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# The Number of Vector Partitions of $n$ with the Crank $m$

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## Abstract

This article shows how to find all vector partitions of any positive integral values of  $n$ , but only all vector partitions of 4, 5, and 6 are shown by algebraically. These must be satisfied by the definitions of crank of vector partitions.

**Keywords:** Vector partitions, Crank, Congruences, Modulo.

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## 1. Introduction

Here we discuss such a crank which in terms of a weighted count of what we call vector partitions. We give the definitions of  $\pi$ ,  $\#(\pi)$ ,  $\sigma(\pi)$ , crank of vector partitions, weight of  $\bar{\pi}$ ,  $N_V(m, n)$ ,  $N_V(m, t, n)$  and prove the partitions congruences moduli 5, 7 and 11 with the help of examples by finding all vector partitions of 4, 5 and 6, respectively. We analyze the generating functions for  $N_V(m, n)$  and  $N_V(m, t, n)$ .

## 2. Definitions

$\pi$  : A partition.

$\#(\pi)$  : The number of parts of  $\pi$ .

$\sigma(\pi)$ : The sum of the parts of  $\pi$ .

Crank of vector partitions: The number of parts of  $\pi_2$  minus the number of parts of  $\pi_3$ , where  $\pi_2$  and  $\pi_3$  are unrestricted partitions in a vector partition  $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$  of  $n$ , if the sum of  $\vec{\pi}$  is  $s(\vec{\pi}) = \sigma(\pi_1) + \sigma(\pi_2) + \sigma(\pi_3) = n$ .

Weight of  $\vec{\pi}$ : Weight of vector partition  $\vec{\pi}$  is defined as;  $\omega(\vec{\pi}) = (-1)^{\#(\pi_1)}$ .

$N_v(m, n)$ : The number of vector partitions of  $n$  (counted according to the weight  $\omega$ ) with the crank  $m$ .

$N_v(m, t, n)$ : The number of vector partitions of  $n$  (counted according to the weight  $\omega$ ) with the crank congruent to  $m$  modulo  $t$ .

### 3. The Crank for Vector Partitions

For a partition  $\pi$ , let  $\#(\pi)$  be the number of parts of  $\pi$  and  $\sigma(\pi)$  be the sum of the parts of  $\pi$  with the convention  $\#(\phi) = \sigma(\phi) = 0$  for the empty partition  $\phi$  of 0 (Andrews 1985, Andrews and Garvan 1988).

Let,  $\vec{V} = \{(\pi_1, \pi_2, \pi_3) \mid \pi_1 \text{ is a partition into unequal parts, } \pi_2, \pi_3 \text{ are unrestricted partitions}\}$ . We shall call the elements of  $\vec{V}$  vector partitions. For  $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$  in  $\vec{V}$  we define the sum of parts,  $s$ , a weight,  $\omega$ , and a crank,  $c$ , by;

$$s(\vec{\pi}) = \sigma(\pi_1) + \sigma(\pi_2) + \sigma(\pi_3).$$

$$\omega(\vec{\pi}) = (-1)^{\#(\pi_1)}.$$

$$c(\vec{\pi}) = \#(\pi_2) - \#(\pi_3).$$

We say  $\vec{\pi}$  is a vector partition of  $n$ , if  $s(\vec{\pi}) = n$ . For example, if  $\vec{\pi} = (1, 1+1, 1)$ , then  $s(\vec{\pi}) = 4$ ,  $\omega(\vec{\pi}) = -1$ ,  $c(\vec{\pi}) = 1$  and  $\vec{\pi}$  is a vector partition of 4.

The number of vector partitions of  $n$  (counted according to the weight  $\omega$ ) with the crank  $m$  is denoted by  $N_v(m, n)$  so that;

$$N_v(m, n) = \sum \omega(\vec{\pi}); \text{ if } \vec{\pi} \in \vec{V}, s(\vec{\pi}) = n, \text{ and } c(\vec{\pi}) = m.$$

We have 41 vector partitions of 4 are given in the following table:

Vector partitions of 4	Weight $\omega(\bar{\pi})$	Crank $(\bar{\pi})$
$\bar{\pi}_1 = (\phi, \phi, 4)$	+1	-1
$\bar{\pi}_2 = (\phi, \phi, 3+1)$	+1	-2
$\bar{\pi}_3 = (\phi, \phi, 2+2)$	+1	-2
$\bar{\pi}_4 = (\phi, \phi, 2+1+1)$	+1	-3
$\bar{\pi}_5 = (\phi, \phi, 1+1+1+1)$	+1	-4
$\bar{\pi}_6 = (\phi, 1, 3)$	+1	0
$\bar{\pi}_7 = (\phi, 1, 2+1)$	+1	-1
$\bar{\pi}_8 = (\phi, 1+1+1+1)$	+1	-2
$\bar{\pi}_9 = (\phi, 2+2)$	+1	0
$\bar{\pi}_{10} = (\phi, 2, 1+1)$	+1	-1
$\bar{\pi}_{11} = (\phi, 1+1, 2)$	+1	1
$\bar{\pi}_{12} = (\phi, 1+1, 1+1)$	+1	0
$\bar{\pi}_{13} = (\phi, 3, 1)$	+1	0
$\bar{\pi}_{14} = (\phi, 2+1, 1)$	+1	1
$\bar{\pi}_{15} = (\phi, 1+1+1, 1)$	+1	2
$\bar{\pi}_{16} = (\phi, 4, \phi)$	+1	1
$\bar{\pi}_{17} = (\phi, 3+1, \phi)$	+1	2
$\bar{\pi}_{18} = (\phi, 2+2, \phi)$	+1	2
$\bar{\pi}_{19} = (\phi, 2+1+1, \phi)$	+1	3
$\bar{\pi}_{20} = (\phi, 1+1+1+1, \phi)$	+1	4
$\bar{\pi}_{21} = (1, \phi, 3)$	-1	-1
$\bar{\pi}_{22} = (1, \phi, 2+1)$	-1	-2
$\bar{\pi}_{23} = (1, \phi, 1+1+1)$	-1	-3
$\bar{\pi}_{24} = (1, 1, 2)$	-1	0
$\bar{\pi}_{25} = (1, 1, 1+1)$	-1	-1
$\bar{\pi}_{26} = (1, 2, 1)$	-1	0
$\bar{\pi}_{27} = (1+1, 1, 1)$	-1	1
$\bar{\pi}_{28} = (1, 3, \phi)$	-1	1
$\bar{\pi}_{29} = (1, 2+1, \phi)$	-1	2
$\bar{\pi}_{30} = (1, 1+1+1, \phi)$	-1	3
$\bar{\pi}_{31} = (2, \phi, 2)$	-1	-1
$\bar{\pi}_{32} = (2, \phi, 1+1)$	-1	-2
$\bar{\pi}_{33} = (2, 1, 1)$	-1	0

$\vec{\pi}_{34} = (2, 2, \phi)$	-1	1
$\vec{\pi}_{35} = (2, 1+1, \phi)$	-1	2
$\vec{\pi}_{36} = (3, \phi, 1)$	-1	-1
$\vec{\pi}_{37} = (2+1, \phi, 1)$	+1	-1
$\vec{\pi}_{38} = (3, 1, \phi)$	-1	1
$\vec{\pi}_{39} = (2+1, 1, \phi)$	+1	1
$\vec{\pi}_{40} = (4, \phi, \phi)$	-1	0
$\vec{\pi}_{41} = (3+1, \phi, \phi)$	+1	0

From the above table we have,

$$\begin{aligned}
N_V(0,4) &= \omega(\vec{\pi}_6) + \omega(\vec{\pi}_9) + \omega(\vec{\pi}_{12}) + \omega(\vec{\pi}_{13}) + \omega(\vec{\pi}_{24}) + \omega(\vec{\pi}_{26}) + \omega(\vec{\pi}_{33}) + \omega(\vec{\pi}_{40}) + \omega(\vec{\pi}_{41}) \\
&= 1+1+1+1-1-1-1-1+1 = 1 \tag{1}
\end{aligned}$$

The number of vector partitions of  $n$  (counted according to the weight  $\omega$ ) with the crank congruent to  $k$  modulo  $t$  is denoted by  $N_V(k, t, n)$ , so that;

$$N_V(k, t, n) = \sum_{m=-\infty}^{\infty} N_V(m, t+k, n) = \sum \omega(\vec{\pi}); \tag{2}$$

if  $\vec{\pi} \in \vec{V}$ ,  $s(\vec{\pi}) = n$ , and  $c(\vec{\pi}) \equiv k \pmod{t}$ .

From the table we get;

$$\begin{aligned}
N_V(1,5,4) &= \omega(\vec{\pi}_5) + \omega(\vec{\pi}_{11}) + \omega(\vec{\pi}_{14}) + \omega(\vec{\pi}_{16}) + \omega(\vec{\pi}_{27}) + \omega(\vec{\pi}_{28}) + \omega(\vec{\pi}_{34}) + \omega(\vec{\pi}_{38}) + \omega(\vec{\pi}_{39}) \\
&= 1+1+1+1-1-1-1-1+1 = 1. \tag{3}
\end{aligned}$$

By considering the transformation that interchanges  $\pi_2$  and  $\pi_3$  we have;

$$N_V(m, n) = N_V(-m, n).$$

We illustrate with an example;

$$\begin{aligned}
N_V(1,4) &= \omega(\vec{\pi}_{11}) + \omega(\vec{\pi}_{14}) + \dots + \omega(\vec{\pi}_{39}) \\
&= 1 + 1 + 1-1-1-1-1+1 = 0.
\end{aligned}$$

and

$$N_V(-1,4) = \omega(\vec{\pi}_1) + \omega(\vec{\pi}_7) + \dots + \omega(\vec{\pi}_{37})$$

$$= 1 + 1 + 1 - 1 - 1 - 1 - 1 + 1 = 0$$

$$\therefore N_V(1,4) = N_V(-1,4).$$

Again,

$$N_V(5-1,5,4) = N_V(4,5,4) = \omega(\vec{\pi}_{20}) = 1$$

$$\therefore N_V(1,5,4) = N_V(5-1,5,4) \quad \text{by (3).}$$

Generally we can write,

$$N_V(m,t,n) = N_V(t-m,t,n)$$

### 3.1. The Generating Function for $N_V(m,n)$

The generating function for  $N_V(m,n)$  is;

$$\prod_{n=1}^{\infty} \frac{(1-x^n)}{(1-zx^n)(1-z^{-1}x^n)} = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_V(m,n) z^n x^n \quad (4)$$

which was proved by Atkin and Swinnerton-Dyer (1954). By putting  $z = 1$  in (4), we get;

$$\begin{aligned} & \prod_{n=1}^{\infty} \frac{(1-x^n)}{(1-x^n)(1-x^n)} \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_V(m,n) x^n \\ &\Rightarrow \sum_{n=0}^{\infty} P(n) x^n = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_V(m,n) x^n \\ &\therefore P(n) = \sum_{m=-\infty}^{\infty} N_V(m,n). \end{aligned} \quad (5)$$

Now we discuss it with an example;

$$\text{R. H. S.} = \sum_{m=-\infty}^{\infty} N_V(m,n)$$

$$\begin{aligned}
&= \sum_{m=-\infty}^{\infty} N_v(m,4) \\
&= \dots + N_v(-4,4) + N_v(-3,4) + N_v(-2,4) + N_v(-1,4) + N_v(0,4) + N_v(1,4) + N_v(2,4) \\
&\quad + N_v(3,4) + N_v(4,4) + \dots
\end{aligned}$$

$$= 0 + 1 + 0 + 1 + 0 + 1 + 0 + 1 + 0 + 1 = 5 = P(4) = \text{L. H. S.}$$

### 3.2. The Generating Function for $N_v(0, n)$

The generating function for  $N_v(0, n)$  is defined as;

$$\begin{aligned}
&(1-x) \sum_{n=0}^{\infty} \frac{x^{n(n+2)}}{(x^2)_n} \\
&= (1-x) \left[ 1 + \frac{x^3}{(1-x)^2} + \frac{x^8}{(1-x)^2(1-x^2)^2} + \frac{x^{15}}{(1-x)^2(1-x^3)^2} + \dots \right] \\
&= 1 - x + 0 \cdot x^2 + x^3 + x^4 + x^5 + x^6 + \dots \\
&= N_v(0,0) + N_v(0,1)x + N_v(0,2)x^2 + N_v(0,3)x^3 + N_v(0,4)x^4 + N_v(0,5)x^5 + N_v(0,6)x^6 + \dots \\
&= \sum_{n=0}^{\infty} N_v(0, n) x^n .
\end{aligned}$$

### 3.4 Result

**3.4.1.** The result is;

$$N_v(k, 5, 5n+4) = \frac{P(5n+4)}{5}; \quad 0 \leq k \leq 4.$$

**Proof:** We prove the result with an example.

From the table 1 we get;

$$N_v(0,5,4) = \omega(\bar{\pi}_6) + \omega(\bar{\pi}_9) + \omega(\bar{\pi}_{12}) + \omega(\bar{\pi}_{13}) + \omega(\bar{\pi}_{24}) + \omega(\bar{\pi}_{26}) + \omega(\bar{\pi}_{33}) + \omega(\bar{\pi}_{40}) + \omega(\bar{\pi}_{41})$$

$$= 1+1+1+1-1-1-1-1+1 = 1,$$

$$N_v(1,5,4) = 1+1+1+1-1-1-1-1+1 = 1,$$

$$N_v(2,5,4) = 1+1+1+1-1-1-1 = 1,$$

$$N_v(3,5,4) = 1+1+1-1-1+1-1 = 1,$$

$$N_v(4,5,4) = 1+1+1-1-1-1-1+1+1 = 1.$$

$$\therefore N_v(0,5,4) = N_v(1,5,4) = N_v(2,5,4) = N_v(3,5,4) = N_v(4,5,4) = 1 = \frac{P(4)}{5}, \text{ where } n = 0.$$

In general we can write;

$$N_v(k,5,5n+4) = \frac{P(5n+4)}{5}; 0 \leq k \leq 4.$$

Hence the Theorem.

**3.2.2.** The result is;

$$N_v(k,7,7n+5) = \frac{P(7n+4)}{7}; 0 \leq k \leq 6.$$

**Proof:** We prove the result with an example.

The vector partitions of 5 are given in the table below:

Vector partitions of 5	Weight $\omega(\vec{\pi})$	Crank $(\vec{\pi})$
$\vec{\pi}_1 = (\phi, \phi, 5)$	+1	-1
$\vec{\pi}_2 = (\phi, \phi, 4+1)$	+1	-2
$\vec{\pi}_3 = (\phi, \phi, 3+2)$	+1	-2
$\vec{\pi}_4 = (\phi, \phi, 3+1+1)$	+1	-3
$\vec{\pi}_5 = (\phi, \phi, 2+2+1)$	+1	-3
$\vec{\pi}_6 = (\phi, \phi, 2+1+1+1)$	+1	-4
$\vec{\pi}_7 = (\phi, \phi, 1+1+1+1+1)$	+1	-5
$\vec{\pi}_8 = (5, \phi, \phi)$	-1	0
$\vec{\pi}_9 = (\phi, 5, \phi)$	+1	1
$\vec{\pi}_{10} = (\phi, 4+1, \phi)$	+1	2
$\vec{\pi}_{11} = (4+1, \phi, \phi)$	+1	0
$\vec{\pi}_{12} = (4, 1, \phi)$	-1	1



$\bar{\pi}_{13} = (1, 4, \phi)$	-1	1
$\bar{\pi}_{14} = (\phi, 4, 1)$	+1	0
$\bar{\pi}_{15} = (\phi, 1, 4)$	+1	0
$\bar{\pi}_{16} = (1, \phi, 4)$	-1	-1
$\bar{\pi}_{17} = (4, \phi, 1)$	-1	-1
$\bar{\pi}_{18} = (3 + 2, \phi, \phi)$	+1	0
$\bar{\pi}_{19} = (\phi, 3 + 2, \phi)$	+1	2
$\bar{\pi}_{20} = (3, 2, \phi)$	-1	1
$\bar{\pi}_{21} = (2, 3, \phi)$	-1	1
$\bar{\pi}_{22} = (\phi, 3, 2)$	+1	0
$\bar{\pi}_{23} = (\phi, 2, 3)$	+1	0
$\bar{\pi}_{24} = (3, \phi, 2)$	-1	-1
$\bar{\pi}_{25} = (2, \phi, 3)$	-1	-1
$\bar{\pi}_{26} = (\phi, 3 + 1 + 1, \phi)$	+1	3
$\bar{\pi}_{27} = (3 + 1, 1, \phi)$	+1	1
$\bar{\pi}_{28} = (1, 3 + 1, \phi)$	-1	2
$\bar{\pi}_{29} = (\phi, 3 + 1, 1)$	+1	1
$\bar{\pi}_{30} = (\phi, 1, 3 + 1)$	+1	-1
$\bar{\pi}_{31} = (3 + 1, \phi, 1)$	+1	-1
$\bar{\pi}_{32} = (1, \phi, 3 + 1)$	-1	-2
$\bar{\pi}_{33} = (3, 1 + 1, \phi)$	-1	2
$\bar{\pi}_{34} = (\phi, 1 + 1, 3)$	+1	1
$\bar{\pi}_{35} = (\phi, 3, 1 + 1)$	+1	-1
$\bar{\pi}_{36} = (3, \phi, 1 + 1)$	-1	-2
$\bar{\pi}_{37} = (\phi, 2 + 2 + 1, \phi)$	+1	3
$\bar{\pi}_{38} = (1, 2 + 2, \phi)$	-1	2
$\bar{\pi}_{39} = (\phi, 2 + 2, 1)$	+1	1
$\bar{\pi}_{40} = (\phi, 1, 2 + 2)$	+1	-1
$\bar{\pi}_{41} = (1, \phi, 2 + 2)$	-1	-2
$\bar{\pi}_{42} = (2 + 1, 2, \phi)$	+1	1
$\bar{\pi}_{43} = (2, 2 + 1, \phi)$	-1	2
$\bar{\pi}_{44} = (\phi, 2, 2 + 1)$	+1	1
$\bar{\pi}_{45} = (\phi, 2 + 1, 2)$	+1	1
$\bar{\pi}_{46} = (2 + 1, \phi, 2)$	+1	-1
$\bar{\pi}_{47} = (2, \phi, 2 + 1)$	-1	-2

$\bar{\pi}_{48} = (\phi, 2+2+1, \phi)$	+1	4
$\bar{\pi}_{49} = (\phi, 2+1+1, 1)$	+1	2
$\bar{\pi}_{50} = (\phi, 1, 2+1+1)$	+1	-2
$\bar{\pi}_{51} = (1, 2+1+1, \phi)$	-1	3
$\bar{\pi}_{52} = (1, \phi, 2+1+1)$	-1	-3
$\bar{\pi}_{53} = (2+1, 1+1, \phi)$	+1	2
$\bar{\pi}_{54} = (\phi, 2+1, 1+1)$	+1	0
$\bar{\pi}_{55} = (\phi, 1+1, 2+1)$	+1	0
$\bar{\pi}_{56} = (2+1, \phi, 1+1)$	+1	-2
$\bar{\pi}_{57} = (\phi, 1+1+1, 2)$	+1	2
$\bar{\pi}_{58} = (\phi, 2, 1+1+1)$	+1	-2
$\bar{\pi}_{59} = (2, 1+1+1, \phi)$	-1	3
$\bar{\pi}_{60} = (2, \phi, 1+1+1)$	-1	-3
$\bar{\pi}_{61} = (\phi, 1+1+1+1+1, \phi)$	+1	5
$\bar{\pi}_{62} = (\phi, 1+1+1+1, 1)$	+1	3
$\bar{\pi}_{63} = (\phi, 1, 1+1+1+1)$	+1	-3
$\bar{\pi}_{64} = (1, \phi, 1+1+1+1)$	-1	-4
$\bar{\pi}_{65} = (1, 1+1+1+1, \phi)$	-1	4
$\bar{\pi}_{66} = (\phi, 1+1, 1+1+1)$	+1	-1
$\bar{\pi}_{67} = (\phi, 1+1+1, 1+1)$	+1	1
$\bar{\pi}_{68} = (1, 1, 1+1+1)$	-1	-2
$\bar{\pi}_{69} = (1, 1+1+1, 1)$	-1	2
$\bar{\pi}_{70} = (1, 1+1, 1+1)$	-1	0
$\bar{\pi}_{71} = (1, 1+1, 2)$	-1	1
$\bar{\pi}_{72} = (1, 2, 1+1)$	-1	-1
$\bar{\pi}_{73} = (2, 1+1, 1)$	-1	1
$\bar{\pi}_{74} = (2, 1, 1+1)$	-1	-1
$\bar{\pi}_{75} = (2, 2, 1)$	-1	0
$\bar{\pi}_{76} = (2, 1, 2)$	-1	0
$\bar{\pi}_{77} = (1, 2, 2)$	-1	0
$\bar{\pi}_{78} = (3, 1, 1)$	-1	0
$\bar{\pi}_{79} = (1, 3, 1)$	-1	0
$\bar{\pi}_{80} = (1, 1, 3)$	-1	0
$\bar{\pi}_{81} = (1+2, 1, 1)$	+1	0
$\bar{\pi}_{82} = (1, 1+2, 1)$	-1	1

$\vec{\pi}_{83} = (1,1,1+2)$	-1	-1
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From this table we have;

$$N_V(0,7,5) = \omega(\vec{\pi}_8) + \omega(\vec{\pi}_{11}) + \omega(\vec{\pi}_{14}) + \omega(\vec{\pi}_{15}) + \omega(\vec{\pi}_{18}) + \omega(\vec{\pi}_{22}) + \omega(\vec{\pi}_{23}) + \omega(\vec{\pi}_{54}) + \omega(\vec{\pi}_{55}) + \omega(\vec{\pi}_{70}) + \omega(\vec{\pi}_{75}) + \omega(\vec{\pi}_{76}) + \omega(\vec{\pi}_{77}) + \omega(\vec{\pi}_{78}) + \omega(\vec{\pi}_{79}) + \omega(\vec{\pi}_{80}) + \omega(\vec{\pi}_{81})$$

$$= -1+1+1+1+1+1+1+1+1-1-1-1-1-1-1-1-1+1 = 1.$$

Similarly,

$$N_V(0,7,5) = N_V(1,7,5) = \dots = N_V(6,7,5) = 1 = \frac{P(5)}{7}.$$

In general we can write;

$$N_V(k,7,7n+5) = \frac{P(7n+5)}{7}; 0 \leq k \leq 6.$$

Hence the result.

**3.2.3.** The result is;

$$N_V(k,11,11n+6) = \frac{P(11n+6)}{11}.$$

**Proof:** We prove the result with an example.

The vector partitions of 6 are given in the table below:

Vector partitions of 6	Weight $\omega(\vec{\pi})$	Crank $(\vec{\pi})$
$\vec{\pi}_1 = (\phi, \phi, 6)$	+1	-1
$\vec{\pi}_2 = (\phi, \phi, 5+1)$	+1	-2
$\vec{\pi}_3 = (\phi, \phi, 4+2)$	+1	-2
$\vec{\pi}_4 = (\phi, \phi, 4+1+1)$	+1	-3
$\vec{\pi}_5 = (\phi, \phi, 3+3)$	+1	-2
$\vec{\pi}_6 = (\phi, \phi, 3+2+1)$	+1	-3
$\vec{\pi}_7 = (\phi, \phi, 3+1+1+1)$	+1	-4
$\vec{\pi}_8 = (\phi, \phi, 2+2+2)$	+1	-3

$\bar{\pi}_9 = (\phi, \phi, 2+2+1+1)$	+1	-4
$\bar{\pi}_{10} = (\phi, \phi, 2+1+1+1+1)$	+1	-5
$\bar{\pi}_{11} = (\phi, \phi, 1+1+1+1+1+1)$	+1	-6
$\bar{\pi}_{12} = (\phi, 6, \phi)$	+1	1
$\bar{\pi}_{13} = (\phi, 5+1, \phi)$	+1	2
$\bar{\pi}_{14} = (\phi, 4+2, \phi)$	+1	2
$\bar{\pi}_{15} = (\phi, 4+1+1, \phi)$	+1	3
$\bar{\pi}_{16} = (\phi, 3+3, \phi)$	+1	2
$\bar{\pi}_{17} = (\phi, 3+2+1, \phi)$	+1	3
$\bar{\pi}_{18} = (\phi, 3+1+1+1, \phi)$	+1	4
$\bar{\pi}_{19} = (\phi, 2+2+2, \phi)$	+1	3
$\bar{\pi}_{20} = (\phi, 2+2+1+1, \phi)$	+1	4
$\bar{\pi}_{21} = (\phi, 2+1+1+1+1, \phi)$	+1	5
$\bar{\pi}_{22} = (\phi, 1+1+1+1+1+1, \phi)$	+1	6
$\bar{\pi}_{23} = (6, \phi, \phi)$	-1	0
$\bar{\pi}_{24} = (5+1, \phi, \phi)$	+1	0
$\bar{\pi}_{25} = (4+2, \phi, \phi)$	+1	0
$\bar{\pi}_{26} = (3+2+1, \phi, \phi)$	-1	0
$\bar{\pi}_{27} = (\phi, 5, 1)$	+1	0
$\bar{\pi}_{28} = (\phi, 1, 5)$	+1	0
$\bar{\pi}_{29} = (\phi, 4, 2)$	+1	0
$\bar{\pi}_{30} = (\phi, 2, 4)$	+1	0
$\bar{\pi}_{31} = (\phi, 4, 1)$	+1	1
$\bar{\pi}_{32} = (\phi, 4, 1+1)$	+1	-1
$\bar{\pi}_{33} = (\phi, 1, 4+1)$	+1	-1
$\bar{\pi}_{34} = (\phi, 1+1, 4)$	+1	1
$\bar{\pi}_{35} = (\phi, 3, 3)$	+1	0
$\bar{\pi}_{36} = (\phi, 3+2, 1)$	+1	1
$\bar{\pi}_{37} = (\phi, 1, 3+2)$	+1	-1
$\bar{\pi}_{38} = (\phi, 3, 2+1)$	+1	-1
$\bar{\pi}_{39} = (\phi, 2+1, 3)$	+1	1
$\bar{\pi}_{40} = (\phi, 1+3, 2)$	+1	1
$\bar{\pi}_{41} = (\phi, 2, 1+3)$	+1	-1
$\bar{\pi}_{42} = (\phi, 3, 1+1+1)$	+1	-2
$\bar{\pi}_{43} = (\phi, 3+1, 1+1)$	+1	0

$\bar{\pi}_{44} = (5, \phi, 1)$	-1	-1
$\bar{\pi}_{45} = (5, 1, \phi)$	-1	1
$\bar{\pi}_{46} = (4, \phi, 2)$	-1	-1
$\bar{\pi}_{47} = (4, 2, \phi)$	-1	1
$\bar{\pi}_{48} = (\phi, 1+1+1, 3)$	+1	2
$\bar{\pi}_{49} = (\phi, 1+1, 3+1)$	+1	0
$\bar{\pi}_{50} = (\phi, 1, 3+1+1)$	+1	-2
$\bar{\pi}_{51} = (\phi, 3+1+1, 1)$	+1	2
$\bar{\pi}_{52} = (\phi, 2+2, 2)$	+1	1
$\bar{\pi}_{53} = (\phi, 2, 2+2)$	+1	-1
$\bar{\pi}_{54} = (\phi, 2, 1+1+1+1)$	+1	-3
$\bar{\pi}_{55} = (\phi, 1+1+1+1, 2)$	+1	3
$\bar{\pi}_{56} = (\phi, 2+1, 1+1+1)$	+1	-1
$\bar{\pi}_{57} = (\phi, 1+1+1, 2+1)$	+1	1
$\bar{\pi}_{58} = (\phi, 2+1+1, 1+1)$	+1	1
$\bar{\pi}_{59} = (\phi, 1+1, 2+1+1)$	+1	-1
$\bar{\pi}_{60} = (\phi, 1+1+1+1, 1+1)$	+1	2
$\bar{\pi}_{61} = (\phi, 1+1, 1+1+1+1)$	+1	-2
$\bar{\pi}_{62} = (\phi, 1+1+1, 1+1+1)$	+1	0
$\bar{\pi}_{63} = (\phi, 1, 1+1+1+1+1)$	+1	-4
$\bar{\pi}_{64} = (\phi, 1+1+1+1+1, 1)$	+1	4
$\bar{\pi}_{65} = (3, 2, 1)$	-1	0
$\bar{\pi}_{66} = (3, 1, 2)$	-1	0
$\bar{\pi}_{67} = (2, 3, 1)$	-1	0
$\bar{\pi}_{68} = (2, 1, 3)$	-1	0
$\bar{\pi}_{69} = (1, 2, 3)$	-1	0
$\bar{\pi}_{70} = (1, 3, 2)$	-1	0
$\bar{\pi}_{71} = (3, 1, 1+1)$	-1	-1
$\bar{\pi}_{72} = (3, 1+1, 1)$	-1	1
$\bar{\pi}_{73} = (2, 2+1, 1)$	-1	1
$\bar{\pi}_{74} = (2, 1, 1+2)$	-1	-1
$\bar{\pi}_{75} = (1, 1+1+1, 1+1)$	-1	1
$\bar{\pi}_{76} = (1, 1+1, 1+1+1)$	-1	-1
$\bar{\pi}_{77} = (1, 1, 1+1+1+1)$	-1	-3
$\bar{\pi}_{78} = (1, 1+1+1+1, 1)$	-1	3

$\bar{\pi}_{79} = (2,1+1,1+1)$	-1	0
$\bar{\pi}_{80} = (4,1,1)$	-1	0
$\bar{\pi}_{81} = (3,\phi,3)$	-1	-1
$\bar{\pi}_{82} = (3,3,\phi)$	-1	1
$\bar{\pi}_{83} = (3,1+1+1,\phi)$	-1	3
$\bar{\pi}_{84} = (3,\phi,1+1+1)$	-1	-3
$\bar{\pi}_{85} = (2,2+2,\phi)$	-1	2
$\bar{\pi}_{86} = (2,\phi,2+2)$	-1	-2
$\bar{\pi}_{87} = (2,2+1+1,\phi)$	-1	3
$\bar{\pi}_{88} = (2,\phi,2+1+1)$	-1	-3
$\bar{\pi}_{89} = (2,\phi,2+2)$	-1	-2
$\bar{\pi}_{90} = (2,\phi,1+1+1+1)$	-1	-4
$\bar{\pi}_{91} = (1,1+1+1+1+1,\phi)$	-1	-4
$\bar{\pi}_{92} = (1,\phi,1+1+1+1+1)$	-1	-5
$\bar{\pi}_{93} = (1+2,3,\phi)$	+1	1
$\bar{\pi}_{94} = (1+2,\phi,3)$	+1	-1
$\bar{\pi}_{95} = (3+1,2,\phi)$	+1	1
$\bar{\pi}_{96} = (3+1,\phi,2)$	+1	-1
$\bar{\pi}_{97} = (3+1,1,1)$	+1	0
$\bar{\pi}_{98} = (4+1,1,\phi)$	+1	1
$\bar{\pi}_{99} = (4+1,\phi,1)$	+1	-1
$\bar{\pi}_{100} = (4,1+1,\phi)$	-1	2
$\bar{\pi}_{101} = (4,\phi,1+1)$	-1	-2
$\bar{\pi}_{102} = (3+1,1+1,\phi)$	+1	2
$\bar{\pi}_{103} = (3+1,\phi,1+1)$	+1	-2
$\bar{\pi}_{104} = (2+1,1+1+1,\phi)$	+1	3
$\bar{\pi}_{105} = (2+1,\phi,1+1+1)$	+1	-3
$\bar{\pi}_{106} = (2+1,1,2)$	+1	0
$\bar{\pi}_{107} = (2+1,2,1)$	+1	0
$\bar{\pi}_{108} = (1,2+1,2)$	-1	1
$\bar{\pi}_{109} = (1,2,2+1)$	-1	-1
$\bar{\pi}_{110} = (1,2+3,\phi)$	-1	2
$\bar{\pi}_{111} = (1,\phi,2+3)$	-1	-2
$\bar{\pi}_{112} = (\phi,4,2)$	+1	1

$\bar{\pi}_{113} = (2 + 3, \phi, 1)$	+1	-1
$\bar{\pi}_{114} = (2, 1 + 3, \phi)$	-1	2
$\bar{\pi}_{115} = (2, \phi, 3 + 1)$	-1	-2
$\bar{\pi}_{116} = (1, 2 + 2 + 1, \phi)$	-1	3
$\bar{\pi}_{117} = (1, \phi, 2 + 2 + 1)$	-1	-3
$\bar{\pi}_{118} = (2 + 1, 1 + 1, 1)$	+1	1
$\bar{\pi}_{119} = (2 + 1, 1, 1 + 1)$	+1	-1
$\bar{\pi}_{120} = (1, 1 + 1, 2 + 1)$	-1	0
$\bar{\pi}_{121} = (1, 2 + 1, 1 + 1)$	-1	0

From this table we have;

$$\begin{aligned}
N_V(0,11,6) &= \omega(\bar{\pi}_{23}) + \omega(\bar{\pi}_{24}) + \omega(\bar{\pi}_{25}) + \omega(\bar{\pi}_{26}) + \omega(\bar{\pi}_{27}) + \omega(\bar{\pi}_{28}) + \omega(\bar{\pi}_{29}) + \omega(\bar{\pi}_{30}) + \omega(\bar{\pi}_{35}) + \\
&\omega(\bar{\pi}_{43}) + \omega(\bar{\pi}_{49}) + \omega(\bar{\pi}_{62}) + \omega(\bar{\pi}_{65}) + \omega(\bar{\pi}_{66}) + \omega(\bar{\pi}_{67}) + \omega(\bar{\pi}_{68}) + \omega(\bar{\pi}_{69}) + \\
&\omega(\bar{\pi}_{70}) + \omega(\bar{\pi}_{79}) + \omega(\bar{\pi}_{80}) + \omega(\bar{\pi}_{97}) + \omega(\bar{\pi}_{106}) + \omega(\bar{\pi}_{107}) + \omega(\bar{\pi}_{120}) + \omega(\bar{\pi}_{121}) \\
&= -1+1+1-1+1+1+1+1+1+1+1-1-1-1-1-1-1-1-1+1+1+1-1-1 = 1.
\end{aligned}$$

$$N_V(0,11,6) = 1 = \frac{P(6)}{11}, \text{ where } n = 0 \text{ and } k = 0.$$

Hence the result.

#### 4. Conclusions

We verified that for any positive integral value of  $n$  in the relation  $P(n) = \sum_{m=-\infty}^{\infty} N_V(m, n)$  and easily can find generating function for  $N_V(m, n)$  in terms of various corresponding cranks of vector partitions.

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