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Generating Functions for P_1^r (n) and P_2^r (n)

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Abstract

In 1970 George E. Andrews defined the generating functions for P_1^r (n) and P_2^r (n). In this article these generating functions are discussed elaborately. This paper shows how to prove the theorem P_2^r (n) = P_3^r (n) with a numerical example when $n = 9$ and $r = 2$. In 1966 Andrews defined the terms A'(n) and B'(n), but this paper proves the remark $A'(n) = B'(n)$ with the help of an example when $n = 10$. In 1961, N. Bourbaki defined the term P(n, m). This paper shows how to prove a Remark in terms of P(n, m), where P(n, m) is the number of partitions of the type of enumerated by P_3^r (n) with the further restrictions that $b_1 < 2m$.

Keywords: Generating functions, number of partitions.

1 Introduction

We give definitions of $P_1^r(n)$, $P_2^r(n)$, $P_3^r(n)$, $A'(n)$, $B'(n)$, and $P(n,m)$. Then we generate the function for $P_1^r(n)$, $P_2^r(n)$, and $P_3^r(n)$, which are collected from George E. Andrews [1] and Hardy and Wright [5] and prove the Theorem $P_2^r(n)=P_3^r(n)$. George E. Andrews [1] has already prove the remark $P_1^r(n)=P_3^r(n)$ and we give a numerical example for $A'(n)=B'(n)$. Finally we prove a remark which is related to the term $P(n,m)$.

2 Definitions

$P_1^r(n)$: The number of partitions of n into part that are either even or not congruent to $4r-2(\mathrm{mod}\,4r)$ or odd and congruent to $2r-1, 4r-1(\mathrm{mod}\,4r)$ [2].

$P_2^r(n)$: The number of partitions of n into parts that are either even or else congruent to $2r-1(\mathrm{mod}\,2r)$ with the further restriction that only even parts may be repeated.

$P_3{}^r(n)$: The number of partitions of *n* of the form $n=b_1+b_2+...+b_s$, where $b_i \geq b_{i+1}$, and for b_i odd, $b_i-b_{i+1} \geq 2r-1$ $(1 \leq i \leq s, \text{where } b_{s+1}=0)$.

$A'(n)$: The number of partitions of *n* into parts congruent to 0, 2, 3, 4, 7 (mod 8).

$B'(n)$: The number of partitions of *n* of the form $n=b_1+b_2+...+b_s$, where $b_s \geq 2$, $b_i \geq b_{i+1}$, and if b_i is odd, $b_i-b_{i+1} \geq 3$.

$P(n,m)$: The number of partitions of the type enumerated by $P_3{}^r(n)$, with the further restriction that $b_1 \leq 2m$ [4].

3 Consider the Generating Function with r ≥ 2, [3]

We have,

$$\prod_{j=1}^{\infty}\left(1-x^{4rj-2}\right)\left(1-x^{2j}\right)^{-1}\left(1-x^{2rj-1}\right)^{-1} \qquad (1)$$

$$=\prod_{j=1}^{\infty}\left(1-x^{4rj-2}\right)\left(1+x^{2j}+x^{4j}+...\infty\right)\times\left(1+x^{2rj-1}+x^{4rj-2}+...\infty\right)$$

$$=1+\sum_{n=1}^{\infty}P_1^r(n)x^n,$$

where the coefficient $P_1{}^r(n)$ is the number of partitions of *n* into parts that are either even and not congruent to $4r-2(\bmod 4r)$ or odd and congruent to $2r-1, 4r-1(\bmod 4r)$.

We consider a function, which is of the form;

$$\prod_{j=1}^{\infty}\left(1+x^{2rj-1}\right)\left(1-x^{2j}\right)^{-1}; r \geq 2$$

$$=\prod_{j=1}^{\infty}\left(1+x^{2rj-1}\right)\left(1+x^{2j}+x^{4j}+...\infty\right)$$

$$=1+\sum_{n=1}^{\infty}P_2^r(n)x^n \qquad (2)$$

where the coefficient $P_2^r(n)$ is the number of partitions of *n* into parts that are either even or else congruent to $2r-1(\bmod 2r)$ with the further restriction that only even parts may be repeated.

From (1) we have;

$$1+\sum_{n=1}^{\infty} P_1^r(n)x^n = \prod_{j=1}^{\infty}\left(1-x^{4rj-2}\right)\left(1-x^{2j}\right)^{-1}\left(1-x^{2rj-1}\right)^{-1}$$

$$=\prod_{j=1}^{\infty}\frac{\left(1+x^{2rj-1}\right)}{\left(1-x^{2j}\right)} = 1+\sum_{n=1}^{\infty} P_2^r(n)x^n \text{ , by (2).}$$

Now equating the coefficient of x^n from the both sides we get;

$P_1^r(n) = P_2^r(n)$.

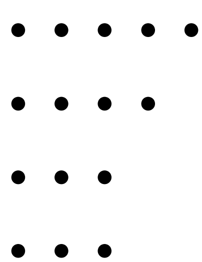
Here we give a Theorem, which is related to the terms $P_2^r(n)$ and $P_3^r(n)$.

Theorem: Let $r \geq 2$ be an integer. Let $P_2^r(n)$ denote the number of partitions of *n* into parts that are either even or else congruent to $2r-1(\text{mod}\, 2r)$ with the further restriction that only even parts may be repeated. Let $P_3^r(n)$ denote the number of partitions of *n* of the form $n = b_1 + b_2 + ... + b_s$, where $b_i \geq b_{i+1}$, and for b_i odd, $b_i - b_{i+1} \geq 2r-1$ $\left(1 \leq i \leq s, \text{where } b_{s+1} = 0\right)$.

Then $P_2^r(n) = P_3^r(n)$.

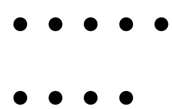
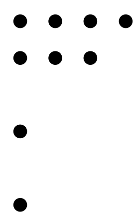
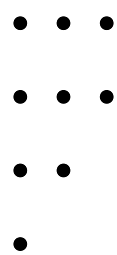
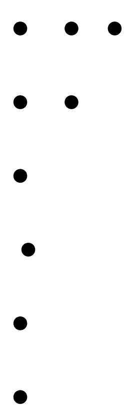
Proof: Let Ψ' be a partition of the type enumerated by $P_3^r(n)$. We represent Ψ' graphically with each even part 2*m* represented by two rows of m nodes and each odd part 2*m* + 1 represented by two rows of *m*+1 nodes and *m* nodes respectively.

Such as 9 + 6 becomes;



Now we may consider the graph vertically with the condition that *r* columns are always to be grouped as a single part, whenever the lowest node in the most right hand column of the group is not presented there. If *r* = 2, we obtain in this manner, the partition 4 +4 + 4 + 3. Now since the condition on partitions enumerated by $P_3^r(n)$ is $b_i - b_{i+1} \geq 2r-1$, whenever b_i is odd, we see that our grouping of *r* columns always have one less node that a rectangle of $r-2v$ nodes, when *v* is any positive integer. Thus a part congruent to $2r-1(\text{mod}\, 2r)$ is produced. Since originally odd parts were distinct, we see that now odd parts will be congruent to $2r-1(\text{mod}\, 2r)$ and will not be repeated and since originally all odd parts were greater or equal to $2r-1$, we see that there will always be *r* columns available for each grouping. Thus in this case we have produced a partition of the type enumerated by $P_2^r(n)$. Clearly our correspondence is one to one, however, the above process is reversible and thus the correspondence is onto. So that $P_2^r(n) = P_3^r(n)$. Hence the Theorem.

Example 1: We take r = 2, n = 9. The corresponding partitions are listed opposite each other in the following table:

$P_3^r(9)$		$P_2^r(9)$
9 with relevant graph		2 + 2 + 2 + 3
7 + 2 " " "		4 + 2 + 3
6 + 3 " " "		7 + 2
5 + 2 + 2 " " "		6 + 3

Now we can write $P_3^r(9) = P_2^r(9) = 4$. Here we give some remarks.

Remark 1: $P_1^r(n) = P_3^r(n)$, if $r \geq 2$ i.e., let $r \geq 2$ be an integer. Let $P_1^r(n)$ denote the number of partitions of n into parts that are either even and not congruent to $4r-2 \pmod{4r}$ or odd and congruent to $2r-1, 4r-1 \pmod{4r}$. Let $P_3^r(n)$ denote the number of partitions of n of the form $n = b_1 + b_2 + ... + b_s$, where $b_i \geq b_{i+1}$, and for b_i odd, $b_i - b_{i+1} \geq 2r-1$ $(1 \leq i \leq s, \text{where } b_{s+1} = 0)$.

Then $P_1^r(n) = P_3^r(n)$. It is proved in George E. Andrews [1]. We can establish the following Remark:

Remark 2: Let $A'(n)$ denote the number of partitions of n into parts congruent to 0, 2, 3, 4, 7 (mod 8). Let $B'(n)$ denote the number of partitions of n of the form $n = b_1 + b_2 + ... + b_s$, where $b_i \geq b_{i+1}$, and for b_i is odd, $b_i - b_{i+1} \geq 3$. Then $A'(n) = B'(n)$.

Example 2: If n = 10, the eight partitions enumerated by $A'(10)$ are 10, 8 + 2, 7 + 3, 4 + 4 + 2, 4 + 2 + 2 + 2, 4 + 3 + 3, 3 + 3 + 2 + 2, 2+ 2 + 2 + 2 + 2.

The eight partitions enumerated by $B'(10)$ are 10, 8 + 2, 7 + 3, 6 + 4, 6 + 2 + 2, 4 + 4 + 2, 4 + 2 + 2 + 2, 2 + 2 + 2 + 2 + 2.

Thus $A'(10) = B'(10)$.

Here we give the Remark which is related to the term $P(n,m)$.

Remark 3:
$P(n,m) - P(n,m-1) = P(n-2m,m) + P(n-2m+1,m-r)$, where $P(n,m)$ is the number of partitions of the type enumerated by $P_3^r(n)$ with the further restriction that $b_1 \leq 2m$.

Proof: Here $P(n,m) - P(n,m-1)$ denotes the number of partitions of the type enumerated by $P(n,m)$ with the further restriction that either $2m$ or $2m$–1 is the largest part. If $2m$ is the largest part, we remove it. We obtain a partition of the type enumerated by $P(n-2m,m)$. If $2m$– 1 is the largest part, then the next largest part does not exceed $2m$–1–($2r$–1) or $2m$–$2r$, since $2m$–1 is an odd part. If $2m$–1 is removed from the partition under consideration, we obtain a partition of the type enumerated by $P(n-2m+1,m-r)$. Hence the above process establishes a (1, 1) correspondence between those partitions enumerated by $P(n,m) - P(n,m-1)$ and the totality of partitions, which are enumerated either by

$P(n-2m,m)$ or by $P(n-2m+1,m-r)$.

Thus, $P(n,m) - P(n,m-1) = P(n-2m,m) + P(n-2m+1,m-r)$.

Hence the Remark.

4 Conclusions

We have seen that for any positive integer of n and $r \geq 2$ the Theorem $P_1^r(n) = P_3^r(n)$ is satisfied. We have shown the Theorem $A'(n) = B'(n)$ is true with the help of example when n = 10.

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