt}$$

$$\frac{d^2r}{dt^2} = -\frac{\frac{2r}{S_0^2}}{1 - \frac{r^2}{S_0^2}}\left(\frac{dr}{dt}\right)^2 + \frac{1}{k^2}\left(1 - \frac{r^2}{S_0^2}\right)^2\left(\frac{r}{S_0^2} + \frac{h^2}{r^3}\right). \tag{81}$$

$$\frac{d^2\phi}{dt^2} = -\frac{2h}{kr^3}\frac{dr}{dt}. \tag{82}$$

From (82) it is obvious that for a particle with radial velocity zero, the radial acceleration $\frac{d^2r}{dt^2}$ at a point r when $0 \leq r \leq S_0$ is necessarily positive. This means that "*A free particle after receding perihelion starts to move away from the perihelion and would never return.*" Also for a particle at rest at origin with $h = 0$, the particle has zero acceleration. Hence, in de Sitter universe a particle at rest at the origin with $h = 0$ would remain at rest forever (Schmidt 1993).

8. Redshift in de Sitter Universe

Let us consider an observer situated at the origin $r = 0$ and the source of light, say a star, at $r = r$ in de Sitter universe. For a light ray emitted from the star travelling along the radial direction we have $ds = 0$, $d\theta = 0 = d\phi$, then (52) becomes (Barata et al. 2017),

$$\left(1 - \frac{r^2}{S_0^2}\right)^{-1} dr^2 = \left(1 - \frac{r^2}{S_0^2}\right) dt^2 \qquad \Rightarrow dt = dr\left(1 - \frac{r^2}{S_0^2}\right)^{-1}. \tag{83}$$

Let, the star emits the light pulse at time t, it would reach the observer at time t', then (83) becomes,

$$\int_t^{t'} dt = \int_0^r \left(1 - \frac{r^2}{S_0^2}\right)^{-1} dr \qquad \Rightarrow t' = t + \int_0^r \left(1 - \frac{r^2}{S_0^2}\right)^{-1} dr. \tag{84}$$

Let, δt be the time interval of two successive wave crests at $r = r$ and $\delta t'$ the corresponding time of their reception by an observer at rest at the origin. Then, by differentiating the equation (84) we get,

$$\delta t' = \delta t + \left(1 - \frac{r^2}{S_0^2}\right)^{-1} \frac{dr}{dt}.dt \qquad \Rightarrow \delta t' \frac{dt'}{dt} = 1 + \left(1 - \frac{r^2}{S_0^2}\right)^{-1} \frac{dr}{dt}, \tag{85}$$

where $\dfrac{dr}{dt}$ is the radial velocity of the source at the time of emission. Further, the proper time interval δs for an observer on the moving particle and the corresponding coordinate time δt, assuming the motion to be in the plane $\theta = \pi/2$, are related by the equation (73),

$$\frac{ds}{dt} = \frac{1}{k}\left(1 - \frac{r^2}{S_0^2}\right), \tag{86}$$

while the proper time interval between two successive wave crests as measured at the origin is $\delta t'$. Thus, there is a change in the frequency at the time of emission and reception. If v and v' are the frequencies at the time of emission and reception respectively, then we have;

$$\frac{v}{v'} = \frac{ds}{dt'} = \frac{ds}{dt}.\frac{dt}{dt'} = \frac{\frac{1}{k}\left(1 - \frac{r^2}{S_0^2}\right)}{1 + \left(1 - \frac{r^2}{S_0^2}\right)^{-1} \frac{dr}{dt}}. \tag{87}$$

If λ and λ' are the wave lengths at the time of emission and reception respectively, then, $v' = \dfrac{c}{\lambda}$ and $v' = \dfrac{c}{\lambda'}$. Hence, from (87) we get;

$$\frac{\lambda'}{\lambda} = \frac{1 + \left(1 - \frac{r^2}{S_0^2}\right)^{-1} \frac{dr}{dt}}{\frac{1}{k}\left(1 - \frac{r^2}{S_0^2}\right)} = k\left(1 - \frac{r^2}{S_0^2}\right)^{-1} + k\left(1 - \frac{r^2}{S_0^2}\right)^{-2} \frac{dr}{dt}. \tag{88}$$

Since $k > 0$, and $1 - \dfrac{r^2}{S_0^2} > 0$ then, $r < S_0^2$,

i) When $\dfrac{dr}{dt} > 0$ then, $\dfrac{\lambda'}{\lambda} > 0$, which means there is a redshift.

ii) When $\dfrac{dr}{dt} < 0$ then, $\dfrac{\lambda'}{\lambda}$ may be greater than or less than zero depending upon the magnitude of velocity of the distance source.

Thus, in this case there may be red or violetshift depending upon the magnitude of the velocity of the source. The violetshift is only possible when $\dfrac{dr}{dt} < 0$ is sufficiently large to make the right hand side of (45) to be negative. At the perihelion $\dfrac{dr}{dt} = 0$ and so, we get,

$$\frac{\lambda}{\lambda'} = \frac{1}{k}\left(1 - \frac{r^2}{S_0^2}\right) \tag{89}$$

which is positive, thus there is a redshift. We see that, in the de Sitter universe there may be both red and violetshifts, but, the possibility of redshift is more prominent.

Now, we consider the Welly's hypothesis "*Nebulae in the actual universe are at rest relative to a certain reference system.*" The line element (52) takes the form;

$$ds^2 = -e^{2kt}\left(dr^2 + r^2 d\Omega^2\right) + dt^2. \tag{90}$$

For a light ray emitted from the star travelling along the radial direction we have $ds = 0$, $d\theta = 0 = d\phi$, then (90) becomes,

$$\frac{dr}{dt} = \pm e^{-kt}. \tag{91}$$

Now let, the star be permanently located at r and, t and t' be the times of emission of light from the star and reception at the origin $r = 0$ then,

$$\int_{t}^{t'} e^{-kt}\, dt = \int_{0}^{r} dr = r = \text{constant} . \tag{92}$$

Differentiating (91) we get for the time interval $\delta t'$ between the reception of two successive wave crests at the origin, and the time interval δt between their emissions as;

$$e^{-kt'}\delta t' - e^{-kt}\delta t \qquad \Rightarrow \frac{\delta t'}{\delta t} = e^{k}\left(t' - t\right). \tag{93}$$

If λ and λ' are the wave lengths at the time of emission and reception respectively, then we get,

$$\frac{\lambda' - \lambda}{\lambda} = \frac{\delta\lambda}{\lambda} = k\left(t' - t\right). \tag{94}$$

If r is the distance travelled in time $\left(t' - t\right)$, then $\left(t' - t\right) = r$, since $c = 1$, then (94) becomes;

$$\frac{\delta\lambda}{\lambda} = kr = \frac{r}{R_0} \tag{95}$$

that is, the redshift is approximately proportional to the distance measured from the origin. This implies that the nebulae apart from small individual velocities are at rest in the system of coordinates used here. de Sitter managed to predict that the light emitted by these sources should be redshifted because; i) they move away from the observer due to the effect of the cosmological constant, and ii) there is a slowing down of time with increasing distance from the observer (de Sitter effect) (Chavanis 2012). The galaxy 'particles' in the de Sitter universe appear to be in accelerated motion away from the observer.

We observe that two types of redshifts exist in the de Sitter universe: i) the Doppler effect, which occurs between the dispersing galaxies, and ii) the cosmological redshift which is based on the structure of space-time. If the real world would be a de Sitter universe, the combination of both effects should be observable in the spectra of distant celestial objects (Duerbeck and Seitter 2001).

Thus, we see that, the de Sitter universe is completely empty, that is, it contains no matter (mysterious), yet it predicts the observed recession of nebulae. On the other hand, Einstein universe is full of matter; but, it does not predict the observed recession of nebulae, that is, unable to explain redshifts (Harvey and Schucking 2000). Hence, we can say that neither Einstein universe nor de Sitter universe represents a true model of the actual universe. But, de Sitter space-time has some special properties, the main one is its expanding and (contracting) property, and the other is, it possesses a horizon. Both models were stood on the assumption that the universe is static.

12. de Sitter Universe at a Glance

The de Sitter solution may be viewed as a universe in which the density of matter and radiation is negligible compared with vacuum energy. The energy density in de Sitter space is constant. In the de Sitter universe redshift for light emitted at an earlier time and detected at a later time. The age of a de Sitter universe is infinite, since only at $t \to \infty$ the scale factor $S(t) \to 0$. The de Sitter space-time is infinitely old, and has no particle horizon. Hence, an observer could in principle see every point in a de Sitter universe. If mass-energy density $\rho_0 = 0$ and pressure $P = 0$, we can write form de Sitter universe $S(t) \, \alpha \, \exp\left((\Lambda/3)^{1/2} ct\right)$, that is, the scale factor in de Sitter universe is proportional to an exponential (Islam 2002). An interesting property is that there is no singularity in the de Sitter universe at a finite time in the past, that is, $S(t)$ does not vanish for any finite value of t (Figure 2).
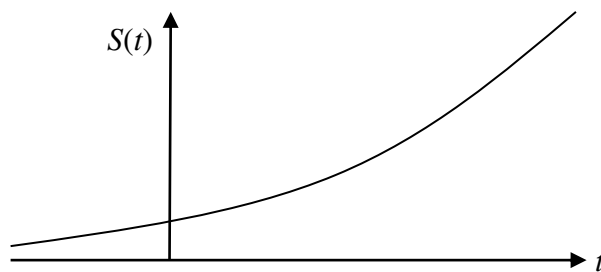

Figure 2: The behavior of $S(t)$ in the de Sitter universe.

13 Conclusions

In this study we have discussed the preliminary concepts of de Sitter universe. It is a static and closed but, empty of matter universe, that is, it contains neither matter nor radiation. It is a maximally symmetric solution of the Einstein field equation with zero density. In this universe a free particle after receding perihelion starts to move away from the perihelion and would never return. On the other hand, in de Sitter universe a particle at rest at the origin would remain at rest forever. In the de Sitter universe there may be both red and violetshifts, but, the possibility of redshift is more prominent. We have observed that de Sitter universe is also singularity free.

References

Anninos, D.; Ng, G.S. and Strominger, A. (2012), Future Boundary Conditions in de Sitter Space, arXiv:1106.1175 [hep-th].

Barata, J.C.A.; Jäkel, C.D. and Mund, J. (2017), Interacting Quantum Fields on de Sitter Space. arXiv:1607.02265v2 [math-ph] 12 Jun 2017.

Bartelmann, M. (2012), Development of Cosmology: From a Static Universe to Accelerated Expansion, In: B. Falkenburg, W. Rhode (Eds.), From Ultra Rays to Astroparticles, Springer.

Carroll, S.M. (2004), *Spacetime and Geometry: An Introduction to General Relativity*, San Francisco: Addison-Wesley.

Chavanis, P.-H. (2012), Models of Universe with a Polytropic Equation of State, arXiv:1208.0797v1 [astro-ph.CO].

de Sitter, W. (1917a), On Einstein's Theory of Gravitation and its Astronomical Consequences, *Monthly Notices of the Royal Astronomical Society*, 78**:** 3–28.

de Sitter, W. (1917b), On the Relativity of Inertia. Remarks Concerning Einstein's Latest Hypothesis, *Proceedings of the Royal Academy of Sciences at Amsterdam (KNAW)*, 19(II): 1217–1225.

Duerbeck, H.W. and Seitter, W.C. (2001), In Hubble's Shadow: Early Research on the Expansion of the Universe, In Miklos Konkoly Thege (1842–1916), 100 Years of Observational Astronomy and Astrophysics; A Collection of Papers on the History of Observational Astrophysics, C. Sterken and J. B. Hearnshaw (Eds.), pp. 231–254, Brussels: VUB Press.

Einstein, A. (1915), The Field Equations of Gravitation (in German), König, Preuss, Akad, Wiss, (Berlin) Sitzungsberichte, 844–847.

Einstein, A. (1917), Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie, Sitzungsber, Preuss, Akad, Wiss, Berlin: 142–152.

Hannabuss, K.C. (1971), The Localizability of Particles in de Sitter Space, *Proceedings of the Cambridge Philosophical Society*, 70: 283–302.

Harvey, A. and Schucking, E. (2000), Einstein's Mistake and the Cosmological Constant**,** *American Journal of Physics*, 68(8): 723–727.

Hawking, S.W. and Ellis, G.F.R. (1973), *The Large Scale Structure of Space-time*, Cambridge University Press, Cambridge.

Hofbaur, T. (2014), On the Stability of a de Sitter Universe with Self-interacting Massive Particles, PhD Thesis, Sigill University, Maximilians, München,

Inomata, A. (1968), The de Sitter Universe and Mach's Principle, *Progress of Theoretical Physics*, 39(5): 1370–1372.

Islam, J.N. (2002), *An Introduction to Mathematical Cosmology*, 2nd Ed., Cambridge University Press, Cambridge.

Joshi, P.S. (1996), *Global Aspects in Gravitation and Cosmology*, 2nd Ed., Clarendon Press, Oxford.

Joshi, P.S. (2008), *Gravitational Collapse and Spacetime Singularities*, Cambridge University Press, Cambridge.

Lemaître, G. (1925), Note on de Sitter's Universe, *Journal of Mathematical Physics*, 4**:** 188–192.

Lord, E.A. (1973), Geometry of the de-Sitter Universe, *International Journal of Theoretical Physics*, 9(2): 117–127.

Mohajan, H.K. (2017a), *Space-Time Singularities in Gravitational Cosmology and Age of the Universe*, Book to Appear.

Mohajan, H.K. (2017b), Development of Einstein's Static Cosmological Model of the Universe, *Journal of Scientific Achievements*, 2(7): 18–30.

Penrose, R. (1968), Structure of Space-Time, in Batelle Rencontres, Lectures in *Mathematics* and *Physics*. In: C. M. DeWitt and J. A. Wheeler *(*Eds.), pp. 121–235, Benjamin*,* New York.

Penrose, R. (1970), Gravitational Collapse: The Role of General Relativity, *General Relativity and Gravitation*, 34(7): 1141–1165.

Perlmutter, S.; Aldring, G.; Goldhaber, G.; Knop. R.A.; Nugent, P.; Castro, P.G.; Deustua, S.; Fabbro, S. Goobar,A.; Groom, D.E.; Hook, I.M.; Kim, A.G.; Kim, M.Y.; Lee, J.C.; Nunes, N.J. et al. (1999), Measurements of Ω and Λ from 42 High-Redshift Supernovae, *The Astrophysical Journal*, 517(2): 565–586.

Schmidt, H.J. (1993), On the de Sitter Space-Time–The Geometric Foundation of Inflationary Cosmology, *Fortschritte der Physik*, 41(3): 179–199.

Strominger, A. (2001), The dS/CFT Correspondence, JHEP 0110, 034, [arXiv:hep-th/0106113].

Tolman, R.C. (1939), Static Solution of Einstein's Field Equation for Spheres of Fluid, *Physical Review*, 55(4): 364–373.

Tolman, R.C. (1969), *Relativity, Thermodynamics and Cosmology*, Clarendon Press, Oxford.

Wikipedia (2017), Wikipedia, the Free Encyclopedia.