Gravitational Collapse of a Massive Star and Black Hole Formation

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ABSTRACT

This paper discusses the final fate of a gravitationally collapse of a massive star and the black hole formation. If the mass of a star exceeds Chandrasekhar limit then it must undergo gravitational collapse. This happens when the star has exhausted its nuclear fuel. As a result a space-time singularity is formed. It is conjectured that singularities must be hidden behind the black hole region which is called the cosmic censorship hypothesis. It has not been possible, to obtain a proof despite many attempts to establish the validity of cosmic censorship and it remains an open problem. An attempt has been taken here to describe causes of black hole formation and nature of singularities therein with easier mathematical calculations.

PACs: 04.70-Bw: Classical black holes.
Keywords: Black hole, Chandrasekhar limit, Gravitational collapse, Singularity.

1. INTRODUCTION

The existence of space-time singularities follows in the form of future or past incomplete non-spacelike geodesics in the space-time. In the approximation of the star composed of homogeneous dust without any pressure, the curvature singularity forming as the end state of collapse will be completely covered by the event horizon and would be invisible to any external observer. The singularities forming in general gravitational collapse should always be covered by the event horizon of the gravity, and remains invisible to any external observer is called the cosmic censorship hypothesis. This hypothesis, originally proposed by Penrose, remain unproved as yet in the general case; despite many attempts towards a proof, and has been recognized as one of the most important open problems in general relativity and gravitational physics. However, this throws the black hole dynamics into serious doubt.

In the Schwarzschild solution such as a singularity was present at $r = 0$ which is the final fate of a massive star, whereas in the Friedmann model it was found at the epoch $t = 0$, which is the beginning of the universe, where the scale factor $S(t)$ also vanishes and all objects are crushed to zero volume due to infinite gravitational tidal force (Hawking and Ellis 1973).

When the star is heavier than a few solar masses, it could undergo an endless gravitational collapse without achieving any equilibrium state. This happens when the star has exhausted its internal nuclear fuel which provides the outwards pressure against the inwards pulling gravitational forces. Then for a wide range of initial data, a space-time singularity must develop. Thus, cosmic censorship implies that the final outcome of gravitational collapse of a massive star must necessarily be a black hole which covers the resulting space-time singularity. So, causal message from the singularity cannot reach the external observer at infinity.

2. MANIFOLD IN DIFFERENTIAL GEOMETRY

Any point p contained in a set S can be surrounded by an open sphere or ball $|x-p|<r$, all of whose points lie entirely in S, where $r>0$; usually it is denoted by;

$$S(p,r)=\{x:d(p,x)<r\}. \qquad (1)$$

Let $\mathscr{M}$ be a non-empty set. A class T of subsets of $\mathscr{M}$ is a topology on $\mathscr{M}$ if T satisfies the following three axioms (Lipschutz 1965):

1. $\mathscr{M}$ and ϕ belong to T,
2. the union of any number of open sets in T belongs to T, and
3. the intersection of any two sets in T belongs to T.

The members of T are open sets, and the space ($\mathscr{M}$, T) is called topological space.

Let p be a point in a topological space $\mathscr{M}$. A subset N of $\mathscr{M}$ is a neighborhood of p iff N is a superset of an open set O containing p, i.e., $p\in O\subset N$.

Let R^n be the set of n-tuples $(x^1,...,x^n)$ of real numbers. A set of points $\mathscr{M}$ is defined to be a manifold if each point of $\mathscr{M}$ has an open neighborhood which is continuous one-one map onto an open set of R^n for some n.

A manifold is essentially a space which is locally similar to Euclidean space in that it can be covered with coordinate patches, but which need not be Euclidean globally. Map $\phi:O\rightarrow O'$ where $O\subset R^n$ and $O'\subset R^m$ is said to be a class $C^r(r\geq 0)$ if the following conditions are satisfied. If we choose a point p of coordinates $(x^1,...,x^n)$ on O and its image $\phi(p)$ of coordinates $(x'^1,...,x'^n)$ on O' then by C^r map we mean that the function ϕ is r-times differential and continuous. If a map is C^r for all $r\geq 0$ then we denote it by C^∞; also by C^0 map we mean that the map is continuous.

An n-dimensional, C^r, real differentiable manifold $\mathscr{M}$ is defined as follows(Hawking and Ellis 1973):

$\mathscr{M}$ has a C^r altas $\{U_\alpha,\phi_\alpha\}$ where U_α are subsets of $\mathscr{M}$ and ϕ_α are one-one maps of the corresponding U_α to open sets in R^n such that (figure 1);

1. U_α cover M i.e., $M=\underset{\alpha}{\cup}U_\alpha$,

2. If $U_\alpha \cap U_\beta \neq \phi$ then the map $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ is a C^r map of an open subset of R^n to an open subset of R^n.

Condition (2) is very important for overlapping of two local coordinate neighborhoods. Now suppose U_α and U_β overlap, and there is a point *p* in $U_\alpha \cap U_\beta$. Now choose a point *q* in $\phi_\alpha(U_\alpha)$ and a point *r* in $\phi_\beta(U_\beta)$. Now $\phi_\beta^{-1}(r) = p$, $\phi_\alpha(p) = (\phi_\alpha \circ \phi_\beta^{-1})(r) = q$. Let coordinates of *q* be $(x^1, \ldots, x^n)$ and those of *r* be $(y^1, \ldots, y^n)$. At this stage, we obtain a coordinate transformation;

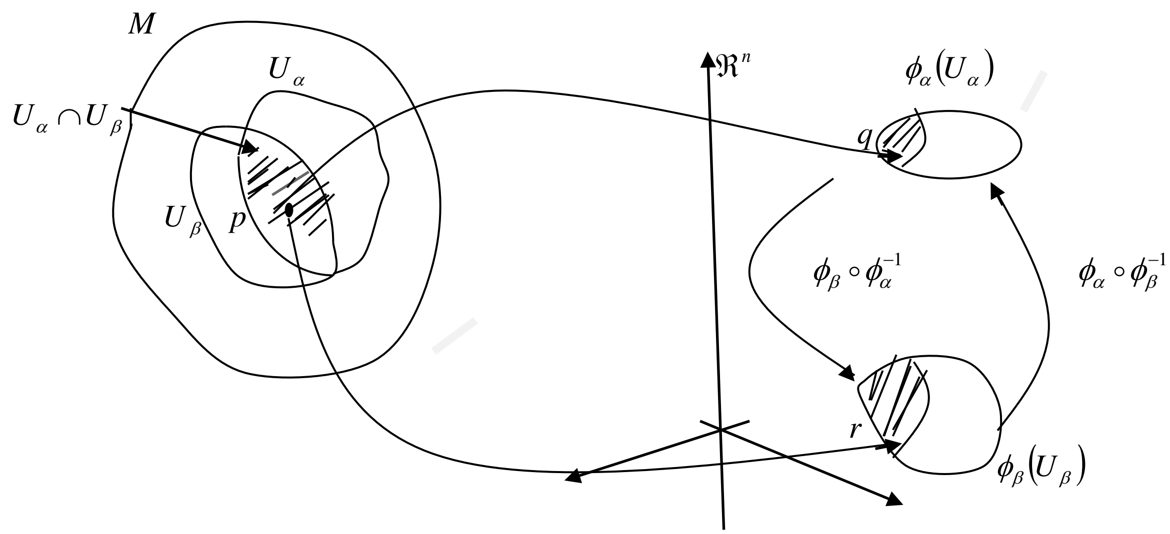

Figure 1: The smooth maps $\phi_\alpha \circ \phi_\beta^{-1}$ on the *n*-dimensional Euclidean space $\Re^n$ giving the change of coordinates in the overlap region.

$$y^1 = y^1(x^1, \ldots, x^n)$$

$$y^2 = y^2(x^1, \ldots, x^n)$$

$$\ldots \quad \ldots \quad \ldots$$

$$y^n = y^n(x^1, \ldots, x^n).$$

The open sets U_α, U_β, and maps $\phi_\alpha \circ \phi_\beta^{-1}$ and $\phi_\beta \circ \phi_\alpha^{-1}$ are all *n*-dimensional, so that C^r manifold $\mathscr{M}$ is *r*-times differentiable and continuous, i.e., $\mathscr{M}$ is a differentiable manifold.

3. GENERAL RELATIVITY AND EINSTEIN EQUATION

By the covariant differentiations, we can write (Mohajan 2013a);

$$A_{\mu;\nu;\sigma} - A_{\mu;\sigma;\nu} = R^\alpha_{\mu\nu\sigma} A_\alpha, \tag{3}$$

where $R^{\alpha}_{\mu\nu\sigma} = \Gamma^{\alpha}_{\mu\sigma;\nu} - \Gamma^{\alpha}_{\mu\nu;\sigma} + \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\sigma} - \Gamma^{\alpha}_{\beta\sigma}\Gamma^{\beta}_{\mu\nu}$ (4)

is a tensor of rank four and called Riemann curvature tensor.

The covariant curvature tensor is defined by;

$$R_{\rho\mu\nu\sigma} = \frac{1}{2}\left(\frac{\partial^2 g_{\rho\sigma}}{\partial x^{\nu}\partial x^{\mu}} + \frac{\partial^2 g_{\mu\nu}}{\partial x^{\sigma}\partial x^{\rho}} - \frac{\partial^2 g_{\mu\sigma}}{\partial x^{\nu}\partial x^{\rho}} - \frac{\partial^2 g_{\rho\nu}}{\partial x^{\sigma}\partial x^{\mu}}\right) + g_{\alpha\lambda}\left(\Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\rho\sigma} - \Gamma^{\alpha}_{\mu\omega}\Gamma^{\lambda}_{\rho\nu}\right). \tag{5}$$

Ricci tensor is defined as;

$$R_{\mu\nu} = g^{\lambda\sigma} R_{\lambda\mu\sigma\nu}. \tag{6}$$

Further contraction of (6) gives Ricci scalar;

$$\hat{R} = g^{\lambda\sigma} R_{\lambda\sigma}. \tag{7}$$

For a perfect fluid, $R_{\mu\nu}$ is defined as;

$$R_{\mu\nu}V^{\mu}V^{\nu} = -4\pi(\rho + 3p) \tag{8}$$

where V^{μ} denotes the timelike tangent vector.

The equation for geodesic is;

$$\frac{du^{\mu}}{dt} + \Gamma^{\mu}_{\nu\lambda}u^{\mu}u^{\nu} = 0. \tag{9}$$

Now we want to derive Jacobi equation that characterizes the coming together or moving away of space-time geodesics from each other as a result of the space-time curvature. Let us consider a smooth one-parameter family of affinely parametrized non-spacelike geodesics, characterized by the parameters (t, v) with $t, v \in \Re$ where t is the affine parameter along a geodesic and v = constant characterizes different geodesics in the family (figure 2).

Let vectors $T = \frac{\partial}{\partial t}$ and $V = \frac{\partial}{\partial v}$ be the coordinate vectors for which, $[T, V] = 0$, and then;

$$\nabla_T V = \nabla_V T \quad \Rightarrow T^{\mu}\nabla_{\mu} V = V^{\mu}\nabla_{\mu} T.$$

Again T is tangent to the geodesics, so,

$$T^{\mu}\nabla_{\mu}T^{\mu}=0.$$

Non-spacelike geodesics

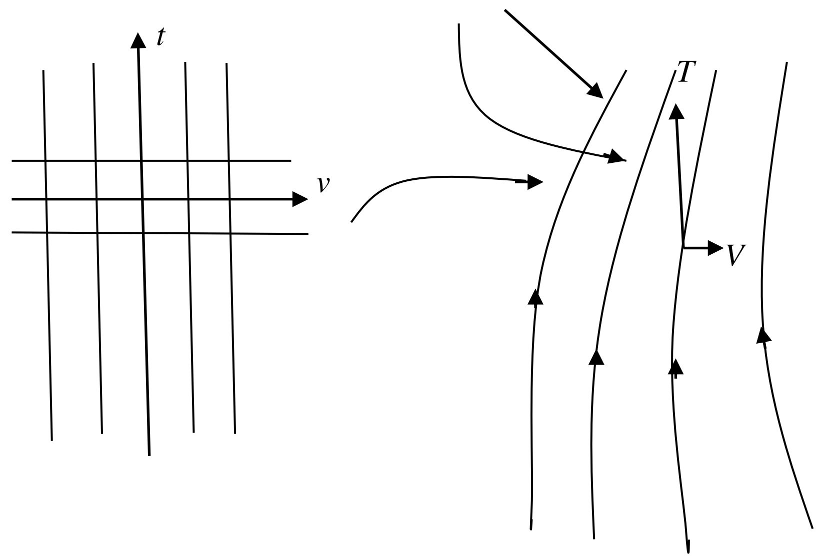

Figure 2: A one-parameter family of non-spacelike geodesics with the tangent vector *T* and separation vector *V*.

Now, define operator *D* by $D=T^{\mu}\nabla_{\mu}$ then;

$$DV^{\mu}=V^{\mu}\nabla_{\mu}T^{\nu}$$

$$D^{2}V^{\nu}=DV^{\mu}\nabla_{\mu}T^{\nu}+V^{\mu}D\left(\nabla_{\mu}T^{\nu}\right)$$

$$=\left(T^{\lambda}\nabla_{\lambda}T^{\mu}\right)\left(\nabla_{\mu}T^{\nu}\right)+V^{\mu}T^{\sigma}\nabla_{\sigma}\nabla_{\mu}T^{\nu}.$$

By the equation (3) we can write;

$$\nabla_{\sigma}\nabla_{\mu}T^{\nu}-\nabla_{\mu}\nabla_{\sigma}T^{\nu}=R^{\nu}_{\lambda\sigma\mu}T^{\lambda},$$

so that,

$$D^{2}V^{\nu}=\left(V^{\lambda}\nabla_{\lambda}T^{\mu}\right)\left(\nabla_{\mu}T^{\nu}\right)+\nabla_{\mu}\nabla_{\sigma}T^{\nu}V^{\mu}T^{\sigma}+R^{\nu}_{\lambda\sigma\mu}T^{\lambda}V^{\mu}T^{\sigma}$$

$$=V^{\lambda}\left\{\left(\nabla_{\lambda}T^{\mu}\right)\left(\nabla_{\mu}T^{\nu}\right)+\left(\nabla_{\lambda}\nabla_{\sigma}T^{\nu}\right)T^{\sigma}\right\}+R^{\nu}_{\lambda\sigma\mu}T^{\lambda}V^{\mu}T^{\sigma}$$

$$=V^{\lambda}\left\{\nabla_{\lambda}\left(T^{\mu}\nabla_{\mu}T^{\nu}\right)\right\}+R^{\nu}_{\lambda\sigma\mu}T^{\lambda}V^{\mu}T^{\sigma}$$

$$= R^{\nu}_{\lambda\sigma\mu} T^{\lambda} V^{\mu} T^{\sigma} .$$

The equation;

$$D^2 V^{\mu} = -R^{\mu}_{\nu\sigma\lambda} T^{\nu} V^{\lambda} T^{\sigma} \tag{10}$$

is called the equation of geodesic deviation or Jacobi equation.

$$D^2 V^{\mu} = -R^{\mu}_{\nu\sigma\lambda} T^{\nu} V^{\lambda} T^{\sigma} . \tag{11}$$

If $R^{\mu}_{\nu\sigma\lambda} = 0$ then $D^2 V^{\mu} = 0$, if $R^{\mu}_{\nu\sigma\lambda} \neq 0$ then the neighboring non-spacelike geodesics will necessarily accelerated towards or away from each other.

Einstein's field equation can be written as;

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{12}$$

where $G = 6.673 \times 10^{-11} m^3 kg^{-1} s^{-2}$ is the gravitational constant and $c = 10^8\, m/s$ is the velocity of light but in relativistic unit $G = c = 1$. Hence in relativistic units (12) becomes (Stephani et al. 2003);

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu} . \tag{13}$$

A perfect fluid is characterized by pressure $p = p(x^{\mu})$, then the energy momentum tensor can be written as;

$$T^{\mu\nu} = (\rho + p) u^{\mu} u^{\nu} + p g^{\mu\nu} \tag{14}$$

where ρ is the scalar density of matter.

It is clear that the divergence of both sides of (12) and (13) is zero. For empty space $T_{\mu\nu} = 0$ and hence $R_{\mu\nu} = \Lambda\, g_{\mu\nu}$, then;

$$R_{\mu\nu} = 0 \text{ for } \Lambda = 0 \tag{15}$$

which is Einstein's law of gravitation for empty space.

4. MINKOWSKI SPACE-TIME

The Minkowski space-time ($\mathscr{M}$, g) is the simplest empty space-time in general relativity and is, in fact, the space-time of the special relativity. Mathematically it is the manifold $\mathscr{M} = \Re^4$ and so the metric can be written as (Mohajan 2013a);

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{16}$$

where $-\infty < t, x, y, z < \infty$. Here coordinate t is timelike and other coordinates x, y, z are spacelike. This is a flat space-time manifold with all the components of the Riemann tensor $R^{\mu}_{\nu\lambda\sigma} = 0$. So the simplest empty space-time solution to Einstein equation is;

$$G_{\mu\nu} = 8\pi T_{\mu\nu} = 0 \tag{17}$$

which underlies of the physics of special theory of relativity. Under Lorentz transformation, the Minkowski metric preserves both time and space orientations. The vector $\frac{\partial}{\partial t}$ provides a time orientation for this model.

In spherical polar coordinates (t, r, θ, ϕ), where $t = t$, $x = r\sin\theta\sin\phi$, $y = r\sin\theta\cos\phi$ and $z = r\cos\theta$ then (16) takes the form;

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{18}$$

with $0 < r < \infty$, $0 < \theta < \pi$, $0 < \phi < 2\pi$ and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. Here coordinate t is timelike and other coordinates r, θ, ϕ are spacelike. There are two apparently singularities for $r = 0$ and $\sin\theta = 0$; however this is because the coordinates used are not admissible coordinates at these points, i.e., we used spherical polar coordinates and the frames are now non-inertial. That is why we have restriction of r, θ, ϕ as above and we need two such coordinate neighborhoods to cover all of the Minkowski space-time, then (18) will be regular. We assumed that all the components of the Riemann curvature tensor vanish for the Minkowski space-time which is a flat space-time. In (t, x, y, z) coordinates this is very clear that all the metric components are constant, i.e., diagonal $(-,+,+,+)$, so all the connection coefficients, Γ's, will vanish. But in spherical polar coordinates (t, r, θ, ϕ), the connection coefficients , Γ's, will not vanish, for example, $\Gamma^1_{22} = r$; however all the Riemann curvature components will still vanishes, $R^{\mu}_{\nu\lambda\sigma} = 0$ i.e., the manifold is still gravitation free, i.e., flat space-time (Hawking and Ellis 1973, Joshi 1996).

5. CAUSALITY AND SPACE-TIME TOPOLOGY

Given an event p in $\mathscr{M}$, the lines at 45^0 to the time axis through that event give null geodesics in $\mathscr{M}$. Such null geodesics form the boundary of the chronological future or past $I^{\pm}(p)$ of an event p

which contains all possible timelike material particle trajectories through p including timelike geodesics (Hawking and Ellis 1973, Joshi 2013, Mohajan 2013c). The causal future $J^+(p)$ is the closure of $I^+(p)$, which includes all the events in $\mathcal{M}$, which are either timelike or null related to p by means of future directed non-spacelike curves from p. An event p chronologically precedes another event q, denoted by $p << q$, if there is a smooth future directed timelike curve from p to q. If such a curve is non-spacelike then, p causally precedes q i.e., $p < q$. The chronological future $I^+(p)$ and past $I^-(p)$ of a point p are defined as (figure–3).

$$I^+(p) = \{q \in M / p << q\}, \text{ and}$$

$$I^-(p) = \{q \in M / q << p\}.$$

The causal future (past) of p can be defined as;

$$J^+(p) = \{q \in M / p < q\}, \text{ and}$$

$$J^-(p) = \{q \in M / q < p\}.$$

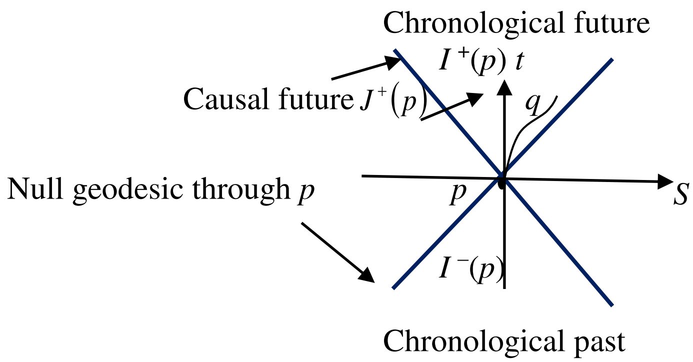

Figure 3: Causality and chronology in $\mathcal{M}$.

Also $p << q$ and $q < r$ or $p < q$ and $q << r$ implies $p << r$. Hence,

$$\overline{I^+(p)} = \overline{J^+(p)} \text{ and } \dot{I}^+(p) = \dot{J}^+(p).$$

6. SCHWARZSCHILD SPACE-TIME MANIFOLD

The Schwarzschild metric which represents the outside metric for a star is given by (Mohajan 2013c);

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{19}$$

If r_0 is the boundary of a star then $r > r_0$ gives the outside metric as in (19). If there is no surface, (19) represents a highly collapsed object viz. a black hole of mass m (will be discussed later). The metric (19) has singularities at $r = 0$ and $r = 2m$, so it represents patches $0 < r < 2m$ or $2m < r < \infty$. If we consider the patches $0 < r < 2m$ then it is seen that as r tends to zero, the curvature scalar,

$$R^{\mu\nu\sigma\lambda} R_{\mu\nu\sigma\lambda} = \frac{48m^2}{r^6} \tag{20}$$

tends to ∞ and it follows that $r = 0$ is a genuine curvature singularity i.e., space-time curvature components tend to infinity (Mohajan 2013a, c, Mohajan 2014a).

At $r = 2m$ the curvature scalars are well behaved at this point, so it is a singularity due to inappropriate choice of coordinates. The maximal extension of the manifold (19) with $2m < r < \infty$ was obtained by Kruskal (1960) and Szekeres (1960). We now discuss about this, which uses suitably defined advanced and retarded null coordinates. For null geodesics (19) takes the form,

$$\left(1 - \frac{2m}{r}\right) dt^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2$$

$$t = \pm \int \frac{r}{r - 2m} dr$$

$$= \pm \left[r + 2m \log\left(\frac{r}{2m} - 1\right)\right] + \text{constant}$$

$$t = r^* + \text{constant}. \tag{21}$$

The null coordinates u and v are defined by;

$$u = t - r^*, \qquad v = t + r^*$$

$$r^* = \frac{v - u}{2} \tag{22}$$

Now;

$$\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2$$

$$= -\frac{2m}{r}\left(\frac{r}{2m} - 1\right)\left(dt^2 - dr^{*2}\right)$$

$$= -\frac{2m}{r} e^{-r/2m} e^{(v-u)/2m} dudv$$

$$ds^2 = -\frac{2m}{r} e^{-r/2m} e^{(v-u)/2m} dudv + r^2 d\Omega^2 . \tag{23}$$

As $r \to 2m$ corresponds to $u \to \infty$ or $v \to \infty$, we define new coordinates U and V by (Mohajan 2014b);

$$U = -e^{u/4m} \ , \ V = e^{v/4m}$$

$$dUdV = \frac{1}{16m^2} e^{(v-u)/4m} dudv \ .$$

Hence, (23) becomes;

$$ds^2 = -\frac{32m^3}{r} e^{-r/2m} dUdV + r^2 d\Omega^2 . \tag{24}$$

Hence there is no singularity at $U = 0$ or $V = 0$ which corresponds to the value at $r = 2m$.

Let us take a final transformation by;

$T = \frac{U+V}{2}$ and $X = \frac{V-U}{2}$, then (23) becomes;

$$ds^2 = \frac{32m^3}{r} e^{-r/2m} \left(-dT^2 + dX^2\right) + r^2 d\Omega^2 \tag{25}$$

which is Kruskal-Szekeres form of Schwarzschild metric. Then transformation (t, r) to (T, X) becomes;

$$X^2 - T^2 = -UV = e^{(v-u)/2m} = e^{r/2m}\left(\frac{r}{2m} - 1\right) \tag{26}$$

$$\frac{T}{X} = \tanh \frac{t}{4m} \quad \Rightarrow t = 4m \tanh^{-1} \frac{T}{X} . \tag{27}$$

From (27), $r > 0$ gives $X^2 - T^2 > -1$. The physical singularity at $r = 0$ gives $X = \pm\left(T^2 - 1\right)^{1/2}$, and we observe that there is no singularity now at $r = 2m$.

7. FRIEDMANN, ROBERTSON–WALKER (FRW) MODEL

In (t, r, θ, ϕ) coordinates the Friedmann, Robertson-Walker line element is given by;

$$ds^2 = -dt^2 + S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \tag{28}$$

where $S(t)$ is the scale factor and k is a constant which denotes the spatial curvature of the three-space and could be normalized to the values +1, 0, –1. When $k = 0$ the three-space is flat and (28) is called Einstein de-Sitter static model, when $k = +1$ and $k = -1$ the three-space are of positive and negative constant curvature; these incorporate the closed and open Friedmann models respectively. Here coordinate t is timelike, and other coordinates r, θ, ϕ are spacelike, θ and ϕ are the corresponding angular coordinates in the co-moving frame (Mohajan 2014b).

8. THE KERR METRIC

We know astronomical bodies are rotating. As a result the solution outside them cannot be exactly spherically symmetric. The Kerr solutions are the stationary axisymmetric asymptotically flat field outside certain rotating object, in fact, a rotating black hole. In Boyer-Lindquist (1967) coordinates (t, r, θ, ϕ) the metric can be written as;

$$ds^2 = -\left(1 - \frac{2mr}{\Sigma}\right)dt^2 - \frac{4amr\sin^2\theta}{\Sigma}dt d\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$
$$+\left(r^2 + a^2 + \frac{2mra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 \tag{29}$$

where $\Sigma \equiv r^2 + a^2\cos^2\theta$, $\Delta \equiv r^2 - 2mr + a^2$ (30)

m and a are constants; m is gravitating mass, J is total angular momentum such that $a = \frac{J}{m}$, so a is angular momentum per unit mass. The Kerr geometry is stationary as it admits a timelike Killing vector field. When a = 0 the solution reduces to the Schwarzschild solution. Again if $r \to \infty$ then the metric components tend to the Minkowski values, which indicate the space-time is asymptotically flat.

The non-removable curvature singularity lies at $\Sigma = r^2 + a^2\cos^2\theta = 0$. This would happen when $r = \cos\theta = 0$ and the singularity has the structure of a ring of radius a, which lies in the equatorial plane z = 0. The square of timelike Killing vector $\xi^2 = \delta_0^\mu$ is given by;

$$\xi^2 = \xi^\mu \xi_\mu = g_{00} = \left(r^2 - 2mr + a^2\cos^2\theta\right)\Sigma^{-1}. \tag{31}$$

The stationary limit surface in Kerr geometry is given as a surface where the Killing vector ξ^{μ} becomes null. Hence $\xi^2 = 0 = r^2 - 2mr + a^2 \cos^2\theta$, which gives;

$$r = m \pm \left(m^2 - a^2 \cos^2\theta\right)^{1/2}. \tag{32}$$

Of the two values given above, generally the outer stationary limit is relevant for most of the discussions. To locate the black hole surface, we note that the metric has a singularity, similar to the $r = 2m$ Schwarzschild case when $\Delta = 0$, then (30) becomes;

$$r_{\pm} = m \pm \left(m^2 - a^2\right)^{1/2} \tag{33}$$

$a^2 < m^2$ gives rise two null event horizons. Here $r = r_+$ is a null surface. Thus, a particle which crosses it in future direction cannot return again to the same region. It forms the boundary of the region in the space-time from which particles can escape to the future null infinity. Here $r = r_+$ is the event horizon or the black hole surface for the future null infinity. Hence the Kerr solution is regular in three regions given by $0 < r < r_-$, $r_- < r < r_+$ and $r_+ < r < \infty$ (Mohajan 2013a).

8. SPACE-TIME SINGULARITY

Let us consider a space-time manifold $\mathcal{M} = \Re^4$. Einstein's empty space equation (17) is $R_{\mu\nu} = 0$. From this we have Schwarzschild metric (19), where there are singularities at $r = 0$ and $r = 2m$, because one of the $g^{\mu\nu}$ or $g_{\mu\nu}$ is not continuously defined. Again we have shown in (20) that $r = 0$ is a real singularity in the sense that along any non-spacelike trajectory falling into the singularity as $r \to \infty$ the Kretschman scalar $\alpha = R^{\mu\nu\gamma\sigma} R_{\mu\nu\gamma\sigma}$ tends to infinity and $r = 2m$ is a coordinate singularity (Kruskal 1960 and Szekeres 1960). Again in FRW models the Einstein equations imply that $\rho + 3p > 0$ at all times, where ρ is the total density and p is the pressure, there is a singularity at $t = 0$, since $S^2(t) \to 0$ when $t \to 0$ in the sense that curvature scalar $\hat{R} = R^{\mu\nu} R_{\mu\nu}$ bends to infinity. Here we consider the time $t = 0$ is the beginning of the universe. Thus there is an essential curvature singularity at $t = 0$ which cannot be transformed away by any coordinate transformation. The existences of real singularities where the curvature scalars and densities diverge imply that all the physical laws break down. Let us consider the metric;

$$ds^2 = -\frac{1}{t^2} dt^2 + dx^2 + dy^2 + dz^2 \tag{34}$$

which is singular on the plane $t = 0$. If any observer starting in the region $t > 0$ tries to reach the surface $t = 0$ by traveling along timelike geodesics, he will not reach at $t = 0$ in any finite time, since the surface is infinitely far into the future. If we put $t' = \log(-t)$ in $t < 0$ then (34) becomes;

$$ds^2 = -dt'^2 + dx^2 + dy^2 + dz^2 \tag{35}$$

with $-\infty < t' < \infty$ which is Minkowski metric, and there is no singularity at all, which is a removable singularity like Schwarzschild singularity at $r = 2m$. Let us consider a non-spacelike geodesic which reaches the singularity in a proper finite time. Such a geodesic will have not any end point in the regular part of the space-time. A timelike geodesic which, when maximally extended, has no end point in the regular space-time and which has finite proper length, is called timelike geodesically incomplete.

9. GRAVITATIONAL COLLAPSE OF A MASSIVE STAR

Here we describe the final fate of a massive star when it has exhausted its nuclear fuels. The star burning hydrogen produces helium, and the volume of the star get contraction in this process. At one stage the gravitational contraction is halted, and the star enters a quasi-static period, when it supports itself against gravity by means of the thermal and radiation pressures. This process may continue for billions of years, depending on the original mass of the star. If $M < M_{\odot}$ (where $M_{\odot} \approx 2\times10^{33}$ gm, the mass of the sun), this period is longer than 10^{10} years, but if $M > 10M_{\odot}$ it has to be less than 2×10^{7} years which means the heavier stars burn out their nuclear fuel much faster. The star has then exhausted much of hydrogen and produces helium, but the collapse must continue further if the star is still sufficiently massive. In the process, the core temperatures rise again to initiate thermonuclear reactions converting helium into carbon, and the core stabilizes again. For heavy enough star this process continues until a large core of stable nuclei, such as iron and nickel, is built up. The final state for such an evolution is either an equilibrium star or state of continual endless gravitational collapse. Two stability states of stars come either from electron degeneracy pressure when the star becomes a white dwarf or from neutron degeneracy pressure giving a neutron star. Indian scientist S. Chandrasekhar introduced maximum mass for a non-rotating star to achieve a white dwarf stable state as (Chandrasekhar 1983);

$$M_c \approx 1.4\left(\frac{2}{\mu_c}\right)^2 M_{\odot} \tag{36}$$

where μ_c is the constant mean molecular weight per electron. The maximum mass for non-rotating white dwarf lies in the range $1.0M_{\odot} - 1.5M_{\odot}$ depending on the composition of matter and for neutron stars this range to be $1.3M_{\odot} - 2.7M_{\odot}$ (Arnett and Bowers 1977).

A typical white dwarf has a radius of ~10^4 km and central density ~10^6 gm cm^{-3} whereas for a neutron star these numbers are given by ~10 km and ~10^{15} gm cm^{-3} respectively. Many examples of white dwarfs are known to exist in the universe and the discovery of pulsars has provided strong support for the existence of neutron stars. If a star has a mass higher than or about $5\,M_{\odot}$, it has exhausted all its nuclear fuel and no equilibrium configurations are possible unless it manages to throw away most of its mass by some process during this evolution, which is discovered by supernova explosion for the star. When the core collapse is halted or slowed down at nuclear densities, a stock wave is produced which propagates outwards in the envelope of the star. While the inner core remains a neutron star, the outer parts are driven away by the shock, releasing enormous mass and energy. But the theory of such ejection of matter is not well understood, and it seems very unlikely that the massive stars will be able to throw away almost all of their masses in such a process. Because for stars of mass $10\,M_{\odot}$, this would amount to throwing away about 90% of the mass of the star, but no mechanism is detected in support of this (Joshi 1996).

Black Holes

The stars contract due to gravitation until all matters in the star collapses to a space-time singularity, when the star is sufficiently massive so that no equilibrium state is possible and the gravitational full overcomes all the internal pressures and stresses which might stop the collapse. This creates a black hole in the space-time which covers the space-time singularity. In the Schwarzschild metric a region of trapped surface form below $r = 2m$, from which no light rays escape to an observer at infinity, so a black hole is created. Again at singularity $r = 0$, the curvature and density is infinite, the singularity is completely hidden within the trapped surface region. So, no signal from the singularity could go out to any observer at infinity, and the singularity is disconnected from the outside observers (Joshi 1996).

The black hole region is formalized by Roger Penrose as trapped surface in M which is closed compact spacelike two-surface T such that the two families of null geodesics orthogonal to it are converting at T i.e., ${}_1\chi_{\mu\nu}g^{\mu\nu} < 0$ and ${}_2\chi_{\mu\nu}g^{\mu\nu} < 0$ where ${}_1\chi_{\mu\nu}$ and ${}_2\chi_{\mu\nu}$ are the two null second fundamental forms of T. In the spherical symmetric case the event horizon is formed marginally trapped surfaces T i.e., for which ${}_1\chi_{\mu\nu}g^{\mu\nu} = 0$ and ${}_2\chi_{\mu\nu}g^{\mu\nu} < 0$ which means one family of null geodesics orthogonal to $\overline{T}$ has zero convergence whereas the other family converges 'inwards'. So the trapped surfaces must be fully contained within the black hole region, not visible from $\mathscr{I}^{+}$, where $\mathscr{I}^{+}$ is future null infinity, defined by imbedding M into another conformal manifold $\overline{M}$. In an asymptotic flat space-time M, a black hole region is defined by (figure 4);

$$B = M - J^{-}(\mathscr{I}^{+}). \tag{37}$$

The boundary of B in M is defined by;

$$H = \dot{J}^{-}(\mathscr{I}^{+}) \cap M \tag{38}$$

is called the event horizon.

In the Minkowski space-time $J^{-}(\mathscr{I}^{+}) = M$ and there is no black hole but in Schwarzschild case $J^{-}(\mathscr{I}^{+})$ is the region for space-time exterior to $r = 2m$ and the event horizon is given by the null hypersurface $r = 2m$ which is the boundary of black hole region $0 < r < 2m$, so $T \cap J^{-}(\mathscr{I}^{+}) = \phi$.

The black hole region $B = M - J^{-}(\mathscr{I}^{+})$ is closed in M. Hence, the event horizon is contained in B. Any event $x \in H$ lies on the boundary of black hole region. So smallest perturbation could make x enter $J^{-}(\mathscr{I}^{+})$, and causally connected to infinity. Then the space-time is no longer asymptotically predictable. This is avoided by demanding that for the partial Cauchy surface S,

$$J^{+}(S)\cap\overline{J^{-}}(\mathscr{I}^{+})\subset D^{+}(S). \tag{39}$$

The area of the horizon increases monotonically until the horizon reaches surface of the star. Outside, this area is a constant and is given by $A=16\pi\, m_0^2$.

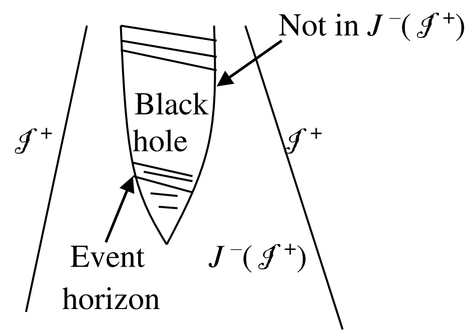

Figure 4: The black hole is a region of space-time from which no causal communications, that is, non-spacelike curves can reach the future null infinity $\mathscr{I}^{+}$.

For the Kerr-metric (29), the event horizon is defined by $r=r_{+}$ in (33). The area is obtained by setting t = constant, $r=r_{+}$ which gives the metric on the surface. Then we have (Mohajan 2013a);

$$A=\int\sqrt{g}\,d\theta d\phi=8\pi m\left[m+\left(m^2-a^2\right)^{1/2}\right]. \tag{40}$$

Hence, the area of the horizon is non-increasing function. For strongly asymptotically predictable space-time in general, the area of the black hole horizon must remain constant or must increase, provided $R_{\mu\nu}K^{\mu}K^{\nu}\geq 0$ for all null vectors K^{μ}. For all null geodesic generators of H, the expansion θ must be everywhere non-negative, $\theta\geq 0$. So, the area of the event horizon must be non-decreasing in future. If two or more black holes merge to form a single black hole, the area of its boundary must be greater than or equal to the sum of the original black hole areas. Consider two black holes with masses m_1 and m_2, and angular momenta a_1 and a_2 which collide to give a third hole with these parameters m_3, and a_3. The area of a single hole is given by (40), so the area theorem implies (Joshi 1996);

$$m_3\left[m_3+\left(m_3^2-a_3^2\right)^{1/2}\right]\geq m_1\left[m_1+\left(m_1^2-a_1^2\right)^{1/2}\right]+m_2\left[m_2+\left(m_2^2-a_2^2\right)^{1/2}\right]. \tag{41}$$

The energy radiated is $m_1+m_2-m_3$, or the function of total energy radiated is;

$$f = 1 - \frac{m_3}{m_1 + m_2}. \tag{42}$$

Then using the inequality (41) it can be seen that $f < \frac{1}{2}$, that is, at most half the initial energy could be released on black hole collisions. We can introduce irreducible mass m_0 for a black hole by the relation;

$$A = 16\pi\, m_0^2. \tag{43}$$

By (43) we can find;

$$m^2 = m_0^2 + \frac{J^2}{4m_0^2} \tag{44}$$

where J is the angular momentum of the black hole. The second term of (44) represents the rotational contribution to the black hole mass. According to the area theorem above, for all physically allowed processes, the total area of black holes cannot decrease, that is, $\delta A \geq 0$. For Schwarzschild black hole, the only way to reduce this area is to extract mass from black hole, which is impossible because no particle or photons can cross the event horizon to come out. Also one can increase area by throwing particles in black hole. Black hole physics is equivalent to the second law of thermodynamics, which states the total entropy of all the matter in the universe is non-decreasing, that is, $\delta S \geq 0$. A black hole is a perfect absorber and does not emit at all, so the thermodynamic temperature of a black hole in classical relativity will be absolute zero. But Hawking has shown that when quantum particle creation effects are taken into account, a black hole actually radiates with a black body spectrum at a temperature proportional to the surface gravity.

CONCLUDING REMARKS

In this article we have discussed the gravitational collapse of a massive star with the help of topology and differential geometry. Chandrasekhar expressed that if the mass of a star is greater than 1.4 times of the solar mass then it must undergo gravitational collapse when it has exhausted its nuclear fuel due to the outwards pressure against the inwards pulling gravitational forces. When the star is heavier than a few solar masses, it could undergo an endless gravitational collapse without achieving any equilibrium state. At this situation the star becomes a black hole in the space-time which covers the space-time singularity, which is called the cosmic censorship hypothesis. Throughout the paper, we have tried to represent every section with easier mathematical calculations.

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