On some decisive players for linear efficient and symmetric values in cooperative games with transferable utility

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Abstract

The main goal of the paper is to shed light on economic allocations issues, in particular by focusing on individuals who receive nothing (that is an amount of zero allocation or payoff). It is worth noting that such individuals may be considered, in some contexts, as poor or socially excluded. To this end, our study relies on the notion of cooperative games with transferable utility and the Linear Efficient and Symmetric values (called LES values) are considered as allocation rules. Null players in Shapley sense are extensively studied ; two broader classes of null players are introduced. The analysis is facilitated by the help of a parametric representation of LES values. It is clearly shown that the control of what a LES value assigns as payoffs to null players gives significant information about the characterization of the value. Several axiomatic characterizations of subclasses of LES values are provided using our approach.

Keywords : TU-game, Linear Efficient and Symmetric value, Null players, Average null players, Shapley value, Solidarity value.

Jel Classification : C71, D61

1 Introduction

One of the main issues in the theory of economic allocation consists in pointing out the group of individuals who receive nothing (zero as allocation or payoff). Such individuals, in some contexts, are considered to be poor or socially excluded. Furthermore, discussion about the identification of these classes of individuals may bring out significant properties of the allocation rule used. Therefore any attempt to analyze and understand the way such a situation arises is an interesting subject about social considerations. In this regard, cooperative games provide useful tools.
The present article deals with cooperative games with transferable utility. A transferable utility game (or cooperative game or coalitional game with side pay-ments or simply a TU-game) is a pair (N, v), where N is a finite set of at least two elements and $v : 2^N \to \mathbb{R}$ is a characteristic function satisfying $v(\emptyset) = 0$. An element of N and a nonempty subset S of N are called a player and a coalition, respectively, and the real number $v(S)$ is called the worth of coalition S.

The solution part of cooperative game theory deals with the allocation problem of how to share the overall earnings the amount of $v(N)$ among the players who participate in the TU-game. There is associated a single allocation rule called the value of the TU-game. A value on $\Gamma(N)$ is a function ψ that assigns a single payoff vector $(\psi_i(N, v))_{i\in N} \in \mathbb{R}^n$ to every game (N, v). $(\psi_i(N, v))_{i\in N}$ is a distribution of the total wealth available to all the players through their participation in the game (N, v). Throughout this paper we focus on the class, denoted LES, of values that satisfy Linearity, Symmetry and Efficiency properties. Nowadays, it is well known that there is a parametric representation of such a class of values that is the core topic in papers by Ruiz et al. (1998), Hermandez-Lamoneda et al. (2008) ; Driessen and Radzik (2003) ; Chameni and Andjiga (2008) and more recently Chameni and Miamo (2016). The foregoing parameterization provides a one to one correspondence between the LES values and the sequences of $n - 1$ constants where n is the number of the players in the game. It turns out that the parameterization expression constitutes a very useful tool which is fundamental for our subsequent considerations.

The famous LES value is certainly the Shapley value (Shapley 1953) which is the unique LES value that satisfies the null player axiom. Here, a null player is the one whom marginal contribution to any coalition that does not contain him is zero and the null player axiom says that the payoff of any null player is always zero, independently of the rest of the game (N, v). Another interesting LES value is the so called Solidarity value (Nowak and Radzik (1994)) which is also characterized by the three LES properties and a null player axiom similar to the Shapley value. But the null player scheme is quite different although it remains linked to the marginal contribution approach. And the null player axiom assigns a zero payoff to any null player in a game (N, v) no matter what the entire game is. The present paper uses the two null players approach to propose and study two broader classes of null players (players who recieve a zero payoff) for LES values.

Based on the null player of Solidarity value approach and the foregoing paramete-

rization of the class of LES value, Chameni Nembua (2012) extended the null player axiom to any LES value. The author showed that every LES value can be characterized by the three LES properties and a specific parametric null player axiom.
The present paper follows Chameni Nembua (2012) reasoning to propose another class of null players. In addition to the Solidarity value null player approach, the paper generalizes the Shapley null player to any regular LES value. The Shapley null player axiom is also extended to a larger one called the Average Null Player axiom. The forms of the expressions of the LES values that satisfy the new axiom are studied. Thus several new axiomatizations of some subclasses of LES values are set up with the help of our finding. Beside, a more general result is established in the sense that two LES values are identical if and only if they coincide on the set of null players of a given LES value. It turns out that this approach is a very useful and convenient tool in analyzing the LES values for TU-games.

The paper is organized as follows. Notations and some basic definitions and results on LES values are given in Section 2. The parametric representations of LES values are recalled. The relationship between the parametric expressions and the classical Shapley value and Solidarity value is etablished. Section 3 is concerned with null players for LES values. Two classes of null players are introduced. One of the main result of the paper provides a characterization of LES in terms of what the value assigns to null players. Section 4 is devoted to the applications of results established in section 3. Several axioms involving LES properties based on their parametric representation are studied as well as the characterizations of subclasses of LES values these axioms induce. Section 5 introduces the Average Null Player concept. It is shown that the Shapley value in fact has a broader class of null players and the result is then extended to all regular LES values. Some final comments conclude the paper in Section 6 and some of the proofs, which are technical calculations, are relegated to the appendix in Section 7.

2 Notations and Preliminaries

The cardinality of any set S is denoted by $|S|$ or simply with the appropriate small letter s. Consider a TU-game (N, v) with N a finite set of players such as $n > 1$. Somtime, when N is fixed and there is no confusion, the TU-game (N, v) is simply denoted by the characteristic function v and we denote $\Gamma(N)$ the set of all transferable utility games v on N, it is well known that $\Gamma(N)$ is a vector space of $2^n - 1$ dimension. Also, for notational convention we will write singleton $\{i\}$ as i.
Player $i \in N$ is a classical Shapley null player in the game (N, v) if $v(S) - v(S \backslash i) = 0$ for all $S \ni i$. Two players $i, j \in N$ are symmetric in the game (N, v) if $v(S + i) = v(S + j)$ for all $S \subseteq N \setminus \{i, j\}$.
For a coalition $T \subseteq N$, the unanimity game associated to T, is the game defined by :
$u_T = \begin{cases} 1 & if \quad T \subseteq S \\ 0 & if \quad otherwise \end{cases}$
It is well known that the family of unanimity games constitutes a basis of the linear

space $\Gamma(N)$ that plays an essential role in TU-games.

A value on $\Gamma(N)$ is a function ψ that assigns a single payoff vector $(\psi_i(N,v))_{i\in N} \in \mathbb{R}^n$ to every game (N,v). When there is not confusion, the vector $(\psi_i(N,v))_{i\in N} \in \mathbb{R}^n$ is simply denoted $\psi(v)$.
The Shapley value (Shapley, 1953) is the value *Shap* given by :
$Shap_i(v) = \sum_{S\ni i} \dfrac{(n-s)!(s-1)!}{n!}(v(S) - v(S\backslash i))$.

The Solidarity value (Nowak and Radzik, 1994) is the value given by :
$Sol_i(v) = \sum_{S\ni i} \dfrac{(n-s)!(s-1)!}{n!} A^v(S)$ where $A^v(S) = \frac{1}{s}\sum_{k\in S}[v(S) - v(S\backslash k)]$.

A value ψ on $\Gamma(N)$ is said to be linear if $\psi_i(N, \alpha v + \beta w) = \alpha\psi_i(N,v) + \beta\psi_i(N,w)$ for all games (N,v), (N,w), for all player $i \in N$ and for all $\alpha, \beta \in \mathbb{R}$. ψ on $\Gamma(N)$ is symmetric if for all games (N,v) and for any automorphism π of v, $\psi_i(N,v) = \psi_{\pi(i)}(N, \pi v)$. Finally a value ψ on $\Gamma(N)$ possesses the efficiency property if $\sum_{i\in N} \psi_i(N,v) = v(N)$.

Now, we quote two results about the parametric representation for LES values and null players.

Proposition 2.1. *Consider a set of player N of cardinality n and $\Gamma(N)$ the set of all transferable utility games (N,v). Then the following statements, for a value ϕ on $\Gamma(N)$, are equivalent :*

i) *ϕ is a LES value on $\Gamma(N)$.*

ii) *There exists a unique collection of $n-1$ constants $a(s)_{s=1}^{n-1}$ such that, for any $i \in N$,*

$$\psi_i(N,v) = \frac{v(N)}{n} + \sum_{s=1}^{n-1} a(s)\left[\frac{(n-s)!(s-1)!}{n!}\sum_{S\ni i} v(S) - \frac{(n-s-1)!s!}{n!}\sum_{i\notin S} v(S)\right] \tag{1}$$

iii) *There exists a unique collection of n constants $a(s)_{s=1}^{n}$ with $a(n) = 1$, such that, for any $i \in N$,*

$$\psi_i(N,v) = \sum_{S\ni i} \frac{(n-s)!(s-1)!}{n!}\left[a(s)v(S) - a(s-1)v(S\backslash i)\right]. \tag{2}$$

Proposition 2.2. *Consider a set of player N of cardinality n and $\Gamma(N)$ the set of all transferable utility games (N,v). Then the following statements, for a value ψ on $\Gamma(N)$, are equivalent :*

i) *ψ is a LES value on $\Gamma(N)$.*

ii) *There exists a unique collection of $n-1$ constants $b(s)_{s=2}^{n}$ such that, for any $i \in N$, and for any $S \ni i$, if*

$$B_i(S) = \begin{cases} b(s)\left[v(S) - v(S\backslash i)\right] + \dfrac{1-b(s)}{s-1}\sum_{j\in S\backslash i}\left[v(S) - v(S\backslash j)\right] & if \quad S > 1 \\ v(i) & if \quad S = 1 \end{cases}$$

$$\psi_i(N,v) = \frac{v(i)}{n} + \sum_{S\ni i}\frac{(n-s)!(s-1)!}{n!}B_i(S). \tag{3}$$

iii) *There exists a unique collection of n constants $b(s)_{s=1}^{n}$ with $b(1) = 1$ such that, for any $i \in N$, and for any $S \ni i$, if*

$$B_i(S) = \begin{cases} b(s)\left[v(S) - v(S\backslash i)\right] + \dfrac{1-b(s)}{s-1}\sum_{j\in S\backslash i}\left[v(S) - v(S\backslash j)\right] & if \quad S > 1 \\ b(s)v(i) & if \quad S = 1 \end{cases}$$

$$\psi_i(N,v) = \sum_{S\ni i}\frac{(n-s)!(s-1)!}{n!}B_i(S). \tag{4}$$

Remark 2.1. *It is easy to see that formula* (2) *generalizes the classical Shapley value. The marginal contribution term $v(s) - v(S\backslash i)$ is replaced by a weighted marginal contribution $a(s)v(s) - a(s-1)v(S\backslash i)$.*

In the literature formula (2) *is attributed to Ruiz et al. (1998) who had established an equivalent expression. But the current form of the formula appeared very recently in the literature (see Chameni and Andjiga (2008), Radzick and Driessen (2013)) for the sake of getting closer to the classical Shapley value expression.*

Remark 2.2. *Formula* (3) *or* (4) *is more general as it provides an extension of the classical representation of the Solidarity value (where $b(s) = \frac{1}{s}$) and Shapley value (where $b(s) = 1$). The formula has been established by Chameni Nembua (2012) and it is worth noting that the sequence (collection of constants) in* (2) *is linked to the sequence in* (4) *by the relation : $b(s) = a(s-1)$ $s = 2, 3, \cdots, n$ and $b(1) = a(n) = 1$.*
The mathematical expression of formulae (3) *and* (4) *may seems complex but their social interpretation are clear : Suppose that $0 \leq b(s) \leq 1$, when player i joins $S\backslash i$ to form S, he/she receives $b(s)[v(S) - v(S\backslash i)]$ which corresponds to a fraction of his/her marginal contribution while the incumbents in $S\backslash i$ receive the rest in equal shares. When player j joins $S\backslash j$ to form S, incumbent i receives additional individual share $\frac{1-b(s)}{s-1}[v(S) - v(S\backslash j)]$. Thus $B_i(S)$ is i's conditional expected payoff given formation of S by the addition of any one player.*

Definition 2.1. *A LES value ψ on $\Gamma(N)$ is considered to be :*

1) *Regular if the constants $a(s)_{s=1}^{n}$ in its representation* (2) *are all different from zero*

2) *Positively regular if the constants $a(s)_{s=1}^{n}$ in its representation* (2) *are all greater than zero.*

3) *Singular if at least one of the constants $a(s)_{s=1}^{n}$ in its representation* (2) *is equal to zero.*

Note that, the positive regularity of a LES value implies that the value is regular. Most of the classical LES values such as Shapley value, Solidarity value and Consensus value are positively regular. The Equal Split value (or Egalitarian value) defined by $E_i(N,v) = \dfrac{v(N)}{n}$ (i.e. the worth $v(N)$ of the grand coalition N is equally divided between all the players) is a special case of singularity since only one, $a(n)$ which equal to 1, of its constants $a(s)_{s=1}^n$ is different from zero.

Also note that regular LES values play a crucial role in TU-games analysis. Any regular LES value gives the possibility of rescaling as explained in the following lemma.

Lemma 2.1. $\Gamma(N)$ *is the linear space of all game* (N,v)*.*

1. *Consider any LES value ϕ (with sequence $a^{\phi}(s)_{s=1}^n$) and any regular LES value ψ (with sequence $a^{\psi}(s)_{s=1}^n$) then, for any Game $(N,v) \in \Gamma(N)$, $\phi(v) = \psi(v^{\frac{\phi}{\psi}})$ where $v^{\frac{\phi}{\psi}}$ is the rescaled game defined by $v^{\frac{\phi}{\psi}}(S) = \dfrac{a^{\phi}(s)}{a^{\psi}(s)}v(S)$ for all $S \subseteq N$.*
2. *A family of games $\{v_T,\ T \subseteq N,\ T \neq \emptyset\}$ constitutes a base of linear space $\Gamma(N)$ if and only if $\left\{v_T^{\frac{\phi}{\psi}},\ T \subseteq N,\ T \neq \emptyset\right\}$ constitutes a base of $\Gamma(N)$ for any regular values ϕ and ψ.*

Proof: See Appendix. ■

The part 1. of the lemma is a strengthened version of result discussed in earlier paper (see Theorem 1 in Chameni and Andjiga, 2008) where the regular value ψ is the Shapley value. Note that part 2. of the lemma is particularly true when the basis $\{v_T,\ T \subseteq N,\ T \neq \emptyset\}$ is such that the games v_T are unanimity games.

3 Null Players for LES values in TU-games

In this section we study two classes of null players in TU-games and their representations. The first class which is based on representation (4) of LES values has been introduced by Chameni Nembua (2012). While the second class, based on the representation (2) of LES values, is new [1] and not studied yet even if it is directly inspired from the classical Shapley null player.

Let us consider a TU-game $(N,v) \in \Gamma(N)$, any player $i \in N$ and any coalition $S \ni i$. Now consider a LES value ψ on $\Gamma(N)$ which the constants in its representation (2) and (4) are $(a(s)_{s=1}^n)$ and $(b(s)_{s=1}^n)$ respectively.

Then set $A_i^v(S) = a(s)v(S) - a(s-1)v(S\backslash i)$ and

1. Radzik and Driessen (2016), in a very recent paper have independently defined and studied some properties of this class of null players.

$$B_i^v(S)=\begin{cases} b(s)\left[v(S)-v(S\backslash i)\right]+\dfrac{1-b(s)}{s-1}\sum_{j\in S\backslash i}\left[v(S)-v(S\backslash j)\right] & if \quad S>1 \\ b(s)v(i) & if \quad S=1 \end{cases}$$

When no confusion arises $A_i^v(S)$ and $B_i^v(S)$ are simply denoted $A_i(S)$ and $B_i(S)$.

Definition 3.1. *In a TU-game $(N,v)\in\Gamma(N)$ and for any LES value ψ on $\Gamma(N)$, a player i is said to be :*

a) *Null player in Shapley sense in the game (N,v) for ψ if $A_i(S)=0$ for all $S\ni i$.*

b) *Null player in Solidarity sense in the game (N,v) for ψ if $B_i(S)=0$ for all $S\ni i$.*

For some classical LES values, the condition for nullity can be simply expressed. For example, in a game $(N,v)\in\Gamma(N)$, a player is null in the both senses for $Shap$ value if $v(S)=v(S\backslash i)$ for all $S\ni i$. Thus the two classes of Null player are generalization of the classical Shapley Null player.

For the Sol value, a player is null in Shapley sense if $\begin{cases} \dfrac{v(S)}{s+1} & = & \dfrac{v(S\backslash i)}{s} \\ v(N) & = & \dfrac{v(N\backslash i)}{n} \end{cases}$ for all $S\ni i$.

While a player is null in Solidarity sense if $v(S)=\dfrac{1}{s}\sum_{j\in S}v(S\backslash j)$ for all $S\ni i$.
Concerning the Egalitarian value, the nullity in Shapley sense seems surprising, this arrives only if $v(N)=0$ and when it is the case, all the players in the game are null. While a player i is null in Solidarity sense if $v(S)=0$ for all $S\ni i$. Sometimes in the literature such a player is called a *Nullifying player* (see Van Den Brink, 2007). Thus, for the Egalitarian value, the nullity of a player in Solidarity sense implies its nullity in Shapley sense ; but not the converse.

Remark 3.1. *Except the particular case of the Shap value for which the two kinds of nullity coincide, for any other LES value ψ on $\Gamma(N)$, the two concepts are distinct meaning that in a TU-game (N,v), a null player in Shapley sense for ψ is not necessary a null player in Solidarity sense for ψ and vice versa.*

For example, in the unanimity game u_T with $t>1$, for any non productive player $i\in N\backslash T$,

$$A_i(S)=\begin{cases} a(s)-a(s-1) & if\ T\subseteq S \\ 0 & otherwise \end{cases} \quad \text{and} \quad B_i(S)=\begin{cases} \dfrac{1-b(s)}{s-1}t & if\ T\subseteq S \\ 0 & otherwise \end{cases}$$

thus, if ψ is a LES value such that $a(s)=1$ for $s=t-1,t,\dots,n$. then player i is null in the both senses for ψ ; this is particularly true for Shapley value. Besides, for any other LES value, i is never null in one or the other sense.

The next lemma stresses some intuitive properties of the two classes of null players.

Lemma 3.1. *Let φ be any LES value on $\Gamma(N)$.*

1) If a player $i\in N$ is null in Shapley sense (respectively, in Solidarity sense) for φ in the two TU-games (N,v) and (N,w) then i is also a null player in Shapley sense (respectively, in Solidarity sense) for φ in the TU-game $(N,\alpha v+\beta w)$ for any $\alpha,\beta\in\mathbb{R}$.

2) If φ is a regular LES value and $i, j \in N$ are two null players in Shapley sense for φ in the TU-game (N, v) then i and j are symmetric players in (N, v).

3) If $i, j \in N$ are two null players in Solidarity sense for the LES value φ (regular or not) in the TU-game (N, v) then i and j are symmetric players in (N, v).

Proof: See Appendix. ■

The first statement of lemma (3.1) clearly shows that, the set of all TU-games in which a given player is null (in one or the other sense) for a fixed *LES* value constitutes a linear subspace of $\Gamma(N)$. While the second and the third statements elucidate the fact that two null players for a fixed *LES* value are always symmetric players. This means in particular that, in a TU-game (N, v) any *LES* value assigns the same amount of payoff to null players for a fixed *LES* value. However, the symmetry of two null players in Shapley sense in a TU-game (N, v) may not hold for some singular values. This is in particular the case of the *Egalitarian* value for which the condition of nullity in Shapley sense in a game (N, v) requieres only that $v(N) = 0$.

Remark 3.2. *It is easy to see that if i is a null player (in one or the other sense) for the LES value ψ in the game $(N, v) \in \Gamma(N)$ then $\psi_i(N, v) = 0$ regardless of the rest of the game . Thus a null player always receives zero as its payoff from the LES value for which he/she is null. Then it would be interesting to focus on what a LES value gives to a null player of another LES value. In the sequel, we show that the control of what any LES value φ assigns to null players (of other values) gives an significant extra information about the characterization of φ.*

Theorem 3.1. ***(Null player equal payoff for two** LES **values)***
Consider two LES values φ and ϕ on $\Gamma(N)$, then

1) *φ and ϕ coincide on $\Gamma(N)$ (i.e. $\varphi = \phi$) if and only if there exists a regular LES value ψ on $\Gamma(N)$ such that for any game (N, v), $\varphi_i(N, v) = \phi_i(N, v)$ whenever i is a null player in Shapley sense for ψ in (N, v).*
But the coincidence of the values φ and ϕ on $\Gamma(N)$ may fail when the LES value ψ is not regular.

2) *φ and ϕ coincide on $\Gamma(N)$ (i.e. $\varphi = \phi$) if and only if there exists a LES value ψ (regular or not) on $\Gamma(N)$ such that for any game (N, v), $\varphi_i(N, v) = \phi_i(N, v)$ whenever i is a null player in Solidarity sense for ψ in (N, v).*

Proof: See Appendix. ■

It worth noting that, the first part of statement 1) of theorem (3.1) concerns only regular *LES* values. The second part of statement 1) says that, the result of the first part is still valid for some singular LES values but not for all of them as demonstrated in the proof. Statement 2) of theorem (3.1) is an equivalent version of a non published result from Chameni (2010, see Theorem 3 there).

The next theorem highlights the case of the *Egalitarian* value that is a singular *LES* value but for which the statement 1) is valid. We remind that in this case, the existence of a null player in Shapley sense in a game (N, v) is equivalent to the condition $v(N) = 0$.

Theorem 3.2. *Two LES value φ and ϕ coincide on $\Gamma(N)$ (i.e. $\varphi = \phi$) if and only if $\varphi_i(N, v) = \phi_i(N, v)$ for all $i \in N$ and for all $(N, v) \in \Gamma(N)$ such that $v(N) = 0$.*

Proof: Suppose that $\varphi_i(N, v) = \phi_i(N, v)$ for all $(N, v) \in \Gamma(N)$ such that $v(N) = 0$ and for all $i \in N$. Thus for all $(N, v) \in \Gamma(N)$ and for all $i \in N$, $\varphi_i(v - v(N)) = \phi_i(v - v(N))$.
$\Leftrightarrow$ for all $(N, v) \in \Gamma(N)$ and for all $i \in N$, $\varphi_i(v) - \frac{v(N)}{n} = \phi_i(v) - \frac{v(N)}{n}$.
$\Leftrightarrow$ for all $(N, v) \in \Gamma(N)$ and for all $i \in N$, $\varphi_i(v) = \phi_i(v)$.
$\Leftrightarrow \varphi = \phi$ ■

Theorem (3.1) immediately leads to next corollaries which are the extension to any *LES* value of some properties hitherto discussed only in the case of the Shapley value.

Corollary 3.1. *Consider two LES values φ and ϕ on $\Gamma(N)$.*

1. *Suppose that ϕ is regular, if for any game (N, v) and for any i null player in Shapley sense for ϕ in (N, v), we have $\varphi_i(N, v) = 0$ then $\varphi = \phi$.*
2. *If for any game (N, v) and for any null player i in Solidarity sense for ϕ (regular or not) in (N, v), we have $\varphi_i(N, v) = 0$ then $\varphi = \phi$.*

Proof: This immediately follows from theorem (3.1). ■

Corollary 3.2. *Consider two LES values φ and ϕ on $\Gamma(N)$.*

1. *If for any games $(N, v), (N, w) \in \Gamma(N)$, and for any $i \in N$ null player in Shapley sense for φ (regular value) in the two games, we have $\phi_i(v) = \phi_i(w)$. Then $\varphi = \phi$.*
2. *$(N, v), (N, w) \in \Gamma(N)$, and for any $i \in N$ null player in Solidarity sense for φ in the two games, we have $\phi_i(v) = \phi_i(w)$. Then $\varphi = \phi$.*

Proof:
1) Suppose that $\phi_i(v) = \phi_i(w)$ whenever $i \in N$ is a null player in Shapley sense for φ in any two games $(N, v), (N, w) \in \Gamma(N)$. This implies in particular that, if $i \in N$ is a null player in Shapley sense for φ in any game (N, v), then $\phi_i(v) = \phi_i(-v) = -\phi_i(v)$ (since a null player in a game (N, v) is also null in the game $(N, -v)$). Thus $\phi_i(v) = 0$ whenever i is a null player in Shapley sense for φ in any game (N, v). Hence, from statement 1) of corollary 3.1 $\varphi = \phi$.

2) Just use the same line of reasoning as in the proof of 1). ■

Applying the same approach to the particular case of the *Egalitarian* value leads to the following corollary.

Corollary 3.3. *Consider two LES values φ and ϕ on $\Gamma(N)$. E denotes the Egalitarian value.*

1. *If for any games $(N, v) \in \Gamma(N)$ with $v(N) = 0$, and for any $i \in N$ we have $\phi_i(v) = 0$. Then $\phi = E$.*
2. *If for any games $(N, v), (N, w) \in \Gamma(N)$ with $v(N) = w(N) = 0$, and for any $i \in N$ we have $\phi_i(v) = \phi_i(w)$. Then $\phi = E$.*

4 Applications

4.1 General properties of LES values.

In this subsection, we introduce axioms involving some properties of LES values based on the coefficients $a(s)_{s=1}^n$ and $b(s)_{s=1}^n$ defined in (2) and (4). Some of them are generalization of properties discussed in earlier papers while others are entirely new. These axioms are used in the next subsection to characterize subclasses of LES values.

In the sequel we suppose that, $a(s)_{s=1}^n$ and $b(s)_{s=1}^n$ are two collections of n constants with $a(s) \neq 0, s = 1, 2, \ldots, n-1$, $a(n) = 1$ and $b(1) = 1$. ϕ is a LES value.

A1) $a(s)_{s=1}^n$ **-Weighted Marginal Contribution Weak Monotonicity** : Let $(N, v) \in \mathit{\Gamma}(N)$, $i \in N$ and $C \in \mathbb{R}$. If $[a(s)v(s) - a(s-1)v(S\backslash i)] \geq C$ for all $S \ni i$, then $\phi_i(v) \geq C$.

A2) $a(s)_{s=1}^n$ **-Weighted Marginal Constant Contribution** : Let $(N, v) \in \mathit{\Gamma}(N)$, $i \in N$ and $C \in \mathbb{R}$. If $[a(s)v(s) - a(s-1)v(S\backslash i)] = C$ for all $S \ni i$, then $\phi_i(v) = C$.

A3) $a(s)_{s=1}^n$ **-Weighted Marginal Contribution Strong Monotonicity** : Let $(N, v), (N, w) \in \mathit{\Gamma}(N), i \in N$. If $[a(s)v(s) - a(s-1)v(S \setminus i)] \geq [a(s)w(s) - a(s-1)w(S\backslash i)]$ for all $S \ni i$, then $\phi_i(v) \geq \phi_i(w)$.

A4) $a(s)_{s=1}^n$ **-Weighted Marginal Contribution Equal Payoffs** : Let $(N, v), (N, w) \in \mathit{\Gamma}(N), i \in N$,. If $[a(s)v(s) - a(s-1)v(S\backslash i)] = [a(s)w(s) - a(s-1)w(S\backslash i)]$ for all $S \ni i$, then $\phi_i(v) = \phi_i(w)$.

A5) $b(s)_{s=1}^n$ **-Marginal Coalitional Gain Weak Monotonicity** : Let $(N, v) \in \Gamma(N)$, $i \in N$ and $C \in \mathbb{R}$.If $B_i^v(S) \geq C$ for all $S \ni i$, then $\phi_i(v) \geq C$.

A6) $b(s)_{s=1}^n$ **-Marginal Coalitional constant Gain** : Let $(N, v) \in \Gamma(N)$, $i \in N$ and $C \in \mathbb{R}$. If $B_i^v(S) = C$ for all $S \ni i$, then $\phi_i(v) = C$.

A7) $b(s)_{s=1}^n$ **-Marginal Coalitional Gain Strong Monotonicity** : Let (N, v), $(N, w) \in \mathit{\Gamma}(N)$, $i \in N$. If $B_i^v(S) \geq B_i^w(S)$ for all $S \ni i$, then $\phi_i(v) \geq \phi_i(w)$.

A8) $b(s)_{s=1}^n$ **-Marginal Coalitional Gain Equal Payoffs** : Let $(N, v), (N, w) \in \mathit{\Gamma}(N), i \in N$. If $B_i^v(S) = B_i^w(S)$ for all $S \ni i$, then $\phi_i(v) = \phi_i(w)$.

A9) φ**- Shapley Null Player Equal Payoffs** (φ is any regular LES value) : If a player i is a null player in Shapley sense for φ in two games $(N, v), (N, w) \in \mathit{\Gamma}(N)$ with $v(N) = w(N)$, then $\phi_i(v) = \phi_i(w)$.

A10) φ**- Solidarity Null Player Equal Payoffs** (φ is any LES value) : If a player i is a null player in Solidarity sense for φ in two games $(N, v), (N, w) \in \mathit{\Gamma}(N)$ with $v(N) = w(N)$, then $\phi_i(v) = \phi_i(w)$.

A11) φ**- Shapley Null Player** $a-$**Average Payoff**(φ is any regular LES value, $a \in \mathbb{R}$) : If a player i is a null player in Shapley sense for φ in the game $(N, v) \in \mathit{\Gamma}(N)$ then $\phi_i(v) = a\frac{v(N)}{n}$.

A12) φ**- Solidarity Null Player** $a-$**Average Payoff**(φ is any LES value, $a \in \mathbb{R}$) : If a player i is a null player in Solidarity sense for φ in the game $(N, v) \in \mathit{\Gamma}(N)$ then $\phi_i(v) = a\frac{v(N)}{n}$.

A13) φ**- Shapley Null Player** $a\psi$ **-Payoff** (φ is regular, ψ is any LES value, $a \in \mathbb{R}$) : If a player i is a null player in Shapley sense for φ in the game $(N, v) \in \Gamma(N)$ then $\phi_i(v) = a\psi_i(v)$.

A14) φ**- Solidarity Null Player** $a\psi$ **-Payoff** (φ, ψ are any LES value, $a \in \mathbb{R}$) : If a player i is a null player in Solidarity sense for φ in the game $(N, v) \in \Gamma(N)$ then $\phi_i(v) = a\psi_i(v)$.

It is clear that these fortheen axioms are not independent, the lemma below highlights some of their links.

Lemma 4.1. *Let ϕ be any LES value on $\Gamma(N)$, $a(s)_{s=1}^n$ and $b(s)_{s=1}^n$ are two collections of n constants with $a(s) \neq 0$, $s = 1, 2, \ldots, n-1$, $a(n) = 1$ and $b(1) = 1$.*

1. *$A1 \Rightarrow A2$:*
   If ϕ verifies $a(s)_{s=1}^n$-Weighted Marginal Contribution Weak Monotonicity then ϕ verifies $a(s)_{s=1}^n$-Weighted Marginal Constant Contribution.
2. *$A3 \Rightarrow A4$:*
   If ϕ verifies $a(s)_{s=1}^n$-Weighted Marginal Contribution Strong Monotonicity then ϕ verifies $a(s)_{s=1}^n$-Weighted Marginal Contributions Equal Payoffs.
3. *$A5 \Rightarrow A6$:*
   If ϕ verifies $b(s)_{s=1}^n$-Marginal Coalitional Gain Weak Monotonicity then ϕ verifies $b(s)_{s=1}^n$-Marginal Coalitional constant Gain.
4. *$A7 \Rightarrow A8$:*
   If ϕ verifies $b(s)_{s=1}^n$-Marginal Coalitional Gain Strong Monotonicity then ϕ verifies $b(s)_{s=1}^n$-Marginal Coalitional Gain Equal Payoffs.
5. *$A13$ is a particular case of $A11$ and $A11 \Longleftrightarrow A9$:*
   ϕ verifies φ-Shapley Null Player Equal Payoffs (φ is any regular LES value) if and only if there exist $a \in \mathbb{R}$, ϕ verifies φ-Shapley Null Player a-Average Payoff.
6. *$A14$ is a particular case of $A12$ and $A12 \Longleftrightarrow A10$:*
   ϕ verifies φ-Solidarity Null Player Equal Payoffs (φ is any regular LES value) if and only if there exist $a \in \mathbb{R}$, ϕ verifies φ-Solidarity Null Player a-Average Payoff.

Proof: See Appendix. ■

4.2 Characterization of LES values.

In this subsection we use previous axioms introduced in subsection 4.1 to characterize several subclasses of LES values.

Theorem 4.1. *A LES value ϕ verifies $a(s)_{s=1}^n$-Weighted Marginal Constant Contribution if and only if the sequence in its representation (2) is $a(s)_{s=1}^n$.*

Proof: Suppose that ϕ is a LES value that verifies $a(s)_{s=1}^n$-Weighted Marginal Constant Contribution and consider ψ^a the LES value with the sequence in its representation (2) is $a(s)_{s=1}^n$. If i is a null player in Shapley sense for ψ^a in any game (N, v) then

$[a(s)v(s) - a(s-1)v(S\backslash i)] = 0$ for all $S \ni i$. Since ϕ verifies $a(s)_{s=1}^n$-Weighted Marginal Constant Contribution,$\phi_i(v) = 0$. Thus from assertion 1) in corollary (3.1), $\phi = \psi^a$. conversely, if $\phi = \psi^a$, using representation (2) of ψ^a directly leads to the result. ■

Corollary 4.1. *A LES value ϕ verifies $a(s)_{s=1}^n$-Weighted Marginal Contribution Weak Monotonicity if and only if the sequence in its representation (2) is $a(s)_{s=1}^n$.*

Proof: If ϕ is a LES value that verifies $a(s)_{s=1}^n$-Weighted Marginal Contribution Weak Monotonicity, then from assertion 1) in lemma (4.1), ϕ verifies $a(s)_{s=1}^n$-Weighted Marginal Constant Contribution, therefore from theorem (4.1), $\phi = \psi^a$, where ψ^a is the LES value with the sequence in its representation (2) equal $a(s)_{s=1}^n$.
The converse is obvious from representation (2) of ϕ. ■

This is a generalization of the strong monotonicity discussed by Young (1985). The author proved, in the case of $a(s) = 1$, $s = 1, 2, \ldots, n$, that the property is sufficient to characterize the Shapley value.

Theorem 4.2. *A LES value ϕ verifies $a(s)_{s=1}^n$-Weighted Marginal Contributions Equal Payoffs if and only if the sequence in its representation* (2) *is $a(s)_{s=1}^n$.*

Proof: Suppose that ϕ is a LES value verifying $a(s)_{s=1}^n$-Weighted Marginal Contributions Equal Payoffs property, consider ψ^a the LES value with the sequence in its representation (2) equal $a(s)_{s=1}^n$. If i is a null player in Shapley sense for ψ^a in any game (N, v) then $[a(s)v(s) - a(s-1)v(S \setminus i)] = 0$ for all $S \ni i$, if (N, w) is the zero game (that is $w(S) = 0$ for all coalition S), then
$[a(s)v(s) - a(s-1)v(S \setminus i)] = [a(s)w(s) - a(s-1)w(S \setminus i)]$ for all $S \ni i$, therefore $\phi_i(v) = \phi_i(w) = 0$. Thus from assertion 1) in corollary 3.1, $\phi = \psi^a$.
The converse is obvious from representation (2) of ϕ. ■

Corollary 4.2. *A LES value ϕ verifies $a(s)_{s=1}^n$-Weighted Marginal Contribution Strong Monotonicity : if and only if the sequence in its representation* (2) *is $a(s)_{s=1}^n$.*

Proof: If ϕ is a LES value that verifies $a(s)_{s=1}^n$-Weighted Marginal Contribution Strong Monotonicity, then from assertion 2) in lemma (4.1), ϕ verifies $a(s)_{s=1}^n$-Weighted Marginal Contributions Equal Payoffs , therefore from Theorem (4.2), $\phi = \psi^a$, where ψ^a is the LES value with the sequence in its representation (2) equal $a(s)_{s=1}^n$.
The converse is obvious from representation (2) of ϕ. ■

Theorem 4.3. *A LES value ϕ verifies $b(s)_{s=1}^n$-Marginal Coalitional constant Gain if and only if the sequence in its representation (4) is $b(s)_{s=1}^n$.*

Proof: Suppose that ϕ is a LES value satisfying $b(s)_{s=1}^n$- Marginal Coalitional constant Gain. and consider ψ^b the LES value with the sequence in its representation (4) is $b(s)_{s=1}^n$. If i is a null player in Solidarity sense for ψ^b in any game (N, v) then $B_i^v(S) = 0$ for all $S \ni i$. Since ϕ verifies $b(s)_{s=1}^n$- Marginal Coalitional Constant Gain, $\phi_i(v) = 0$. Thus from assertion 2) in corollary 3.1, $\phi = \psi^b$.
The converse is obvious from representation (4) of ϕ. ■

Corollary 4.3. *A LES value ϕ verifies $b(s)_{s=1}^{n}$-Marginal Coalitional Gain Weak Monotonicity if and only if the sequence in its representation (4) is $b(s)_{s=1}^{n}$.*

Proof: If ϕ is a LES value that verifies $b(s)_{s=1}^{n}$-Marginal Coalitional Gain Weak Monotonicity, then from assertion 3) in lemma (4.1), ϕ verifies $b(s)_{s=1}^{n}$- Marginal Coalitional Constant Gain, therefore from theorem (4.3), $\phi = \psi^b$, where ψ^b is the LES value with the sequence in its representation (4) equal $b(s)_{s=1}^{n}$.
The converse is obvious from representation (4) of ϕ. ■

Theorem 4.4. *A LES value ϕ verifies $b(s)_{s=1}^{n}$- Marginal Coalitional Gain Equal Payoffs if and only if the sequence in its representation (4) is $b(s)_{s=1}^{n}$.*

Proof: Suppose that ϕ is a LES value verifying $b(s)_{s=1}^{n}$- Marginal Coalitional Gain Equal Payoffs property, consider ψ^b the LES value with the sequence in its representation (4) equal $b(s)_{s=1}^{n}$. If i is a null player in Solidarity sense for ψ^b in any game (N, v) then $B_i^v(S) = 0$ for all $S \ni i$, if (N, w) is the zero game (that is $w(S) = 0$ for all coalition S), then $B_i^v(S) = B_i^w(S)$ for all $S \ni i$, therefore $\phi_i(v) = \phi_i(w) = 0$. Thus from assertion 2) in corollary (3.1), $\phi = \psi^b$.
The converse is obvious from representation (4) of ϕ. ■

Corollary 4.4. *A LES value ϕ verifies $b(s)_{s=1}^{n}$-Marginal Coalitional Gain Weak Monotonicity if and only if the sequence in its representation* (4) *is $b(s)_{s=1}^{n}$.*

Proof: If ϕ is a LES value that verifies $b(s)_{s=1}^{n}$-Marginal Coalitional Gain Weak Monotonicity, then from assertion 4) in lemma (4.1), ϕ verifies $b(s)_{s=1}^{n}$- Marginal Coalitional Gain Equal Payoffs, therefore from theorem (4.4), $\phi = \psi^b$, where ψ^b is the LES value with the sequence in its representation (4) equal $b(s)_{s=1}^{n}$.
The converse is obvious from representation (4) of ϕ. ■

Theorem 4.5. *A LES value ϕ verifies φ-Shapley Null Player $a\psi$-Payoffs (φ is regular, ψ is any LES value, $a \in \mathbb{R}$), if and only if $\phi = a\psi + (1-a)\varphi$.*

Proof: ϕ verifies φ-Shapley Null Player $a\psi$-Payoffs $\Leftrightarrow \phi_i(v) = a\psi_i(v)$ for any null player i in Shapley sense for φ in any game $(N, v) \in \Gamma(N) \Leftrightarrow \phi_i(v) = a\psi_i(v) + (1-a)\varphi_i(v)$ for any null player i in Shapley sense for φ in any game $(N, v) \in \Gamma(N) \Leftrightarrow \phi = a\psi + (1-a)\varphi$ according to assertion 1) of theorem (3.1). ■

Theorem (4.5) is a general finding that set up the characterization of any regular LES value φ (when the constant $a = 0$) in the same way as the classical characterization of the Shapley value (Shapley, 1953). Note that when $a = 1$ the outcome of the theorem (4.5) coincides with the first part of theorem (3.1). Also note that, according to the discussion in Section 3 about the *Egalitarian* value, the result of theorem (4.5) is still valid when the value φ is the *Egalitarian* value.
Besides, Theorem (4.5) engenders many others results discussed in earlier papers. For instance, the theorem leads to :

a. The characterization of the Shapley value (Shapley,1953) when $a = 0$ and $\varphi = Shap$.

b. The characterization of the a- consensus value (due to Ju, Born and Ruys, 2007) when φ is the Equal Surplus value and ψ is the *Egalitarian* value.

c. theorem of Yang (1997) when φ is the Shapley value and ψ is the *Egalitarian* value.

d. And more recently, the characterization of LES values satisfying the Per-Capita Null Player a-Average Payoffs axiom (Radzik and Driessen, 2016) when φ is the Per-Capita value and ψ is the *Egalitarian* value.

Theorem 4.6. *E denotes the Egalitarian value.*
A LES value ϕ verifies φ-Shapley Null Player a-Average Payoff (φ is any regular LES value,$a \in \mathbb{R}$), if and only if $\phi = aE + (1-a)\varphi$.

Proof: This is a particular case of theorem (4.5) with ψ equal *Egalitarian* value. ∎

Theorem 4.7. *E denotes the Egalitarian value.*
A LES value ϕ verifies φ-Shapley Null Player Equal Payoffs (φ is any regular LES value), if and only if ϕ is of the form $\phi = aE + (1-a)\varphi$, for $a \in \mathbb{R}$.

Proof: This is a direct outcome of Theorem (4.6) and assertion 5) of lemma (4.1). ∎

Theorem 4.8. *A LES value ϕ verifies φ-Solidarity Null Player φ-Payoff (φ and ψ are any LES value, $a \in \mathbb{R}$), if and only if $\phi = a\psi + (1-a)\varphi$.*

Proof: ϕ verifies φ-Solidarity Null Player $a\psi$-Payoff $\Leftrightarrow \phi_i(v) = a\psi_i(v)$ for any null player i in Solidarity sense for φ in any game $(N,v) \in \Gamma(N) \Leftrightarrow \phi_i(v) = a\psi_i(v) + (1-a)\varphi_i$ for any null player i in Solidarity sense for φ in any game $(N,v) \in \Gamma(N) \Leftrightarrow \phi = a\psi + (1-a)\varphi$ according to assertion 2) of theorem (3.1). ∎

Clearly, Theorem (4.8) provides a solid approach to characterize any LES value. Theorem (4.8) is at least as general as theorem (4.5). The theorem is based on the *Null player in Solidarity sense* which is clearly more demanding than the *Null player in Shapley sense*. However the improvement is the free regularity of the value φ. Thus the result of the theorem is valid not only for the regular LES values but also for singular LES values.

Theorem 4.9. *E denotes the Egalitarian value.*
A LES value ϕ verifies φ-Solidarity Null Player a-Average Payoff (φ is any LES value, $a \in \mathbb{R}$), if and only if $\phi = aE + (1-a)\varphi$.

Proof: This is a particular case of theorem (4.8) with ψ equal to *Egalitarian* value. ∎

Theorem 4.10. *E denotes the Egalitarian value.*
A LES value ϕ verifies φ-Solidarity Null Player Equal Payoffs (φ is any LES value), if and only if ϕ is of the form $\phi = aE + (1-a)\varphi$, for $a \in \mathbb{R}$.

Proof: This is a direct outcome of Theorem (4.9) and assertion 6) of lemma (4.1). ∎

To end the section, let us state two results that are strengthened versions, in terms of convex combinations, of theorem (4.8) and theorem (4.5).

Corollary 4.5. *ϕ, $\psi^1, \psi^2, ..., \psi^p$ are any $p+1$ LES value on $\Gamma(N)$; $c_1, c_2, ..., c_p$ are any p constant numbers.*

1) Consider any regular LES value φ on $\Gamma(N)$.
If for any TU-game (N, v) and for any $i \in N$ null player for φ in Shapley sense in (N, v), $\phi_i(v) = \sum_{s=1}^{p} c_s \psi_i^s(v)$ then ϕ is of the form $\phi = (1 - \sum_{s=1}^{p} c_s)\varphi + \sum_{s=1}^{p} c_s \psi^s$.

2) Consider any LES value φ on $\Gamma(N)$.
If for any TU-game (N, v) and for any $i \in N$ null player for φ in Solidarity sense in (N, v), $\phi_i(v) = \sum_{s=1}^{p} c_s \psi_i^s(v)$ then ϕ is of the form $\phi = (1 - \sum_{s=1}^{p} c_s)\varphi + \sum_{s=1}^{p} c_s \psi^s$.

5 Average null players

In this section, a new class of Null players in TU-game is introduced. The class is directely inspired from Shapley (1953) Null player. But rather than considering marginal contribution in each coalition, the mean of marginal contributions is taken into the account. It turn out that, the class contains the Shapley null players and thus many of the results found in the literature based on Shapley null players are still valid for the new concepts of null players.

5.1 Average null players for Shapley value

Let (N, v) be a TU-game in $\Gamma(N)$. For any player $i \in N$, set $u_k(i) = \sum_{S \ni i; |S|=k} v(S)$. $u_k(i)$ reflects the productivity of player i in the game (N, v) when only the coalitions of size k are taking into account.

$u(i) = \begin{pmatrix} u_1(i) \\ u_2(i) \\ \vdots \\ u_n(i) \end{pmatrix}$ is the productivity vector of the player i in the game.

Now, set $m_{ik}^v = \frac{1}{\binom{n-1}{k-1}} u_k(i)$. m_{ik}^v is the average productivity of the player i in the game (N, v) when only the coalitions of size k are taking into account.
In the same spirit we set :

$\overline{m}_{ik}^v = \frac{1}{\binom{n-1}{k}} \sum_{i \notin S; |S|=k} v(S)$. $\overline{m}_{ik}^v$ is the mean of worths of all coalitions size k non containing player i.

$m_k^v = \frac{1}{\binom{n}{k}} \sum_{|S|=k} v(S)$. m_k^v is the mean of worths of all coalitions size k.

When no confusion is possible, m_{ik}^v, $\overline{m}_{ik}^v$ and m_k^v are simply respectively written m_{ik}, $\overline{m}_{ik}$ and m_k.
We can now state two results in the next corollary which will be used in our subsequent considerations.

Corollary 5.1. *For any TU-game (N, v), if ψ is any LES value with the constants $a(k)_{k=1}^{n-1}$ in its representation (2), then for all $i \in N$,* *1)* $\psi_i(N,v) = \dfrac{v(N)}{n} + \sum_{k=1}^{n-1} a(k)\frac{m_{ik}-m_k}{n-k}$.

2) $\psi_i(N,v) = \dfrac{v(N)}{n} + \sum_{k=1}^{n-1} a(k)\frac{m_{ik}-\overline{m}_{ik}}{n} = \sum_{k=1}^{n} a(k)\frac{m_{ik}-\overline{m}_{ik}}{n}$, *with* $a_n = 1$ *and* $\overline{m}_{in} = 0$.

Proof: See Appendix. ■

Corollary (5.1) gives another representation of LES value in terms of average productivity of players instead of marginal contribution of players as it is usualy the case since the famous work of Shapley (1953). This representation directly leads to new *symmetric players* and new *null player* definitions.

Definition 5.1. *Two players* $i, j \in N$ *are said to be average symmetric in the game* (N,v)*, if they have the same productivity vecteur in* (N,v)*. Formally,* i *and* j *are average symmetric if :* $m_{ik} = m_{jk}$ *for all* $k = 1, 2, ..., n$.

Note that if i and j are two symmetric players in the game (N,v) (that is $v(S+i) = v(S+j)$ for all $S \subseteq N \setminus \{i,j\}$) then players i and j are also average symmetric in (N,v). The converse is always true when the number of the players in the game is $n \leq 3$. However, two players can be average symmetric without being symmetric if $n \geq 4$ as it is illustrated in the following game where players 1 and 2 are average symmetric but not symmetric.

$N = \{1,2,3,4\}$, $v(1) = 1$, $v(2) = 1$, $v(3) = 2$, $v(4) = 1$, $v(1,2) = 2$, $v(1,3) = 1$, $v(1,4) = 3$, $v(2,3) = 4$, $v(2,4) = 0$, $v(3,4) = 5$, $v(1,2,3) = 4$, $v(1,2,4) = 1$, $v(2,3,4) = 4$.

Here, the productivity vector of players 1 and 2 are $u(1) = u(2) = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix}$.

The next result characterizes average symmetric players in term of LES values.

Proposition 5.1. *Consider a TU-game* (N,v) *and any two players* $i, j \in N$*. Then the following statements are equivalent :*

1) *players* i *and* j *are average symmetric in* (N,v).

2) $\overline{m}_{ik} = \overline{m}_{jk}$ *for all* $k = 1, 2, ..., n$.

3) *For any* LES *value* φ, $\varphi_i(N,v) = \varphi_j(N,v)$.

Proof: See Appendix. ■

The particular property of a symmetric value is to assign the same amount of payoff to symmetric players in a TU-game. Proposition (5.1) clearly shows that, when the value is a LES one, the property is extended to the larger class of average symmetric players.

Definition 5.2. *Consider a TU-game* (N,v)*. A player* $i \in N$ *is an average null player in* (N,v) *if for all* $k = 1, 2, \cdots, n$, $m_{ik} = \overline{m}_{ik-1}$ *with* $\overline{m}_{i0} = 0$ *and* $m_{in} = v(N)$.

Note that, the average null player concept is an extension of the classical null player defined by Shapley (1953). If i is a null player in a TU-game (N, v) then i is also an average null player in (N, v). However, the converse does not hold in the sense that, some average null players are not null player. In other words, the condition for being an average null player is weaker than the one required to be a null player.

For example, in the following game, $N = \{1, 2, 3\}$, $v(N) = 1$, $v(1) = 0$, $v(2) = 1$, $v(3) = 3$, $v(1, 2) = 4$, $v(1, 3) = 0$, $v(2, 3) = 1$.

Player 1 is an average null player but not a null player. Nevertheless, in some class of games, the two concepts coincide. It is for instance the case if the game (N, v) is monotonic and particulary in unanimity games as stated in the next proposition.

Proposition 5.2. *Let $T \subseteq N$ be any coalition of players and considers (N, v_T) the unanimity game associated to T. Then for any player $i \in N$, the following statements are equivalent.*

1) i is a null player in (N, v_T).

2) i is an average null player in (N, v_T).

3) i belongs to $N \setminus T$.

Proof: See Appendix. ■

Proposition (5.2) is certainly one of the main key results of the subsection as it clearly establishes that *null player* and *average null player* are two concepts that coincide in unanimity game. Knowing that the set of all unanimity games constitutes a basis of the linear space $\Gamma(N)$ that plays a central role in the characterization of LES values, most results concerning the *null player* property will be also valid for *average null player* property as we will see it in the sequel.

Definition 5.3. *A value φ defined on $\Gamma(N)$ possesses the average null player property if for all TU-game (N, v), for all average null-player i in (N, v), $\varphi_i(v) = 0$.*

Proposition 5.3. *Given a TU-game (N, v),*

1) If two players $i, j \in N$ are average null in (N, v), then i and j are average symmetric in (N, v).

2) If a player $i \in N$ is an average null palyer in (N, v), then $Shap_i(N, v) = 0$.

Proof: See Appendix. ■

It is well known that null player axiom is one of the characteristic properties of Shapley value. However, null players are not solely to receive zero as shapley payoff. Proposition (5.3) reveals that in fact, there is a broader class of players, constitued by average null players, to whom Shapley value always assigns zero as payoff. Therefore, the following corollary holds.

Corollary 5.2. *Shapley value is the unique LES value satisfying the average null player property.*

5.2 General average null players for LES values

Definition 5.4. *Consider a TU-game (N, v) and let φ be any LES value with the constants $a(k)_{k=1}^{n-1}$ in its representation (2). A player $i \in N$ is an average null player for φ in (N, v) if for all $k = 1, 2, \cdots, n$,*
$a_k m_{ik} = a_{k-1}\overline{m}_{ik-1}$*, with* $a_n = 1$, $\overline{m}_{i0} = 0$ *and* $m_{in} = v(N)$.

Note that, when the LES value φ in the definition equal to the Shapley value, *the average null player for* φ coincides with *the average null player* defined in subsection 5.1. Thus this is an extension of the Shapley average null player to any LES value.

Proposition 5.4. *Consider a TU-game $(N, v) \in \Gamma(N)$ and any LES value φ defined on $\Gamma(N)$.*

1) If φ is regular, any two players $i, j \in N$ average null for φ in (N, v), are average symmetric in (N, v).

2) If a player $i \in N$ is average null for φ in (N, v), then $\varphi_i(N, v) = 0$.

Proof: See Appendix. ■

Definition 5.5. *Let φ be any LES value on $\Gamma(N)$. A value ϕ defined on $\Gamma(N)$ possesses the φ-average null player property if for all TU-game (N, v) and for all $i \in N$ average null player for φ in (N, v), $\phi_i(v) = 0$.*

Theorem 5.1. *Let φ be any regular LES value on $\Gamma(N)$. Then for any LES value ϕ on $\Gamma(N)$, the following statements are equivalent.*

1) For any TU-game $(N, v) \in \Gamma(N)$ and for any $i \in N$ null player in Shapley sense for φ in (N, v), $\phi_i(v) = 0$.

2) $\phi = \varphi$.

3) ϕ possesses the φ-average null player property.

The incidence of theorem (5.1) is clear. The theorem shows that one can easly replace along the whole text and in particular in section 4, the *null player for φ in Shapley sense in (N, v)* by the *φ-average null player in (N, v)* and then obtain exactly the same results. We do it in the sequel, the axioms in subsection 4.1 are rewritten in term of avarage null player and the following corollary summarizes the impact on results in subsection 4.2. The proof of the corollary is immediate and left to the readers.

We suppose that, $a(k)_{k=1}^{n}$ is a collection of n constants with $a(k) \neq 0, k = 1, 2, \ldots, n-1$, $a(n) = 1$. ϕ is any LES value.

A15) $a(k)_{k=1}^n$ **-Weighted average contribution Weak Monotonicity** : Let $(N, v) \in \Gamma(N)$, $i \in N$ and $C \in \mathbb{R}$. If $[a(k)m_{ik} - a(k-1)\overline{m}_{ik-1}] \geq C$ for all $k = 1, 2, ...n.$, then $\phi_i(v) \geq C$.

A16) $a(k)_{k=1}^n$ **-Weighted average Constant Contribution** : Let $(N, v) \in \Gamma(N)$, $i \in N$ and $C \in \mathbb{R}$. If $[a(k)m_{ik} - a(k-1)\overline{m}_{ik-1}] = C$ for all $k = 1, 2, ...n.$, then $\phi_i(v) = C$.

A17) $a(k)_{k=1}^n$ **-Weighted Average Contribution Strong Monotonicity** : Let $(N, v), (N, w) \in \Gamma(N), i \in N$. If $[a(k)m_{ik}^v - a(k-1)\overline{m}_{ik-1}^v] \geq [a(k)m_{ik}^w - a(k-1)\overline{m}_{ik-1}^w]$ for all $k = 1, 2, ...n.$, then $\phi_i(v) \geq \phi_i(w)$.

A18) $a(k)_{k=1}^n$ **-Weighted Average Contribution Equal Payoffs** : Let $(N, v), (N, w) \in \Gamma(N), i \in N$,. If $[a(k)m_{ik}^v - a(k-1)\overline{m}_{ik-1}^v] = [a(k)m_{ik}^w - a(k-1)\overline{m}_{ik-1}^w]$ for all $k = 1, 2, ...n.$, then $\phi_i(v) = \phi_i(w)$.

A19) φ**- Average Null Player Equal Payoffs** (φ is any regular LES value) : If a player i is an average null player in for φ in two games $(N, v), (N, w) \in \Gamma(N)$ with $v(N) = w(N)$, then $\phi_i(v) = \phi_i(w)$.

A20) φ**- Average Null Player** $a-$**Average Payoff**(φ is any regular LES value, $a \in \mathbb{R}$) : If a player i is an average null player for φ in the game $(N, v) \in \Gamma(N)$ then $\phi_i(v) = a\frac{v(N)}{n}$.

A21) φ**- Average Null Player** $a\psi$ **-Payoff** (φ is regular, ψ is any LES value, $a \in \mathbb{R}$) : If a player i is an average null player for φ in the game $(N, v) \in \Gamma(N)$ then $\phi_i(v) = a\psi_i(v)$.

Corollary 5.3.

1) A LES value ϕ verifies $a(k)_{k=1}^n$-Weighted Average Contribution Weak Monotonicity if and only if the sequence in its representation (2) *is $a(k)_{k=1}^n$.*

2) A LES value ϕ verifies $a(k)_{k=1}^n$-Weighted Average Constant Contribution if and only if the sequence in its representation (2) *is $a(k)_{k=1}^n$.*

3) A LESvalue ϕ verifies $a(k)_{k=1}^n$-Weighted Average Constant Strong Monotonicity if and only if the sequence in its representation (2) *is $a(k)_{k=1}^n$.*

4) A LES value ϕ verifies $a(k)_{k=1}^n$-Weighted Average Contribution Equal Payoffs if and only if the sequence in its representation (2) *is $a(k)_{k=1}^n$.*

5) A LES value ϕ verifies φ-Average Null Player Equal Payoffs(φ is any regular LES value) if and only if ϕ is of the form $\phi = aE + (1-a)\varphi$, for $a \in \mathbb{R}$.

6) A LES value ϕ verifies φ-Average Null Player $a-$Average Payoffs(φ is any regular LES value, $a \in \mathbb{R}$) if and only if ϕ is of the form $\phi = aE + (1-a)\varphi$.

7) A LES value ϕ verifies φ-Average Null Player $a\psi$-Payoffs(φ is regular, ψ is any LES value, $a \in \mathbb{R}$) if and only if $\phi = a\psi + (1-a)\varphi$.

6 Concluding Remarks

In the present paper, we have conducted a general study of null players for LES values. Basing on two representations formula for such values, we have set up two classes of null players. An interesting open question is how to know if there do not exist other classes of null players for LES values. The concept of average null player have been defined. In this regard, it has been noted that, in monotonic games, a player is average null if and only if he/she is a classical Shapley null player. Thus, it is interesting to characterize the whole class of games in which the two concepts of null players coincide. The paper has clearly shown that the control of what a LES value assigns to null players constitutes a good tool to characterize the value. We have provided many results characterizing several families of LES values using this approach. But some of the results are establihsed only for regular values. It seems interesting to see how to extend the forgoing results to a broader class of LES values and why not to the whole set of LES values.

7 Appendix

Proof of lemma 2.1

1. We prove that, for any player $i \in N$, $\phi_i(v) = \psi_i(v^{\frac{\phi}{\psi}})$. According to formula (2), for any $i \in N$,

$$\begin{aligned} \psi_i(v^{\frac{\phi}{\psi}}) &= \textstyle\sum_{S \ni i} \frac{(n-s)(s-1)}{n!} \left[a^{\psi}(s) \frac{a^{\phi}(s)}{a^{\psi}(s)} v(S) - a^{\psi}(s-1) \frac{a^{\phi}(s-1)}{a^{\psi}(s-1)} v(S \backslash i) \right] \\ &= \textstyle\sum_{S \ni i} \frac{(n-s)(s-1)}{n!} \left[a^{\phi}(s) v(S) - a^{\phi}(s-1) v(S \backslash i) \right] \\ &= \phi_i(v). \end{aligned}$$

2. Suppose that ϕ and ψ are two regular LES values. Thus $\frac{a^{\phi}(s)}{a^{\psi}(s)} \neq 0$ for all $s = 1, 2, \ldots, n$. We need only to prove that $\left\{ v_T^{\frac{\phi}{\psi}},\ T \subseteq N,\ T \neq \emptyset \right\}$ constitutes a set of linear independent vectors.

   Assume that $\sum_{T \subseteq N} \alpha_T v_T^{\frac{\phi}{\psi}} = 0$ with $\alpha_T \in \mathbb{R}$ for all $T \subseteq N$, then

$$\begin{aligned} \textstyle\sum_{T \subseteq N} \alpha_T v_T^{\frac{\phi}{\psi}}(S) = 0 \ for\ all\ S \subseteq N \quad &\Leftrightarrow \quad \textstyle\sum_{T \subseteq N} \alpha_T \frac{a^{\phi}(s)}{a^{\psi}(s)} v_T(S) = 0 \ for\ all\ S \subseteq N \\ &\Leftrightarrow \quad \textstyle\sum_{T \subseteq N} \alpha_T v_T(S) = 0 \ for\ all\ S \subseteq N \\ &\Leftrightarrow \quad \textstyle\sum_{T \subseteq N} \alpha_T v_T = 0 \ for\ all\ S \subseteq N \\ &\Leftrightarrow \quad \alpha_T = 0 \ for\ all\ T \subseteq N. \end{aligned}$$

   Since $\{v_T,\ T \subseteq N,\ T \neq \emptyset\}$ constitutes a base of $\Gamma(N)$. Thus $\left\{ v_T^{\frac{\phi}{\psi}},\ T \subseteq N,\ T \neq \emptyset \right\}$

constitutes a set of linear independent vectors of $\Gamma(N)$. Hence $\left\{ v_T^{\frac{\phi}{\psi}},\ T \subseteq N,\ T \neq \emptyset \right\}$ is a basis of $\Gamma(N)$. ■

Proof of lemma 3.1
1) The proof is direct since the condition for being null player (in the both senses) in a TU-game (N, v) is a linear expression relatively to v. The details are left to the readers.

2) Suppose φ is a regular value and denote $a(s)_{s=1}^{n}$ the sequence in its representation (2). As φ is regular, $a_s \neq 0$ for $s = 1, 2, ...n$. Consider any two null players in Shapley sense for φ in (N, v). If S is a coalition such that $S \subseteq N \setminus \{i, j\}$. i and j null player in Shapley sense value for φ in $(N, v) \Rightarrow a(s+1)v(S+i) = a(s)v(S)$ and $a(s+1)v(S+j) = a(s)v(S) \Rightarrow a(s+1)v(S+i) = a(s)v(S) = a(s+1)v(S+j)$ $\Rightarrow v(S+i) = v(S+j)$as $a(s+1) \neq 0$. Hence, since this is valid for any coalition $S \subseteq N \setminus \{i, j\}$, the two players i and j are symmetric in (N, v).

3) Consider any i, j, two null players in Solidarity sense for a LES value φ in (N, v). Denote $b(s)_{s=1}^{n}$ the sequence in the representation (4) of φ. If S is a coalition such that $S \subseteq N \setminus \{i, j\}$. Then,
$b(s+1)\left[v(S+i) - v(S)\right] + \dfrac{1-b(s+1)}{s} \sum_{i' \in S} \left[v(S+i) - v(S+i \backslash i')\right] = 0$ and $v(i) = 0$
$b(s+1)\left[v(S+j) - v(S)\right] + \dfrac{1-b(s+1)}{s} \sum_{i' \in S} \left[v(S+j) - v(S+j \backslash i')\right] = 0$ and $v(j) = 0$
rearrange the terms leads to :
$v(S+i) = \dfrac{1-b(s+1)}{s} \sum_{i' \in S} v(S+i \backslash i') + b_{s+1}v(S)$ and $v(i) = 0$.
$v(S+j) = \dfrac{1-b(s+1)}{s} \sum_{i' \in S} v(S+j \backslash i') + b_{s+1}v(S)$ and $v(j) = 0$.
Now, let us show by induction relatively to the size of coalition S that $v(S+i) = v(S+j)$ for all $S \subseteq N \setminus \{i, j\}$. If $s = 0$, $v(S+i) = v(i) = v(j) = v(S+j) = 0$, thus the property is satisfied.
Suppose that the property is satisfied for all S of size k and let us show that the property is also satisfied for all S with size $k+1$. If S is any coalition of size $k+1$ such that $S \subseteq N \setminus \{i, j\}$,
$v(S+i) = \dfrac{1-b(s+1)}{s} \sum_{i' \in S} v(S+i \backslash i') + b_{s+1}v(S)$.
$v(S+j) = \dfrac{1-b(s+1)}{s} \sum_{i' \in S} v(S+j \backslash i') + b_{s+1}v(S)$.
Considering that $S \backslash i'$ for $i' \in S$ is a coalition fo size k, the property is valid for $S \backslash i'$, therefore $v(S+i \backslash i') = v(S+j \backslash i')$ for all $i' \in S$, hence $v(S+i) = v(S+j)$.
■

Proof of theorem 3.1
The approach here consist to define a base for the linear space $\Gamma(N)$ in which the two *LES* values φ and ϕ coincide.
1) Suppose that φ and ϕ are such that $\varphi_i(N, v) = \phi_i(N, v)$ whenever i is a null player in Shapley sense for a regular *LES* value ψ in any $(N, v) \in \Gamma(N)$. Let us consider that $a(s)_{s=1}^{n}$ are the constants in the representation (2) of ψ. Since ψ is regular, $a(s) \neq 0$, $s = 1, 2, \ldots, n$. Now for any coalition $T \subseteq N$, $T \neq \emptyset$, consider the game

$$w_T(S)=\begin{cases} \dfrac{1}{a(s)} & if\ T\subseteq S \\ 0 & if\ T\nsubseteq S \end{cases}$$

We show that :

a) $\{w_T,\ T\subseteq N, T\neq\emptyset\}$ constitutes a base of $\Gamma(N)$.

b) For any non empty coalition T, i is a null player in Shapley sense for ψ in w_T if and only if $i\in N\backslash T$.

c) φ and ϕ coincide in the base $\{w_T,\ T\subseteq S, T\neq\emptyset\}$.

Let us start,
a) From Lemma 2.1 statement 2) $\{w_T,\ T\subseteq N, T\neq\emptyset\}$ constitutes a base of $\Gamma(N)$ since $w_T=v_T^{\frac{\phi}{\psi}}$ where v_T is the unanimity game $v_T(S)=\begin{cases} 1 & if\ T\subseteq S \\ 0 & if\ T\nsubseteq S \end{cases}$ and ϕ is the Shapley value.

b) Consider a player $i\in N\backslash T$, for any coalition $S\ni i$, if $T\subseteq S$ then $T\subseteq S\backslash i$ thus, $A_i(S)=a(s)w_T(S)-a(s-1)w_T(S\backslash i)=\dfrac{a(s)}{a(s)}-\dfrac{a(s-1)}{a(s-1)}=1-1=0.$
If $T\nsubseteq S$ then $T\nsubseteq S\backslash i$ thus, $A_i(S)=a(s)w_T(S)-a(s-1)w_T(S\backslash i)=0-0=0.$
Hence, i is a null player in Shapley sense for ψ in w_T.
Conversely, if $i\notin N\backslash T$, then $T\ni i$ and $T\nsubseteq T\backslash i$ thus,
$A_i(T)=a(t)w_T(T)-a(s-1)w_T(T\backslash i)=a(t)w_T(T)=1$. Therefore, player i is not a null player in Shapley sense for ψ in the game w_T.

c) Suppose that φ and ϕ are two LES values such that for any game (N,v), $\varphi_i(N,v)=\phi_i(N,v)$ whenever i is a null player in Shapley sense for ψ in (N,v).
Thus for any non empty coalition T, we have $\varphi_i(w_T)=\phi_i(w_T)$ for any $i\in N\backslash T$. Setting $h(w_T)=\varphi_i(w_T)=\phi_i(w_T)$ for any $i\in N\backslash T$, by Symmetry and Efficiency we have, for any non empty coalition T,

$$\varphi_i(w_T)=\phi_i(w_T)=\begin{cases} \dfrac{1-(n-t)h(w_T)}{t} & if\ i\in T \\ h(w_T) & if\ i\notin T \end{cases}$$

Thus φ and ϕ coincide on the basis $\{w_T,\ T\subseteq S, T\neq\emptyset\}$. Hence $\varphi=\phi$.

To prove the second part of statement 1), suppose that ψ is not regular and defined such that there exists $p<n$ with $a^{\psi}(p)=1$, $a^{\psi}(p-1)=a^{\psi}(p+1)=0$. Consider the LES value ϕ defined such that $a^{\phi}(p)=0$ and $a^{\phi}(s)=a^{\psi}(s)$ elsewhere. Let us prove that, if $i\in N$ is a null player in Shapley sense for ψ in a game (N,v) then i is also null in Shapley sense for ϕ in (N,v) :
Assume that $i\in N$ is null in Shapley sense for ψ in a game (N,v). Thus $a^{\psi}(s)v(S)-a^{\psi}(s-1)v(S\backslash i)=0$ for all $S\ni i$.
for all $S\ni i$,

- if $s<p$ or $s\geq p+2$, then
  $a^{\phi}(s)v(S)-a^{\phi}(s-1)v(S\backslash i)=a^{\psi}(s)v(S)-a^{\psi}(s-1)v(S\setminus i)=0.$

- if $s = p$ or $s = p+1$, then $a^{\phi}(s)v(S) - a^{\phi}(s-1)v(S \setminus i) = 0 - 0 = 0$.

Finally, i is a null player in Shapley sense for ϕ in (N,v).
Thus, for any game (N,v) and for any null player $i \in N$ in Shapley sense for ψ in the game (N,v), $\psi_i(N,v) = \phi_i(N,v) = 0$, while $\psi \neq \phi$.

2) Suppose that φ and ϕ are such that $\varphi_i(N,v) = \phi_i(N,v)$ whenever i is a null player in Solidarity sense for a LES value ψ in any $(N,v) \in \Gamma(N)$. Let us consider that $b(k)_{k=1}^{n}$ are the constants in the representation (4) of ψ. Now for any coalition $T \subseteq N$, $T \neq \emptyset$, consider the game

$$w_T(S) = \begin{cases} \prod_{k=t+1}^{s} \left(1 - \frac{1-b(k)}{k-1} t\right) & \text{if } T \subset S,\ |S| = s > |T| = t \\ 1 & \text{if } T = S \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

a) $\{w_T,\ T \subseteq N, T \neq \emptyset\}$ constitutes a base of $\Gamma(N)$ since :
If $T_1, T_2, ..., T_K$ (with $K = 2^n - 1$ and $|T_k| = t_k$) is a sequence of all the non empty subsets of N such that $1 = t_1 \leq t_2 \leq, ..., \leq t_K = n$ then the $K \times K$ matrix $M = [m_{lq}]$ defined by $m_{lq} = w_{T_l}(T_q)$ $l, q = 1, 2, ..., K$. is a triangle matrix with all diagonal entries equal to 1. Thus $\{w_T,\ T \subseteq N, T \neq \emptyset\}$ constitutes a set of K linear independant vectors of $\Gamma(N)$, thus a basis of $\Gamma(N)$.
b) We prove that, For any non empty coalition T, i is a null player in Solidarity sense for ψ in w_T if and only if $i \in N \setminus T$:

First, it is easy to see that, for any player $i \in N \setminus T$ and for any coalition $S \ni i$, with $|S| = s > 1$,
$w_T(S) = \left(1 - \frac{1-b(s)}{s-1} t\right) w_T(S \setminus i)$ and $w_T(S) = 0$ if $s = 1$

Second, we prove that, for any player $i \in N \setminus T$ and for all coalition $S \ni i$, $B_i^{w_T}(S) = 0$.
For any player $i \in N \setminus T$, for any coalition $S \ni i$, with $|S| = s$, we have :

If $s = 1$,
- $T \nsubseteq S \Rightarrow w_T(S) = 0$, and thus $B_i^{w_T}(S) = w_T(S) = 0$.
- $T \subseteq S$ is impossible since $i \notin T$.

If $s > 1$,
- $T \nsubseteq S \Rightarrow T \nsubseteq S \setminus j$ for all $j \in N$, thus $w_T(S) = w_T(S \setminus j) = 0$.
$B_i^{w_T}(S) = b(s)\left[w_T(S) - w_T(S \setminus i)\right] + \left(\frac{1-b(s)}{s-1}\right) \sum_{j \in S \setminus i} \left[w_T(S) - w_T(S \setminus j)\right] = 0$.
- $T \subseteq S$
$B_i^{w_T}(S) = b(s)\left[w_T(S) - w_T(S \setminus i)\right] + \left(\frac{1-b(s)}{s-1}\right) \sum_{j \in S \setminus i} \left[w_T(S) - w_T(S \setminus j)\right] = w_T(S) - b(s)w_T(S \setminus i) - \left(\frac{1-b(s)}{s-1}\right) \sum_{j \in S \setminus i} w_T(S \setminus j)$.
since $w_T(S \setminus j) = w_T(S \setminus i)$ for all $j \in S \setminus T$, and $w_T(S \setminus j) = 0$ for all $j \in T$,
$B_i^{w_T}(S) = w_T(S) - b(s)w_T(S \setminus i) - \left(\frac{1-b(s)}{s-1}\right)(s-1-t)w_T(S \setminus i)$.
$= \mathrm{w}_T(S) - w_T(S \setminus i) + t\left(\frac{1-b(s)}{s-1}\right) w_T(S \setminus i)$.
$= \mathrm{w}_T(S) - \left(1 - \frac{1-b(s)}{s-1} t\right) w_T(S \setminus i)$.

$= 0$ according to the property in the first point of b).
c) We refer the reader to the point c) of 1).
■

Proof of lemma 4.1
1) Assume $A1$ holds.
Let $i \in N$, $C \in \mathbb{R}$ and $(N,v) \in \Gamma(N)$.
If $[a(s)v(s) - a(s-1)v(S \setminus i)] = C$ for all $S \ni i \Leftrightarrow [a(s)v(s) - a(s-1)v(S \setminus i)] \geq C$ and $[a(s)v(s) - a(s-1)v(S \setminus i)] \leq C$ for all $S \ni i \Leftrightarrow [a(s)v(s) - a(s-1)v(S \setminus i)] \geq C$ and $[a(s)(-v(s)) - a(s-1)(-v(S \setminus i))] \geq -C$ for all $S \ni i$.
Applying $A1$ leads to, $\phi_i(v) \geq C$ and $\phi_i(-v) \geq -C \Leftrightarrow \phi_i(v) \geq C$ and $\phi_i(v) \leq C \Leftrightarrow \phi_i(v) = C$.

2) Assume $A3$ holds.
Let $i \in N$ and $(N,v),(N,w) \in \Gamma(N)$.
If $[a(s)v(s)-a(s-1)v(S\setminus i)] = [a(s)w(s)-a(s-1)w(S\setminus i)]$ for all $S \ni i \Leftrightarrow [a(s)v(s)-a(s-1)v(S\setminus i)] \geq [a(s)w(s) - a(s-1)w(S\setminus i)]$ and $[a(s)v(s) - a(s-1)v(S\setminus i)] \leq [a(s)w(s) - a(s-1)w(S\setminus i)]$ for all $S \ni i \Leftrightarrow [a(s)v(s)-a(s-1)v(S\setminus i)] \geq [a(s)w(s)-a(s-1)w(S\setminus i)]$ and $[a(s)(-v(s)) - a(s-1)(-v(S \setminus i))] \geq [a(s)(-w(s)) - a(s-1)(-w(S \setminus i))]$ for all $S \ni i$.
Applying A3 leads to $\phi_i(v) \geq \phi_i(w)$ and $\phi_i(-v) \geq \phi_i(-w)$. Since ϕ is linear, $\phi_i(v) \geq \phi_i(w)$ and $\phi_i(v) \leq \phi_i(w) \Leftrightarrow \phi_i(v) = \phi_i(w)$.
3) If $A5$ holds then, for $i \in N$, $C \in \mathbb{R}$ and $(N,v) \in \Gamma(N)$. If $B_i^v(S) = C$ for all $S \ni i$, then $[B_i^v(S) \geq C]$ and $[B_i^v(S) \leq C] \Leftrightarrow [B_i^v(S) \geq C]$ and $[B_i^{-v}(S) \geq -C]$. Applying $A5$ leads to $\phi_i(v) \geq C$ and $\phi_i(-v) \geq -C \Leftrightarrow \phi_i(v) \geq C$ and $\phi_i(v) \leq C \Leftrightarrow \phi_i(v) = C$.
4) If $A7$ holds then, for $i \in N$,$(N,v),(N,w) \in \Gamma(N)$. If $[B_i^v(S) = B_i^w(S)]$ for all $S \ni i$, then $[B_i^v(S) \geq B_i^w(S)]$ and $[B_i^v(S) \leq B_i^w(S)]$ for all $S \ni i \Leftrightarrow [B_i^v(S) \geq B_i^w(S)]$ and $[B_i^{-v}(S) \geq B_i^{-w}(S)]$ for all $S \ni i$. Applying $A5$ leads to $\phi_i(v) \geq \phi_i(w)$ and $\phi_i(v) \leq \phi_i(w)$ $\Leftrightarrow \phi_i(v) = \phi_i(w)$.
5) $A13 \Rightarrow A11$
It obvious that $A13 \Rightarrow A11$ since $A11$ is a particular case of $A13$ with $\varphi =$ Shapley value and $\psi =$ *Egalitarian* value.
$A9 \Rightarrow A11$
Suppose that $A9$ holds, consider a null player i in Shapley sense for φ in the game (N,v) and let us show that, there exists $a \in \mathbb{R}$, such that $\phi_i(v) = a\dfrac{v(N)}{n}$.

- if $v(N) = 0$ then $v(N) = w(N)$ where (N,w) is the zero game (that is $w(S) = 0$ for all coalition S). Thus applying $A9$ leads to $\phi_i(v) = \phi_i(w) = 0$. Therefore $\phi_i(v) = a\dfrac{v(N)}{n}$ for any $a \in \mathbb{R}$.
- If $v(N) \neq 0$, considering that ϕ is efficient, there exists $a \in \mathbb{R}$, such that$\phi_i(v) = a\dfrac{v(N)}{n}$.
  Let us shows that a is independent of the game (N,v).
  Consider another game (N,w) with $w(N) \neq 0$ and a null player i in Shapley sense for φ in the game (N,w). There exists $\beta \in \mathbb{R}$ such that $v(N) = \beta w(N)$.

Considering that i is still a null player in Shapley sense for φ in the game $(N, \beta w)$, applying $A9$ we have, $\phi_i(v) = \phi_i(\beta w) \Rightarrow a\dfrac{v(N)}{n} = a\dfrac{\beta w(N)}{n} = \phi_i(\beta w)$, since ϕ is linear, $\Rightarrow a\dfrac{v(N)}{n} = a\dfrac{\beta w(N)}{n} = \beta\phi_i(w) \Rightarrow \phi_i(w) = a\dfrac{w(N)}{n}$.

$A9 \Rightarrow A11$ Obvious.

6) The same demonstration approach used in the proof of 5) directly leads to the proof of 6). ∎

Proof of Corollary 5.1

If ψ is any LES value with the constants $a(k)_{k=1}^{n-1}$ in its representation (2), then for any TU-game (N, v) and for any $i \in N$,

$\psi_i(N, v) = \dfrac{v(N)}{n} + \sum_{k=1}^{n-1} a(k) \left[\dfrac{(n-k)!(k-1)!}{n!} \sum_{S\ni i} v(S) - \dfrac{(n-k-1)!k!}{n!} \sum_{i\notin S} v(S)\right]$

Noting that, $m_{ik} = \frac{1}{\binom{n-1}{k-1}} \sum_{S\ni i;|S|=k} v(S)$, $\overline{m}_{ik} = \frac{1}{\binom{n-1}{k}} \sum_{i\notin S;|S|=k} v(S)$ and

$m_k = \frac{1}{\binom{n}{k}} \sum_{|S|=k} v(S)$, it is clear that, $\psi_i(N, v) = \dfrac{v(N)}{n} + \sum_{k=1}^{n-1} a(k) \frac{m_{ik}-\overline{m}_{ik}}{n}$.

To achieve the proof of the corollary, we need only to show that $\frac{m_{ik}-\overline{m}_{ik}}{n} = \frac{m_{ik}-m_k}{n-k}$ for all $k = 1, 2, ..., n-1$.

$\sum_{|S|=k} v(S) = \sum_{S\ni i;|S|=k} v(S) + \sum_{S\not\ni i;|S|=k} v(S) \Leftrightarrow$

$\frac{1}{\binom{n}{k}} \sum_{|S|=k} v(S) = \frac{1}{\binom{n}{k}} \sum_{S\ni i;|S|=k} v(S) + \frac{1}{\binom{n}{k}} \sum_{S\not\ni i;|S|=k} v(S)$. $\Leftrightarrow$

$m_k = \frac{k}{n} m_{ik} + \frac{n-k}{n} \overline{m}_{ik}$. $\Leftrightarrow$ $\frac{m_{ik}-\overline{m}_{ik}}{n} = \frac{m_{ik}-m_k}{n-k}$. ∎

Proof of Proposition 5.1

1) $\Leftrightarrow$ 2). For any TU-game (N, v) and any two players $i, j \in N$. The property is obvious for $k = n$. If $k \neq n$, then :

i, j average symmetric $\Leftrightarrow m_{ik} = m_{jk}$ for all $k = 1, 2, ..., n-1$.$\Leftrightarrow m_k - \frac{k}{n} m_{ik} = m_k - \frac{k}{n} m_{jk}$ for all $k = 1, 2, ..., n-1$.$\Leftrightarrow$ $\frac{n-k}{n}\overline{m}_{ik} = \frac{n-k}{n}\overline{m}_{jk}$ for all $k = 1, 2, ..., n-1$.$\Leftrightarrow$ $\overline{m}_{ik} = \overline{m}_{jk}$ for all $k = 1, 2, ..., n-1$.

1) $\Leftrightarrow$ 3). For any TU-game (N, v) and any two players $i, j \in N$. Players i and j are average symmetric in (N, v).$\Leftrightarrow$ $m_{ik} = m_{jk}$ for all $k = 1, 2, ..., n$ $\Leftrightarrow$ $\frac{m_{ik}-m_k}{n} = \frac{m_{jk}-m_k}{n}$ for all $k = 1, 2, ..., n$. $\Leftrightarrow$ $a(k)\frac{m_{ik}-m_k}{n} = a(k)\frac{m_{jk}-m_k}{n}$ for all $k = 1, 2, ..., n$ and for all $a(k) \in \mathbb{R}$.$\Leftrightarrow$ $\sum_{k=1}^{n} a(k)\frac{m_{ik}-m_k}{n} = \sum_{k=1}^{n} a(k)\frac{m_{jk}-m_k}{n}$ for all $a(k) \in \mathbb{R}$ $\Leftrightarrow$ $\varphi_i(N, v) = \varphi_j(N, v)$ for all LES value φ. ∎

Proof of Proposition 5.2

In an unanimity game (N, v_T) (with|T|=t), a direct computation leads to :

$m_{ik} = \frac{\binom{n-t}{k-t}}{\binom{n-1}{k-1}}$ if $k \geq t$ and $i \in T$, and $m_{ik} = \frac{\binom{n-t-1}{k-t-1}}{\binom{n-1}{k-1}}$ if $k > t$ and $i \notin T$, and $m_{ik} = 0$ elsewhere.

$\overline{m}_{ik} = \frac{\binom{n-1-t}{k-t}}{\binom{n-1}{k}}$ if $i \notin T$ and $k \geq t$, and $\overline{m}_{ik} = 0$ elsewhere. It comes that,

if $i \notin T$ then, $m_{ik} = \overline{m}_{ik-1} = 0$ if $k \leq t$ and $m_{ik} = \overline{m}_{ik-1} = \frac{\binom{n-t-1}{k-t-1}}{\binom{n-1}{k-1}}$ if $k > t$

if $i \in T$ then, $\overline{m}_{ik} = 0$ for all k while $m_{ik} = \frac{\binom{n-t}{k-t}}{\binom{n-1}{k-1}}$ if $k \geq t$.
Thus, $m_{ik} = \overline{m}_{ik-1}$ for all $k = 1, 2, ...n$ if and only if $i \notin T$. ■

Proof of Proposition 5.3
1) Suppose that i, j are two average null players in a TU-game (N, v), then $m_{ik} = \overline{m}_{ik-1}$ and $m_{jk} = \overline{m}_{jk-1}$ for all $k = 1, 2, ...n$ Considering that, for any player $i \in N$, and for all $k = 1, 2, ..., n$ we have : $m_k = \frac{k}{n} m_{ik} + \frac{n-k}{n} \overline{m}_{ik}$ thus,
$k(m_{ik} - m_{jk}) = (n-k)(\overline{m}_{jk} - \overline{m}_{ik})$ for all $k = 1, 2, ...n$.
Since i and j are average null player, we obtain :
$k(m_{ik} - m_{jk}) = (n-k)(m_{jk+1} - m_{ik+1})$ for all $k = 0, 1, 2, ..., n-1$ with $m_{i0} = m_{j0} = 0$.
Thus by induction it is clear that, for all $k = 1, 2, ..., n$ $m_{ik} = m_{jk}$.
2) Consider a TU-game (N, v) and an average null player $i \in N$, $Shap_i(N, v) = \sum_{k=1}^{n} \frac{m_{ik} - \overline{m}_{ik}}{n}$, with $\overline{m}_{in} = 0$ and since i is average null $m_{i1} = 0$. Thus,
$Shap_i(N, v) = \frac{1}{n}(\sum_{k=1}^{n} m_{ik} - \sum_{k=1}^{n} \overline{m}_{ik}) = \frac{1}{n}(\sum_{k=2}^{n} m_{ik} - \sum_{k=2}^{n} \overline{m}_{ik-1}) =$
$\sum_{k=2}^{n} \frac{m_{ik} - \overline{m}_{ik-1}}{n} = 0$ ■

Proof of Proposition 5.4
1) Suppose that i, j are two average null players for the regular value φ in a TU-game (N, v), and denote $a(k)_{k=1}^{n-1}$ the constants in the representation (1) of φ ; $a(k) \neq 0$ for all $k = 1, 2, ..., n$.
Then $a_k m_{ik} = a_{k-1} \overline{m}_{ik-1}$ and $a_k m_{jk} = a_{k-1} \overline{m}_{jk-1}$ for all $k = 1, 2, ...n$. Considering that, for any player $i \in N$, and for all $k = 1, 2, ..., n$ we have : $m_k = \frac{k}{n} m_{ik} + \frac{n-k}{n} \overline{m}_{ik}$ thus,
$k a_k (m_{ik} - m_{jk}) = (n-k) a_k (\overline{m}_{jk} - \overline{m}_{ik})$ for all $k = 1, 2, ...n$.
Since i and j are average null player for φ, we obtain :
$k a_k (m_{ik} - m_{jk}) = (n-k) a_{k+1} (m_{jk+1} - m_{ik+1})$ for all $k = 0, 1, 2, ..., n-1$ with $m_{i0} = m_{j0} = 0$. Thus by induction it is clear that, for all $k = 1, 2, ..., n$ $m_{ik} = m_{jk}$.
2) Consider a TU-game (N, v) and $i \in N$ an average null player for φ in (N, v) , $\varphi_i(N, v) = \sum_{k=1}^{n} a_k \frac{m_{ik} - \overline{m}_{ik}}{n}$, with $\overline{m}_{in} = 0$ and since i is average null for φ, $m_{i1} = 0$. Thus,
$\varphi_i(N, v) = \frac{1}{n}(\sum_{k=1}^{n} a_k m_{ik} - \sum_{k=1}^{n} a_k \overline{m}_{ik}) = \frac{1}{n}(\sum_{k=2}^{n} a_k m_{ik} - \sum_{k=2}^{n} a_{k-1} \overline{m}_{ik-1}) =$
$\sum_{k=2}^{n} \frac{a_k m_{ik} - a_{k-1} \overline{m}_{ik-1}}{n} = 0.$ ■

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