Matrix representation of TU-games for Linear Efficient and Symmetric values

Miamo, Clovis Wendji

University of Dschang

1 January 2017

Matrix representation of TU-games for Linear Efficient and Symmetric values

C. Miamo Wendji

Abstract

The aim of this article is to present a new tool for assessing TU-game based on a matrix representation. We focus on TU-games with coalition structures and provide a general matrix form of TU-game. We shed light on some useful properties of the matrix representation of TU-game and the general form obtained is applied to describe the representation for some classical TU-game. The facilities provided by such a representation are used to characterize subclasses of Linear Efficient and Symmetric (LES) values.

JEL classification : D63 D31 C71

Keywords : Cooperative games ; Matrix ; LES value.

1 Introduction and notations

A cooperative game in the transferable utility form (or a coalitional game with side payments, or simply a TU-game) is any (N, v), where N is a finite set (of players) with at least two elements, and $v : 2^N \to \mathbb{R}$ is an application called the characteristic function satisfying $v(\emptyset) = 0$. A nonempty subset S of N is called a coalition while $v(S)$ is the worth of the coalition S. For any set of players N, n is the number of players (or the cardinality of N). Given such a set N, we denote by $\Gamma(N)$ the set of all TU-games (N, v). It is well known that $\Gamma(N)$ is a $2^n - 1$ dimensional linear space.

A value on $\Gamma(N)$ is a function ψ that assigns a single payoff vector $(\psi_i(N, v))_{i\in\mathbb{N}} \in \mathbb{R}^n$ to every game (N, v). $(\psi_i(N, v))_{i\in\mathbb{N}}$ is a distribution of the total wealth available to all the players through their participation in the game (N, v).

A famous solution value for TU-games, *Shapley value*, is widely used in economic modeling. Its axiomatization are provided in theoritical game analysis, from which three main characteristic axioms relate to linearity, efficiency and symmetry. Moreover, the class of values which verify the three properties is wide, playing an essential role in TU-games theoretical litterature. Among other members of this class of LES values, we can cite the Solidarity value (Nowak and Radzik, 1994), Egalitarian value (Van Den Brink, 2007), Equal surplus value (Driessen and Funaki, 1991), Consensus value (Ju, Born and Ruys, 2007) etc.

A value ψ on $\Gamma(N)$ is said to be linear if $\psi_i(N, \alpha v+\beta w) = \alpha\psi_i(N, v)+\beta\psi_i(N, w)$ for all games (N, v), (N, w), for all player $i \in N$ and for all $\alpha, \beta \in \mathbb{R}$. ψ on $\Gamma(N)$ is symmetric if for all games (N, v) and for any automorphism π of v, $\psi_i(N, v) = \psi_{\pi(i)}(N, \pi v)$. A value ψ on $\Gamma(N)$ possesses the efficiency property if $\sum_{i\in\mathbb{N}} \psi_i(N, v) = v(N)$. A value ψ is covariant if $\psi_i(N, w) = k\psi_i(N, w) + p_i$ for every $(N, v) \in \Gamma(N), k \in \mathbb{R}_+$ and $p \in \mathbb{R}^n$ where w is given by $w(S) = kv(S) + \sum_{j\in S} p_j$ for all $S \subseteq N$.

The main purpose of our note is to propose a matrix representation for LES values computation in TU-games. We start with the representation formula of LES values found in Ruiz et al. (1998) and Chameni and Andjiga (2008) which establishes some one to one correspondence between LES values and collections of $n-1$ constants where n is the number of the player in the TU-game. It turns out that the matrix representation we provide, is a very useful and convenient tool in computation of LES values, analyzing LES values and TU-games. We show that many of the properties of TU-games and LES values can be obtained using very simple conditions on the matrix representation.

The paper is organized as follows. In Section 2 we state some preliminaries on LES values and give the matrix representation of such values. The properties of matrix are studied. In Section 3 we show how using the matrix representation can be usefull in analysing TU-games and LES values. We illustrate it by characterizing some well-kwown class of TU-games and LES values. Finally Section 4 contains some concluding remarks.

2 Parameterization of LES values and matrix representation

To start with, we quote a result about the parametric representation for LES.

Proposition 2.1. *Consider a set of player N of cardinality n and $\Gamma(N)$ the set of all transferable utility games (N, v). Then the following statements, for a value ψ on $\Gamma(N)$, are equivalent :*

1) *ψ is a LES value on $\Gamma(N)$.*

2) *There exists a unique collection of $n-1$ constants $a(s)_{s=1}^{n-1}$ such that, for any $i \in N$,*

$$\psi_i(N,v) = \frac{v(N)}{n} + \sum_{s=1}^{n-1} a(s) \left[\frac{(n-s)!(s-1)!}{n!} \sum_{S \ni i} v(S) - \frac{(n-s-1)!s!}{n!} \sum_{i \notin S} v(S) \right] \tag{1}$$

3) *There exists a unique collection of n constants $a(s)_{s=1}^{n}$ with $a(n) = 1$, such that, for any $i \in N$,*

$$\psi_i(N,v) = \sum_{S \ni i} \frac{(n-s)!(s-1)!}{n!} \left[a(s)v(S) - a(s-1)v(S \backslash i) \right]. \tag{2}$$

Remark 2.1. *It is easy to see that the formula given in (1) generalizes the classical Shapley value. The marginal contribution term $v(s) - v(S\backslash i)$ is replaced by a weighted marginal contribution $a(s)v(s) - a(s-1)v(S\backslash i)$.*

In the literature the formula defined in (2) is attributed to Ruiz et al. (1998) who had established an equivalent expression. But the current form of the formula appeared very recently in the literature (see Chameni and Andjiga (2008), Radzick and Driessen (2013)) for the sake of getting closer to the classical Shapley value expression. We refer the lector to Chameni (2012) for the economic interpretation of the coefficient $a(s)_{s=1}^{n-1}$.

Corollary 2.1. *Let (N, v) be a TU-game of $\Gamma(N)$ and ψ any LES value on $\Gamma(N)$, if we set for any player $i \in N$ and for any k with $1 \leq k \leq n-1$,*

$$t_k(i) = \frac{(n-k)!(k-1)!}{n!} \sum_{S \ni i; |S|=k} v(S) - \frac{(n-k-1)!k!}{n!} \sum_{S \not\ni i; |S|=k} v(S) \tag{3}$$

Then, $\psi_i(N,v) = \frac{v(N)}{n} + \sum_{k=1}^{n-1} a_\psi(k) t_k(i)$
where $a_\psi(k)_{k=1}^{n-1}$ is the collection of constants in the representation of the LES value ψ given by (1).

Proposition 2.2. *(Matrix representation of TU-game and LES value)*
Consider a TU-game (N, v) of $\Gamma(N)$ and ψ any LES value on $\Gamma(N)$, if we set for any player $i \in N$ and for any k with $1 \leq k \leq n-1$,
$t_k(i) = \frac{(n-k)!(k-1)!}{n!} \sum_{S \ni i; |S|=k} v(S) - \frac{(n-k-1)!k!}{n!} \sum_{S \not\ni i; |S|=k} v(S).$

Then we have the matrix representation,

$$\begin{pmatrix} \psi_1(v) \\ \psi_2(v) \\ \vdots \\ \psi_n(v) \end{pmatrix} = \begin{pmatrix} \frac{v(N)}{n} \\ \frac{v(N)}{n} \\ \vdots \\ \frac{v(N)}{n} \end{pmatrix} + \begin{pmatrix} t_1(1) & \cdots & t_1(n-1) \\ \vdots & \ddots & \vdots \\ t_n(1) & \cdots & t_n(n-1) \end{pmatrix} \begin{pmatrix} a_\psi(1) \\ a_\psi(2) \\ \vdots \\ a_\psi(n-1) \end{pmatrix}. \tag{4}$$

In the sequel, for any TU-game $(N,v) \in \Gamma(N)$ the matrix representation in (4) is denoted $M_v = \left((t_{ik})_{\substack{i=1\dots n \\ k=1\dots n-1}}\right)$ and it is called the matrix of the game (N,v) while the vector $A_\psi = \begin{pmatrix} a_\psi(1) \\ a_\psi(2) \\ \vdots \\ a_\psi(n-1) \end{pmatrix}$ is called the associated vector of the value ψ.

It is worth noting that each LES value is characterized by its associated vector while the link between a game and its matrix is not a one to one correspondence. It is easy to see that two different TU-games may have the same matrix. For more details we give some properties of the matrix of the game below. In this regard let us introduce a new subclass of TU-game.

Definition 2.1. *A TU-game (N,v) is said to be weakly symmetric if for all players $i,j \in N$ and for all k with $1 \leq k \leq n$,*

$$\sum_{S \ni i; |S|=k} v(S) = \sum_{S \ni j; |S|=k} v(S) \tag{5}$$

The economic interpretation of a weakly symmetric game is done as follow : in a TU-game (N,v) the productivity of a player i is evaluated by the production vector

$$u(i) = (u_k(i)_{k=1,2,\dots,n-1}) \qquad where \qquad u_k(i) = \sum_{S \ni i, |S|=k} v(S) \tag{6}$$

Then, a weakly symmetric game is a game where all the players have the same level of productivity. The reason why the term *weakly* is used is that all symmetric games (i.e. games (N,v) such that $v(S) = v(T)$ iff $|T| = |S|$), are weakly symmetric. Thus, any additive game (N,v) with $v(i) = v(j)$ for all players $i,j \in N$, is weakly symmetric.

Properties of the matrix M_v

Consider a TU-game (N,v) and a set of any player $i \in N$ and for any k with $1 \leq k \leq n-1$, $t_k(i) = \frac{(n-k)!(k-1)!}{n!} \sum_{S \ni i; |S|=k} v(S) - \frac{(n-k-1)!k!}{n!} \sum_{S \not\ni i; |S|=k} v(S)$.

Property 1 : $t_k(i) = \frac{m_{ik} - m_k}{n-k} = \frac{m_{ik} - \overline{m}_{ik}}{n}$, for all k with $1 \leq k \leq n-1$
where :
m_{ik} = mean of worths of all coalitions size k containing player i.

m_k = mean of worths of all coalitions size k.
$\overline{m}_{ik}$ = mean of worths of all coalitions size k non containing player i.

Property 2 : Each column entries of the matrix M_v sum to zero and each LES value is obtained as linear combination of vectors column plus the egalitarian value E (where $E(N,v) = \frac{V(N)}{n}$ for all $i \in N$).

Property 3 : The application H defined by $H(N,v) = M_v$ verifies :

a) H is a linear transformation.

b) The kernel of H is formed by the familly of weakly symmetric games. In other words $M_v = 0$ iff (N,v) is a weakly symmetric game.

c) For two TU-games (N,v) and (N,w) in $\Gamma(N)$, $M_v = M_w$ iff there exist a weakly symmetric game (N,u) such that w=v+u.

Proof: See Appendix. ■

Definition 2.2. *Two TU-games (N,v) and (N,w) are said to be similar if their respective matrix M_v and M_w coincide, that is $M_v = M_w$.*

Note that the binary relation $\sim$ defined in $\Gamma(N)$ by $(N,v) \sim (N,w)$ iff (N,v) and (N,w) are similar, is an equivalent relation. In other words, the relation $\sim$ satisfies : reflexivity, symmetry and transitivity.

The next proposition sheds light on the cosets of the equivalence relation.

Proposition 2.3. *Let (N,v) and (N,w) be any TU-games. Then the following statements are equivalent :*

1) *(N,v) and (N,w) are similar.*

2) *There exists a weakly symmetric game (N,u) such that $v = w + u$.*

3) *For any LES value φ and for any player $i \in N$, $\varphi_i(N,v) - \frac{V(N)}{n} = \varphi_i(N,w) - \frac{w(N)}{n}$.*

Proof: See Appendix. ■

3 Applications

In this section we study the matrix representation for some classical TU-games. We use the facilities provided by the representation to characterize subclasses of values.

3.1 Matrix representation of some classical TU-games

3.1.1 Characteristic game

The characteristic game of a coalition $T \subseteq N$ is the game $v_T(S) = \begin{cases} 1 & \text{if } S = T, \\ 0 & \text{otherwise.} \end{cases}$

The set of all characteristic games $\{v_T(S), T \subseteq N, T = \emptyset\}$ constitutes a base of $\Gamma(N)$. If we suppose that $T = \{1, 2, ..., t\}$, the matrix of the game v_T is of the form :

$$M_{v_T} = \begin{array}{c} \\ 1 \\ \vdots \\ t \\ t+1 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & \cdots & t-1 & t & t+1 & \cdots & n-1 \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & \frac{1}{\binom{n-1}{t-1}} & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots & 0 & \vdots \\ 0 & \cdots & 0 & \frac{1}{\binom{n-1}{t-1}} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{-1}{\binom{n-1}{t}} & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots & 0 & \vdots \\ 0 & \cdots & 0 & \frac{-1}{\binom{n-1}{t}} & 0 & \cdots & 0 \end{pmatrix} \end{array} \tag{7}$$

3.1.2 Unanimity game

Another famous basis of TU-game consist of unanimity games. Unanimity game associated to a coalition $T \subseteq N$ is defined by $v_T(S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$

The matrix of unanimity game v_T is given by :

$$M_{v_T}(i,k) = \begin{cases} 0 & \text{if } k < t, \\ (\frac{1}{n})\frac{\binom{n-t}{k-t}}{\binom{n-1}{k-1}} & \text{if } k \geq t \text{ and } i \in T \\ (\frac{1}{n})\frac{\binom{n-t-1}{k-t-1}}{\binom{n-1}{k-1}} - (\frac{1}{n})\frac{\binom{n-t-1}{k-t}}{\binom{n-1}{k-1}} & \text{if } k \geq t \text{ and } i \notin T \end{cases} \tag{8}$$

3.1.3 Additive game

A game (N, v) is additive or inessential if for all $S \subseteq N$, $v(S) = \sum_{i \in S} v(i)$.

If a game (N, v) is additive then the terms of it matrix M_v are :

$$M_v(i,k) = \frac{v(i)}{n} - \frac{\sum_{j \neq i} v(j)}{n(n-1)} = \frac{\sum_{j=1}^{n}(v(i) - v(j))}{n(n-1)} \tag{9}$$

It is worth noting that in this case, $M_v(i,k)$ is independent to k, which also means that the $n-1$ columns of the matrix are identical. Besides, the following result holds :

Proposition 3.1. *Let φ be any LES value on $\Gamma(N)$. Then, the following statements are equivalent :*

1) *φ is covariant.*

2) *For any additive game (N, v) and for any $i \in N$, $\varphi_i(N, v) = v(i)$.*

3) *The associated vector $\begin{pmatrix} a_\psi(1) \\ a_\psi(2) \\ \vdots \\ a_\psi(n-1) \end{pmatrix}$ of the value φ is such that $\sum_{k=1}^{n-1} a_\varphi(k) = n-1$.*

Note that, the result stated in proposition 3.1 is already obtained in Chameni and Andjiga (2008). However, here the proof is easily obtained by using the matrix representation of TU-game.

Proposition 3.2. *If M is a $n \times (n-1)$ matrix with all its $n-1$ columns identical and each column entries sum to zero, then there exists an additive game (N, v) such that $M_v = M$.*

Proposition 3.3. *The $n-1$ columns of a matrix M_v of a TU-game (N, v) are identical iff there exists an additive game (N, w) such that (N, v) and (N, w) are similar.*

Proof: See Appendix. ■

3.1.4 Weakly symmetric game

In this subsection, we use previous results to give a general characterization of weakly symmetric games.

Proposition 3.4. *For any TU-game (N, v), the following statements are equivalent.*

1) *(N, v) is weakly symmetric.*

2) *M_v=0.*

3) *For any LES value φ and ϕ, for any player $i \in N$, $\varphi_i(N, v) = \phi_i(N, v)$.*
In other words, all LES values coincide in (N, v).

4) *For any player $i \in N$, for any LES value ϕ, $\phi_i(N, v) = \frac{V(N)}{n}$.*

Proof: See Appendix. ■

3.1.5 Weakly Linear game (Freixas, 2010)

Let (N, v) be a TU-game. For any player $i \in N$ we consider the vector $u(i)$ defined in (6) whose components are respectively $u_k(i)$. Clearly, the process defines a function $u : N \to \mathbb{R}^{n-1}$. Therefore, the binary relation $\succcurlyeq_u$ defined by $i \succcurlyeq_u j$ iff for all k, $k = 1, 2, \cdots, n-1$, $u_k(i) \geq u_k(j)$ is a preordering in N that is not always complete.

Note that, generaly speaking, the canonical preordering defined in $\mathbb{R}^n$ by :

$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \leq \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ iff $x_i \leq y_i$ for all $i = 1, 2, \cdots, n$, is not a complete preordering in $\mathbb{R}^n$. However, it could be the case in particular subsets on $\mathbb{R}^n$.

Definition 3.1. *A TU-game (N, v) is weakly linear if the binary relation $\succcurlyeq_u$ is a complete preordering in N.*

Definition 3.2. *Consider a TU-game (N, v) and let $\succcurlyeq_1$ and $\succcurlyeq_2$ be two binary relations in N that are preordering. We say that $\succcurlyeq_1$ and $\succcurlyeq_2$ are compatible if, for all $i, j \in N$, $i \succ_1 j \Rightarrow i \succcurlyeq_2 j$ and $i \succ_2 j \Rightarrow i \succcurlyeq_1 j$*

Note that, two preordering $\succcurlyeq_1$ and $\succcurlyeq_2$ are compatible does not necessarely means that they coincide, since it is possible to obtain $i \succ_1 j$ and $i \sim_2 j$. However, if the compatibility holds, the two preordering can not opposite.

Proposition 3.5. *For any TU-game (N, v), the following statements are equivalent.*

1) *(N, v) is weakly Linear.*

2) *The canonical preordering of $\mathbb{R}^{n-1}$ is complete in the set of the n vectors line of M_v.*

3) *All the $n-1$ preordering $\succcurlyeq_k$ $(k = 1, 2, \cdots, n-1)$ defined in N by : $i \succcurlyeq_k j$ iff $M_v(i,k) \geq M_v(j,k)$ are compatible.*

Proof: See Appendix. $\blacksquare$

4 Concluding remark

In this article, we have proposed a matrix representation of TU-games which is a useful tool for handling such cooperative games and computing LES values. We have shown that many of the properties of TU-games and LES values can be obtained using very simple conditions on the matrix representation. In particular a new class of TU-games have been introduced, the so called weakly symmetric game. This class of game contains the subclass of symmetric games and the question at this stage is whether the two classes of game coincide. It is easy to observe that in games of small size ($n < 5$), weakly symmetric game and symmetric game coincide. However for games of large size ($n \geq 5$) the question is still pending.

5 Appendix

Proof of properties

Property 1

a) We show that, for $1 \leq k \leq n-1$, $t_k(i) = \frac{m_{ik} - \overline{m}_{ik}}{n}$

$$
\begin{aligned}
t_k(i) &= \frac{(n-k)!(k-1)!}{n!} \sum_{S \ni i; |S|=k} v(S) - \frac{(n-k-1)!k!}{n!} \sum_{S \not\ni i; |S|=k} v(S) \\
&= \frac{1}{n}\Big[\frac{(n-k)!(k-1)!}{(n-1)!} \sum_{S \ni i; |S|=k} v(S)\Big] - \frac{1}{n}\Big[\frac{(n-k-1)!k!}{(n-1)!} \sum_{S \not\ni i; |S|=k} v(S)\Big] \\
&= \frac{1}{n}\Big[\frac{1}{\binom{n-1}{k-1}} \sum_{S \ni i; |S|=k} v(S) - \frac{1}{\binom{n-1}{k}} \sum_{S \not\ni i; |S|=k} v(S)\Big] \\
&= \frac{1}{n}(m_{ik} - \overline{m}_{ik}).
\end{aligned}
$$

b) We show that, $t_k(i) = \frac{m_{ik}-m_k}{n-k}$

$$
\begin{aligned}
t_k(i) &= \frac{(n-k)!(k-1)!}{n!} \sum_{S\ni i;|S|=k} v(S) - \frac{(n-k-1)!k!}{n!} \sum_{S\not\ni i;|S|=k} v(S) \\
&= \frac{(n-k)!(k-1)!}{(n)!} \sum_{S\ni i;|S|=k} v(S) - \frac{(n-k-1)!k!}{(n)!} [\sum_{|S|=k} v(S) - \sum_{S\ni i;|S|=k} v(S)] \\
&= \frac{(n-k)!(k-1)! + (n-k-1)!k!}{(n)!} \sum_{S\ni i;|S|=k} v(S) - \frac{(n-k-1)!k!}{(n)!} \sum_{|S|=k} v(S) \\
&= \frac{(n-k-1)!(k-1)!}{(n-1)!} \sum_{S\ni i;|S|=k} v(S) - \frac{(n-k)!k!}{(n-k)n!} \sum_{|S|=k} v(S) \\
&= \frac{1}{(n-k)} [\frac{(n-k-1)!(k-1)!}{(n-1)!} \sum_{S\ni i;|S|=k} v(S) - \frac{(n-k)!k!}{(n-k)n!} \sum_{|S|=k} v(S)] \\
&= \frac{m_{ik}-m_k}{n-k}.
\end{aligned}
$$

Property 2

We show that, for all k, column k entries sum to zero,

$$
\begin{aligned}
\sum_{i=1}^{n} \frac{m_{ik}-m_k}{n-k} &= \frac{1}{(n-k)} [\sum_{i=1}^{n} m_{ik} - nm_k] \\
&= \frac{1}{(n-k)} [\sum_{i=1}^{n} \frac{1}{\binom{n-1}{k-1}} \sum_{S\ni i;|S|=k} v(S) - nm_k] \\
&= \frac{1}{(n-k)} [\frac{1}{\binom{n-1}{k-1}} \sum_{i=1}^{n} \sum_{S\ni i;|S|=k} v(S) - nm_k] \\
&= \frac{1}{(n-k)} [\frac{1}{\binom{n-1}{k-1}} \sum_{|S|=k} v(S) - nm_k] \\
&= \frac{1}{(n-k)} [nm_k - nm_k] = 0.
\end{aligned}
$$

Property 3

a) $H(\alpha v + \beta w) = M_{\alpha v+\beta w}$ For all $i \in N$ and $1 \le k \le n-1$,
$M_{\alpha v+\beta w}(i,k) = \frac{1}{(n-k)}[\frac{1}{\binom{n-1}{k-1}} \sum_{S\ni i;|S|=k} (\alpha v+\beta w)(S) - \frac{1}{\binom{n-1}{k}} \sum_{|S|=k} (\alpha v+\beta w)(S)]$
$= [\frac{1}{\binom{n-1}{k-1}} \sum_{S\ni i;|S|=k} \alpha v(S) - \frac{1}{\binom{n-1}{k}} \sum_{|S|=k} \alpha v(S)] + [\frac{1}{\binom{n-1}{k-1}} \sum_{S\ni i;|S|=k} \beta w(S) - \frac{1}{\binom{n-1}{k}} \sum_{|S|=k} \beta w(S)]$
$= \alpha M_v(i,k) + \beta M_w(i,k)$
$\Rightarrow M_{\alpha v+\beta w} = \alpha M_v + \beta M_w = \alpha H(v) + \beta H(w)$.

b) If (N,v) is weakly symmetric, then $\sum_{S\ni i;|S|=k} v(S) = \sum_{S\ni j;|S|=k} v(S)$ for all $i,j \in N$ and for all $1 \le k \le n-1$, thus for all $i \in N$, and for all k,

$1 \leq k \leq n-1$, we have :
If we set $N = \{i, j_1, j_2, \cdots, j_{n-1}\}$,
$\sum_{S \ni i; |S|=k} v(S) = \sum_{S \ni i; |S|=k} v(S)$
$\sum_{S \ni i; |S|=k} v(S) = \sum_{S \ni j_1; |S|=k} v(S)$
$\vdots\vdots$
$\sum_{S \ni i; |S|=k} v(S) = \sum_{S \ni j_{n-1}}$; $|S| = kv(S)$, where $N = \{i, j_1, j_2, \cdots, j_{n-1}\}$

Summing each side of the equations leads to :
$n \sum_{S \ni i; |S|=k} v(S) = \sum_{j \in N} \sum_{S \ni j; |S|=k} v(S)$
$\Rightarrow n \sum_{S \ni i; |S|=k} v(S) = k \sum_{|S|=k} v(S) \Rightarrow \frac{1}{\binom{n-1}{k-1}} \sum_{S \ni i; |S|=k} = \frac{k}{n\binom{n-1}{k-1}} \sum_{|S|=k}$

$\Rightarrow m_{i,k} = m_k \Rightarrow m_{i,k} - m_k = 0$
$\Rightarrow$ For all $i \in N$, for all $1 \leq k \leq n-1$, $M_v(i,k) = 0 \Rightarrow M_v = 0$

Conversely, suppose that (N, v) is such that M_v=0.
$\Rightarrow$ for all $i \in N$, and for all $1 \leq k \leq n-1$, $M_v(i,k) = 0$
$\Rightarrow$ for all $i \in N$, and for all $1 \leq k \leq n$, $m_{i,k} = m_k \Rightarrow m_{i,k} = m_{j,k} = m_k$ for all $i, j \in N$, and for all $1 \leq k \leq n-1 \Rightarrow (N, v)$ is weakly linear.

■

Proof of proposition 2.3
(**1) $\Rightarrow$ 2)**) Suppose that (N, v) and (N, w) are two similar TU-games $\Rightarrow M_v = M_w \Rightarrow M_{v-w} = 0$. Setting $u = v - w$, we have $M_u = 0 \Rightarrow (N, u)$ is weakly symmetric $\Rightarrow v = w + u$, with u weakly symmetric.
(**2) $\Rightarrow$ 3)**) Suppose (N, v) and (N, w) are such that $v = w + u$ with u a weakly symmetric game $\Rightarrow M_v = M_{w+u} = M_w + M_u = M_w + 0 = M_w$, thus the matrix representation implies $\varphi_i(N, v) - \frac{V(N)}{n} = \varphi_i(N, w) - \frac{w(N)}{n}$ for any LES value φ and for any $i \in N$.
(**3) $\Rightarrow$ 1)**) Consider M_v and M_w the matrix of (N, v) and (N, w). According to property (**3)**), for any vector $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix} \in \mathbb{R}^{n-1}$, we have $M_v \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix} = M_w \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix} \Rightarrow$ $M_v = M_w \Rightarrow (N, v)$ and (N, w) are similar. ■

Proof of proposition 3.1
(**1) $\Rightarrow$ 2)**) Obvious from definition of covariant value.
(**2) $\Rightarrow$ 3)**) According to the matrix representation of an additive game, we have :
$\varphi_i(N, v) = \frac{V(N)}{n} + \sum_{k=1}^{n-1} a_k \left(\frac{v(i)}{n} - \frac{\sum_{j \neq i} v(j)}{n-1} \right) = v(i)$ for all $v(j) \in \mathbb{R}$
$\Rightarrow \frac{v(i)}{n} \sum_{k=1}^{n-1} a_k - \frac{\sum_{j \neq i} v(j)}{(n-1)n} \sum_{k=1}^{n-1} a_k = v(i) - \frac{v(i)}{n} - \frac{\sum_{j \neq i} v(j)}{n}$ for all $v(j) \in \mathbb{R}$
$\Rightarrow v(i) \frac{\sum_{k=1}^{n-1} a_k}{n} - \frac{\sum_{j \neq i} v(j)}{n} \frac{\sum_{k=1}^{n-1} a_k}{n-1} = \frac{n-1}{n} v(i) - \frac{\sum_{j \neq i} v(j)}{n}$
$\Rightarrow \frac{\sum_{k=1}^{n-1} a_k}{n} = \frac{n-1}{n}$

$\Rightarrow \sum_{k=1}^{n-1} a_k = n-1.$
(3) ⇒ 1)) (See corollary 2 in Chameni and Andjiga, 2008). ■

Proof of proposition 3.2
Suppose that M is a $n \times (n-1)$ matrix with the $n-1$ columns identical and each column entries sum to zero. Setting $E = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n / \sum_{i=1}^{n} x_i = 0\}$.

It is clear that E is a hyperplane of $\mathbb{R}^n$, thus $dimE = n-1$ and every column of M belongs to E. Now, suppose that $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is equal to the $n-1$ identical columns of M.

We have to prove that, there exists an additive TU-game (N, v) such that,$M_v = M \Leftrightarrow$ there exists $v(1), v(2), \cdots, v(n)$ such that $\frac{v(i)}{n} - \frac{\sum_{j \neq i} v(j)}{n(n-1)} = x_i$ for all $i = 1, 2, \cdots, n$ (1)

$$(1) \Leftrightarrow \frac{1}{n} \begin{pmatrix} 1 & -\frac{1}{n-1} & \cdots & -\frac{1}{n-1} \\ -\frac{1}{n-1} & 1 & \cdots & -\frac{1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n-1} & -\frac{1}{n-1} & \cdots & 1 \end{pmatrix} \begin{pmatrix} v(1) \\ v(2) \\ \vdots \\ v(n) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Consider F the subspace of $\mathbb{R}^n$ generated by the n columns of the matrix A such that :

$$A = \begin{pmatrix} 1 & -\frac{1}{n-1} & \cdots & -\frac{1}{n-1} \\ -\frac{1}{n-1} & 1 & \cdots & -\frac{1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n-1} & -\frac{1}{n-1} & \cdots & 1 \end{pmatrix} = -\frac{1}{n-1} \begin{pmatrix} -n-1 & 1 & \cdots & 1 \\ 1 & -n-1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -n-1 \end{pmatrix}$$

To obtain the exixtence of $v(1)$, $v(2)$, $\cdots$, $v(n)$, it is sufficient to prove that $E = F$. Since $F \subseteq E$, we need only to prove that $dimF = rank(A) = n-1 = dimE$.

Let us prove that $rank(A) = n-1$ by induction on n.

If $n = 2$

$$A = \begin{pmatrix} -1 & +1 \\ +1 & -1 \end{pmatrix} \Rightarrow rank(A) = rank \begin{pmatrix} -1 & +1 \\ 0 & 0 \end{pmatrix} = 1 = n-1.$$

Suppose that the property is true for n and let us prove that the property holds for $n+1$,

$$\text{with } A = \begin{pmatrix} -n & 1 & \cdots & 1 \\ 1 & -n & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -n \end{pmatrix},$$

$$rank(A) = rank \begin{pmatrix} -n & 1 & \cdots & 1 \\ 0 & \frac{1}{n} - n & \cdots & \frac{1}{n} + 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{n} + 1 & \cdots & \frac{1}{n} - n \end{pmatrix}$$

$$= 1 + rank \begin{pmatrix} \frac{1}{n} - n & \frac{1}{n} + 1 & \cdots & \frac{1}{n} + 1 \\ \frac{1}{n} + 1 & \frac{1}{n} - n & \cdots & \frac{1}{n} + 1 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} + 1 & \frac{1}{n} + 1 & \cdots & \frac{1}{n} - n \end{pmatrix}$$

$$= 1 + rank \begin{pmatrix} \frac{(1-n)(n+1)}{n} & \frac{(n+1)}{n} & \cdots & \frac{(n+1)}{n} \\ \frac{(n+1)}{n} & \frac{(1-n)(n+1)}{n} & \cdots & \frac{(n+1)}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{(n+1)}{n} & \frac{(n+1)}{n} & \cdots & \frac{(1-n)(n+1)}{n} \end{pmatrix}$$

$$= 1 + rank \begin{pmatrix} -(n-1) & 1 & \cdots & 1 \\ 1 & -(n-1) & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -(n-1) \end{pmatrix}$$

=1+(n-1)

=n ■

Proof of proposition 3.3

Suppose that the $n-1$ columns of the matrix M_v of the game (N, v) are identical. According to the proposition 3.1, there exits an additive game (N, w) such that :

$M_w = M_v$
$\Leftrightarrow M_{w-v} = 0$
$\Leftrightarrow u = v - w$, (N, u) is weakly symmetric
$\Leftrightarrow v = w + u$ with w additive and u weakly symmetric.

■

Proof of proposition 3.4

(1) $\Rightarrow$ **2))** See property 3 of the matrix M_v.

(2) $\Rightarrow$ **3))** If $M_v = 0$ then for any LES value φ, $\varphi_i(N, v) = \frac{V(N)}{n}$ for all $i \in N$. Thus for any two LES value φ and ϕ, $\varphi_i(N, v) = \phi_i(N, v) = \frac{V(N)}{n}$ for all $i \in N$.

(3) $\Rightarrow$ **4))** if for any player $i \in N$ and for any LES value φ and ϕ, we have $\varphi_i(N, v) = \phi_i(N, v)$. Thus for any LES value φ and for $\phi = E$ (Egalitarian Value), we have $\varphi_i(N, v) = E_i(N, v) = \frac{V(N)}{n}$.

(4) $\Rightarrow$ **2))** If for any $i \in N$ and for any LES value ϕ we have $\phi_i(N, v) = \frac{V(N)}{n}$. Thus, using the matrix representation, we have for any $V \in \mathbb{R}^{n-1}$ $M_v V = 0 \Rightarrow M_v = 0 \Leftrightarrow (N, v)$ is weakly symmetric. ■

Proof of proposition 3.5

(1) $\Rightarrow$ **2))** (N, v) is a weakly linear $\Leftrightarrow$ the binary relation $\succcurlyeq_u$ defined by $i \succcurlyeq_u j$ iff, for all $k = 1, 2, \cdots, N-1$, $u_k(i) \geq u_k(j)$ is a complete preordering in N.

Noting that, $u_k(i) \geq u_k(j) \Leftrightarrow m_{ik} \geq m_{jk} \Leftrightarrow \frac{m_{ik} - m_k}{n-k} \geq \frac{m_{jk} - m_k}{n-k} \Leftrightarrow M_v(i, k) \geq M_v(j, k)$.

This proves that, the preordering $\succcurlyeq_u$ is equivalent to the canonical preordering in the subset of the n vectors line of the matrix M_v.

Thus $\succcurlyeq_u$ complete in $N \Leftrightarrow$ the canonical preordering is complete in the subset of the n

vectors line of M_v.
($\mathbf{2}$) $\Rightarrow$ $\mathbf{3}$)) Suppose that, for $k = 1, 2, \cdots, n-1$, we have $i \succ_k j$ $(i, j \in N)$ and there exists $k' \neq k$ $(k' = 1, 2, \cdots n-1)$ such that $j \succ_{k'} i$. Thus $M_v(i,k) > M_v(j,k)$ and $M_v(i,k') > M_v(j,k')$.
Thus the vector of $\mathbb{R}^{n-1}$ corresponding to the line of i and the vector of the line of j are not comparable by the canonical preordering of $\mathbb{R}^{n-1}$. This is in contradiction with ($\mathbf{2}$).
Hence, $i \succ_k j \Rightarrow i \succcurlyeq_{k'} j$ for all $k, k' = 1, 2, \cdots, n-1$.
($\mathbf{3}$) $\Rightarrow$ $\mathbf{2}$)) Suppose that (2) is not satisfied, that is, the canonical preordering is not complete in the set of the n vectors line of M_v. Therefore, there exists two vectors line, corresponding to two players i and j, which are not comparable. Thus, there exists k and k' such that $M_v(i,k) > M_v(j,k)$ and $M_v(i,k') < M_v(j,k')$, hence $i \succ_k j$ and $i \prec_{k'} j$. This implies that $\succcurlyeq_k$ and $\succcurlyeq_{k'}$ are not compatible, which is in contradiction with ($\mathbf{3}$). ■

References

Chameni, Nembua, C., Andjiga, Nicolas, 2008. Linear, efficient and symmetric values for TU-games. *Economic Bulletin* 3 (71), 1-10.

Chameni Nembua, C., 2012. Linear efficient and symmetric values for TU-games : sharing the joint gain of cooperation. *Games and Economic Behavior* 74, 431-433.

Driessen, Theo, S., Funaki, Yukihiko, 1991. Coincidence of and Collinearity between game theoretic solutions.*OR Spektrum* 13, 15-30.

Driessen, Theo, S., Radzik Tadeusz, (2003). Extension of Hart and Mas-Colell's consistensy toefficient, linear, and symmetric values for TU-games. In : Petrosyan LA, Yeung DWK (eds) ICM millennium lectures on games, Springer, Heidelberg, Germany, 147-166. Volume dedicated to the international congress of mathematicians, game theory and applications satellite conference, August 14-17, 2002, Quindao, China.

Freixas, Josep, 2010. On ordinal equivalence of the Shapley and Banzhaf values for cooperative games. *International Journal of Game Theory* 39, 513-527.

Ju, Yuan, Born, Peter, Ruys, Pieter, H., 2007.The Concensus Value : a new solution concept for cooperative games. Soc. *Choice Welfare* 28, 685-703.

Nowak, Andrzej, S., Radzik, Tadeusz, 1994. A solidarity value for n-person transferable utility games. *International Journal of Game Theory* 23, 43-48.

Van Den Brink, R., 2007. Null or nullifying players : the difference between Shapley value and the equal division solutions. *J. Econ. Theory* 136, 767–775.

Shapley, Lloyd, S., 1953. A Value for n-Person Games. *Princeton University Press*, 307-317.