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Sion's minimax theorem and Nash equilibrium of symmetric multi-person zero-sum game**

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Abstract

About a symmetric multi-person zero-sum game we will show the following results.

- (1) The existence of a symmetric Nash equilibrium is proved by the Glicksberg fixed point theorem.
- (2) Sion's minimax theorem and the coincidence of the maximin strategy and the minimax strategy are proved by the existence of a symmetric Nash equilibrium.
- (3) The existence of a symmetric Nash equilibrium is proved by Sion's minimax theorem and the coincidence of the maximin strategy and the minimax strategy.

If a zero-sum game is asymmetric, maximin strategies and minimax strategies of players do not correspond to Nash equilibrium strategies. However, if it is symmetric, the maximin strategies and the minimax strategies constitute a Nash equilibrium. With only the minimax theorem there may exist an asymmetric equilibrium in a symmetric multi-person zero-sum game.

Keywords: multi-person zero-sum game, Nash equilibrium, Sion's minimax theorem.

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1. Introduction

We consider the relation between Sion's minimax theorem for a continuous function and the existence of Nash equilibrium in a symmetric multi-person zero-sum game. We will show the following results.

- (1) The existence of a symmetric Nash equilibrium is proved by the Glicksberg fixed point theorem.
- (2) Sion's minimax theorem and the coincidence of the maximin strategy and the minimax strategy are proved by the existence of a symmetric Nash equilibrium.
- (3) The existence of a symmetric Nash equilibrium is proved by Sion's minimax theorem and the coincidence of the maximin strategy and the minimax strategy.

An example of such a game is a relative profit maximization game in a Cournot oligopoly. Suppose that there are $n \geq 3$ firms in an oligopolistic industry. Let $\bar{\pi}_i$ be the absolute profit of the i -th firm. Then, its relative profit is

$$\pi_i = \bar{\pi}_i - \frac{1}{n-1} \sum_{j=1, j \neq i}^n \bar{\pi}_j.$$

We see

$$\sum_{i=1}^n \pi_i = \sum_{i=1}^n \bar{\pi}_i - \frac{1}{n-1} (n-1) \sum_{j=1}^n \bar{\pi}_j = 0.$$

Thus, the relative profit maximization game in a Cournot oligopoly is a zero-sum game². If the oligopoly is asymmetric because the demand function is not symmetric (in a case of differentiated goods) or firms have different cost functions (in both homogeneous and differentiated goods cases), maximin strategies and minimax strategies of firms do not correspond to Nash equilibrium strategies. However, if the demand function is symmetric and the firms have the same cost function, the maximin strategies and the minimax strategies constitute a Nash equilibrium. With only the minimax theorem there may exist an asymmetric equilibrium in a symmetric multi-person zero-sum game.

In Section 3 we will show the main results, and in Section 4 we present an example of an asymmetric n -person zero-sum game.

²About relative profit maximization under imperfect competition please see Matsumura, Matsushima and Cato (2013), Satoh and Tanaka (2013), Satoh and Tanaka (2014a), Satoh and Tanaka (2014b), Tanaka (2013a), Tanaka (2013b) and Vega-Redondo (1997)

2. The model and Sion's minimax theorem

Consider a symmetric n -person zero-sum game with $n \geq 3$ as follows. There are n players, $1, 2, \dots, n$. The set of players is denoted by N . A vector of strategic variables is $(s_1, s_2, \dots, s_n) \in S_1 \times S_2 \times \dots \times S_n$. S_i is a convex and compact set in a linear topological space for each $i \in N$. The payoff functions of the players are $u_i(s_1, s_2, \dots, s_n)$ for $i \in N$. We assume

u_i for each $i \in N$ is continuous and quasi-concave on S_i for each $s_j \in S_j$, $j \in N$, $j \neq i$. It is continuous and quasi-convex on S_j for $j \in N$, $j \neq i$ for each $s_i \in S_i$.

Symmetry of a game means that all players have the same payoff function and in the payoff function of each Player i , Players j and k , $j, k \neq i$, are interchangeable. If the game is symmetric and zero-sum, we have

$$\sum_{i=1}^n u_i(s_1, s_2, \dots, s_n) = 0, \quad (1)$$

for given (s_1, s_2, \dots, s_n) . Also all S_i 's are identical. Denote them by S .

Sion's minimax theorem (Sion (1958), Komiya (1988), Kindler (2005)) for a continuous function is stated as follows.

Lemma 1. *Let X and Y be non-void convex and compact subsets of two linear topological spaces, and let $f : X \times Y \rightarrow \mathbb{R}$ be a function that is continuous and quasi-concave in the first variable and continuous and quasi-convex in the second variable. Then*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

We follow the description of this theorem in Kindler (2005).

Suppose that $s_k \in S_k$ for all $k \in N$ other than i and j , $j \neq i$, are given. Denote a vector of such s_k 's by $s_{-i,j}$. Then, $u_i(s_1, s_2, \dots, s_n)$ is written as $u_i(s_i, s_j, s_{-i,j})$, and it is a function of s_i and s_j . We can apply Lemma 1 to such a situation, and get the following lemma.

Lemma 2. *Let $j \neq i$, and S_i and S_j be non-void convex and compact subsets of two linear topological spaces, and let $u_i : S_i \times S_j \rightarrow \mathbb{R}$ given $s_{-i,j}$ be a function that is continuous and quasi-concave on S_i and continuous and quasi-convex on S_j . Then*

$$\max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, s_{-i,j}) = \min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j, s_{-i,j}).$$

We assume that $\arg \max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j, s_{-i,j})$ and $\arg \min_{s_j \in S_j} \max_{s_i \in S_i} u_i(s_i, s_j, s_{-i,j})$ are single-valued for any pair of i and j . By the maximum theorem they are continuous in $s_{-i,j}$.

Since we consider a symmetric game, by Lemma 2 we can assume that when $s_{-i,j} = s_{-k,l}$,

$$\begin{aligned} \max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, s_{-i,j}) &= \max_{s_k \in S} \min_{s_l \in S} u_k(s_k, s_l, s_{-k,l}) \\ &= \min_{s_l \in S} \max_{s_k \in S} u_k(s_k, s_l, s_{-k,l}) = \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, s_{-i,j}), \end{aligned}$$

$$\arg \max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, s_{-i,j}) = \arg \max_{s_k \in S} \min_{s_l \in S} u_k(s_k, s_l, s_{-k,l}),$$

and

$$\arg \min_{s_l \in S} \max_{s_k \in S} u_k(s_k, s_l, s_{-k,l}) = \arg \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, s_{-i,j}) \text{ for } i, j, k, l \in N.$$

They mean

$$\begin{aligned} \max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, s_{-i,j}) &= \max_{s_j \in S} \min_{s_i \in S} u_j(s_i, s_j, s_{-i,j}) \\ &= \min_{s_i \in S} \max_{s_j \in S} u_j(s_i, s_j, s_{-i,j}) = \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, s_{-i,j}), \end{aligned}$$

$$\arg \max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, s_{-i,j}) = \arg \max_{s_j \in S} \min_{s_i \in S} u_j(s_i, s_j, s_{-i,j}),$$

and

$$\arg \min_{s_l \in S} \max_{s_k \in S} u_k(s_k, s_l, s_{-k,l}) = \arg \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, s_{-i,j}) \text{ for any } i, j.$$

Assume $(s_1, s_2, \dots, s_n) = (s, s, \dots, s)$, and let $\mathbf{s}_{-i,j}$ be a vector of s_k , $k \in N$, $k \neq i, j$ such that $s_k = s$. Then, for a symmetric game Lemma 2 is rewritten as follows.

Lemma 3. *Let $j \neq i$, and S_i and S_j be non-void convex and compact subsets of two linear topological spaces, let $u_i : S_i \times S_j \rightarrow \mathbb{R}$ given $\mathbf{s}_{-i,j}$ be a function that is continuous and quasi-concave on S_i and continuous and quasi-convex on S_j , and assume $S_i = S_j = S$. Then*

$$\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}) = \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}) \text{ for any } i, j.$$

3. The main results

Consider a Nash equilibrium of a symmetric multi-person zero-sum game. Let s_i^* , $i \in N$, be the values of s_i 's which, respectively, maximize u_i , $i \in N$, given s_j^* , $j \neq i$, in a neighborhood around $(s_1^*, s_2^*, \dots, s_n^*)$ in $S_1 \times S_2 \times \dots \times S_n = S^n$. Then,

$$u_i(s_1^*, \dots, s_i^*, \dots, s_n^*) \geq u_i(s_1^*, \dots, s_i, \dots, s_n^*) \text{ for all } s_i \neq s_i^*, i \in N.$$

If the Nash equilibrium is symmetric, all s_i^* 's are equal at equilibria. Then, $u_i(s_1^*, \dots, s_i^*, \dots, s_n^*)$'s for all i are equal, and by the property of zero-sum game they are zero.

We show the following theorem.

Theorem 1. (1) *The existence of Nash equilibrium in a symmetric multi-person zero-sum game is proved by the Glicksberg fixed point theorem.*

(2) *The existence of Nash equilibrium in a symmetric multi-person zero-sum game implies Sion's minimax theorem, and implies that the maximin strategy and the minimax strategy for each pair of players coincide at the symmetric Nash equilibrium.*

Proof. (1) Let $(s_1, s_2, \dots, s_n) = (s, s, \dots, s)$. Consider a function;

$$s \rightarrow \max_{s_i \in S} u_i(s_i, \mathbf{s}_{-i}), \quad s \in S, \quad i \in N.$$

\mathbf{s}_{-i} is a vector of s_k , $k \in N$, $k \neq i$ such that $s_k = s$. This is the same function for each $i \in N$. Since u_i is continuous and S is compact, it is continuous in s . Thus, there exists a fixed point. Denote it by s^* . Then,

$$s^* = \max_{s_i \in S} u_i(s_i, \mathbf{s}_{-i}^*).$$

\mathbf{s}_{-i}^* is a vector of s_k , $k \in N$, $k \neq i$ such that $s_k = s^*$. This means that $\mathbf{s}^* = (s^*, s^*, \dots, s^*)$ is a symmetric Nash equilibrium.

(2) Let (s^*, s^*, \dots, s^*) be a symmetric Nash equilibrium of an n -person zero-sum game. Then,

$$u_i(s^*, s^*, \mathbf{s}_{-i,j}^*) = \max_{s_i \in S} u_i(s_i, s^*, \mathbf{s}_{-i,j}^*) \geq u_i(s_i, s^*, \mathbf{s}_{-i,j}^*). \quad (2)$$

$\mathbf{s}_{-i,j}^*$ is a vector of s_k , $k \in N$, $k \neq i, j$ such that $s_k = s^*$. Since the game is zero-sum,

$$u_i(s_i, s^*, \mathbf{s}_{-i,j}^*) + (n-1)u_j(s_i, s^*, \mathbf{s}_{-i,j}^*) = 0, \quad j \neq i$$

imply

$$u_i(s_i, s^*, \mathbf{s}_{-i,j}^*) = -(n-1)u_j(s_i, s^*, \mathbf{s}_{-i,j}^*).$$

Thus,

$$\arg \max_{s_i \in S} u_i(s_i, s^*, \mathbf{s}_{-i,j}^*) = \arg \min_{s_i \in S} u_j(s_i, s^*, \mathbf{s}_{-i,j}^*).$$

By the symmetry of the game,

$$\arg \max_{s_i \in S} u_i(s_i, s^*, \mathbf{s}_{-i,j}^*) = \arg \min_{s_j \in S} u_i(s^*, s_j, \mathbf{s}_{-i,j}^*) = s^*.$$

Therefore,

$$u_i(s^*, s^*, \mathbf{s}_{-i,j}^*) = \min_{s_j \in S} u_i(s^*, s_j, \mathbf{s}_{-i,j}^*) \leq u_i(s^*, s_j, \mathbf{s}_{-i,j}^*).$$

With (2), we get

$$\max_{s_i \in S} u_i(s_i, s^*, \mathbf{s}_{-i,j}^*) = u_i(s^*, s^*, \mathbf{s}_{-i,j}^*) = \min_{s_j \in S} u_i(s^*, s_j, \mathbf{s}_{-i,j}^*).$$

This means

$$\begin{aligned} \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}^*) &\leq \max_{s_i \in S} u_i(s_i, s^*, \mathbf{s}_{-i,j}^*) \\ &= \min_{s_j \in S} u_i(s^*, s_j, \mathbf{s}_{-i,j}^*) \leq \max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}^*). \end{aligned} \quad (3)$$

On the other hand, since

$$\min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}^*) \leq u_i(s_i, s_j, \mathbf{s}_{-i,j}^*),$$

we have

$$\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}^*) \leq \max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}^*).$$

Thus,

$$\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}^*) \leq \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}^*).$$

With (3), we obtain

$$\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}^*) = \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}^*).$$

From

$$\min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}^*) \leq u_i(s_i, s^*, \mathbf{s}_{-i,j}^*),$$

and

$$\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}^*) = \max_{s_i \in S} u_i(s_i, s^*, \mathbf{s}_{-i,j}^*),$$

we have

$$\arg \max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}^*) = \arg \max_{s_i \in S} u_i(s_i, s^*, \mathbf{s}_{-i,j}^*) = s^*.$$

Also, from

$$\max_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}^*) \geq u_i(s^*, s_j, \mathbf{s}_{-i,j}^*),$$

and

$$\min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}^*) = \min_{s_j \in S} u_i(s^*, s_j, \mathbf{s}_{-i,j}^*),$$

we get

$$\arg \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}^*) = \arg \min_{s_j \in S} u_i(s^*, s_j, \mathbf{s}_{-i,j}^*) = s^*.$$

Therefore,

$$\arg \max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}^*) = \arg \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}^*).$$

□

Next we show the following theorem.

Theorem 2. *Sion's minimax theorem and the coincidence of the maximin strategy and the minimax strategy imply the existence of a symmetric Nash equilibrium.*

Proof. Let $\mathbf{s} = (s, s, \dots, s)$. By the minimax theorem

$$\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}) = \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}).$$

Assume

$$\arg \max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}) = \arg \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}). \quad (4)$$

Consider the following function;

$$s \rightarrow \arg \max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}).$$

Since u_i is continuous and S is compact, this function is also continuous. Thus, by the Glicksberg fixed point theorem there exists a fixed point. Denote it by \tilde{s} . Let $\tilde{\mathbf{s}} = (\tilde{s}, \tilde{s}, \dots, \tilde{s})$. Then, from the minimax theorem and

$$\tilde{s} = \arg \max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \arg \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}),$$

we have

$$\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \min_{s_j \in S} u_i(\tilde{s}, s_j, \tilde{\mathbf{s}}_{-i,j}) = \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \max_{s_i \in S} u_i(s_i, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}).$$

$\tilde{\mathbf{s}}_{-i,j}$ is a vector of s_k , $k \in N$, $k \neq i, j$ such that $s_k = \tilde{s}$. Since

$$u_i(\tilde{s}, s_j, \tilde{\mathbf{s}}_{-i,j}) \leq \max_{s_i \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}),$$

and

$$\min_{s_j \in S} u_i(\tilde{s}, s_j, \tilde{\mathbf{s}}_{-i,j}) = \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}),$$

we get

$$\arg \min_{s_j \in S} u_i(\tilde{s}, s_j, \tilde{\mathbf{s}}_{-i,j}) = \arg \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \tilde{s}.$$

Also, since

$$u_i(s_i, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}) \geq \min_{s_j \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}),$$

and

$$\max_{s_i \in S} u_i(s_i, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}) = \max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}),$$

we obtain

$$\arg \max_{s_i \in S} u_i(s_i, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}) = \arg \max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \tilde{\mathbf{s}}_{-i,j}) = \tilde{s}.$$

Therefore,

$$u_i(\tilde{s}, s_j, \tilde{\mathbf{s}}_{-i,j}) \geq u_i(\tilde{s}, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}) \geq u_i(s_i, \tilde{s}, \tilde{\mathbf{s}}_{-i,j}),$$

and so $(\tilde{s}, \tilde{s}, \dots, \tilde{s})$ is a symmetric Nash equilibrium of an n -person zero-sum game. \square

Note on the case where (4) is not assumed.

Let $\mathbf{s} = (s, s, \dots, s)$, and define

$$s^1 = \arg \max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}),$$

$$s^2 = \arg \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \mathbf{s}_{-i,j}).$$

Let \bar{s} be the fixed point of the following function;

$$s \rightarrow s^1(s).$$

Then, by the minimax theorem

$$\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \bar{\mathbf{s}}_{-i,j}) = \min_{s_j \in S} u_i(\bar{s}, s_j, \bar{\mathbf{s}}_{-i,j}) = \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \bar{\mathbf{s}}_{-i,j}).$$

$\bar{\mathbf{s}}_{-i,j}$ is a vector of s_k , $k \in N$, $k \neq i$ such that $s_k = \bar{s}$. Since

$$\max_{s_i \in S} u_i(s_i, s_j, \bar{\mathbf{s}}_{-i,j}) \geq u_i(\bar{s}, s_j, \bar{\mathbf{s}}_{-i,j}),$$

and

$$\min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \bar{\mathbf{s}}_{-i,j}) = \min_{s_j \in S} u_i(\bar{s}, s_j, \bar{\mathbf{s}}_{-i,j}),$$

we have

$$\arg \min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \bar{\mathbf{s}}_{-i,j}) = \arg \min_{s_j \in S} u_i(\bar{s}, s_j, \bar{\mathbf{s}}_{-i,j}) = s^2.$$

Then,

$$\min_{s_j \in S} \max_{s_i \in S} u_i(s_i, s_j, \bar{\mathbf{s}}_{-i,j}) = \max_{s_i \in S} u_i(s_i, s^2, \bar{\mathbf{s}}_{-i,j}).$$

Since

$$\min_{s_j \in S} u_i(s_i, s_j, \bar{\mathbf{s}}_{-i,j}) \leq u_i(s_i, s^2, \bar{\mathbf{s}}_{-i,j}),$$

and

$$\max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \bar{\mathbf{s}}_{-i,j}) = \max_{s_i \in S} u_i(s_i, s^2, \bar{\mathbf{s}}_{-i,j}),$$

we have

$$\arg \max_{s_i \in S} \min_{s_j \in S} u_i(s_i, s_j, \bar{\mathbf{s}}_{-i,j}) = \arg \max_{s_i \in S} u_i(s_i, s^2, \bar{\mathbf{s}}_{-i,j}) = \bar{s}. \quad (5)$$

Because the game is symmetric and zero-sum,

$$(n-1)u_i(\bar{s}, s_j, \bar{\mathbf{s}}_{-i,j}) + u_j(\bar{s}, s_j, \bar{\mathbf{s}}_{-i,j}) = 0.$$

Thus,

$$u_i(\bar{s}, s_j, \bar{s}_{-i,j}) = -(n-1)u_j(\bar{s}, s_j, \bar{s}_{-i,j}).$$

This means

$$\arg \min_{s_j \in S} u_i(\bar{s}, s_j, \bar{s}_{-i,j}) = \arg \max_{s_j \in S} u_j(\bar{s}, s_j, \bar{s}_{-i,j}) = s^2. \quad (6)$$

(5) is applicable to each player other than one player denoted by j in (6). Therefore, if $s^2 \neq \bar{s}$, there may exist an asymmetric Nash equilibrium denoted as follows.

$$(\bar{s}, \dots, \bar{s}, s^2, \bar{s}, \dots, \bar{s})$$

In which only $s_j = s^2$. Of course, Theorem 1 means that there always exist a symmetric Nash equilibrium. Thus, in this case we have multiple equilibria.

4. Example of asymmetric multi-person zero-sum game

Consider a three-person game. Suppose that the payoff functions of players are

$$\pi_1 = (a - s_1 - s_2 - s_3)s_1 - c_1 s_1 - \frac{1}{2}[(a - s_2 - s_1 - s_3)s_2 - c_2 s_2 + (a - s_3 - s_2 - s_1)s_3 - c_3 s_3],$$

$$\pi_2 = (a - s_2 - s_1 - s_3)s_2 - c_2 s_2 - \frac{1}{2}[(a - s_1 - s_2 - s_3)s_1 - c_1 s_1 + (a - s_3 - s_2 - s_1)s_3 - c_3 s_3],$$

and

$$\pi_3 = (a - s_3 - s_2 - s_1)s_3 - c_3 s_3 - \frac{1}{2}[(a - s_1 - s_2 - s_3)s_1 - c_1 s_1 + (a - s_2 - s_1 - s_3)s_2 - c_2 s_2].$$

This is a model of relative profit maximization in a three firms Cournot oligopoly with constant marginal costs and zero fixed cost producing a homogeneous good. $s_i, i = 1, 2, 3$, are the outputs of the firms. The conditions for maximization of $\pi_i, i = 1, 2, 3$, are

$$\frac{\partial \pi_1}{\partial s_1} = a - 2s_1 - (s_2 + s_3) - c_1 + \frac{1}{2}(s_2 + s_3) = 0,$$

$$\frac{\partial \pi_2}{\partial s_2} = a - 2s_2 - (s_1 + s_3) - c_2 + \frac{1}{2}(s_1 + s_3) = 0,$$

and

$$\frac{\partial \pi_3}{\partial s_3} = a - 2s_3 - (s_2 + s_1) - c_3 + \frac{1}{2}(s_2 + s_1) = 0.$$

The Nash equilibrium strategies are

$$s_1 = \frac{3a - 5c_1 + c_2 + c_3}{9}, \quad s_2 = \frac{3a - 5c_2 + c_1 + c_3}{9}, \quad s_3 = \frac{3a - 5c_3 + c_2 + c_1}{9}. \quad (7)$$

We consider maximin and minimax strategy about Player 1 and 2. The condition for minimization of π_1 with respect to s_2 is $\frac{\partial \pi_1}{\partial s_2} = 0$. Denote s_2 which satisfies this condition by $s_2(s_1, s_3)$, and substitute it into π_1 . Then, the condition for maximization of π_1 with respect to s_1 given $s_2(s_1, s_3)$ and s_3 is

$$\frac{\partial \pi_1}{\partial s_1} + \frac{\partial \pi_1}{\partial s_2} \frac{ds_2}{ds_1} = 0.$$

We call the strategy of Player 1 obtained from these conditions the maximin strategy of Player 1 to Player 2. It is denoted by $\arg \max_{s_1} \min_{s_2} \pi_1$. The condition for maximization of π_1 with respect to s_1 is $\frac{\partial \pi_1}{\partial s_1} = 0$. Denote s_1 which satisfies this condition by $s_1(s_2, s_3)$, and substitute it into π_1 . Then, the condition for minimization of π_1 with respect to s_2 given $s_1(s_2, s_3)$ is

$$\frac{\partial \pi_1}{\partial s_2} + \frac{\partial \pi_1}{\partial s_1} \frac{ds_1}{ds_2} = 0.$$

We call the strategy of Player 2 obtained from these conditions the minimax strategy of Player 2 to Player 1. It is denoted by $\arg \min_{s_2} \max_{s_1} \pi_1$. In our example we obtain

$$\begin{aligned} \arg \max_{s_1} \min_{s_2} \pi_1 &= \frac{3a - 4c_1 + c_2}{9}, \\ \arg \min_{s_2} \max_{s_1} \pi_1 &= \frac{6a - 9s_3 - 2c_1 - 4c_2}{9}. \end{aligned}$$

Similarly, we get the following results.

$$\begin{aligned} \arg \max_{s_2} \min_{s_1} \pi_2 &= \frac{3a - 4c_2 + c_1}{9}, \\ \arg \min_{s_1} \max_{s_2} \pi_2 &= \frac{6a - 9s_3 - 2c_2 - 4c_1}{9}, \\ \arg \max_{s_1} \min_{s_3} \pi_1 &= \frac{3a - 4c_1 + c_3}{9}, \\ \arg \min_{s_3} \max_{s_1} \pi_1 &= \frac{6a - 9s_2 - 2c_1 - 4c_3}{9}, \end{aligned}$$

$$\begin{aligned}
\arg \max_{s_3} \min_{s_1} \pi_3 &= \frac{3a - 4c_3 + c_1}{9}, \\
\arg \min_{s_1} \max_{s_3} \pi_3 &= \frac{6a - 9s_2 - 2c_3 - 4c_1}{9}, \\
\arg \max_{s_2} \min_{s_3} \pi_2 &= \frac{3a - 4c_2 + c_3}{9}, \\
\arg \min_{s_3} \max_{s_2} \pi_2 &= \frac{6a - 9s_1 - 2c_2 - 4c_3}{9}, \\
\arg \max_{s_3} \min_{s_2} \pi_3 &= \frac{3a - 4c_3 + c_2}{9}, \\
\arg \min_{s_2} \max_{s_3} \pi_3 &= \frac{6a - 9s_1 - 2c_3 - 4c_2}{9}.
\end{aligned}$$

If the game is asymmetric, for example, $c_2 \neq c_3$, $\arg \max_{s_1} \min_{s_2} \pi_1 \neq \arg \max_{s_1} \min_{s_3} \pi_1$, $\arg \max_{s_2} \min_{s_3} \pi_2 \neq \arg \max_{s_3} \min_{s_2} \pi_3$, $\arg \min_{s_3} \max_{s_2} \pi_2 \neq \arg \min_{s_2} \max_{s_3} \pi_3$, and so on. However, if the game is symmetric, we have $c_2 = c_3 = c_1$ and

$$\begin{aligned}
\arg \max_{s_1} \min_{s_2} \pi_1 &= \arg \max_{s_2} \min_{s_1} \pi_2 = \arg \max_{s_1} \min_{s_3} \pi_1 = \arg \max_{s_3} \min_{s_1} \pi_3 \\
&= \arg \max_{s_2} \min_{s_3} \pi_2 = \arg \max_{s_3} \min_{s_2} \pi_3 = \frac{a - c_1}{3}.
\end{aligned}$$

All of the Nash equilibrium strategies of the players in (7) are also equal to $\frac{a-c_1}{3}$. Assume $s_2 = s_3 = s_1$ as well as $c_2 = c_3 = c_1$. Then,

$$\begin{aligned}
\arg \min_{s_2} \max_{s_1} \pi_1 &= \arg \min_{s_1} \max_{s_2} \pi_2 = \arg \min_{s_3} \max_{s_1} \pi_1 = \arg \min_{s_1} \max_{s_3} \pi_3 \\
&= \arg \min_{s_3} \max_{s_2} \pi_2 = \arg \min_{s_2} \max_{s_3} \pi_3 = \frac{2a - 3s_1 - 2c_1}{3}.
\end{aligned}$$

Further, if

$$s_1 = \arg \min_{s_1} \max_{s_2} \pi_2 = \arg \min_{s_1} \max_{s_3} \pi_3,$$

we obtain

$$\begin{aligned}
\arg \min_{s_2} \max_{s_1} \pi_1 &= \arg \min_{s_1} \max_{s_2} \pi_2 = \arg \min_{s_3} \max_{s_1} \pi_1 = \arg \min_{s_1} \max_{s_3} \pi_3 \\
&= \arg \min_{s_3} \max_{s_2} \pi_2 = \arg \min_{s_2} \max_{s_3} \pi_3 = \frac{a - c_1}{3}.
\end{aligned}$$

Therefore, the maximin strategy, the minimax strategy and the Nash equilibrium strategy for all players are equal.

5. Concluding Remark

In this paper we have shown that Sion's minimax theorem plus coincidence of the maximin strategy and the minimax strategy is equivalent to the existence of a symmetric Nash equilibrium in a symmetric multi-person zero-sum game. As we have shown in Section 4, if a game is asymmetric, the equivalence result does not hold.

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